Envelopes, indicators and conservativeness

Andrés Cordón-Franco, Alejandro Fernández-Margarit, and **F.Félix Lara-Martín**

Departamento de Ciencias de la Computación e Inteligencia Artificial, Facultad de Matemáticas, Universidad de Sevilla, C/ Tarfia s/n, 41012 Sevilla, Spain

Key words Collection principles, initial segments, indicators, envelopes, conservation results.

well known theorem proved (independently) by J. Paris and H. Friedman states that $\mathbf{B}\Sigma_{n+1}$ (the fragment of Arithmetic given by the collection scheme restricted to Σ_{n+1}-formulas) is a Π_{n+2}-conservative extension of $\mathbf{I}\Sigma_n$ (the fragment given by the induction scheme restricted to Σ_n-formulas). In this paper, as a continuation of our previous work on collection schemes for $\Delta_{n+1}(T)$-formulas (see [4]), we study a general version of this theorem and characterize theories T such that $T + \mathbf{B}\Sigma_{n+1}$ is a Π_{n+2}-conservative extension of T. We prove that this conservativeness property is equivalent to a model-theoretic property relating Π_n-envelopes and Π_n-indicators for T. The analysis of Σ_{n+1}-collection we develop here is also applied to Σ_{n+1}-induction using Parsons' conservativeness theorem instead of Friedman-Paris' theorem.

As a corollary, our work provides new model-theoretic proofs of two theorems of R. Kaye, J. Paris and C. Dimitracopoulos (see [8]): $\mathbf{B}\Sigma_{n+1}$ and $\mathbf{I}\Sigma_{n+1}$ are Σ_{n+3}-conservative extensions of their parameter free versions, $\mathbf{B}\Sigma^-_{n+1}$ and $\mathbf{I}\Sigma^-_{n+1}$.

1 Introduction

In studying the class of provably total functions of a fragment of Arithmetic a useful idea is to isolate a subclass of those functions with good growth properties. Typically, one obtains a sequence of increasing functions of ever-faster rate of growth such that each provably total function is bounded by some function in the sequence. This kind of constructions not only provides descriptions of the class of provably total functions but also of the class of the Π_2-consequences of the considered fragment. In this way, they also constitute a useful tool to obtain Π_2-conservation results.

A basic principle involved in those arguments is the collection principle. In one of its formulations this principle asserts that every total function can be bounded by an increasing one. Hence, there exist natural relations between conservation properties for collection principles and conditions under which the above mentioned constructions can be carried out. In a model-theoretic setting previous ideas occur notably in the theory of *indicators* and *envelopes* of fragments of Arithmetic but, in some aspects, only in an implicit and ad hoc form. In this paper we shall make explicit this relationship and develop a general theory on this topic.

Indicators were introduced by L. Kirby and J. Paris in [9] (see also [12]) as a tool to study the distribution in models of Peano Arithmetic of initial segments satisfying a given property. Paris and Kirby (see [14] or [10]) developed the theory of indicators in countable models of Peano Arithmetic. However, as noticed, for example, in [11], it is possible to develop a general theory of indicators in countable models of theories weaker than Peano Arithmetic. Indeed, in [15] Paris stated the main results on indicators for theories extending $\mathbf{I}\Sigma_1$, see [15, Theorem 0]. Here, we shall consider a more concrete (but still general enough) notion of indicator in (countable) models of theories that can prove a weak form of collection. In [4], these theories have been called Π_n-*functional*.

On the other hand, the notion of an envelope was first isolated by K. McAloon in [13]. There envelopes were used to give an alternative treatment to some results previously formulated in terms of indicators. However, envelopes were implicit in Paris' first works on indicators (see [14]) as a tool to characterize the class of provably total recursive functions of a theory.

Since that early work it is well known that it is possible to obtain an envelope from an indicator. A general version of this fact, for indicators for theories extending $\mathbf{I}\Sigma_1$, is proved in P. Hájek and P. Pudlák's book [5]. Conversely, in certain cases, it is possible to get an indicator from an envelope. This suggests that in some sense envelopes and indicators are equivalent notions. The principal aim of this work is to provide a precise formulation of this equivalence by explicitly stating the conditions under which it holds.

In order to get those conditions a general theory of envelopes is needed. Such a general theory was developed in [4] elaborating on the material on indicators included by R. Kaye in his book [7]. There he introduced the concept of a Π_n-envelope of a theory T in another theory T_0. As it is apparent from our work in [4], Π_n-functional theories are the proper context to deal with Π_n-envelopes. Roughly speaking a theory is Π_n-functional if its Π_{n+2}-consequences can be described by $\mathbf{I}\Sigma_n$ plus a class of nondecreasing Π_n-functions. In this case, (if there exists) a Π_n-envelope provides a uniform description of such a family of nondecreasing functions. Here we study the equivalence between Π_n-envelopes and Π_n-indicators in this context. The key property to obtain that equivalence is a general version of Friedman-Paris' Π_{n+2}-conservativeness theorem between $\mathbf{B}\Sigma_{n+1}$ and $\mathbf{I}\Sigma_n$. That property was considered in [4], where we asked the following question (see [4, Problem 7.3]):

$(\mathbf{P})$ Let T be a theory such that $T + \mathbf{B}\Sigma_{n+1}$ is consistent. Are the following conditions equivalent?
- (a) T is a Π_n-functional theory.
- (b) $T + \mathbf{B}\Sigma_{n+1}$ is Π_{n+2}-conservative over T.

Since property (b) will play a central role through this paper, we introduce the following definition.

Definition 1.1 We say that a theory T is $\Pi^{\mathbf{B}}_{n+2}$*-conservative* if $T + \mathbf{B}\Sigma_{n+1}$ is Π_{n+2}-conservative over T; that is, $\mathrm{Th}_{\Pi_{n+2}}(T + \mathbf{B}\Sigma_{n+1}) = \mathrm{Th}_{\Pi_{n+2}}(T)$.

Using this definition Friedman-Paris' theorem can be stated as: $\mathbf{I}\Sigma_n$ is $\Pi^{\mathbf{B}}_{n+2}$-conservative. L. Beklemishev showed in [2, Theorem 3.2] that if T is a Π_{n+3}-axiomatizable theory (proving that exponential function is total) and it is closed under Σ_{n+1}-collection rule, then it is $\Pi^{\mathbf{B}}_{n+2}$-conservative. (Collection rule is defined by

$$\text{CR}: \qquad \frac{\forall x \exists y\, \varphi(x,y)}{\forall z \exists u\, (\forall x \leq z)\, (\exists y \leq u)\, \varphi(x,y)}.$$

For a theory T, $T + \Sigma_{n+1}$-CR denotes the closure of T under first-order logic and applications of CR restricted to formulas $\varphi \in \Sigma_{n+1}$.)

It is easy to see that every $\Pi^{\mathbf{B}}_{n+2}$-conservative theory is closed under Σ_{n+1}-collection rule. Therefore, $\Pi^{\mathbf{B}}_{n+2}$-conservativeness is equivalent to the closure under Σ_{n+1}-collection rule, for Π_{n+3}-axiomatizable theories. This fact is used in [2] to derive a proof-theoretic proof of Friedman-Paris' theorem. In addition, it also provides a partial answer to problem $(\mathbf{P})$, as long as the theories closed under Σ_{n+1}-collection rule are exactly the Π_n-functional ones (see Theorems 1.3 and 1.4 below).

In this paper we address a general answer to problem $(\mathbf{P})$ and provide a model-theoretic analysis of this question. Our work relates $\Pi^{\mathbf{B}}_{n+2}$-conservativeness to a model-theoretic property of Π_n-envelopes. Roughly speaking, that property (that we call Π_n-IND) asserts that Π_n-envelopes define (Π_n-)indicators, for precise definitions see Section 2. The analysis of Π_n-IND was initiated in [4] in connection with our work on the collection scheme for $\Delta_{n+1}(T)$-formulas. Here, we continue that analysis pointing out a close relation between conservation results on Σ_{n+1}-collection (including both axioms and rules) and natural conditions to develop a general theory of indicators and envelopes.

In order to present more precisely the results obtained in this paper and their connection with the above conservation property we state some general notation and recall some concepts and results from [4]. We shall work in the first-order language of Arithmetic, $\mathcal{L}$. As usual, we denote by $\mathbf{exp}$ a Π_2-sentence expressing that exponential function is total. As above, T will denote a consistent theory in the language $\mathcal{L}$. For such a theory we introduced the classes of formulas

$$\Delta_{n+1}(T) = \{\varphi(x) \in \Sigma_{n+1} \,:\, \text{there exists } \psi(x) \in \Pi_{n+1},\, T \vdash \varphi \leftrightarrow \psi\}.$$

When the schemes of induction and minimization are restricted to these classes of formulas we obtain the theories $\mathbf{I}\Delta_{n+1}(T)$ and $\mathbf{L}\Delta_{n+1}(T)$. We also consider the following version of the collection scheme:

$$\mathbf{B}^*\Delta_{n+1}(T) = \mathbf{I}\Delta_0 + \{B_{\varphi,x,y} \,:\, \varphi \in \Pi_n, \exists y\, \varphi(x,y) \in \Delta_{n+1}(T)\},$$

where $B_{\varphi,x,y}(z,\vec{v}\,)$ denotes the collection axiom for $\varphi(x,y,\vec{v}\,)$ with respect to x, y; that is, the formula

$$(\forall x \leq z)\exists y\, \varphi(x,y,\vec{v}\,) \rightarrow \exists u\, (\forall x \leq z)\, (\exists y \leq u)\, \varphi(x,y,\vec{v}\,).$$

In [4] and [3] properties and relative strength of these schemes were studied. We introduced, among others, the following notion:

Definition 1.2 We say that a theory T *has* Δ_{n+1}*-collection* if $T \Rightarrow \mathbf{B}^*\Delta_{n+1}(T)$.

The main feature of theories having Δ_{n+1}-collection is that for every such a theory T, the class of $\Delta_{n+1}(T)$-formulas is closed in T under bounded quantification. From this basic fact we derived in [4] that if T has Δ_{n+1}-collection, then T extends $\mathbf{L}\Delta_{n+1}(T)$ and $\mathbf{I}\Delta_{n+1}(T)$.

It is not difficult to prove (see [4, Remark 2.8]) that an extension of $\mathbf{I}\Delta_0$ has Δ_{n+1}-collection if and only if it is closed under Σ_{n+1}-collection rule. Besides this proof-theoretic characterization of Δ_{n+1}-collection we obtained a "functional" characterization more amenable to model-theoretic investigation. This is precisely the notion of a Π_n-functional theory. We need some notation to state it.

In what follows, Γ will denote a class of formulas of $\mathcal{L}$ with two free variables, x, y say. For a formula $\varphi(x,y)$, $\mathrm{IPF}(\varphi)$ will denote a formula expressing that $\varphi(x,y)$ defines a nondecreasing (partial) function. Let

$$\Gamma^* = \{\mathrm{IPF}(\varphi(x,y)) \,:\, \varphi(x,y) \in \Gamma\} \cup \{\forall x \exists! y\, \varphi(x,y) \,:\, \varphi(x,y) \in \Gamma\}.$$

Let us observe that if $\Gamma \subseteq \Pi_n$, then Γ^* is a set of Π_{n+2}-sentences expressing that each formula in Γ defines a nondecreasing total function. We consider the following concepts:

1. We say that $\Gamma \subseteq \Pi_n$ is a Π_n*-functional class* if $\mathbf{I}\Sigma_n + \Gamma^*$ is consistent.
2. A theory T is Π_n*-functional* if there exists a Π_n-functional class Γ such that

$$\mathrm{Th}_{\Pi_{n+2}}(T) = \mathrm{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma^*).$$

The class of formulas $\Delta_{n+1}(T)$ is determined by the Π_{n+2}-consequences of T, so it is natural to look for a characterization of the theories having Δ_{n+1}-collection which relies on the description of $\mathrm{Th}_{\Pi_{n+2}}(T)$. This is achieved through the notion of a Π_n-functional theory. The next theorem sums up the equivalence among the three notions we have introduced (see [4, Theorem 3.5 and Remark 2.8]). For $n = 0$ a similar result was obtained by L. Beklemishev in [1, Proposition 5.4].

Theorem 1.3 *Let T be a consistent theory. The following conditions are equivalent:*

1. *T has Δ_{n+1}-collection.*
2. *T is Π_n-functional.*
3. *T is an extension of $\mathbf{I}\Delta_0$ closed under Σ_{n+1}-collection rule; that is, $T \Leftrightarrow T + \Sigma_{n+1}$-*CR.

In view of this result and our previous remarks on Beklemishev's work ([2, Theorem 3.2]) we can state the following theorem.

Theorem 1.4 *Let T be a Π_{n+3}-axiomatizable and consistent theory. The following conditions are equivalent:*

(1) *T is Π_n-functional.*

(2) *T is $\Pi^{\mathbf{B}}_{n+2}$-conservative.*

In this paper we present a model-theoretic proof of this result. In [4, Remark 3.7.2], we proved this theorem for Π_{n+2}-axiomatizable theories. Here, we give an alternative proof for the Π_{n+2} case and we extend it to Π_{n+3}-axiomatizable theories as an application of our work on Π_n-envelopes. Since $T + \mathbf{B}\Sigma_{n+1}$ has Δ_{n+1}-collection, part (2) $\Rightarrow$ (1) easily follows from Theorem 1.3. Hence, our concern here will be the converse. Moreover, we will show that Theorem 1.4 is optimal with respect to the syntactical complexity of the theory (see Remark 3.10) and this answers problem ($\mathbf{P}$).

The paper is organized as follows. In Section 2 we introduce the basic notions we will deal with in this paper: Π_n-envelopes, Π_n-indicators and the property Π_n-IND. We describe the relationship between property Π_n-IND and $\Pi^{\mathbf{B}}_{n+2}$-conservativeness and obtain the results on Π_n-envelopes that constitute the real core of this paper.

Firstly, without restrictions on the syntactical complexity of the theory, we characterize recursively axiomatizable and Π_n-functional theories (proving that exponential function is total) which are $\Pi^{\mathbf{B}}_{n+2}$-conservative. They are, namely, those theories whose Π_n-envelopes define (well-behaved) Π_n-indicators. The next theorem sums up the results obtained in Theorems 2.4 and 2.7.

Theorem 1.5 *Let T be a recursively axiomatizable and Π_n-functional theory such that $T \vdash \mathbf{exp}$. The following conditions are equivalent:*

1. *T is $\Pi^{\mathbf{B}}_{n+2}$-conservative.*
2. *There exists a Π_n-envelope of T in T satisfying* Π_n-IND.
3. *There exists a well-behaved Π_n-indicator, $Y(x, y) = z$, for T in T such that for all $k \in \omega$,*

$$T \vdash \forall x \exists y\, (Y(x, y) > k).$$

Secondly (see Theorem 2.10), we prove (a slightly sharper variation of) the following theorem.

Theorem 1.6 *Let T be a Π_n-functional theory such that $T \vdash \mathbf{exp}$. If T is Π_{n+3}-axiomatizable, then every Π_n-envelope of T in T satisfies* Π_n-IND.

From these results Theorem 1.4 can be derived for theories proving that exponential function is total (see Corollary 2.11). However, because known existence theorems for envelopes use the exponential function, the proof of Theorem 1.4 in its whole generality requires some additional work that will be done in Section 3. To this end, the central result is Theorem 3.7 there. The proof of this theorem deals with the notion of a pseudo-Π_n-envelope and provides a model-theoretic tool for the analysis of collection rule. As we have noticed, the first condition of Theorem 1.4 is equivalent to the closure of T under Σ_{n+1}-collection rule; so, the work presented here (and in [4]) can be considered as a model-theoretic analysis of collection rule (and, therefore, it is closely tied to Beklemishev's work in [2]). As a matter of fact, from Theorem 3.7 we derive [2, Theorem 3.2] (proved there by a cut-elimination argument) and a new model-theoretic proof of a conservativeness result between $\mathbf{B}\Sigma_{n+1}$ and $\mathbf{B}\Sigma^-_{n+1}$ firstly proved by R. Kaye, J. Paris and C. Dimitracopoulos (see [8, Theorem 2.4]).

Finally, in Section 4 we apply the previously developed analysis of Σ_{n+1}-collection rule to Σ_{n+1}-induction. This can be done with minor changes. Now the basic result that plays the role of Friedman-Paris' theorem is Parsons' conservation theorem on Σ_{n+1}-induction rule. Induction rule is given by

$$\text{IR}: \qquad \frac{\varphi(0), \quad \forall x\, (\varphi(x) \rightarrow \varphi(x+1))}{\forall x\, \varphi(x)}.$$

As for collection rule, $T + \Sigma_{n+1}$-IR denotes the closure of T under first-order logic and applications of IR restricted to formulas $\varphi \in \Sigma_{n+1}$. C. Parsons proved that (see [16]):

Theorem 1.7 (Parsons) *$\mathbf{I}\Sigma_{n+1}$ is a Π_{n+2}-conservative extension of* $\mathbf{I}\Delta_0 + \Sigma_{n+1}$-IR.

Since $\mathbf{I}\Sigma_n$ is equivalent to $\mathbf{I}\Delta_0 + \Sigma_{n+1}$-CR (see [2, Corollary 4.2]), Friedman-Paris' theorem can be stated as: $\mathbf{B}\Sigma_{n+1}$ is a Π_{n+2}-conservative extension of $\mathbf{I}\Delta_0 + \Sigma_{n+1}$-CR.

So, it is natural to introduce a notion of $\Pi^{\mathbf{I}}_{n+2}$-conservativeness and study it along the lines just described for the collection case.

Definition 1.8 A theory T is $\Pi^{\mathbf{I}}_{n+2}$-*conservative* if $T + \mathbf{I}\Sigma_{n+1}$ is Π_{n+2}-conservative over T.

The main result on Σ_{n+1}-induction rule proved here is the following one:

Theorem 1.9 (see Theorem 4.1) *Let T be a Π_{n+3}-axiomatizable theory. Then we have:*

$$T \text{ is closed under } \Sigma_{n+1}\text{-induction rule} \Leftrightarrow T \text{ is } \Pi^{\mathbf{I}}_{n+2}\text{-conservative}.$$

As for collection, Theorem 1.9 is optimal with respect to the syntactical complexity of the theory. It is easy to offer examples along the lines given in Remark 3.10 for the collection case. Moreover, Theorem 1.9 provides a new proof of [8, Theorem 2.1]: $\mathbf{I}\Sigma_{n+1}$ is a Σ_{n+3}-conservative extension of $\mathbf{I}\Sigma^-_{n+1}$.

The work presented in this paper shows the relationship between conservation (for $\mathbf{B}\Sigma_{n+1}$ and $\mathbf{B}\Sigma^-_{n+1}$) and consistency of theories of the form $T + \mathbf{B}\Sigma_{n+1}$, where T is Π_n-functional. We conclude this section with a related question.

Problem 1.10 *Let $\varphi \in \Pi_{n+2}$ be a sentence such that $\mathbf{L}\Delta^-_{n+1} + \varphi$ (or $\mathbf{L}\Delta^-_{n+1} + \varphi + \mathbf{I}\Sigma_n$) is a consistent theory. Is there a Π_n-functional theory T such that $T \vdash \varphi$?*

If yes, then $\mathbf{B}\Sigma_{n+1}$ is Σ_{n+2}-conservative over $\mathbf{L}\Delta^-_{n+1}$ (or over $\mathbf{L}\Delta^-_{n+1} + \mathbf{I}\Sigma_n$). Let us prove it.

Let $\varphi \in \Sigma_{n+2}$ be a sentence such that $\mathbf{B}\Sigma_{n+1} \vdash \varphi$. If $\mathbf{L}\Delta^-_{n+1}(+\mathbf{I}\Sigma_n) \nvdash \varphi$, then $\mathbf{L}\Delta^-_{n+1} + \neg\varphi\ (+\mathbf{I}\Sigma_n)$ is consistent. By hypothesis, there exists a Π_n-functional theory T such that $T \vdash \neg\varphi$. Since $\neg\varphi$ is Π_{n+2}, we may assume that T is Π_{n+2}-axiomatizable. Hence, by Theorem 1.4, $T + \mathbf{B}\Sigma_{n+1}$ is consistent. Since $T \vdash \neg\varphi$ and $\mathbf{B}\Sigma_{n+1} \vdash \varphi$, this gives the desired contradiction.

2 Π_n-envelopes and Π_n-indicators

In what follows we introduce the basic notions we will deal with in this paper: Π_n-envelopes, Π_n-indicators and the property Π_n-IND. We describe the relationship between property Π_n-IND and $\Pi^{\mathbf{B}}_{n+2}$-conservativeness and obtain the main results relating Π_n-envelopes and Π_n-indicators.

2.1 The property Π_n-IND

In this section we give a short review of some relevant facts of the theory of Π_n-envelopes as developed in [4]. In this way, in addition to present the exact definitions of the notions involved we will make explicit the connection between Π_n-envelopes and $\Pi^{\mathbf{B}}_{n+2}$-conservativeness.

To get a uniform description of Π_n-functional classes, in [4] we introduced the concept of a Π_n-envelope.

Definition 2.1 Let T, T_0 be consistent theories, $\varphi(u,x,y) \in \Sigma_{n+1}$ and $\Gamma_\varphi = \{\varphi(k,x,y) : k \in \omega\}$. We say:

(1) $\varphi(u,x,y)$ is a Π_n*-q-envelope of* T *in* T_0 if

(a) $T \vdash \Gamma^*_\varphi$; that is, for all $k \in \omega$, $T \vdash \mathrm{IPF}(\varphi(k,x,y)) \wedge \forall x \exists y\, \varphi(k,x,y)$;

(b) for all $k \in \omega$, $T_0 \vdash \varphi(k+1,x,y) \rightarrow (\exists z < y)\, \varphi(k,x,z)$.

(2) $\varphi(u,x,y)$ *satisfies* Π_n-ENV *for* T *and* T_0 if for each $\psi(x,y) \in \Pi_n$ such that $T \vdash \forall x \exists y\, \psi(x,y)$, there exists $k \in \omega$ such that $T_0 \vdash \varphi(k,x,y) \rightarrow (\exists z < y)\, \psi(x,z)$. (By a standard quantifier contraction argument this property also holds for $\psi(x,y) \in \Sigma_{n+1}$.)

(3) $\varphi(u,x,y)$ *is a* Π_n*-envelope of* T *in* T_0 if $\varphi(u,x,y)$ is a Π_n-q-envelope of T in T_0 and satisfies Π_n-ENV for T and T_0.

This definition describes a very general notion and, at a first sight, the role of T_0 could be seen as artificial; so, some comments are in order. Actually, in most of the cases, it can be assumed that T_0 is $\mathbf{I}\Sigma_n + \mathbf{exp}$. This assumption would simplify the statements of the results in this section; and the reader can assume that this is the case until Section 3. However, when dealing with the weaker notion of a pseudo-Π_n-envelope in Section 3, it will be useful to consider different choices of T_0. So, we prefer to make explicit the role of T_0 from the very beginning.

Let us denote the formula $\varphi(u,x,y)$ by $F_u(x) = y$. Then, roughly speaking, a Π_n-envelope is a Σ_{n+1}-definable sequence of provably total nondecreasing functions, $\{F_k\}_{k\in\omega}$. Its essential feature is property Π_n-ENV expressed in item (2): Every Π_n-definable (or even Σ_{n+1}-definable) function provably total in T is (provably in T_0) majorized by a function in the sequence. The remaining properties stated in (1) are of technical character. Sequences satisfying these properties have been called Π_n-q-envelopes in order to discuss alternatives to Π_n-ENV. The following property can be considered as a model-theoretic strong version of Π_n-ENV. It captures a common property of envelopes defined from indicators.

Definition 2.2 Let $\varphi(u,x,y) \in \Sigma_{n+1}$. We say $\varphi(u,x,y)$ *satisfies* Π_n-IND *for* T *and* T_0 if for every $\mathfrak{A} \models T_0$ countable, nonstandard and $a, b \in \mathfrak{A}$, the following conditions are equivalent:

IND-(i): For all $k \in \omega$, $\mathfrak{A} \models (\exists y < b)\, \varphi(k,a,y)$.

IND-(ii): There exists $I \models T$ such that $I \prec^{\mathrm{e}}_n \mathfrak{A}$ and $a < I < b$.

In [4] we obtained conditions under which Π_n-ENV is equivalent to Π_n-IND. As we noticed there, if T_0 extends $\mathbf{I}\Sigma_n$, then every Π_n-q-envelope of T in T_0 that satisfies Π_n-IND is a Π_n-envelope. The converse also holds for $\Pi^{\mathbf{B}}_{n+2}$-conservative theories. We shall reformulate these facts in the following two results.

Proposition 2.3 *Assume that* $T \Rightarrow T_0 \Rightarrow \mathbf{I}\Sigma_n + \mathbf{exp}$ *and* T *is a consistent recursively axiomatizable theory. The following conditions are equivalent:*

(1) T *is* Π_n*-functional.*

(2) *There exists a* Π_n*-envelope of* T *in* T_0.

P r o o f.

(1) ⇒ (2): Since $T \vdash \mathbf{exp}$ and it is recursively axiomatizable, this part follows by [4, Theorem 5.8].

(2) ⇒ (1): By Theorem 1.3 it is enough to prove that T is closed under Σ_{n+1}-collection rule. Therefore, let $\psi(x, y) \in \Sigma_{n+1}$ such that $T \vdash \forall x \exists y\, \psi(x, y)$ and $\varphi(u, x, y) \in \Sigma_{n+1}$ a Π_n-envelope of T in T_0. Then, there is $k \in \omega$ such that

$$T_0 \vdash \varphi(k, x, y) \rightarrow (\exists z < y)\, \psi(x, z).$$

Since $T \vdash \forall x \exists y\, \varphi(k, x, y)$ and T extends T_0, we have

$$(\star) \qquad T \vdash \forall x \exists y \exists z\, (z < y \wedge \varphi(k, x, y) \wedge \psi(x, z)).$$

In addition, T also proves that $\varphi(k, x, y)$ defines a nondecreasing function, so, from $(\star)$ it follows that

$$T \vdash \forall u \exists v\, (\forall x \leq u)\, (\exists y \leq v)\, \psi(x, y),$$

as required. □

Theorem 2.4 *Assume that $T \Rightarrow T_0 \Rightarrow \mathbf{I}\Sigma_n + \mathbf{exp}$ and T is a consistent recursively axiomatizable theory. The following conditions are equivalent:*

(1) *T is $\Pi^{\mathbf{B}}_{n+2}$-conservative.*

(2) *There exists a (and therefore every) Π_n-envelope of T in T_0 that satisfies* Π_n-IND.

P r o o f. This result is essentially proved in [4, Theorem 5.5]. We will include the proof here for the sake of completeness.

(1) ⇒ (2): In the proof we use some basic results on recursive saturation and the standard system of a model, $\mathrm{SSy}(\mathfrak{A})$, see [7] for details. By (1), T is Π_n-functional; so, by Proposition 2.3 there exists $\varphi(u, x, y) \in \Sigma_{n+1}$, a Π_n-envelope of T in T_0. In order to show that $\varphi(u, x, y)$ satisfies property Π_n-IND it is enough to see that IND-(i) ⇒ IND-(ii). We follow the proof of [7, Theorem 14.7]. Let $\mathfrak{A} \vDash T_0$ countable, nonstandard and let $a, b \in \mathfrak{A}$ such that $\mathfrak{A} \vDash (\exists y < b)\, \varphi(k, a, y)$, for all $k \in \omega$. Let

$$T' = T + \mathbf{B}\Sigma_{n+1} + \mathbf{c} + \{\forall \vec{z}\, \psi(\mathbf{c}, \vec{z}) \,:\, \psi(x, \vec{z}) \in \Sigma_n, \mathfrak{A} \vDash (\forall \vec{z} \leq b)\, \psi(a, \vec{z})\}.$$

By (1), as in [7, proof of Theorem 14.7], it follows that T' is consistent. Since $\mathfrak{A} \vDash \mathbf{I}\Sigma_n + \mathbf{exp}$, the Σ_n-type of a, b in $\mathfrak{A}$ belongs to $\mathrm{SSy}(\mathfrak{A})$; hence,

$$\{\ulcorner \forall \vec{z}\, \psi(\mathbf{c}, \vec{z}) \urcorner \,:\, \psi \in \Sigma_n, \mathfrak{A} \vDash (\forall \vec{z} \leq b)\, \psi(a, \vec{z})\} \in \mathrm{SSy}(\mathfrak{A}).$$

So, as T is recursively axiomatizable, $T' \in \mathrm{SSy}(\mathfrak{A})$. Since $\mathrm{SSy}(\mathfrak{A})$ is a Scott set, there exists $\mathfrak{B} \vDash T'$ countable and $\mathrm{SSy}(\mathfrak{A})$-saturated (hence, $\mathfrak{B}$ is recursively saturated). Let $c = \mathfrak{B}(\mathbf{c})$. Then, for each $\theta(x, \vec{z}) \in \Pi_n$, if $\mathfrak{B} \vDash \exists \vec{z}\, \theta(c, \vec{z})$, then $\mathfrak{A} \vDash (\exists \vec{z} \leq b)\, \theta(a, \vec{z})$. So, by Friedman's theorem, there exists $H : \mathfrak{B} \preceq^{\mathrm{e}}_n \mathfrak{A}$ such that $H(c) = a$ and $b \notin H(\mathfrak{B})$. Let $I = H(\mathfrak{B})$. Then $I \vDash T$, $I \prec^{\mathrm{e}}_n \mathfrak{A}$ and $a < I < b$.

(2) ⇒ (1): Let $\varphi(u, x, y) \in \Sigma_{n+1}$ be a Π_n-envelope of T in T_0 that satisfies Π_n-IND. Let $\psi(x, y) \in \Pi_n$ such that $T + \mathbf{B}\Sigma_{n+1} \vdash \forall x \exists y\, \psi(x, y)$. We must prove that $T \vdash \forall x \exists y\, \psi(x, y)$. Since $T \vdash \forall x \exists y\, \varphi(k, x, y)$ for all $k \in \omega$ and $T \Rightarrow T_0$, it suffices to see that there is $k \in \omega$ such that

$$T_0 \vdash \varphi(k, x, y) \rightarrow (\exists z < y)\, \psi(x, z).$$

Towards a contradiction, assume that for all $k \in \omega$, $T_0 \nvdash \varphi(k, x, y) \rightarrow (\exists z < y)\, \psi(x, z)$. Then, for each $k \in \omega$, there exist $\mathfrak{A}_k \vDash T_0$, and $a, b \in \mathfrak{A}_k$ such that $\mathfrak{A}_k \vDash \varphi(k, a, b) \wedge (\forall z < b)\, \neg\psi(a, z)$. Let $\mathbf{c}$ and $\mathbf{d}$ be two new constants and T' the theory

$$T_0 + (\forall z < \mathbf{d})\, \neg\psi(\mathbf{c}, z) + \{(\exists y < \mathbf{d})\, \varphi(j, \mathbf{c}, y) \,:\, j \in \omega\}.$$

By compactness, T' is consistent. Let $\mathfrak{A}' \vDash T'$ countable, $\mathfrak{A}$ the restriction of $\mathfrak{A}'$ to $\mathcal{L}$, $a = \mathfrak{A}'(\mathbf{c})$ and $b = \mathfrak{A}'(\mathbf{d})$. Then $\mathfrak{A} \vDash T_0$ and, for each $k \in \omega$, $\mathfrak{A} \vDash (\exists y < b)\, \varphi(k, a, y)$. By hypothesis, there exists $I \vDash T$ such that $I \prec^{\mathrm{e}}_n \mathfrak{A}$ and $a < I < b$. As a consequence, we get $I \vDash T + \mathbf{B}\Sigma_{n+1}$; so, there exists $e \in I$ such that $I \vDash \psi(a, e)$. Hence, $\mathfrak{A} \vDash (\exists z < b)\, \psi(a, z)$. Contradiction. □

Remark 2.5 The proof of (1) $\Rightarrow$ (2) in Theorem 2.4 yields:

1. Condition IND-(ii) is equivalent to the following one:

IND-(ii)$'$: There exists $I \models T$ recursively saturated such that $I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $a < I < b$.

This well known fact of the general theory of indicators will be extensively used through this work.

2. If there exists a Π_n-envelope of T in T_0 that satisfies Π_n-IND, then *every* Π_n-envelope of T in T_0 satisfies Π_n-IND.

We close this section with some remarks on a special kind of Π_n-functional classes which are useful in order to get a uniform treatment of Π_n-envelopes. We recall the following definitions from [4].

Let Γ be a Π_n-functional class. We say that Γ is a *strong* Π_n*-functional class* if for every model $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*$ and $I \subset^{\mathrm{e}} \mathfrak{A}$, it holds that

$$I \prec_n^{\mathrm{e}} \mathfrak{A} \Leftrightarrow \text{for each } \varphi(x,y) \in \Gamma \text{ and } a \in I \text{ there exists } b \in I \text{ such that } \mathfrak{A} \models \varphi(a,b).$$

That is, if Γ is a strong Π_n-functional class and $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*$, then every initial segment of $\mathfrak{A}$ closed (in $\mathfrak{A}$) under the family of functions defined by Γ is a Σ_n-elementary substructure of $\mathfrak{A}$ (the converse is obviously true). Notice that the notions of Π_0-functional class and strong Π_0-functional class coincide since each initial substructure of a model of $\mathbf{I}\Delta_0$ is Δ_0-elementary. However, these notions differ for $n > 0$. In [4], we studied properties of strong Π_n-functional classes and proved existence results. In particular, elaborating on a construction by R. Kaye in [6], we showed (see [4, Theorem 5.13]) that for each $n \in \omega$, there exists a Π_n-formula $\mathbb{K}_n(x) = y$ such that:

1. $\mathbf{I}\Sigma_n \vdash \mathrm{IPF}(\mathbb{K}_n) \wedge \forall x \exists y\, (\mathbb{K}_n(x) = y)$.

2. $\Gamma_n = \{\mathbb{K}_n(x) = y\}$ is a strong Π_n-functional class and $\mathbf{I}\Sigma_n \vdash (x+1)^2 \leq \mathbb{K}_n(x)$. So, if $\mathfrak{A} \models \mathbf{I}\Sigma_n$ and I is an initial segment of $\mathfrak{A}$ closed (in $\mathfrak{A}$) under the function defined by $\mathbb{K}_n$, then $I \prec_n^{\mathrm{e}} \mathfrak{A}$.

Strong Π_n-functional classes provide us with a special kind of Π_n-envelopes. Let $\varphi(u,x,y) \in \Pi_n$ be a Π_n-envelope of T in T_0. We say that $\varphi(u,x,y)$ is a *strong* Π_n*-envelope* if Γ_φ is a strong Π_n-functional class. In this definition we restrict ourselves to Π_n-formulas, while our definition of a Π_n-envelope deals with Σ_{n+1}-formulas. This is not an essential restriction. By a straightforward construction, from any Π_n-envelope we can obtain a new Π_n-envelope given by a Π_n-formula (see [4, Remark 5.9]).

Bearing in mind this fact and the formulas $\mathbb{K}_n(x) = y$, it is easy to get strong Π_n-envelopes. Indeed, if T_0 extends $\mathbf{I}\Sigma_n$ and $\varphi(u,x,y) \in \Pi_n$ is a Π_n-envelope of T in T_0, then the following Π_n-formula

$$(u = 0 \wedge \mathbb{K}_n(x) = y) \vee (u > 0 \wedge (\exists y_1, y_2 \leq y)\, (\mathbb{K}_n(x) = y_1 \wedge \varphi(u,x,y_2) \wedge y = y_1 + y_2))$$

is a strong Π_n-envelope of T in T_0. So, in what follows we can assume without loss of generality that the Π_n-envelopes involved are strong or are given by Π_n-formulas.

2.2 Π_n-indicators

In this section we characterize Π_n-functional theories which are $\Pi_{n+2}^{\mathbf{B}}$-conservative by means of indicators, envelopes and the distribution, in countable and recursively saturated models of T, of the initial segments which, in turn, are models of T.

Let $\mathfrak{A}$ be a countable $\mathcal{L}$-structure and Q a class of initial segments. In [9] an indicator for Q in $\mathfrak{A}$ is defined as a formula $Y(x,y) = z$ such that

1. $\mathfrak{A} \models \forall x, y \exists! z\, (Y(x,y) = z)$;

2. for each $a, b \in \mathfrak{A}$, $(\exists I \in Q)\, (I \subset^{\mathrm{e}} \mathfrak{A} \wedge a < I < b) \Leftrightarrow Y(a,b) > \omega$.

We shall consider the following notion of indicator.

Definition 2.6 We say that a Σ_{n+1}-formula $Y(x,y) = z$ is a Π_n*-indicator for* T *in* T_0 if

1. $T_0 \vdash \forall x, y \exists! z\, (Y(x,y) = z)$;

2. for each (countable) $\mathfrak{A} \models T_0$ and $a, b \in \mathfrak{A}$, $(\exists I \models T)\, (I \prec_n^{\mathrm{e}} \mathfrak{A} \wedge a < I < b) \Leftrightarrow Y(a,b) > \omega$.

In order to get a more elegant theory it is usual to consider a stronger notion of indicator, namely well-behaved indicators (see [9] or [7]). We say that $Y(x,y) = z$ is a *well-behaved* Π_n*-indicator* if

$$T_0 \vdash \forall x, y\, (Y(x,y) \leq y) \wedge \forall x, y, x', y'\, [x \leq x' \leq y' \leq y \rightarrow Y(x',y') \leq Y(x,y)].$$

It is easy to prove that if T_0 is a Π_n-functional theory, then there exists a Π_n-indicator for T in T_0 if and only if there exists a well-behaved one. As a matter of fact, for Π_n-functional theories a general theory of Π_n-indicators can be developed as in [9, 12, 14]. The basic results on indicators can be stated in this context as in those papers.

As noticed in the introduction, given a (well-behaved) indicator $Y(x, y) = z$ it is possible to define an envelope $\varphi_Y(u, x, y)$. The following result shows that the converse is also true for Π_n-functional theories: given an envelope $\psi(u, x, y)$, we can get a (well-behaved) indicator $Y_\psi(x, y) = z$. The key property that allows us to get Π_n-envelopes from Π_n-indicators is $\Pi^{\mathbf{B}}_{n+2}$-conservativeness. Along the proof we shall describe explicitly the construction of φ_Y and Y_ψ.

Theorem 2.7 *Let T be a Π_n-functional theory such that $T \vdash \mathbf{exp}$. The following conditions are equivalent:*

(1) *There exists a Π_n-envelope of T in T satisfying* Π_n-IND.

(2) *There exists a well-behaved Π_n-indicator, $Y(x, y) = z$, for T in T such that for all $k \in \omega$,*

$$T \vdash \forall x \exists y\, (Y(x, y) > k).$$

Proof.

(2) ⇒ (1): Let $\varphi_Y(u, x, y)$ be a Σ_{n+1}-formula equivalent (in T) to

$$y = u + \min\{v \geq x \,:\, Y(x, v) > u\}.$$

By definition, for every $k \in \omega$, $T \vdash \varphi_Y(k{+}1, x, y) \rightarrow (\exists z < y)\, \varphi_Y(k, x, z)$. Let us see now that for every $k \in \omega$, $T \vdash \mathrm{IPF}(\varphi_Y(k, x, y))$. Indeed, uniqueness follows from the definition of $\varphi_Y(u, x, y)$, and in order to prove (in T) that $\varphi_Y(k, x, y)$ defines a nondecreasing function it is enough to observe that if $x_1 \leq x_2$ and $\varphi_Y(k, x_2, y_2)$, then $Y(x_2, y_2 - k) > k$ and, since $Y(x, y) = z$ is a well-behaved Π_n-indicator,

$$k < Y(x_2, y_2 - k) \leq Y(x_1, y_2 - k).$$

So, if $\varphi_Y(k, x_1, y_1)$, then $y_1 - k \leq y_2 - k$; hence, $y_1 \leq y_2$.

Let $k \in \omega$. Then $Y(x, y) > k$ is a $\Delta_{n+1}(T)$-formula. Since T is a Π_n-functional theory, $T \Rightarrow \mathbf{L}\Delta_{n+1}(T)$; so, as $T \vdash \forall x \exists y\, (Y(x, y) > k)$, $T \vdash \forall x \exists y\, \varphi_Y(k, x, y)$. This proves that $\varphi_Y(u, x, y)$ is a Π_n-q-envelope of T in T.

Next we show that $\varphi_Y(u, x, y)$ satisfies Π_n-IND for T and T. Let $\mathfrak{A} \models T$ and $a, b \in \mathfrak{A}$ such that for every $k \in \omega$, $\mathfrak{A} \models (\exists y < b)\, \varphi_Y(k, a, y)$. By the definition of $\varphi_Y(u, x, y)$, for each $k \in \omega$ there is $b_k \in \mathfrak{A}$ such that $\mathfrak{A} \models b_k < b \wedge Y(a, b_k) > k$. Since $Y(x, y) = z$ is a well-behaved Π_n-indicator for T in T, for every $k \in \omega$,

$$\mathfrak{A} \models k < Y(a, b_k) \leq Y(a, b).$$

Hence, $Y(a, b) > \omega$. So, there exists $I \models T$ such that $I \prec^{\mathrm{e}}_n \mathfrak{A}$ and $a < I < b$.

(1) ⇒ (2): Let $\varphi(u, x, y) \in \Pi_n$ be a Π_n-envelope of T in T that satisfies Π_n-IND. Let $\psi(u, x, y) \in \Pi_n$ be the formula

$$\mathrm{Seq}(y) \wedge \mathrm{lg}(y) = x + 1 \wedge (\forall j \leq x)\, \varphi(u, j, (y)_j).$$

That is, $\psi(u, x, y)$ expresses that y is a sequence of length $x + 1$ and for each $j \leq x$, its jth component, $(y)_j$, satisfies $\varphi(u, j, (y)_j)$. Then $\psi(u, x, y)$ is a Π_n-envelope of T in T satisfying Π_n-IND. Moreover,

(•) $\quad T \vdash x_1 \leq x_2 \wedge \psi(u, x_2, y_2) \rightarrow (\exists y_1 \leq y_2)\, \psi(u, x_1, y_1).$

Let $Y_\psi(x, y) = z$ be a Σ_{n+1}-formula equivalent (in T) to

$$[y \leq x \wedge z = 0] \vee [x < y \wedge z = \max\{w \leq y \,:\, (\forall u < w)\, (\exists v \leq y)\, \psi(u, x, v)\}].$$

Let us see that $Y_\psi(x, y) = z$ is a well-behaved Π_n-indicator for T in T.

First of all, observe that, as $T \Rightarrow \mathbf{I}\Sigma_n$, $T \vdash \forall x, y \exists! z\, (Y_\psi(x, y) = z)$. Now, let $\mathfrak{A} \models T$ countable and $a, b \in \mathfrak{A}$. Then

$$(\exists I \models T)\, (I \prec^{\mathrm{e}}_n \mathfrak{A}) \wedge a < I < b \Leftrightarrow (\forall k \in \omega)\, (\mathfrak{A} \models (\exists y < b)\, \psi(k, a, y)) \Leftrightarrow Y_\psi(a, b) > \omega.$$

This proves that $Y_\psi(x,y)=z$ is a Π_n-indicator for T in T. By the definition of $Y_\psi(x,y)=z$, it follows that $T \vdash Y_\psi(x,y) \leq y$. Let $a', b', c' \in \mathfrak{A}$ such that $a \leq a' < b' \leq b$ and $\mathfrak{A} \models Y_\psi(a',b') = c'$. Then, by ($\bullet$),

$$\mathfrak{A} \models (\forall u < c')\,(\exists y \leq b')\,\psi(u, a', y).$$

Hence, $\mathfrak{A} \models Y_\psi(a,b) \geq c'$. Consequently, $Y_\psi(x,y)=z$ is a well-behaved Π_n-indicator. To conclude the proof let us see that:

Claim 2.7.1 *For every $k \in \omega$, $T + \mathbf{B}\Sigma_{n+1} \vdash \forall x \exists y\, (Y_\psi(x,y) > k)$.*

Proof. Let $\mathfrak{A} \models T + \mathbf{B}\Sigma_{n+1}$. Without loss of generality we may assume that $\mathfrak{A}$ is countable, recursively saturated and $T \in \mathrm{SSy}(\mathfrak{A})$. Let $\mathbf{c}$ be a new constant and let T' be the theory

$$T + \{(\exists \vec{y} < \mathbf{c})\,\psi(\vec{y}) \,:\, \psi(\vec{y}) \in \Pi_n, \mathfrak{A} \models \exists \vec{y}\,\psi(\vec{y})\}.$$

By compactness T' is consistent. Since $\mathfrak{A}$ is recursively saturated,

$$\{\psi(\vec{y}) \in \Pi_n \,:\, \mathfrak{A} \models \exists \vec{y}\,\psi(\vec{y})\} \in \mathrm{SSy}(\mathfrak{A}).$$

Hence, as $T \in \mathrm{SSy}(\mathfrak{A})$, $T' \in \mathrm{SSy}(\mathfrak{A})$. So, there exists $\mathfrak{B} \models T'$ countable and recursively saturated verifying $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$. Let $c = \mathfrak{B}(\mathbf{c})$. Then,

$$\mathfrak{A} \models \exists \vec{y}\,\psi(\vec{y}) \Rightarrow \mathfrak{B} \models (\exists \vec{y} < c)\,\psi(\vec{y}).$$

So, by Friedman's immersion theorem, there is $H : \mathfrak{A} \preceq_n^{\mathrm{e}} \mathfrak{B}$ such that $c \notin H(\mathfrak{A})$. Therefore, identifying $H(\mathfrak{A})$ and $\mathfrak{A}$, we get $\mathfrak{A} \prec_n^{\mathrm{e}} \mathfrak{B}$ and $\mathfrak{B} \models T$.

Let $a \in \mathfrak{A}$. Then for each $b \in \mathfrak{B}$ such that $b > \mathfrak{A}$ and $k \in \omega$, $\mathfrak{B} \models Y_\psi(a,b) > k$. Then, as $Y_\psi(x,y) > k$ is a $\Delta_{n+1}(T)$-formula and T extends $\mathbf{I}\Delta_{n+1}(T)$, by underspill, there exists $d \in \mathfrak{A}$ such that $\mathfrak{B} \models Y_\psi(a,d) > k$; hence, $\mathfrak{A} \models Y_\psi(a,d) > k$. □ (Claim 2.7.1)

By (1) and Theorem 2.4, T is $\Pi_{n+2}^{\mathbf{B}}$-conservative; therefore, by the above claim, we get that for all $k \in \omega$,

$$T \vdash \forall x \exists y\, (Y_\psi(x,y) > k),$$

as required. □

By Theorems 2.4 and 2.7, consistent and recursively axiomatizable $\Pi_{n+2}^{\mathbf{B}}$-conservative theories (proving $\mathbf{exp}$) correspond to those theories whose Π_n-envelopes define Π_n-indicators. Moreover, from the above results, we can also derive the following model-theoretic characterization of $\Pi_{n+2}^{\mathbf{B}}$-conservativeness.

Corollary 2.8 *Let T be a recursively axiomatizable theory such that $T \vdash \mathbf{exp}$. The following conditions are equivalent:*

(1) *T is $\Pi_{n+2}^{\mathbf{B}}$-conservative.*

(2) *For every countable and recursively saturated $\mathfrak{A} \models T$ and $a \in \mathfrak{A}$ there is $I \prec_n^{\mathrm{e}} \mathfrak{A}$ (proper) such that $I \models T$ and $a \in I$.*

Proof.

(1) ⇒ (2): Let $\mathfrak{A} \models T$ recursively saturated and $a \in \mathfrak{A}$. By Theorems 2.4 and 2.7, there exists a well-behaved Π_n-indicator, $Y(x,y)=z$, for T in T such that for all $k \in \omega$,

$$T \vdash \forall x \exists y\, (Y(x,y) > k).$$

Clearly, $\mathbf{p}(y) = \{Y(a,y) > k \,:\, k \in \omega\}$ is a recursive type over $\mathfrak{A}$. Then, as $\mathfrak{A}$ is recursively saturated, there exists $b \in \mathfrak{A}$ such that $\mathfrak{A} \models \mathbf{p}(b)$; that is, for each $k \in \omega$, $\mathfrak{A} \models Y(a,b) > k$. So, $Y(a,b) > \omega$ and, since $Y(x,y)=z$ is a Π_n-indicator, there exists $I \models T$ such that $I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $a < I < b$.

(2) ⇒ (1): Let $\varphi(x,y) \in \Pi_n$ such that $T + \mathbf{B}\Sigma_{n+1} \vdash \forall x \exists y\, \varphi(x,y)$. Let $\mathfrak{A} \models T$ countable and recursively saturated and $a \in \mathfrak{A}$. By (2), there exists $I \models T$ such that $a \in I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $I \neq \mathfrak{A}$. Then we have $I \models T + \mathbf{B}\Sigma_{n+1}$; hence $I \models \exists y\, \varphi(a,y)$. So, $\mathfrak{A} \models \exists y\, \varphi(a,y)$. □

2.3 Π_{n+3}-axiomatizable theories

In view of Proposition 2.3 and Theorem 2.4, our problem $(\mathbf{P})$ on the equivalence between Π_n-functional and $\Pi^{\mathbf{B}}_{n+2}$-conservative theories can be reformulated as follows:

$(\mathbf{P}')$ Assume that $T + \mathbf{B}\Sigma_{n+1}$ is consistent. Are the following conditions equivalent?

(a) There is a Π_n-envelope of T in T.

(b) There is a Π_n-envelope of T in T that satisfies Π_n-IND.

Here we obtain a positive answer to $(\mathbf{P}')$ for recursively Π_{n+3}-axiomatizable theories (proving $\mathbf{exp}$). As a corollary, we also obtain a model-theoretic proof of Theorem 1.4 for theories proving $\mathbf{exp}$. Our starting point is a positive answer to $(\mathbf{P}')$ for Π_{n+2}-axiomatizable theories.

Lemma 2.9 *Let T be a Π_n-functional theory and T_0 an extension of $\mathbf{I}\Sigma_n + \mathbf{exp}$. If T is Π_{n+2}-axiomatizable, then every Π_n-envelope of T in T_0 satisfies* Π_n-IND.

Proof. Let $\varphi(u, x, y) \in \Pi_n$ be a Π_n-envelope of T in T_0. In the proof we will denote the formula $\varphi(u, x, y)$ by $F_u(x) = y$.

Let $\mathfrak{A} \models T_0$ countable and $a, b \in \mathfrak{A}$ such that, for all $k \in \omega$, $\mathfrak{A} \models (\exists y < b)\,(F_k(a) = y)$. We must prove that there exists $I \models T$ such that $I \prec^{\mathrm{e}}_n \mathfrak{A}$ and $a < I < b$.

Let $I = \{c \in \mathfrak{A} \,:\, \text{there exists } k \in \omega \text{ such that } \mathfrak{A} \models c < F_k(a)\}$. Obviously, $a < I < b$. Let us see that $I \models T$ and $I \prec_n \mathfrak{A}$. First we prove that I is closed (in $\mathfrak{A}$) under functions F_m, $m \in \omega$; that is

$(\star)$ for each $m \in \omega$ and $c \in I$ there exists $d \in I$ such that $\mathfrak{A} \models F_m(c) = d$.

Now let $c \in I$ and $k \in \omega$ be such that $c < F_k(a)$. Since $T \vdash \forall x \exists z\,(\forall x_0 < F_k(x))\,(\exists y_0 < z)\,(F_m(x_0) = y_0)$ and $(\forall x_0 < F_k(x))\,(\exists y_0 < z)\,(F_m(x_0) = y_0)$ is a Π_n-formula, there exists $j \in \omega$ such that

$$T_0 \vdash F_j(x) = y \rightarrow (\exists z < y)\,(\forall x_0 < F_k(x))\,(\exists y_0 < z)\,(F_m(x_0) = y_0).$$

Hence, for $x = a$ it holds that $\mathfrak{A} \models (\exists z < F_j(a))\,(\forall x_0 < F_k(a))\,(\exists y_0 < z)\,(F_m(x_0) = y_0)$. So, now for $x_0 = c$, $\mathfrak{A} \models (\exists y_0 < F_j(a))\,(F_m(c) = y_0)$, as required.

We show that $I \prec_n \mathfrak{A}$. Since T is Π_n-functional, T extends $\mathbf{I}\Sigma_n$; so, $T \vdash \forall x \exists y\,(\mathbb{K}_n(x) = y)$. Hence, there exists $k \in \omega$ such that

$$T_0 \vdash F_k(x) = y \rightarrow (\exists z < y)\,(\mathbb{K}_n(x) = z).$$

Hence, by $(\star)$, I is closed under the function $\mathbb{K}_n$. But, as remarked at the end of Section 2.1, $\{\mathbb{K}_n(x) = y\}$ is a strong Π_n-functional class, so $I \prec_n \mathfrak{A}$.

Next we show that $I \models T$. Let $\psi(x, y) \in \Pi_n$ such that $\forall x \exists y\,\psi(x, y)$ is an axiom of T. Then there is $k \in \omega$ such that

$$T_0 \vdash F_k(x) = y \rightarrow (\exists z < y)\,\psi(x, z).$$

Let $c \in I$. Then $\mathfrak{A} \models (\exists z < F_k(c))\,\psi(c, z)$. By $(\star)$, $F_k(c) \in I$; so, it follows that $I \models \exists z\,\psi(c, z)$ since $I \prec_n \mathfrak{A}$. Hence, $I \models \forall x \exists y\,\psi(x, y)$, as required.

This completes the proof of the lemma. □

Theorem 2.10 *Let T be a Π_n-functional theory and let T_0 be a Π_{n+2}-axiomatizable theory such that*

$$T \Rightarrow T_0 \Rightarrow \mathbf{I}\Sigma_n + \mathbf{exp}.$$

If T is Π_{n+3}-axiomatizable, then every Π_n-envelope of T in T_0 satisfies Π_n-IND.

Proof. Let $\varphi(u, x, y) \in \Pi_n$ be a strong Π_n-envelope of T in T_0 and $T^* = \mathrm{Th}_{\Pi_{n+2}}(T)$. First of all let us observe that $\varphi(u, x, y)$ is a Π_n-envelope of T^* in T_0 and also, as T^* extends T_0, of T^* in T^*.

Let $\mathfrak{A} \models T_0$ countable and $a, b \in \mathfrak{A}$ such that for all $k \in \omega$, $\mathfrak{A} \models (\exists y < b)\,\varphi(k, a, y)$. We must prove that there exists $I \models T$ such that $I \prec^{\mathrm{e}}_n \mathfrak{A}$ and $a < I < b$.

Let $\{a_k : k \in \omega\}$ be an enumeration of all the elements of $\mathfrak{A}$ less than b such that $a_0 = a$, and let

$$\{\theta_k(x, y, z) \in \Sigma_n : k \in \omega\}$$

be an enumeration of the Σ_n-formulas such that $T \vdash \forall x \exists y \forall z\, \theta_k(x, y, z)$. We shall define a sequence of initial segments $\{I_k \prec_n^{\mathrm{e}} \mathfrak{A} : k \in \omega\}$ and two sequences of elements of $\mathfrak{A}$,

$$\{b_k \in \mathfrak{A} : a \leq b_k < b\} \quad \text{and} \quad \{d_k \in \mathfrak{A} : a \leq d_k < b\}$$

such that for each $k \in \omega$,

1. $d_{k+1} \geq d_k$ and $a, d_k \in \bigcap_{j \in \omega} I_j$;
2. $b_{k+1} < I_k < b_k$;
3. $I_{k+1} \prec_n^{\mathrm{e}} I_k \prec_n^{\mathrm{e}} \mathfrak{A}$;
4. $I_k \models T^*$ is recursively saturated;
5. for each $i \leq k$, if $a_i \in I_k$, then $I_k \models (\exists y_0, \ldots, y_k \leq d_k) \forall z\, (\bigwedge_{j \leq k} \theta_j(a_i, y_j, z))$.

Properties 1. – 5. will turn evident from the definition. We proceed by recursion on $k \in \omega$.

$k = 0$: Since $\varphi(u, x, y)$ is a Π_n-envelope of T^* in T_0, by Lemma 2.9, there exists $I_0' \models T^*$ such that $I_0' \prec_n^{\mathrm{e}} \mathfrak{A}$ and $a < I_0' < b$. By Remark 2.5 we can assume that I is recursively saturated. We denote the formula $\varphi(u, x, y)$ by $F_u(x) = y$. Since $T \vdash \forall x \exists y \forall z\, \theta_0(x, y, z)$, for each $m \in \omega$,

$$T^* \vdash \forall x \exists z\, (\exists y < z)\, [F_m(\max(x, y)) < z \wedge (\forall z' < z)\, \theta_0(x, y, z')].$$

Let $\psi(u, a_0, z) \in \Sigma_0(\Sigma_n)$ be the formula

$$(\exists y < z)\, (F_u(\max(a_0, y)) < z \wedge (\forall z' < z)\, \theta_0(a_0, y, z') \\ \wedge (\forall u' < u)\, [F_{u'}(\max(a_0, y)) < F_u(\max(a_0, y))]).$$

Then $\mathbf{p}_0(z) = \{\psi(m, a_0, z) : m \in \omega\}$ is a recursive type over I_0'; hence, as I_0' is recursively saturated, there exists $b_0 \in I_0'$ realizing $\mathbf{p}_0(z)$ in I_0'; that is, for all $m \in \omega$, $I_0' \models \psi(m, a_0, b_0)$. Since ψ is $\Sigma_0(\Sigma_n)$, by overspill, there exists $q > \omega$ such that $I_0' \models \psi(q, a_0, b_0)$. Let $d_0 < b_0$ such that

$$I_0' \models F_q(\max(a_0, d_0)) < b_0 \wedge (\forall z' < b_0)\, \theta_0(a_0, d_0, z') \\ \wedge (\forall u' < q)\, [F_{u'}(\max(a_0, d_0)) < F_q(\max(a_0, d_0))].$$

Then, for each $m \in \omega$, $I_0' \models (\exists y < b_0)\, [F_m(\max(a_0, d_0)) = y]$. Therefore, as $\varphi(u, x, y)$ is a Π_n-envelope of T^* in T^*, by Lemma 2.9 and Remark 2.5, there exists $I_0 \models T^*$ recursively saturated such that $I_0 \prec_n^{\mathrm{e}} I_0' \prec_n^{\mathrm{e}} \mathfrak{A}$ and $\max(a_0, d_0) < I_0 < b_0$. Clearly, $I_0 \models (\exists y \leq d_0) \forall z\, \theta_0(a_0, y, z)$.

$k \to k+1$: Assume that I_k, b_k and d_k have been defined. We define I_{k+1}, b_{k+1} and d_{k+1} as follows. Let $c_0, \ldots, c_r$ be an enumeration of $\{a_j \in I_k : j \leq k+1\}$. Since

$$T \vdash \forall x_0 \cdots \forall x_r \exists y_{0,0} \cdots \exists y_{r,k+1}\, (\bigwedge_{i=0}^{r} \bigwedge_{j=0}^{k+1} \forall z\, \theta_j(x_i, y_{i,j}, z)),$$

for each $m \in \omega$, T^* proves that

$$\forall \vec{x} \forall v \exists z\, (\exists \vec{y} < z)\, [F_m(\max(\vec{x}, v, \vec{y})) < z \wedge \bigwedge_{i=0}^{r} \bigwedge_{j=0}^{k+1} (\forall z' < z)\, \theta_j(x_i, y_{i,j}, z')].$$

Let $\psi_{k+1}(u, c_0, \ldots, c_r, d_k, z) \in \Sigma_0(\Sigma_n)$ be the following formula

$$(\exists \vec{y} < z)\, (F_u(\max(\vec{c}, d_k, \vec{y})) < z \wedge \bigwedge_{i=0}^{r} \bigwedge_{j=0}^{k+1} (\forall z' < z)\, \theta_j(c_i, y_{i,j}, z') \\ \wedge (\forall u' < u)\, [F_{u'}(\max(\vec{c}, d_k, \vec{y})) < F_u(\max(\vec{c}, d_k, \vec{y}))]).$$

Then $\mathbf{p}_{k+1}(z) = \{\psi_{k+1}(m, \vec{c}, d_k, z) : m \in \omega\}$ is a recursive type over I_k; hence, as I_k is recursively saturated, there exists $b_{k+1} \in I_k$ realizing $\mathbf{p}_{k+1}(z)$; that is, for all $m \in \omega$, $I_k \models \psi_{k+1}(m, \vec{c}, d_k, b_{k+1})$. By overspill, as ψ_{k+1} is a $\Sigma_0(\Sigma_n)$-formula, there exists $q > \omega$ such that $I_k \models \psi_{k+1}(q, \vec{c}, d_k, b_{k+1})$. Let $e_{i,j} < b_{k+1}$, $i \leq r$, $j \leq k+1$, such that

$$I_k \models F_q(\max(\vec{c}, d_k, \vec{e})) < b_{k+1} \wedge \bigwedge_{i=0}^{r} \bigwedge_{j=0}^{k+1} (\forall z' < b_{k+1})\, \theta_j(c_i, e_{i,j}, z') \\ \wedge (\forall u' < q)\, [F_{u'}(\max(\vec{c}, d_k, \vec{e})) < F_q(\max(\vec{c}, d_k, \vec{e}))].$$

Let $d_{k+1} = \max(d_k, \vec{e})$. Then $b_{k+1} < I_k < b_k$ and $I_k \vDash (\exists y < b_{k+1})\,[F_m(\max(\vec{c}, d_{k+1})) = y]$, for each $m \in \omega$. Therefore, as $\varphi(u, x, y)$ is a Π_n-envelope of T^* in T^*, by Lemma 2.9 and Remark 2.5, there exists $I_{k+1} \vDash T^*$ recursively saturated such that $I_{k+1} \prec_n^{\mathrm{e}} I_k$ and $\max(\vec{c}, d_{k+1}) < I_{k+1} < b_{k+1}$. Moreover, property 5. follows from

$$I_{k+1} \vDash (\exists \vec{y} \le d_{k+1})\,(\textstyle\bigwedge_{i=0}^{r} \bigwedge_{j=0}^{k+1} \forall z\, \theta_j(c_i, y_{i,j}, z)).$$

This concludes the definition of I_{k+1}, b_{k+1} and d_{k+1}. Let $I = \bigcap_{j \in \omega} I_j$. Then $a < I < b$. It holds:

Claim 2.10.1 *$I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $I \vDash T$.*

P r o o f. First of all let us recall that $\varphi(u, x, y)$ is a strong Π_n-envelope and that each I_k is closed in $\mathfrak{A}$ under the functions F_m defined in $\mathfrak{A}$ by the formulas $\varphi(m, x, y)$. As a consequence, I is also closed (in $\mathfrak{A}$) under those functions. Since Γ_φ is a strong Π_n-functional class, $I \prec_n^{\mathrm{e}} \mathfrak{A}$.

Let us see that $I \vDash T$. Let $\theta(x, y, z) \in \Sigma_n$ such that $T \vdash \forall x \exists y \forall z\, \theta(x, y, z)$ and let $c \in I$. Then there exist $i, m \in \omega$ such that $a_i = c$ and $\theta(x, y, z)$ is $\theta_m(x, y, z)$. Let $k > m, i$. Then $a_i = c \in I_k$ and, by 5.,

$$I_k \vDash (\exists y_0, \dots, y_k \le d_k) \forall z\, (\textstyle\bigwedge_{j=0}^{k} \theta_j(a_i, y_j, z)).$$

In particular, $I_k \vDash (\exists y \le d_k) \forall z\, \theta_m(c, y, z)$. Since $d_k \in I$, there is $e \in I$ such that $I_k \vDash \forall z\, \theta_m(c, e, z)$. Then we have $I \prec_n^{\mathrm{e}} I_k$, since $I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $I \subset^{\mathrm{e}} I_k \prec_n^{\mathrm{e}} \mathfrak{A}$; hence, $I \vDash \forall z\, \theta_m(c, e, z)$. It follows that $I \vDash \forall x \exists y \forall z\, \theta(x, y, z)$; so, $I \vDash T$ as required. □ (Claim 2.10.1)

The previous claim concludes the proof of the theorem. □

Corollary 2.11 *Let T be a Π_{n+3}-axiomatizable and consistent extension of $\mathbf{I}\Sigma_n + \mathbf{exp}$. Then*

T is Π_n-functional $\Leftrightarrow$ T is $\Pi_{n+2}^{\mathbf{B}}$-conservative.

P r o o f. By Theorem 1.3, it easily follows that every $\Pi_{n+2}^{\mathbf{B}}$-conservative theory is Π_n-functional. Let us prove the converse. First of all, observe that, by Proposition 2.3 and Theorems 2.4 and 2.10, the result holds if T is recursively axiomatizable. Now we address the general case.

Let $\psi \in \Pi_{n+2}$ be such that $T + \mathbf{B}\Sigma_{n+1} \vdash \psi$. Then there exists a sentence $\theta \in \Pi_{n+3}$ such that $T \vdash \theta$ and $\mathbf{B}\Sigma_{n+1} + \theta \vdash \psi$. Let T' be the theory $(\mathbf{I}\Sigma_n + \mathbf{exp} + \theta) + \Sigma_{n+1}$-CR. Then T' is recursively Π_{n+3}-axiomatizable, proves $\mathbf{exp}$ and, as it is closed under Σ_{n+1}-collection rule, is Π_n-functional. So, as noticed above, by Proposition 2.3 and Theorems 2.4 and 2.10, T' is $\Pi_{n+2}^{\mathbf{B}}$-conservative; hence, as $T' + \mathbf{B}\Sigma_{n+1} \vdash \psi$, it holds that $T' \vdash \psi$. Since T extends T', we get that $T \vdash \psi$, as required. □

Corollary 2.12 *Let T be a recursively Π_{n+3}-axiomatizable and Π_n-functional theory such that $T \vdash \mathbf{exp}$ and $\mathfrak{A} \vDash T$ is countable and recursively saturated. Then for every $a \in \mathfrak{A}$ there exists $I \prec_n^{\mathrm{e}} \mathfrak{A}$ (proper) such that $I \vDash T$ and $a \in I$.*

3 Collection rule vs. collection axioms

Corollary 2.11 gives Theorem 1.4 for $n \ge 1$. Nevertheless, known existence results for Π_0-envelopes use that exponential function is total. This fact forced us to develop an alternative proof for $n = 0$. However, the method used in this new proof does work for every $n \in \omega$. This is why we present our arguments for every $n \in \omega$ and not only for $n = 0$. The main idea is to adapt the proof of Theorem 2.10 giving a direct construction of the proper Σ_n-elementary initial segment required there. To this end, the key tool is the following weak notion of Π_n-envelope.

Definition 3.1 A *pseudo-Π_n-envelope of T in T_0* is a sequence $\Gamma = \{\varphi_k(x, y) : k \in \omega\}$ of Σ_{n+1}-formulas satisfying

1. $T \vdash \Gamma^*$ and for all $k \in \omega$, $T_0 \vdash \varphi_{k+1}(x, y) \rightarrow (\exists z < y)\, \varphi_k(x, z)$;
2. for each $\psi(x, y) \in \Pi_n$ such that $T \vdash \forall x \exists y\, \psi(x, y)$ there exists $k \in \omega$ such that

$$T_0 \vdash \varphi_k(x, y) \rightarrow (\exists z < y)\, \psi(x, z).$$

Pseudo-Π_n-envelopes will provide us with a reduction of $\mathbf{B}\Sigma_{n+1}$ to Σ_{n+1}-collection rule. We shall thus obtain a model-theoretic proof of [2, Theorem 3.2], proved there by a cut-elimination argument.

Remark 3.2 A pseudo-Π_n-envelope is, essentially, a "non-uniform" Π_n-envelope. So, results on Π_n-envelopes not depending on their uniform character also hold for pseudo-Π_n-envelopes. In particular, if Γ is a pseudo-Π_n-envelope of T in T_0, by the very definition it follows that:

(i) If $T \Rightarrow T_0$, then $\mathrm{Th}_{\Pi_{n+2}}(T) = \mathrm{Th}_{\Pi_{n+2}}(T_0 + \Gamma^*)$.

(ii) If $\Gamma \subseteq \Pi_n$ and $T + \mathbf{I}\Sigma_n$ is consistent, then Γ is a Π_n-functional class.

Definition 3.3 Let $\Gamma = \{\varphi_k(x,y) \,:\, k \in \omega\}$ be a sequence of Σ_{n+1}-formulas. We say that Γ *satisfies* Π_n-IND *for T and T_0 in recursively saturated models* if for each $\mathfrak{A} \models T_0$ countable and recursively saturated and $a, b \in \mathfrak{A}$, the following conditions are equivalent:

IND-(i): For all $k \in \omega$, $\mathfrak{A} \models (\exists y < b)\, \varphi_k(a,y)$.

IND-(ii): There exists $I \models T$ such that $I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $a < I < b$.

Remark 3.4 By the proof of (2) $\Rightarrow$ (1) in Theorem 2.4, if Γ is a sequence of Σ_{n+1}-formulas that verifies condition 1. in the definition of pseudo-Π_n-envelope and satisfies Π_n-IND in recursively saturated models, then Γ is a pseudo-Π_n-envelope and T is $\Pi_{n+2}^{\mathbf{B}}$-conservative.

Remark 3.5 (The class $\Gamma(\theta)$) The next construction associates a pseudo-Π_n-envelope to each Σ_n-formula $\theta(x,y,z)$ such that $\forall x \exists y \forall z\, \theta(x,y,z)$ is provable in a Π_n-functional theory. In order to ensure that initial segments closed under the functions defined by the pseudo-Π_n-envelope are Σ_n-elementary substructures, we shall use the auxiliary functions $\mathbb{K}_n(x) = y$ discussed at the end of Section 2.1.

Let $\theta(x,y,z) \in \Sigma_n$ such that $\forall x \exists y \forall z\, \theta(x,y,z)$ is provable in some Π_n-functional theory. Let

$$\Gamma(\theta) = \{\varphi_k(x,y) \,:\, k \in \omega\},$$

where, for each $k \in \omega$, $\varphi_k(x,y)$ is a $\Sigma_0(\Sigma_n)$-formula defining the function $G_k^\theta(x) = y$ given by

$$\begin{aligned} G_0^\theta(x) &= (\mu z)((\forall x_0 \le x)\,(\exists y_0 < z)\,[\mathbb{K}_n(\max(x,y_0)) < z \wedge (\forall z' < z)\,\theta(x_0,y_0,z')]), \\ G_{k+1}^\theta(x) &= (\mu z)((\forall x_0,\dots,x_{k+1} \le x)\,(\exists y_0,\dots,y_{k+1} < z)\,(G_k^\theta(\mathbb{K}_n(\max(x,\vec{y}))) < z \\ &\qquad\qquad \wedge \textstyle\bigwedge_{j=0}^{k+1} (\forall z' < z)\,\theta(x_j,y_j,z'))). \end{aligned}$$

We shall prove that $\Gamma(\theta)$ is a pseudo-Π_n-envelope of $\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x,y,z)$ in $\mathbf{I}\Sigma_n + \Gamma(\theta)^*$ and satisfies Π_n-IND in recursively saturated models.

Lemma 3.6 *Let $\theta(x,y,z) \in \Sigma_n$, T' be a Π_n-functional theory such that $T' \vdash \forall x \exists y \forall z\, \theta(x,y,z)$ and $\Gamma(\theta)$ as above. Then we have:*

(1) *$T' \vdash \Gamma(\theta)^*$. So, T' extends $\mathbf{I}\Sigma_n + \Gamma(\theta)^*$.*

(2) *For all $k \in \omega$, $\mathbf{I}\Sigma_n + \Gamma(\theta)^* \vdash G_{k+1}^\theta(x) = y \rightarrow (\exists z < y)\,(G_k^\theta(x) = z)$.*

Proof. Property (2) easily follows from definition. Let us prove (1). We show by induction on m that for all $m \in \omega$, $T' \vdash \mathrm{IPF}(G_m^\theta(x) = y) \wedge \forall x \exists y\, (G_m^\theta(x) = y)$.

$m = 0$: Let $\theta_0(x, x_0, z)$ be the $\Sigma_0(\Sigma_n)$-formula

$$(\exists y_0 < z)\,[\mathbb{K}_n(\max(x,y_0)) < z \wedge (\forall z' < z)\,\theta(x_0,y_0,z')].$$

Since $T' \vdash \forall x \exists y \forall z\, \theta(x,y,z)$, $T' \vdash \forall x \forall x_0 \exists z\, \theta_0(x,x_0,z)$ and, as a consequence, $\exists z\, (\theta_0(x,x_0,z) \in \Delta_{n+1}(T'))$. Since T' extends $\mathbf{B}^*\Delta_{n+1}(T')$, it holds that $T' \vdash \forall u \exists z\, (\forall x_0 \le u)\,(\exists z_0 \le z)\,\theta_0(x,x_0,z_0)$.

In particular, $T' \vdash \forall x \exists z\, (\forall x_0 \le x)\,(\exists z_0 \le z)\,\theta_0(x,x_0,z_0)$. Since T' is an extension of $\mathbf{I}\Sigma_n$,

$$T' \vdash \forall x \exists v\, (v = (\mu z)((\forall x_0 \le x)\,(\exists z_0 \le z)\,\theta_0(x,x_0,z_0))).$$

This proves that $T' \vdash \forall x \exists y\, (G_0^\theta(x) = y)$. Uniqueness follows from the definition of G_0^θ.

Finally, we must prove now that G_0^θ is a nondecreasing function. We reason in T'. Let x_0, x_1, x_2 be such that $x_0 \le x_1 \le x_2$. Then $x_0 \le x_2$; thus, there exists $y < G_0^\theta(x_2)$ such that

$$\mathbb{K}_n(\max(x_2,y)) < G_0^\theta(x_2) \wedge (\forall z' < G_0^\theta(x_2))\,\theta_0(x_0,y,z').$$

Since $\mathbb{K}_n(x) = y$ defines a nondecreasing function, $\mathbb{K}_n(\max(x_1,y)) < G_0^\theta(x_2) \wedge (\forall z' < G_0^\theta(x_2))\,\theta_0(x_0,y,z')$. Hence, $G_0^\theta(x_1) \le G_0^\theta(x_2)$.

$m \rightarrow m+1$: To prove that $T' \vdash \mathrm{IPF}(G_{m+1}^\theta(x) = y) \wedge \forall x \exists y\, (G_{m+1}^\theta(x) = y)$, we can easily adapt our previous argument for $m = 0$. $\square$

Theorem 3.7 *Let $\theta(x,y,z) \in \Sigma_n$ and $\Gamma(\theta)$ as above. Then we have:*

(a) $\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x,y,z)$ *is consistent.*

(b) *The class $\Gamma(\theta)$ is a pseudo-Π_n-envelope of $\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x,y,z)$ in $\mathbf{I}\Sigma_n + \Gamma(\theta)^*$ satisfying Π_n-IND in recursively saturated models.*

(c) $\mathrm{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x,y,z)) = \mathrm{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma(\theta)^*)$.

Proof. Part (c) follows from the previous ones: By (a), $\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x,y,z)$ is Π_n-functional; so, by Lemma 3.6, $\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x,y,z)$ extends $\mathbf{I}\Sigma_n + \Gamma(\theta)^*$. Hence, (c) follows by part (b) and Remark 3.2(i). Now we prove parts (a) and (b).

Let us denote $\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x,y,z)$ by T_1 and $\mathbf{I}\Sigma_n + \Gamma(\theta)^*$ by T_0. First we prove that

($\bullet$) $\quad\Gamma(\theta)$ satisfies Π_n-IND for T_1 and T_0 in recursively saturated models.

Let $\mathfrak{A} \models T_0$ be countable and recursively saturated and $a, b \in \mathfrak{A}$, such that $\mathfrak{A} \models (\exists y < b)\,(G_k^\theta(a) = y)$, for all $k \in \omega$. In order to get ($\bullet$) we prove that there exists $I \models T_1$ such that $I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $a < I < b$.

Let $\{a_k : k \in \omega\}$ be an enumeration of all the elements of $\mathfrak{A}$ less than b such that $a_0 = a$. We shall define a sequence of elements $\{c_k < b : k \in \omega\}$ and two sequences $\{b_k \in \mathfrak{A} : a \le b_k < b\}$ and $\{d_k \in \mathfrak{A} : a \le d_k < b\}$ such that for each $k \in \omega$,

1. $d_{k+1} \ge d_k$ and $b_{k+1} \le b_k$;
2. $a_0 = c_0$, $c_k < b_k$ and, for all $m \in \omega$, $G_m^\theta(\max(c_0, \dots, c_k, d_k)) < b_k$;
3. $\mathfrak{A} \models (\exists y_0, \dots, y_k \le d_k)\,(\forall z < b_k)\,(\bigwedge_{j=0}^k \theta(c_j, y_j, z))$.

We proceed by recursion on $k \in \omega$.

$k = 0$: Let $c_0 = a_0$. Observe that for each $m \in \omega$,

$$T_0 \vdash \forall x \exists z\, (\forall x_0 \le x)\,(\exists y_0 < z)\,[G_m^\theta(\mathbb{K}_n(\max(x, y_0))) < z \wedge (\forall z' < z)\,\theta(x_0, y_0, z')].$$

For each $m \in \omega$, let $\psi(m, c_0, y_0, z, b) \in \Sigma_0(\Sigma_n)$ be the formula

$$y_0 < z < b \wedge G_m^\theta(\mathbb{K}_n(\max(c_0, y_0))) < z \wedge (\forall z' < z)\,\theta(c_0, y_0, z').$$

Then $\mathbf{p}_0(y_0, z) = \{\psi(m, c_0, y_0, z, b) : m \in \omega\}$ is a recursive type over $\mathfrak{A}$; hence, as $\mathfrak{A}$ is recursively saturated, there exist $d_0, b_0 \in \mathfrak{A}$ realizing $\mathbf{p}_0(y_0, z)$ in $\mathfrak{A}$; that is, for all $m \in \omega$, $\mathfrak{A} \models \psi(m, c_0, d_0, b_0, b)$. Then, for all $m \in \omega$, $G_m^\theta(\max(c_0, d_0)) < b_0$ and, clearly, $\mathfrak{A} \models (\exists y_0 \le d_0)\,(\forall z' < b_0)\,\theta(c_0, y_0, z')$.

$k \to k+1$: Assume that c_k, b_k and d_k have been defined. We define c_{k+1}, b_{k+1} and d_{k+1} as follows. Let $a' = \max(\{c_0, \dots, c_k, d_k, a_{k+1}\})$. We distinguish two cases:

Case 1: There exists $m \in \omega$ such that $\mathfrak{A} \models G_m^\theta(a') \ge b_k$. Then we define $b_{k+1} = b_k$, $c_{k+1} = c_k$, $d_{k+1} = d_k$.

Case 2: For all $m \in \omega$, $\mathfrak{A} \models G_m^\theta(a') < b_k$. Let $c_{k+1} = a_{k+1}$. For each $m \in \omega$, T_0 proves that

$$\forall x \forall v \exists z\, (\forall \vec{x} \le x)\,(\exists \vec{y} < z)\,[G_m^\theta(\mathbb{K}_n(\max(x, v, \vec{y}))) < z \wedge \textstyle\bigwedge_{j=0}^{k+1} (\forall z' < z)\,\theta(x_j, y_j, z')].$$

Let $\psi_{k+1}(m, c_0, \dots, c_{k+1}, d_k, b_k, y_0, \dots, y_{k+1}, z) \in \Sigma_0(\Sigma_n)$ be the formula

$$z < b_k \wedge y_0 < z \wedge \dots \wedge y_{k+1} < z \wedge G_m^\theta(\mathbb{K}_n(\max(\vec{c}, d_k, \vec{y}))) < z \wedge \textstyle\bigwedge_{j=0}^{k+1} (\forall z' < z)\,\theta(c_j, y_j, z').$$

Then $\mathbf{p}_{k+1}(\vec{y}, z) = \{\psi_{k+1}(m, \vec{c}, d_k, b_k, \vec{y}, z) : m \in \omega\}$ is a recursive type over $\mathfrak{A}$; hence, as $\mathfrak{A}$ is recursively saturated, there exist $e_0, \dots, e_{k+1}, b_{k+1} \in \mathfrak{A}$ realizing $\mathbf{p}_{k+1}(\vec{y}, z)$ in $\mathfrak{A}$; that is, for all $m \in \omega$,

$$\mathfrak{A} \models \psi_{k+1}(m, \vec{c}, d_k, b_k, \vec{e}, b_{k+1}).$$

Let $d_{k+1} = \max(d_k, \vec{e})$.

Then $b_{k+1} < b_k$ and for every $m \in \omega$, $G_m^\theta(\max(c_0, \dots, c_{k+1}, d_{k+1})) < b_{k+1}$. Moreover, we can check that 3. holds since we have

$$\mathfrak{A} \models (\exists \vec{y} \le d_{k+1})\,(\textstyle\bigwedge_{j=0}^{k+1} (\forall z < b_{k+1})\,\theta(c_j, y_j, z)).$$

This concludes the definition of b_{k+1}, c_{k+1} and d_{k+1}. Let $I = \{c_k : k \in \omega\}$. It holds that $a < I < b$ and

Claim 3.7.1

(i) $I \prec_n^{\mathrm{e}} \mathfrak{A}$. *Hence,* $I \models \mathbf{B}\Sigma_{n+1}$.

(ii) *For all* $k \in \omega$, $d_k \in I$.

(iii) $I \models T_1$.

Proof.

(i) First we prove that I is an initial segment of $\mathfrak{A}$. Let $e \in I$ and $c < e$. Then there is $k \in \omega$ such that $c_k = e$. Assume that there is no $i < k$ such that $c_i = c$. Then, there exists $j \geq k$ such that $a_{j+1} = c$. So,

$$a_{j+1} \leq \max(c_0, \ldots, c_j, d_j).$$

By 2., for each $m \in \omega$,

$$G_m^\theta(\max(c_0, \ldots, c_j, d_j)) < b_j.$$

Since $a' = \max(c_0, \ldots, c_j, d_j) = \max(c_0, \ldots, c_j, a_{j+1}, d_j)$, then by definition $c = a_{j+1} = c_{j+1} \in I$.

Now we prove that $I \prec_n \mathfrak{A}$. It is enough to show that for each $k \in \omega$, $\mathbb{K}_n(c_k) \in I$, recall that $(x+1)^2 \leq \mathbb{K}_n(x)$ and $\{\mathbb{K}_n(x) = y\}$ is a strong Π_n-functional class. For each $k, m \in \omega$, it holds that

$$\mathbb{K}_n(c_k) \leq G_0^\theta(c_k) \leq G_m^\theta(c_k) \leq G_m^\theta(\max(c_0, \ldots, c_k, d_k)) < b_k.$$

Moreover, for all $m, k \in \omega$,

$$G_m^\theta(\mathbb{K}_n(c_k)) \leq G_{m+1}^\theta(c_k) \leq G_{m+1}^\theta(\max(c_0, \ldots, c_k, d_k)) < b_k.$$

Hence, we prove that $\mathbb{K}_n(c_k) \in I$ as previously: If there is no $i < k$ such that $c_i = \mathbb{K}_n(c_k)$, then there is $j \geq k$ such that $a_{j+1} = \mathbb{K}_n(c_k)$. Since $a' = \max(c_0, \ldots, c_j, d_j, a_{j+1}) = \max(c_0, \ldots, c_j, d_j)$, then, by definition, $\mathbb{K}_n(c_k) = a_{j+1} = c_{j+1} \in I$.

(ii) Since for all $m \in \omega$, $G_m^\theta(d_k) < b_k$, we have $d_k \in I$.

(iii) Let us see that $I \models T_1$. Since $I \models \mathbf{B}\Sigma_{n+1}$, it is enough to prove that $I \models \forall x \exists y \forall z\, \theta(x, y, z)$.

Let $c \in I$. Then there exists $k \in \omega$ such that $c_k = c$ and $c_k < b_k$. By 3.,

$$\mathfrak{A} \models (\exists y_0, \ldots, y_k \leq d_k)\, (\forall z < b_k)\, (\textstyle\bigwedge_{j=0}^k \theta(c_j, y_j, z)).$$

In particular,

$$\mathfrak{A} \models (\exists y \leq d_k)\, (\forall z < b_k)\, \theta(c, y, z).$$

Since $d_k \in I$ and $I \subset^{\mathrm{e}} \mathfrak{A}$, there exists $e \in I$ such that $\mathfrak{A} \models (\forall z < b_k)\, \theta(c, e, z)$. Since $I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $I < b_k$, it holds that $I \models \forall z\, \theta(c, e, z)$. This proves that $I \models \forall x \exists y \forall z\, \theta(x, y, z)$, as required. □ (Claim 3.7.1)

This completes the proof of (•).

Since there exists a Π_n-functional theory (so, consistent) T' such that $T' \vdash \forall x \exists y \forall z\, \theta(x, y, z)$, by Lemma 3.6, T' extends T_0; so, T_0 is consistent. As a consequence, there exists $\mathfrak{A} \models T_0$ countable and recursively saturated and $a, b \in \mathfrak{A}$, such that $\mathfrak{A} \models (\exists y < b)\, (G_k^\theta(a) = y)$, for all $k \in \omega$. Hence, by (•) there exists $I \models T_1$ and this proves (a).

Finally, part (b) follows from (•) and Remark 3.4. □

Proof of Theorem 1.4. We only prove (1) ⇒ (2). Let T be a Π_{n+3}-axiomatizable and Π_n-functional theory and $\psi \in \Pi_{n+2}$ a sentence such that $T + \mathbf{B}\Sigma_{n+1} \vdash \psi$. We must prove that $T \vdash \psi$. Let $\theta(x, y, z) \in \Sigma_n$ such that $T \vdash \forall x \exists y \forall z\, \theta(x, y, z)$ and $\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x, y, z) \vdash \psi$. By Theorem 3.7,

$$\mathrm{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x, y, z)) = \mathrm{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma(\theta)^*);$$

so, $\mathbf{I}\Sigma_n + \Gamma(\theta)^* \vdash \psi$; hence, by Lemma 3.6(1), $T \vdash \psi$. □

As a corollary we get [2, Theorem 3.2]:

Corollary 3.8 *Let* T *be a* Π_{n+3}*-axiomatized extension of* $\mathbf{I}\Delta_0$. *Then*

$$\mathrm{Th}_{\Pi_{n+2}}(T + \mathbf{B}\Sigma_{n+1}) = \mathrm{Th}_{\Pi_{n+2}}(T + \Sigma_{n+1}\text{-CR}).$$

P r o o f. If $T + \Sigma_{n+1}$-CR is inconsistent, then there is nothing to prove. So assume that it is consistent and, thus, by Theorem 1.3, Π_n-functional. Let $\psi \in \Pi_{n+2}$ be a sentence such that $T + \mathbf{B}\Sigma_{n+1} \vdash \psi$. Then there exists $\theta(x, y, z) \in \Sigma_n$ such that $T \vdash \forall x \exists y \forall z\, \theta(x, y, z)$ and $\mathbf{B}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x, y, z) \vdash \psi$. Then, by Theorem 3.7, $\mathbf{I}\Sigma_n + \Gamma(\theta)^* \vdash \psi$. By Lemma 3.6(1), $T' = T + \Sigma_{n+1}$-CR is an extension of $\mathbf{I}\Sigma_n + \Gamma(\theta)^*$. Hence,

$$T + \Sigma_{n+1}\text{-CR} \vdash \psi,$$

as required. □

In [8], R. Kaye, J. Paris and C. Dimitracopoulos proved that $\mathbf{B}\Sigma_{n+1}$ is a Σ_{n+3}-conservative extension of its parameter free version, $\mathbf{B}\Sigma^-_{n+1}$. A proof-theoretic proof of this result was obtained by L. Beklemishev as a consequence of his analysis of collection rule in [2]. Our work provides a new model-theoretic proof of that conservation result.

Corollary 3.9 $\mathbf{B}\Sigma_{n+1}$ *is a* Σ_{n+3}*-conservative extension of* $\mathbf{B}\Sigma^-_{n+1}$.

P r o o f. Let $\psi \in \Sigma_{n+3}$ be a sentence such that $\mathbf{B}\Sigma^-_{n+1} \nvdash \psi$. Then $T' = \mathbf{B}\Sigma^-_{n+1} + \neg\psi$ is a Π_n-functional theory and $\neg\psi$ is a Π_{n+3}-sentence provable in T'. Let $\theta(x, y, z) \in \Sigma_n$ such that $\neg\psi$ is $\forall x \exists y \forall z\, \theta(x, y, z)$. By Theorem 3.7(a), $\mathbf{B}\Sigma_{n+1} + \neg\psi$ is consistent. Hence, $\mathbf{B}\Sigma_{n+1} \nvdash \psi$. □

Remark 3.10 Theorem 1.4 is optimal with respect to the syntactical complexity of the axioms of the theory T. Below we give examples of Σ_{n+3}-axiomatizable theories which are Π_n-functional, but not $\Pi^{\mathbf{B}}_{n+2}$-conservative.

1. Let $\varphi \in \Pi_{n+3}$ be a sentence such that $\mathbf{B}\Sigma_{n+1} \vdash \varphi$ and $\mathbf{B}\Sigma^-_{n+1} \nvdash \varphi$. Let $T = \mathbf{B}\Sigma^-_{n+1} + \neg\varphi$. Since T is a consistent extension of $\mathbf{B}\Sigma^-_{n+1}$, then it is Π_n-functional. It is obvious that $T + \mathbf{B}\Sigma_{n+1}$ is inconsistent. So, T is not $\Pi^{\mathbf{B}}_{n+2}$-conservative.

2. Now we give an example where $T + \mathbf{B}\Sigma_{n+1}$ is consistent. Let $\psi \in \Pi_1$ such that $\mathbf{I}\Sigma_n + \mathbf{exp} \nvdash \psi$ and $\mathbf{B}\Sigma_{n+1} + \mathbf{exp} + \psi$ is consistent. Let T be the theory

$$\mathbf{I}\Sigma_n + \mathbf{exp} + \{\Theta_B \rightarrow \psi\},$$

where Θ_B is a Π_{n+3}-sentence axiomatizing $\mathbf{B}\Sigma_{n+1} + \mathbf{exp}$. It is obvious that $T + \mathbf{B}\Sigma_{n+1}$ is consistent and T is Σ_{n+3}-axiomatizable. It also holds that

$$(\star) \qquad \mathrm{Th}_{\Pi_{n+2}}(T) = \mathrm{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \mathbf{exp}).$$

To prove $(\star)$, it is enough to see that $\mathbf{I}\Sigma_n + \mathbf{exp} \Rightarrow \mathrm{Th}_{\Pi_{n+2}}(T)$. By way of contradiction, suppose that there is a sentence $\varphi \in \Pi_{n+2}$ such that $T \vdash \varphi$, but $\mathbf{I}\Sigma_n + \mathbf{exp} \nvdash \varphi$. We may assume that φ is $\forall x\, \varphi_0(x)$, with $\varphi_0 \in \Sigma_{n+1}$. Then there exists $\mathfrak{A} \vDash \mathbf{I}\Sigma_n + \mathbf{exp} + \neg\varphi$ and $p \in \mathfrak{A}$ such that $\mathfrak{A} \vDash \neg\varphi_0(p)$. Let us denote by $\mathcal{K}_{n+1}(\mathfrak{A}, p)$ the substructure of $\mathfrak{A}$ determined by the Σ_{n+1}-definable elements of $\mathfrak{A}$. It is well known that $\mathcal{K}_{n+1}(\mathfrak{A}, p) \prec_n \mathfrak{A}$ and $\mathcal{K}_{n+1}(\mathfrak{A}, p) \nvDash \mathbf{B}\Sigma_{n+1}$. So,

$$\mathcal{K}_{n+1}(\mathfrak{A}, p) \vDash \mathbf{I}\Sigma_n + \mathbf{exp} + \neg\varphi.$$

Since $\mathcal{K}_{n+1}(\mathfrak{A}, p) \nvDash \mathbf{B}\Sigma_{n+1}$, we get $\mathcal{K}_{n+1}(\mathfrak{A}, p) \vDash T + \neg\varphi$ and this contradicts $T \vdash \varphi$.

Since $\mathbf{I}\Sigma_n + \mathbf{exp}$ is Π_n-functional, by $(\star)$, so is T. However, T is not $\Pi^{\mathbf{B}}_{n+2}$-conservative since $T + \mathbf{B}\Sigma_{n+1} \vdash \psi$ and, by $(\star)$, $T \nvdash \psi$.

4 On Σ_{n+1}-induction rule

The analysis of collection rule we have presented in the previous section can be applied with minor changes to Σ_{n+1}-induction rule. Now the basic result that plays the role of Friedman-Paris' theorem is Parsons' conservation theorem. Actually, here we shall obtain a general version of Parsons' theorem, namely

Theorem 4.1 *Let* T *be a* Π_{n+3}*-axiomatizable extension of* $\mathbf{I}\Delta_0$*. Then we have:*

T is closed under Σ_{n+1}-induction rule $\Leftrightarrow$ T is $\Pi^{\mathbf{I}}_{n+2}$-conservative.

We will derive Theorem 4.1 along the lines used in Section 3 to prove Theorem 1.4. Only two additional facts are needed:

1. If T is closed under Σ_{n+1}-induction rule, then the class of Σ_{n+1}-definable functions which are provably total in T is closed under iteration. That is: Let $F(x) = y$ denote a Σ_{n+1}-formula such that $T \vdash \forall x \exists! y\,(F(x) = y)$. Then there exists a Σ_{n+1}-formula $F'(x, z) = y$ such that T proves the formulas $\forall x \forall z \exists! y\,(F'(x, z) = y)$ and

$$\forall x \forall z\,(F'(x, 0) = x \wedge F'(x, z+1) = F(F'(x, z))).$$

The formula $F'(x, z) = y$ will be denoted by $F^z(x) = y$. Since T is closed under Σ_{n+1}-induction rule, $T \vdash \mathbf{exp}$ and the usual definition of $F^z(x) = y$ works. Moreover, if $T \vdash \mathrm{IPF}(F(x) = y)$, then T proves that the formula $F^{x+1}(x) = y$ also defines a nondecreasing total function.

2. By [2, Lemma 5.1], every theory closed under Σ_{n+1}-IR is also closed under Σ_{n+1}-CR; thus, by Theorem 1.3, if T is consistent and closed under Σ_{n+1}-IR, then T is Π_n-functional.

Bearing in mind these facts, our arguments in Section 3 can be adapted to prove Theorem 4.1. First we associate a pseudo-Π_n-envelope to each Σ_n-formula, $\theta(x, y, z)$, such that $\forall x \exists y \forall z\, \theta(x, y, z)$ is provable in some consistent theory closed under Σ_{n+1}-induction rule. For each $k \in \omega$, $\varphi_k(x, y)$ is a Σ_{n+1}-formula defining the function $H_{\theta,k}(x) = y$ given by

$$\begin{aligned} H_{\theta,0}(x) &= (\mu z)((\forall x_0 \leq x)\,(\exists y_0 < z)\,(\mathbb{K}_n(\max(x, y_0)) < z \wedge (\forall z' < z)\,\theta(x_0, y_0, z'))), \\ H_{\theta,k+1}(x) &= (\mu z)((\forall \vec{x} \leq x)\,(\exists \vec{y} < z)\,(H_{\theta,k}^{x+2}(\mathbb{K}_n(\max(x, \vec{y}\,))) < z \\ &\qquad\qquad \wedge \textstyle\bigwedge_{j=0}^{k+1} (\forall z' < z)\,\theta(x_j, y_j, z'))). \end{aligned}$$

We shall prove that $\Gamma_I(\theta) = \{\varphi_k(x, y) : k \in \omega\}$ is a pseudo-Π_n-envelope of $\mathbf{I}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x, y, z)$ in $\mathbf{I}\Sigma_n + \Gamma_I(\theta)^*$ and satisfies Π_n-IND in recursively saturated models.

Lemma 4.2 *Let $\theta(x, y, z) \in \Sigma_n$ and let T' be a consistent extension of $\mathbf{I}\Delta_0$ closed under Σ_{n+1}-induction rule such that $T' \vdash \forall x \exists y \forall z\, \theta(x, y, z)$ and $\Gamma_I(\theta)$ as above. Then we have:*

(1) *$T' \vdash \Gamma_I(\theta)^*$. So, T' extends $\mathbf{I}\Sigma_n + \Gamma_I(\theta)^*$.*

(2) *For all $k \in \omega$, $\mathbf{I}\Sigma_n + \Gamma_I(\theta)^* \vdash H_{\theta,k+1}(x) = y \to (\exists z < y)\,(H_{\theta,k}^{x+2}(x) = z)$.*

Proof.

(1) We show by induction on m that for all $m \in \omega$,

$$T' \vdash \mathrm{IPF}(H_{\theta,m}(x) = y) \wedge \forall x \exists y\,(H_{\theta,m}(x) = y).$$

$m = 0$: Notice that $H_{\theta,0} = G_0^\theta$ and by Theorem 1.3, T' is Π_n-functional since it is consistent and closed under Σ_{n+1}-collection rule. So, the result follows by Lemma 3.6.

$m \to m+1$: Recall that T' is closed under Σ_{n+1}-IR; so, $H_{\theta,k}^z(x) = y$ defines a total function in T'. Now, we can proceed as in Lemma 3.6.

(2) By (1), $H_{\theta,m}$ is a nondecreasing function; so, part (2) follows from definition. □

Theorem 4.3 *Let $\theta(x, y, z) \in \Sigma_n$ and $\Gamma_I(\theta)$ as above. Then we have:*

(a) *$\mathbf{I}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x, y, z)$ is consistent.*

(b) *The class $\Gamma_I(\theta)$ is a pseudo-Π_n-envelope of $\mathbf{I}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x, y, z)$ in $\mathbf{I}\Sigma_n + \Gamma_I(\theta)^*$ satisfying Π_n-IND in recursively saturated models.*

(c) $\mathrm{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x, y, z)) = \mathrm{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma_I(\theta)^*)$.

Proof. We follow the proof of Theorem 3.7. Part (c) follows from parts (a) and (b) as there. Let us denote $\mathbf{I}\Sigma_{n+1} + \forall x \exists y \forall z\, \theta(x, y, z)$ by T_1 and $\mathbf{I}\Sigma_n + \Gamma_I(\theta)^*$ by T_0. Now, we prove that

(•) $\Gamma_I(\theta)$ satisfies Π_n-IND for T_1 and T_0 in recursively saturated models.

Parts (a) and (b) follow from (•) as in Theorem 3.7.

Let $\mathfrak{A} \models T_0$ countable and recursively saturated and $a, b \in \mathfrak{A}$, such that $\mathfrak{A} \models (\exists y < b)\,(H_{\theta,k}(a) = y)$, for all $k \in \omega$. In order to get (•) we show that there exists $I \models T_1$ such that $I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $a < I < b$.

Let $\{\beta_k(x,y,v) \in \Pi_n : k \in \omega\}$ be an enumeration of all Π_n-formulas with exactly those free variables. Let $\{(a_k, r_k, s_k) : k \in \omega\}$ be an enumeration of all triples of elements of $\mathfrak{A}$ such that $a_0 = a$ and $a_k, r_k, s_k < b$. We can assume that for each $c < b$ the set $\{\beta_j(x, r_j, s_j) : c = a_j\}$ is the set of all Π_n-formulas with only one free variable and two parameters $< b$. Moreover, we can assume that for all $m \in \omega$,

$$\mathfrak{A} \models (\exists y < b)\,(H_{\theta,m}(\max(a_0, r_0, s_0)) = y).$$

We shall define a sequence of elements $\{c_k < b : k \in \omega\}$ and four sequences $\{b_k \in \mathfrak{A} : a \leq b_k < b\}$, $\{d_k \in \mathfrak{A} : a \leq d_k < b\}$, $\{t_k \in \mathfrak{A} : t_k < b\}$ and $\{p_k \in \mathfrak{A} : p_k < b\}$ such that for each $k \in \omega$,

1. $d_{k+1} \geq d_k$ and $b_{k+1} \leq b_k$;
2. $a_0 = c_0$, $c_k < b_k$ and, for all $m \in \omega$, $H_{\theta,m}(\max(c_0, \ldots, c_k, d_k)) < b_k$;
3. $\mathfrak{A} \models (\exists \vec{y} \leq d_k)\,(\bigwedge_{j=0}^{k} (\forall z < b_k)\,\theta(c_j, y_j, z))$;
4. $\mathfrak{A} \models \beta_k(0, t_k, p_k) \wedge (\forall u < c_k)\,((\exists w < b_k)\,\beta_k(u, w, p_k) \to (\exists w < b_k)\,\beta_k(u+1, w, p_k))$
$\to (\forall u \leq c_k)\,(\exists w \leq d_k)\,\beta_k(u, w, p_k)$.

We proceed by recursion on $k \in \omega$. We assume that, for all $m \in \omega$, $H_{\theta,m}(\max(a_0, r_0, s_0)) < b$.

$k = 0$: Let $a' = \max(a_0, r_0, s_0)$ and $c_0 = a_0$, $t_0 = r_0$ and $p_0 = s_0$. Observe that, for each $m \in \omega$,

$$T_0 \vdash \forall x \exists z\,(\forall x_0 \leq x)\,(\exists y_0 < z)\,(H_{\theta,m}(\mathbb{K}_n(\max(x, y_0))) < z \wedge (\forall z' < z)\,\theta(x_0, y_0, z')).$$

For each $m \in \omega$, let $\psi_0(m, y_0, z)$ be the formula $H_{\theta,m}(\mathbb{K}_n(\max(a', y_0))) < z < b \wedge (\forall z' < z)\,\theta(x_0, y_0, z')$.

Then $\mathbf{p}_0(y_0, z) = \{\psi_0(m, y_0, z) : m \in \omega\}$ is a recursive type over $\mathfrak{A}$; hence, as $\mathfrak{A}$ is recursively saturated, there exist $d_0', b_0' \in \mathfrak{A}$ realizing $\mathbf{p}_0(y_0, z)$ in $\mathfrak{A}$; that is, for all $m \in \omega$, $\mathfrak{A} \models \psi(m, d_0', b_0')$. Then, for all $m \in \omega$, $H_{\theta,m}(\max(a', d_0')) < b_0'$ and, clearly, $\mathfrak{A} \models (\exists y_0 \leq d_0')\,(\forall z' < b_0')\,\theta(c_0, y_0, z')$.

Now, there are three cases. Properties 1. – 3. will follow from definitions. We pay attention to 4.

Case A: $\mathfrak{A} \models \neg\beta_0(0, t_0, p_0)$. Then, we set $d_0 = d_0'$ and $b_0 = b_0'$. Obviously, 4. holds.

Case B: $\mathfrak{A} \models \beta_0(0, t_0, p_0)$ and there exists $j \in \omega$ such that

$$(\star) \qquad \mathfrak{A} \models (\forall u < c_0)\,((\exists w_1 < b_0')\,\beta_0(u, w_1, p_0) \to (\exists w_2 < H_{\theta,j}(\max(a', d_0', w_1)))\,\beta_0(u+1, w_2, p_0)).$$

Let $d = \max(a', d_0')$. Observe that T_0 is Π_n-functional; hence, T_0 extends $\mathbf{I}\Delta_{n+1}(T_0)$. On the other hand, the formula $(\exists w < H_{\theta,j}^{x+1}(z))\,\beta_0(u, w, v)$ is $\Delta_{n+1}(T_0)$; so, by induction on $u < c_0$ we prove that

$$(\star\star) \qquad \mathfrak{A} \models (\forall u \leq c_0)\,(\exists w < H_{\theta,j}^{u+1}(d))\,\beta_0(u, w, p_0).$$

$u = 0$: It is enough to observe that $\mathfrak{A} \models \beta_0(0, t_0, p_0) \wedge t_0 \leq a' \leq H_{\theta,j}^{u+1}(d)$.

$u \to u+1$: From the induction hypothesis and $(\star)$ follows that there exists $w < H_{\theta,j}(H_{\theta,j}^{u+1}(d)) = H_{\theta,j}^{u+2}(d)$ such that $\beta_0(u+1, w, p_0)$, as required.

We define $d_0 = H_{\theta,j}^{c_0+1}(d)$ and $b_0 = b_0'$. Observe that $c_0 \leq d_0$ and, by Lemma 4.2(2), for all $m \geq j$,

$$H_{\theta,m}(d_0) \leq H_{\theta,m}(H_{\theta,j}^{d+2}(d)) \leq H_{\theta,m}(H_{\theta,m+1}(d)) \leq H_{\theta,m+1}^2(d) \leq H_{\theta,m+2}(d) < b_0.$$

This yields property 3. Moreover, $H_{\theta,j}$ is nondecreasing and, by $(\star\star)$, $\mathfrak{A} \models (\forall u \leq c_0)\,(\exists w \leq d_0)\,\beta_0(u, w, p_0)$; so, property 4. also holds.

Case C: $\mathfrak{A} \models \beta_0(0, t_0, p_0)$ and for all $j \in \omega$, $(\star)$ does not hold. Let $\psi_0'(m, u, w_1, z)$ be the formula

$$H_{\theta,m}(\max(c_0, d_0', w_1)) < z < b_0' \wedge u < c_0 \wedge \beta_0(u, w_1, p_0) \wedge (\forall w_2 < z)\,\neg\beta_0(u+1, w_2, p_0).$$

Then $\mathbf{q}_0(u, w_1, z) = \{\psi_0'(m, u, w_1, z) : m \in \omega\}$ is a recursive type (since each $H_{\theta,m}$ is a non-decreasing function). So, there exist e, d_0'', b_0'' such that $\mathfrak{A} \models \mathbf{q}_0(e, d_0'', b_0'')$. We define $b_0 = b_0''$ and $d_0 = \max(d_0', d_0'')$. Clearly 4. holds since $\mathfrak{A} \models (\exists u < c_0)\,((\exists w_1 < b_0)\,\beta_0(u, w_1, p_0) \wedge (\forall w_2 < b_k)\,\neg\beta_0(u+1, w_2, p_0))$.

$k \to k+1$: Assume that c_k, b_k, d_k, p_k and t_k have been defined. We define $c_{k+1}, b_{k+1}, d_{k+1}, p_{k+1}$ and t_{k+1} as follows. Let $a' = \max\{c_0, \ldots, c_k, d_k, a_{k+1}, r_{k+1}, s_{k+1}\}$. We distinguish two cases:

Case 1: There exists $m \in \omega$ such that $\mathfrak{A} \models H_{\theta,m}(a') \geq b_k$. Then define $b_{k+1} = b_k$, $c_{k+1} = c_k$, $d_{k+1} = d_k$, $p_{k+1} = p_k$ and $t_{k+1} = t_k$.

Case 2: For all $m \in \omega$, $\mathfrak{A} \models H_{\theta,m}(a') < b_k$. Let $c_{k+1} = a_{k+1}$, $p_{k+1} = s_{k+1}$ and $t_{k+1} = r_{k+1}$. Then for each $m \in \omega$, T_0 proves that

$$\forall x \exists z\, (\forall \vec{x} \leq x)\, (\exists \vec{y} < z)\, (H_{\theta,k}(\mathbb{K}_n(\max(x, \vec{y}))) < z \wedge \bigwedge_{j=0}^{k+1} (\forall z' < z)\, \theta(x_j, y_j, z')).$$

Let $\psi_{k+1}(m, \vec{y}, z)$ be the formula $H_{\theta,m}(\mathbb{K}_n(\max(a', \vec{y}))) < z < b_k \wedge \bigwedge_{j=0}^{k+1} (\forall z' < z)\, \theta(c_j, y_j, z')$. Then

$$\mathbf{p}_{k+1}(\vec{y}, z) = \{\psi_{k+1}(m, \vec{y}, z) \,:\, m \in \omega\}$$

is a recursive type over $\mathfrak{A}$; as a consequence, there exist $e_0, \ldots, e_{k+1}, b'_{k+1} \in \mathfrak{A}$, such that for all $m \in \omega$, $\mathfrak{A} \models \psi_{k+1}(m, \vec{e}, d, b_{k+1})$. Let $d'_{k+1} = \max(d_k, \vec{e})$.

Now, we distinguish three cases:

Case A: $\mathfrak{A} \models \neg\beta_{k+1}(0, t_{k+1}, p_{k+1})$. Then, we set $d_{k+1} = d'_{k+1}$ and $b_{k+1} = b'_{k+1}$.

Case B: $\mathfrak{A} \models \beta_{k+1}(0, t_{k+1}, p_{k+1})$ and there exists $j \in \omega$ such that

$$\mathfrak{A} \models (\forall u < c_{k+1})\, ((\exists w_1 < b'_{k+1})\, \beta_{k+1}(u, w_1, p_{k+1}) \\ \rightarrow (\exists w_2 < H_{\theta,j}(\max(a', d'_{k+1}, w_1)))\, \beta_{k+1}(u+1, w_2, p_{k+1})).$$

Then, we set $d_{k+1} = H_{\theta,j}^{c_{k+1}+1}(\max(a', d'_{k+1}))$ and $b_{k+1} = b'_{k+1}$. Properties 1. – 4. follow as for $k = 0$.

Case C: $\mathfrak{A} \models \beta_{k+1}(0, t_{k+1}, p_{k+1})$ and Case B fails. Let $\psi'_{k+1}(m, u, w_1, z)$ be the formula

$$H_{\theta,m}(\max(a', d'_{k+1}, w_1)) < z < b'_{k+1} \wedge u < c_{k+1} \wedge \beta_{k+1}(u, w_1, p_{k+1}) \\ \wedge (\forall w_2 < z)\, \neg\beta_{k+1}(u+1, w_2, p_{k+1}).$$

As for $k = 0$, $\mathbf{q}_{k+1}(u, w_1, z) = \{\psi'_{k+1}(m, u, w_1, z) \,:\, m \in \omega\}$ is a recursive type. So, there exist e, d''_{k+1}, b''_{k+1} such that $\mathfrak{A} \models \mathbf{q}_{k+1}(e, d''_{k+1}, b''_{k+1})$. We define $b_{k+1} = b''_{k+1}$ and $d_{k+1} = \max(d'_{k+1}, d''_{k+1})$.

This concludes the definition of the five sequences. Let $I = \{c_k \,:\, k \in \omega\}$. Obviously $a < I < b$.

Claim 4.3.1

(i) $I \prec_n^{\mathrm{e}} \mathfrak{A}$.

(ii) *For all* $k \in \omega$, $d_k, t_k, p_k \in I$.

(iii) $I \models T_1$.

Proof. (i) and (ii) are proved as in Theorem 3.7.

(iii) We only prove $I \models \mathbf{I}\Sigma_{n+1}$ since $I \models \forall x \exists y \forall z\, \theta(x, y, z)$ can be checked as in Theorem 3.7.

Let $\beta(x, y, p)$ be a Π_n-formula with parameter $p \in I$ and $t \in I$ such that

(H) $\quad I \models \beta(0, t, p) \wedge \forall x\, (\exists y\, \beta(x, y, p) \rightarrow \exists y\, \beta(x+1, y, p)).$

Let $c \in I$. We must prove that $I \models \exists y\, \beta(c, y, p)$. By construction, there exists $k \in \omega$ such that $c_k = c$, $t_k = t$, $p_k = p$ and $\beta_k = \beta$. Since $I \prec_n^{\mathrm{e}} \mathfrak{A}$, by (H), at step k of the construction we are in Case B. So, it follows that $\mathfrak{A} \models (\forall u \leq c_k)\, (\exists w \leq d_k)\, \beta_k(u, w, p_k)$. In particular, $\mathfrak{A} \models (\exists w \leq d_k)\, \beta_k(c_k, w, p_k)$; hence,

$$I \models (\exists w \leq d_k)\, \beta(c, w, p)$$

since $I \prec_n^{\mathrm{e}} \mathfrak{A}$ and $d_k \in I$. □ (Claim 4.3.1)

This completes the proof of (•). Parts (a) and (b) follow from (•) as in Theorem 3.7. □

Now Theorem 4.1 can be derived from Theorem 4.3 following the proof of Theorem 1.4 in Section 3. Moreover, modifying accordingly the proofs of Corollaries 3.8 and 3.9 we get

Corollary 4.4

1. $\mathbf{I}\Sigma_{n+1}$ *is a* Σ_{n+3}*-conservative extension of* $\mathbf{I}\Sigma_{n+1}^{-}$.
2. *Let* T *be a* Π_{n+3}*-axiomatizable extension of* $\mathbf{I}\Delta_0$. *Then* $T + \Sigma_{n+1}$-IR *is* $\Pi_{n+2}^{\mathbf{I}}$*-conservative.*

Acknowledgements The present work was partially supported by the Andalusian Government, grant TIC-137.

References

[1] L. Beklemishev, Induction rules, reflection principles, and provably total recursive functions. Ann. Pure Applied Logic **85**, 193 – 242 (1997).
[2] L. Beklemishev, A proof-theoretic analysis of collection. Archive Math. Logic **37**, 275 – 296 (1998).
[3] A. Cordón-Franco, A. Fernández-Margarit, and F. F. Lara-Martín, On the quantifier complexity of $\Delta_{n+1}(T)$-induction. Archive Math. Logic **43**, 371 – 398 (2004).
[4] A. Fernández-Margarit and F. F. Lara-Martín, Induction, minimization and collection for $\Delta_{n+1}(T)$-formulas. Archive Math. Logic **43**, 505 – 541 (2004).
[5] P. Hájek and P. Pudlák, Metamathematics of First Order Arithmetic. Perspectives in Mathematical Logic (Springer-Verlag, 1993).
[6] R. Kaye, Diophantine and parameter-free induction. Ph. D. thesis, University of Manchester, 1987.
[7] R. Kaye, Models of Peano Arithmetic (Clarendon Press, 1991).
[8] R. Kaye, J. Paris, and C. Dimitracopoulos, On parameter free induction schemas. J. Symbolic Logic **53**, 1082 – 1097 (1988).
[9] L. Kirby, Initial segments of models of arithmetic. Ph. D. thesis, University of Manchester, 1977.
[10] L. Kirby, La méthode des indicatrices et le théorèm d'incompletude. In: Modèles de l'arithmétique (K. McAloon, ed.), Astérisque 73, pp. 5 – 18 (Société Mathématique de France, 1980).
[11] L. Kirby, K. McAloon, and R. Murawski, Indicators, recursive saturation and expandability. Fundamenta Mathematicae **114**, 127 – 139 (1981).
[12] L. Kirby and J. Paris, Initial segments of models of Peano's axioms. In: Conference on Set Theory and Hierarchy Theory, Bierutowice, 1976, Lecture Notes in Mathematics 619, pp. 211 – 226 (Springer-Verlag, 1977).
[13] K. McAloon, Models of arithmetic and complexity theory. In: Studies in Complexity Theory (R. Book, ed.), pp. 119 – 226 (Pitman, 1986).
[14] J. Paris, Some independence results for Peano arithmetic. J. Symbolic Logic **43**, 725 – 731 (1978).
[15] J. Paris, A hierarchy of cuts in models of Arithmetic. In: Lecture Notes in Mathematics 834, pp. 312 – 337 (Springer-Verlag, 1979).
[16] C. Parsons, On n-quantifier induction. J. Symbolic Logic **37** (3), 466 – 482 (1972).

