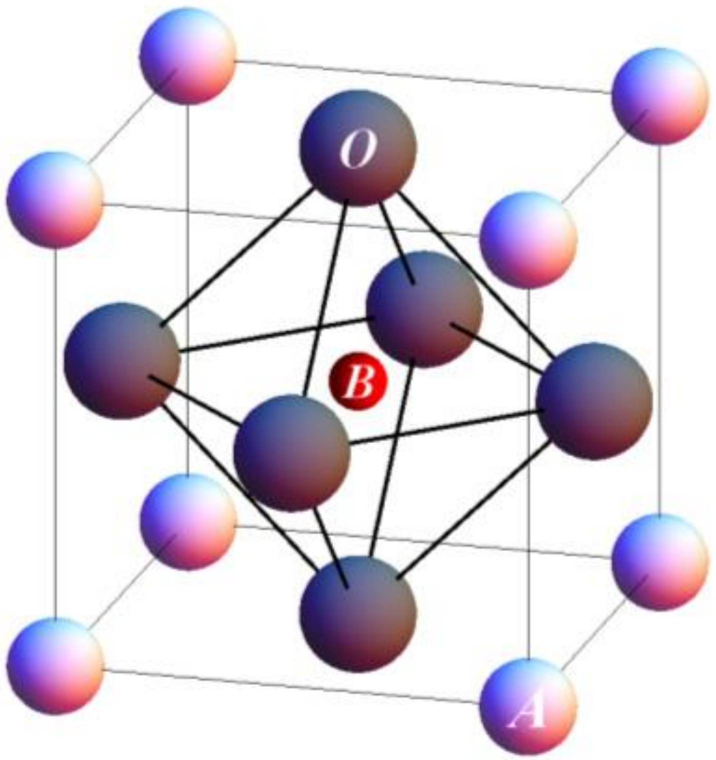
Multicritical behavior near the structural phase transitions in the perovskites

Amnon Aharony, Tel Aviv University



ABO_3

$A = Sr,\ La,\ Ba,\ \ldots$

$B = Ti,\ Al,\ ,Mn,\ \ldots$

$SrTiO_3:$ *cubic to tetragonal*

$LaAlO_3:$ *cubic to trigonal*

https://www.tau.ac.il/~aharony/

Talk at Bootstat2021

Rockets on Israel – 10 min break on alarm

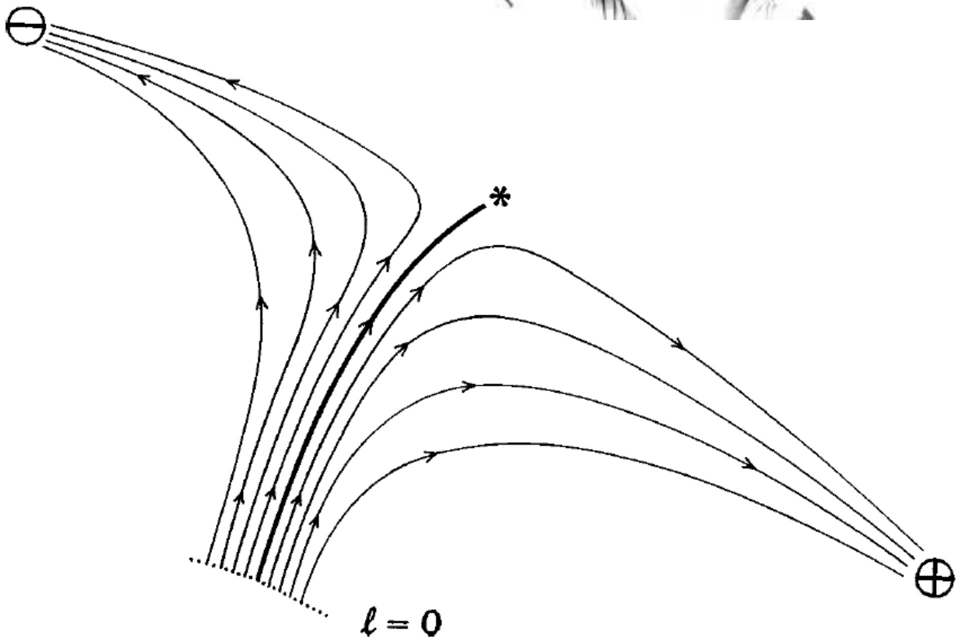
Reviews of Modern Physics, Vol. 70, No. 2, April 1998

Renormalization group theory: Its basis and formulation in statistical physics*

Michael E. Fisher

FOREWORD

"In March 1996 the Departments of Philosophy and of Physics at Boston University cosponsored a Colloquium 'On the Foundations of Quantum Field Theory.' But in the full title, this was preceded by the phrase 'A Historical Examination and Philosophical Reflections,' which set the aims of the meeting. The participants were mainly high-energy physicists, experts in field theories, and interested philosophers of science.[1] I was called on to speak, essentially in a service role, presumably because I had witnessed and had some hand in the development of renormalization group concepts and because I have played a role in applications where these ideas really mattered. It is hoped that this article, based on the talk I presented in Boston, may prove of interest to a wider audience."



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The participants here are mainly experts in conformal bootstrap techniques. I was called on to speak, essentially in a service role, presumably because I witnessed and had some hand in the history of multicritical phenomena in perovskites. This talk presents my **subjective history** of 50 years on this topic, with simulating theory-experiment feedbacks.

Table 2. *Examples of known tilt systems*

Substance	Cation displacements	Observed space group	Space group of tilted framework alone
$NaNbO_3(S)$	Na?	*Pnmm*	*Pnmm*
$YAlO_3$	Y	*Pbnm*	*Pbnm*
$SmAlO_3$	Sm	*Pbnm*	*Pbnm*
$EuAlO_3$*	Eu?	*Pbnm*	*Pbnm*
$GdAlO_3$*	Gd?		
$DyAlO_3$	Dy	*Pbnm*	*Pbnm*
$BaCeO_3$	Ba	*Pbnm*	*Pbnm*
$YCrO_3$,	Y		
$YFeO_3$, $LaFeO_3$,	Y, La,		
$PrFeO_3$, $NdFeO_3$,	Pr, Nd,	*Pbnm*	*Pbnm*
$SmFeO_3$, $EuFeO_3$,	Sm, Eu,		
$GdFeO_3$	Gd		
$TbFeO_3$, $DyFeO_3$,	Tb, Dy		
$HoFeO_3$, $ErFeO_3$,	Ho, Er	*Pbnm*	*Pbnm*
$TmFeO_3$, $YbFeO_3$,	Tm, Yb		
$LuFeO_3$	Lu		
$NaMgF_3$(< 760 °C)*		*Pbnm*	*Pbnm*
$YNiO_3$, $SmNiO_3$,	Y, Sm,		
$EuNiO_3$, $GdNiO_3$,	Eu, Gd,		
$DyNiO_3$, $HoNiO_3$,	Dy, Ho,	*Pbnm*	*Pbnm*
$ErNiO_3$, $TmNiO_3$,	Er, Tm		
$YbNiO_3$, $LuNiO_3$	Yb, Lu		
$BaPrO_3$	Ba	*Pbnm*	*Pbnm*
$CaTiO_3$		*Pbnm*	*Pbnm*
$CdTiO_3$	Cd, Ti	*Pbn*2_1	*Pbnm*
$LaAlO_3$	La	$R\bar{3}c$	$R\bar{3}c$
$PrAlO_3$(172–293 °K)		$R\bar{3}c$	$R\bar{3}c$
$NdAlO_3$		$R\bar{3}c$	$R\bar{3}c$
$LaCoO_3$		$R\bar{3}c$	$R\bar{3}c$
FeF_3, CoF_3,			
RuF_3, RhF_3,		$R\bar{3}c$	$R\bar{3}c$
PdF_3, IrF_3			
VF_3		$R\bar{3}c$	$R\bar{3}c$
$BiFeO_3$	Bi, Fe	$R3c$	$R\bar{3}c$
$LiNbO_3$	Li, Nb	$R3c$	$R\bar{3}c$
$NaNbO_3(N)$	Na, Nb	$R3c$	$R\bar{3}c$
$LiTaO_3$	Li, Ta	$R3c$	$R\bar{3}c$
$BaTbO_3$		$R\bar{3}c$	$R\bar{3}c$
$PbZr_{0.9}Ti_{0.1}O_3$	Pb, (Zr, Ti)	$R3c$	$R\bar{3}c$

Minerology:

Perovskite ($CaTiO_3$)

lueshite($NaNbO_3$),

tausonite ($SrTiO_3$),

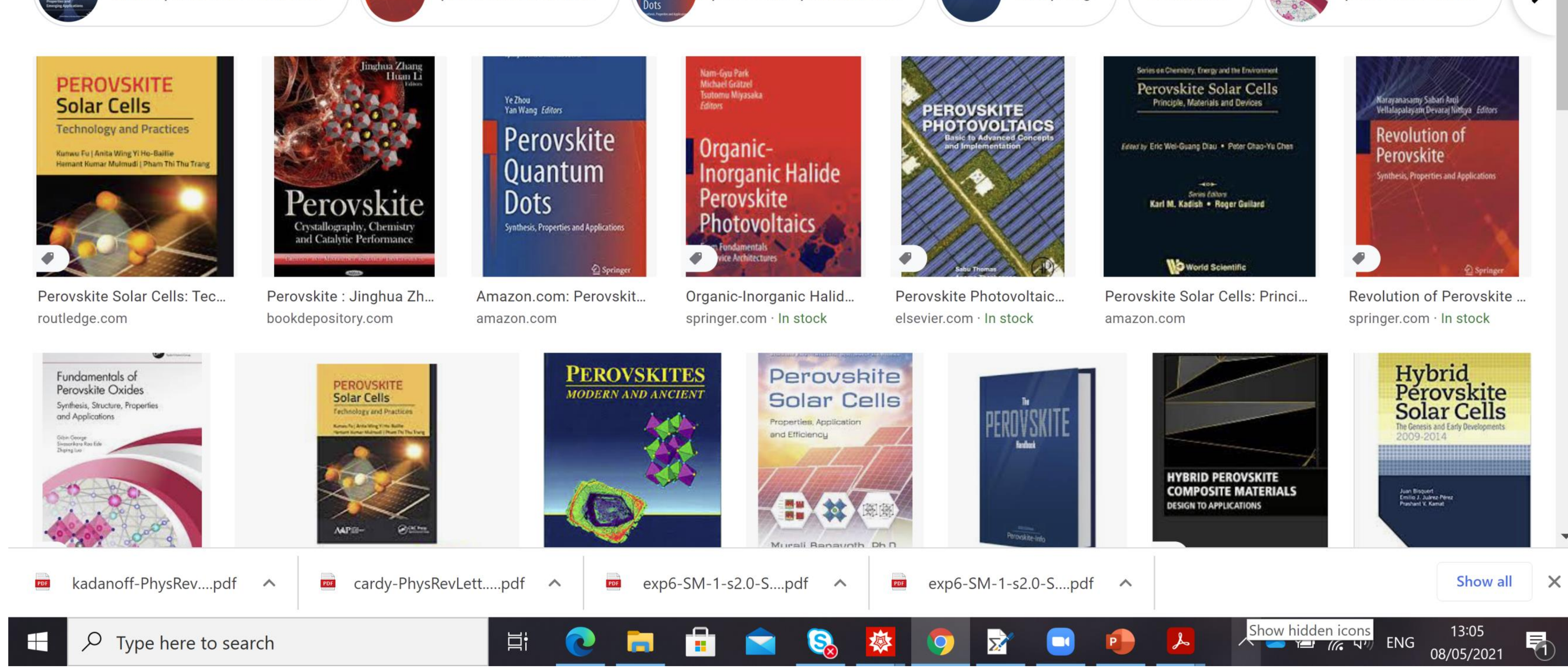
macedonite ($PbTiO_3$),

lakargiite ($CaZrO_3$),

barioperovskite ($BaTiO_3$),

neighborite ($NaMgF_3$)

Ferroelectrics,
Magnets,
Multiferroics,
Colossal magnetoresistance,
Superconductivity,
Charge ordering,
Solar cells,
Memory devices in spintronics,
Catalysts, …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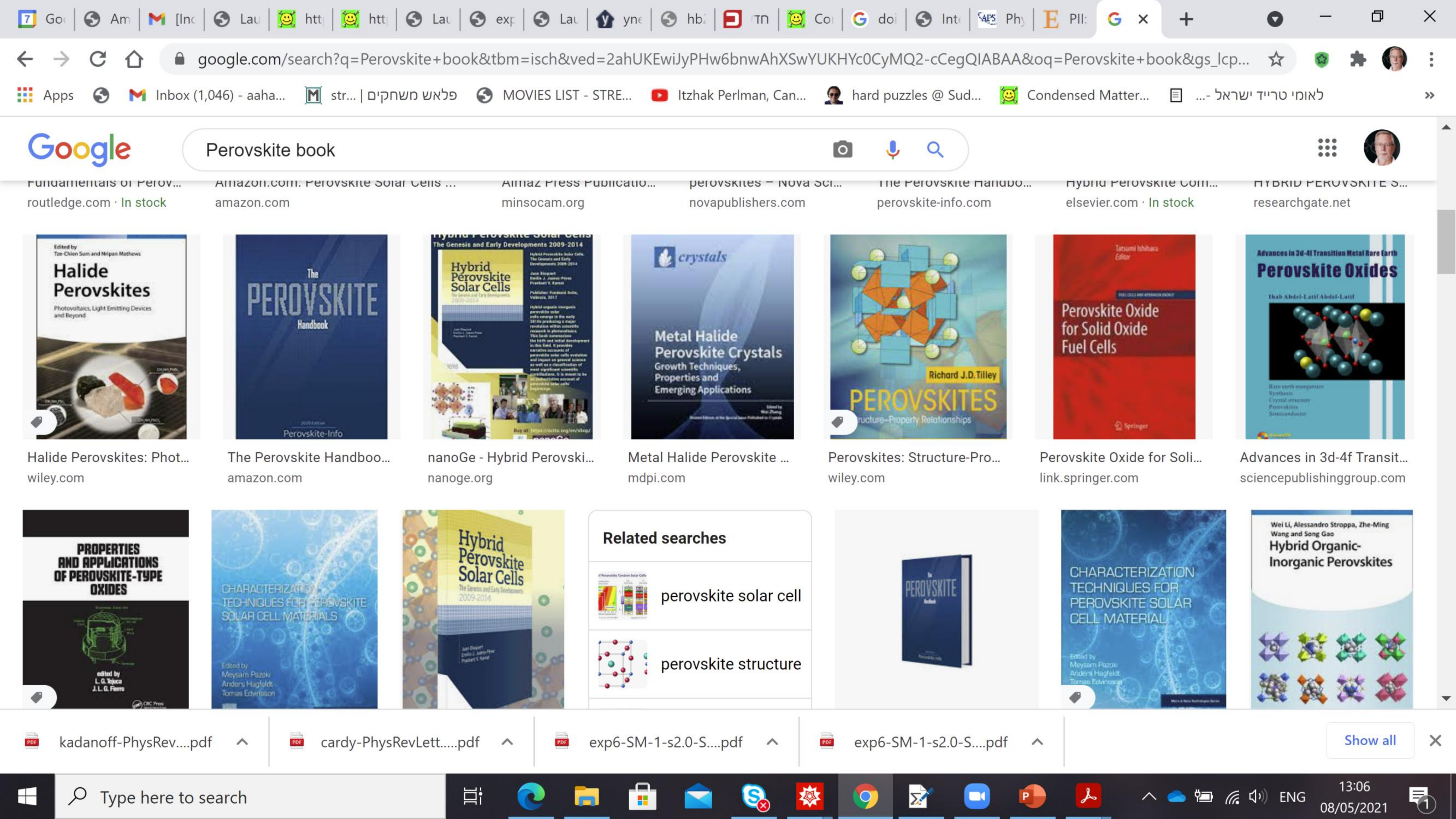
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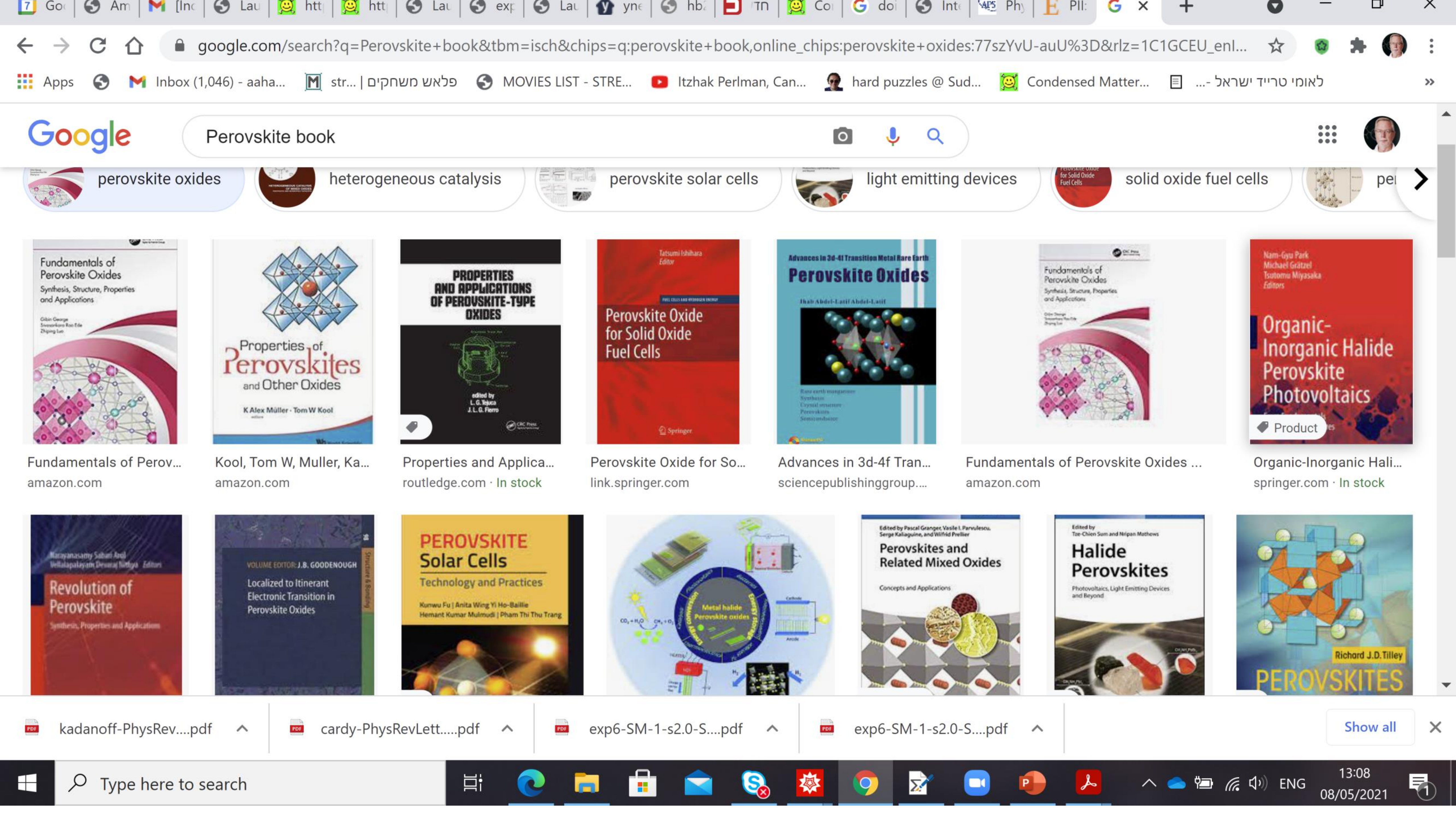
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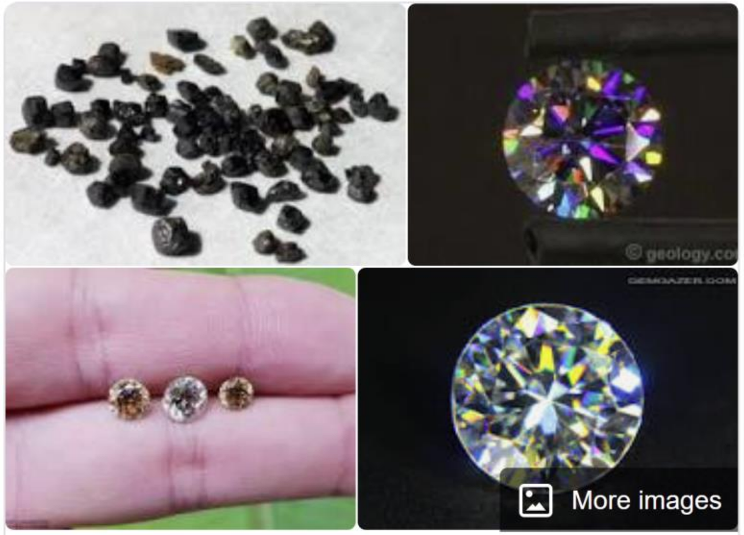
Strontium titanate - Wikipedia

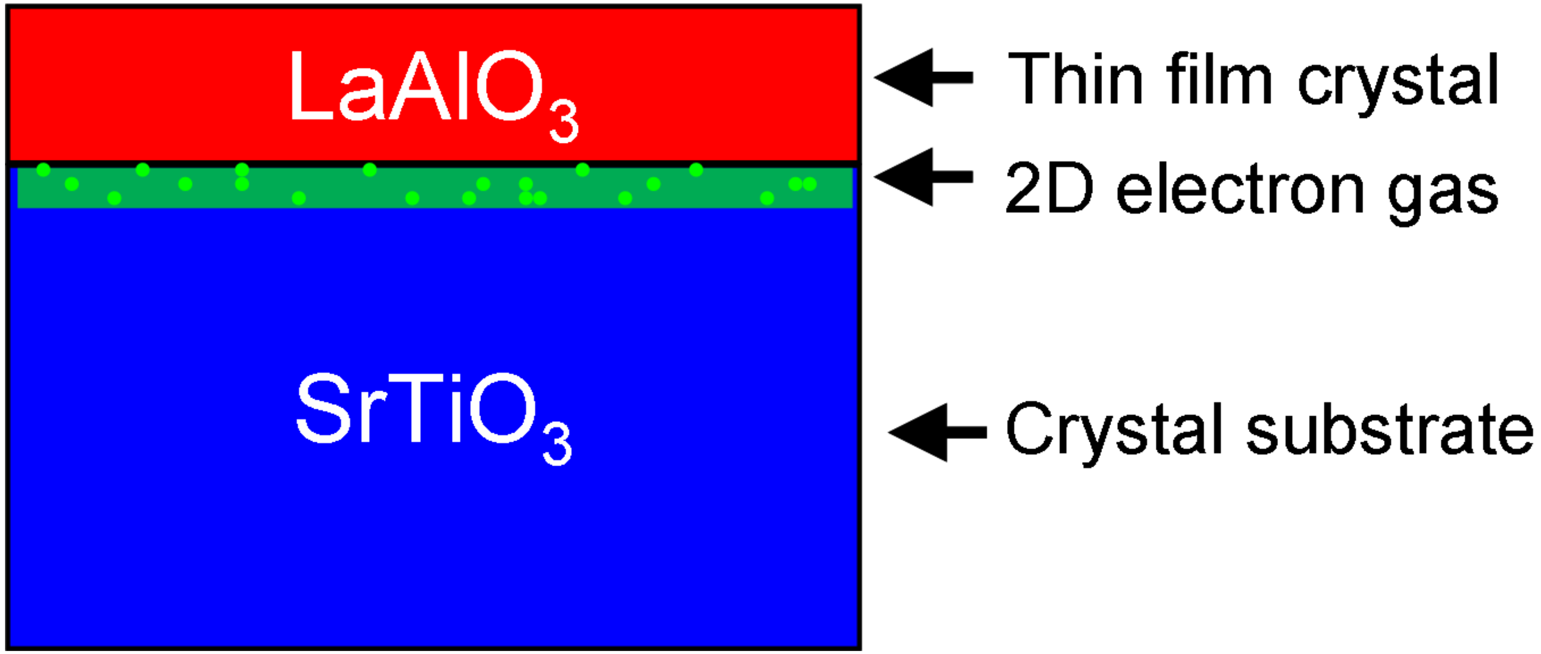
Strontium titanate is an oxide of strontium and titanium with the chemical formula SrTiO3. At room temperature, it is a centrosymmetric paraelectric material with a ...



Strontium titanate

Chemical compound

Strontium titanate is an oxide of strontium and titanium with the chemical formula $SrTiO_3$. At room



Conductive interfaces[edit]

- $GdTiO_3/SrTiO_3$[41]
- $LaTiO_3/SrTiO_3$[42]
- $LaVO_3/SrTiO_3$[42]
- $LaGaO_3/SrTiO_3$[43]
- $PrAlO_3/SrTiO_3$[44]
- $NdAlO_3/SrTiO_3$[44]
- $NdGaO_3/SrTiO_3$[44]
- $GdAlO_3/SrTiO_3$[45]
- $Nd_{0.35}Sr_{0.65}MnO_3/SrTiO_3$[46]
- $Al_2O_3/SrTiO_3$[47]
- amorphous-$YAlO_3/SrTiO_3$[40]
- $La_{0.5}Al_{0.5}Sr_{0.5}Ti_{0.5}O_3/SrTiO_3$[10]
- $DyScO_3/SrTiO_3$[48]
- $KTaO_3/SrTiO_3$[49]
- $CaZrO_3/SrTiO_3$[50]

Insulating interfaces[edit]

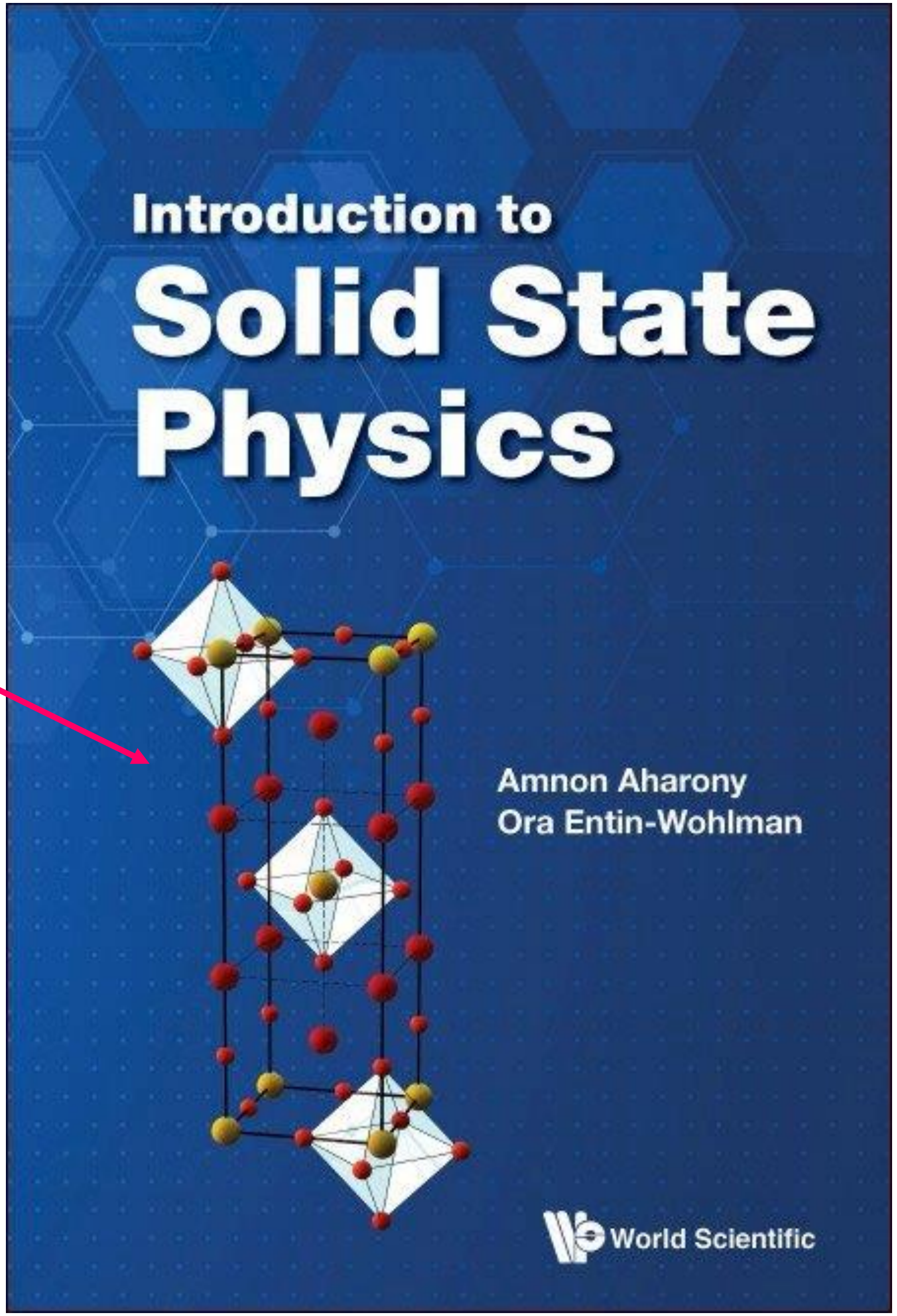
- $LaCrO_3/SrTiO_3$[51]
- $LaMnO_3/SrTiO_3$[43]
- $La_2O_3/SrTiO_3$[40]
- $Y_2O_3/SrTiO_3$[40]
- $LaYO_3/SrTiO_3$[40]
- $EuAlO_3/SrTiO_3$[45]
- $BiMnO_3/SrTiO_3$[52]

Double perovskite

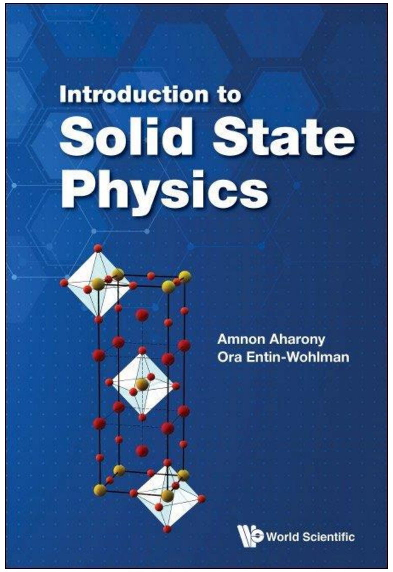
$$A_2BO_4$$

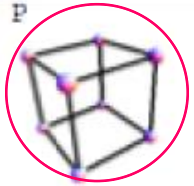
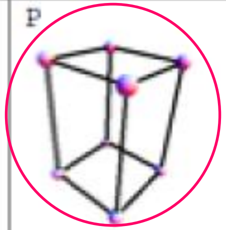
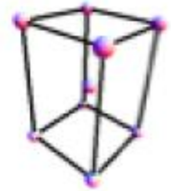
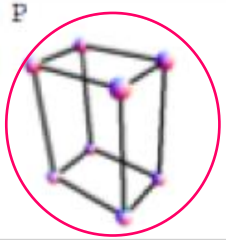
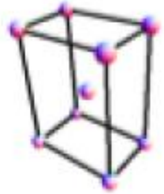
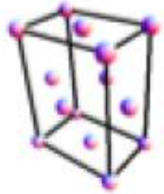
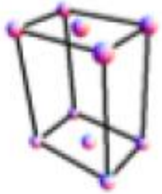
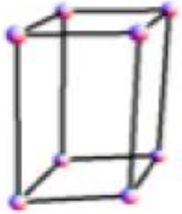
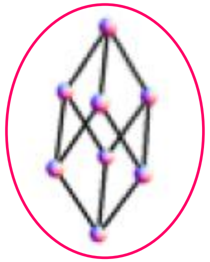
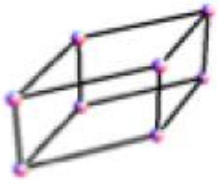
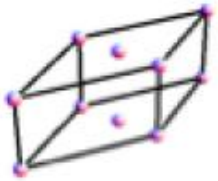
$$La_2CuO_4$$

Parent material of high Temperature superconductors



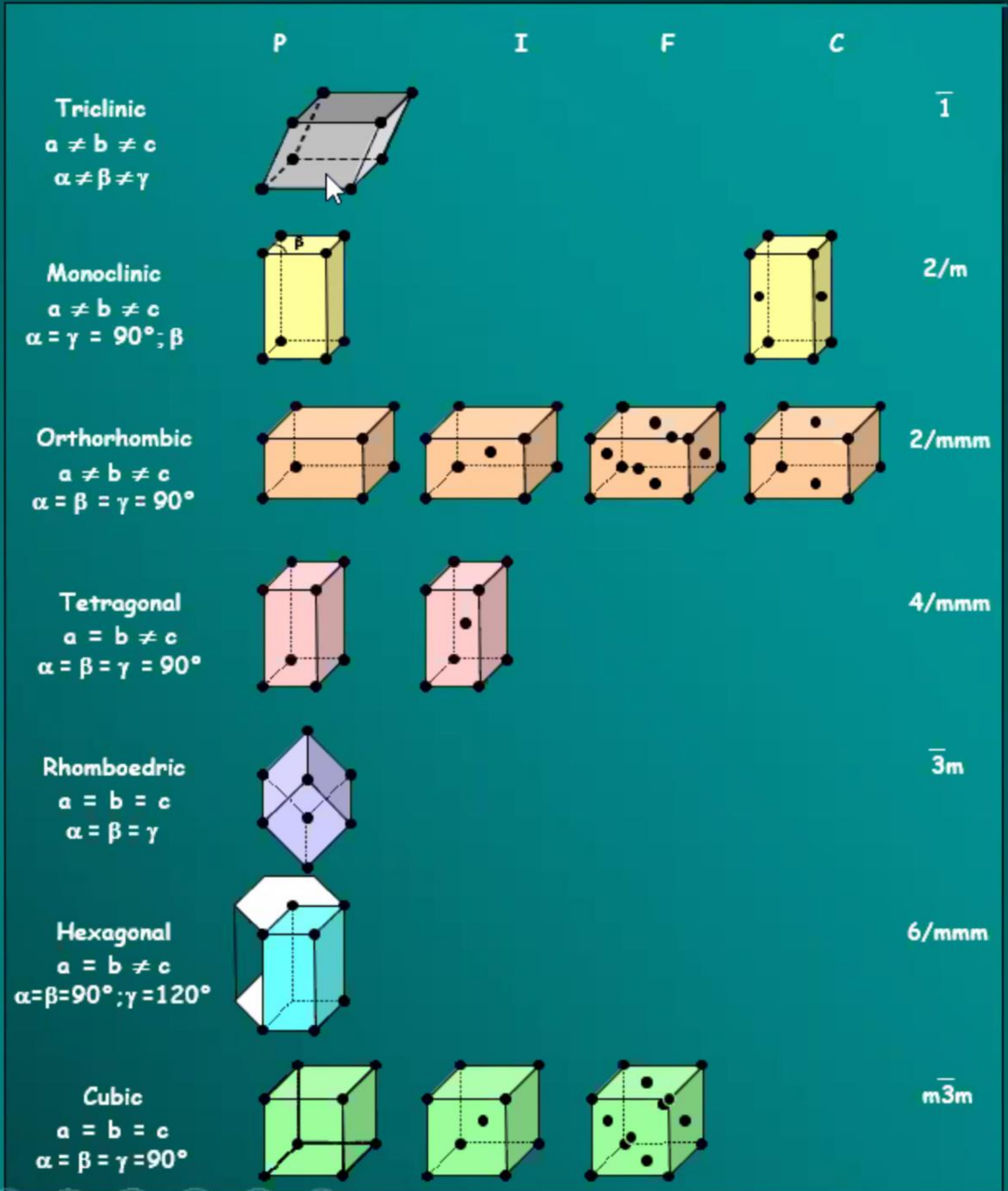
Tetragonal to Orthorhombic



Cubic lattices $a_1 = a_2 = a_3$ $\alpha = \beta = \gamma = 90^\circ$	P 	B 	F 	
Tetragonal lattices $a_1 = a_2 \neq a_3$ $\alpha = \beta = \gamma = 90^\circ$	P 	B 		
Orthorhombic lattices $a_1 \neq a_2 \neq a_3$ $\alpha = \beta = \gamma = 90^\circ$	P 	B 	F 	C 
Hexagonal lattice $a_1 = a_2 \neq a_3$ $\alpha = \beta = 90^\circ$ $\gamma = 120^\circ$			Trigonal lattice $a_1 = a_2 = a_3$ $\alpha = \beta = \gamma \neq 90^\circ$	
Monoclinic lattices $a_1 = a_2 \neq a_3$ $\alpha = \beta = 90^\circ$ $\gamma = 120^\circ$	P 	C 	Triclinic lattice $a_1 \neq a_2 \neq a_3$ $\alpha \neq \beta \neq \gamma \neq 90^\circ$	

Recording

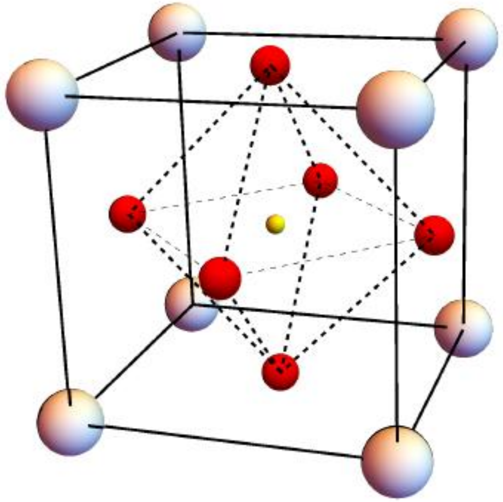
Bravais lattices

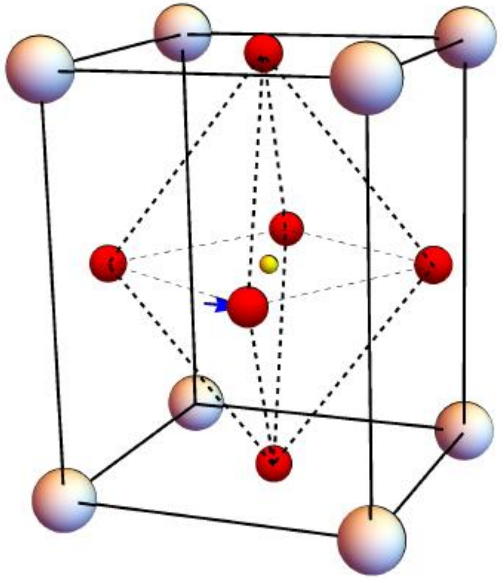


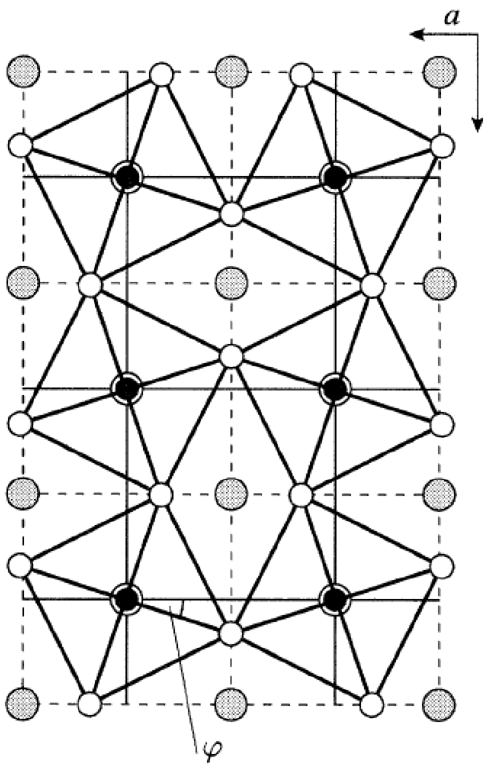
- In 3D
  - 7 systems (symmetry)
  - 14 lattice modes

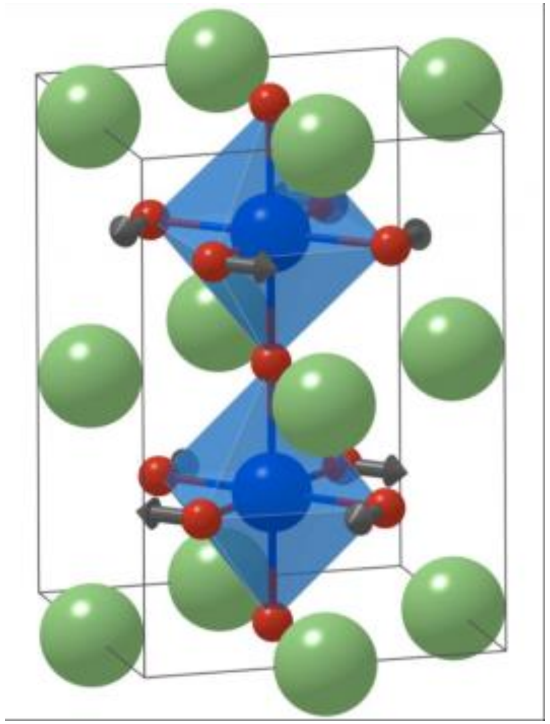
$SrTiO_3$: cubic to tetragonal

Also $RbCaF_3$





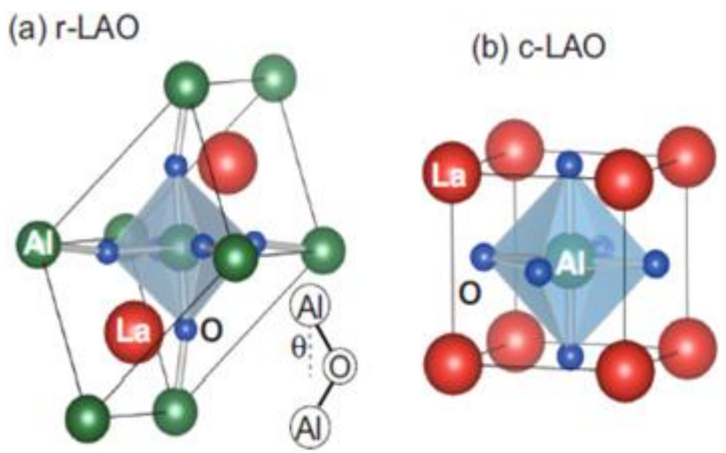


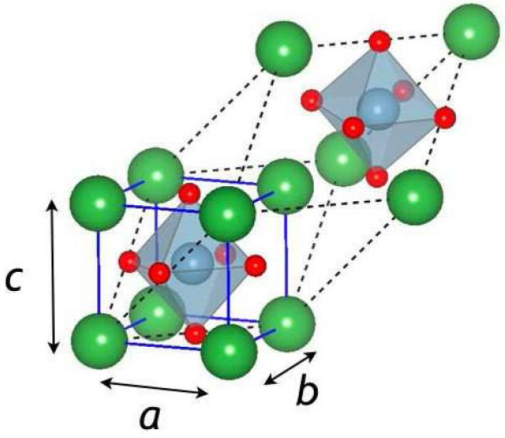


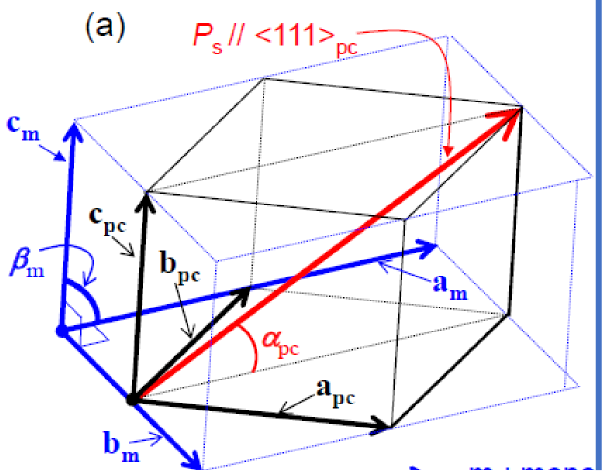
Octahedra rotate – move Sr towards vertical axis – tetragonal

antiferrodistortive

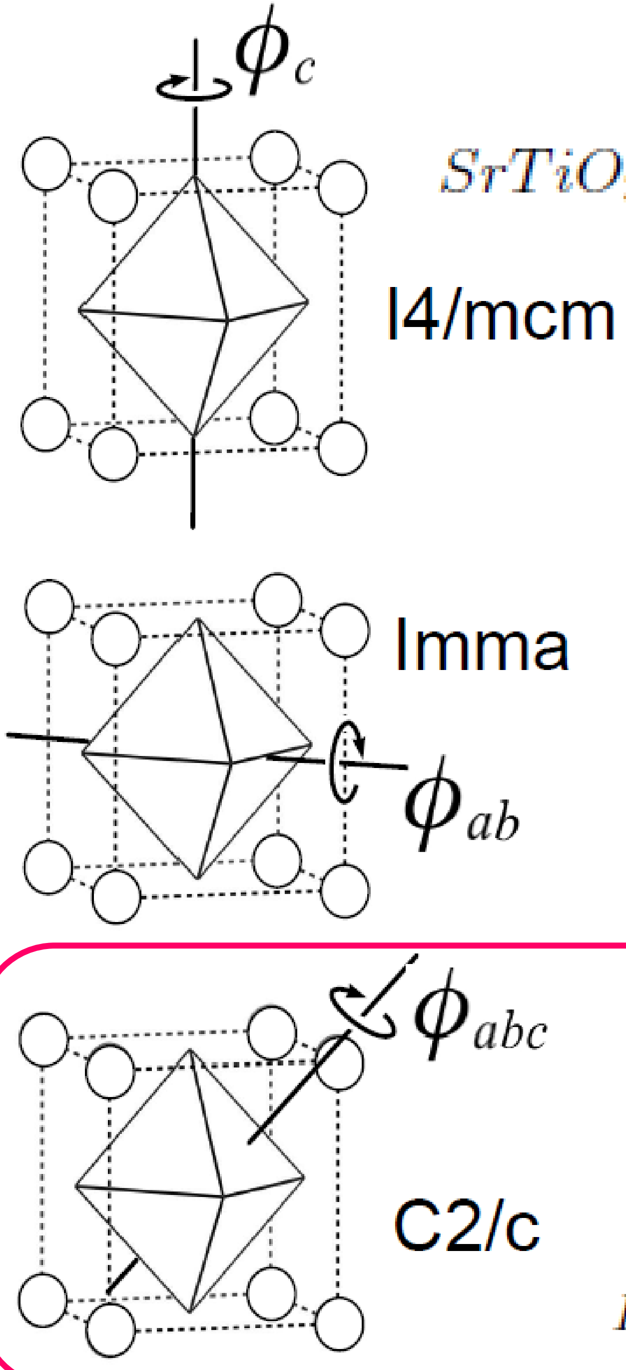
$LaAlO_3$: *cubic to trigonal*







$SrTiO_3$: cubic to tetragonal



$LaAlO_3$: cubic to trigonal

Landau theory

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Interaction of Elastic Strain with the Structural Transition of Strontium Titanate*

J. C. Slonczewski
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598
AND
H. Thomas

Order parameter – vector of rotation ***Q***

Q has n=3=d components

$$G = G_0 + \tfrac{1}{2}a(T - T_C)\boldsymbol{Q}^2 + (\boldsymbol{Q}^2)^2 + v(Q_1^4 + Q_2^4 + Q_3^4) + e_1(\eta_1 + \eta_2 + \eta_3)\boldsymbol{Q}^2$$
$$+ e_2[\eta_1(2Q_1^2 - Q_2^2 - Q_3^2) + \eta_2(2Q_2^2 - Q_1^2 - Q_3^2) + \eta_3(2Q_3^2 - Q_1^2 - Q_2^2)]$$
$$+ e_3(Q_1Q_2\eta_6 + Q_1Q_3\eta_5 + Q_2Q_3\eta_4) + \tfrac{1}{2}\sum_{\alpha\beta} C_{\alpha\beta}\eta_\alpha\eta_\beta,$$

Elastic strains

v>0 ➡ order along [111], trigonal.

u+v/3>0

v<0 ➡ order along [001], tetragonal.

u+v>0

$$|\mathbf{Q}|^2 = 0 \ \ for \ T > T_c, \quad |\mathbf{Q}|^2 \propto (T_c - T), \quad T < T_c$$

Soft mode

$$G = G_0 + \tfrac{1}{2}a(T - T_{\mathrm{C}})\boldsymbol{Q}^2 + u(\boldsymbol{Q}^2)^2 + v(Q_1^4 + Q_2^4 + Q_3^4) + e_1(\eta_1 + \eta_2 + \eta_3)\boldsymbol{Q}^2$$
$$+ e_2[\eta_1(2Q_1^2 - Q_2^2 - Q_3^2) + \eta_2(2Q_2^2 - Q_1^2 - Q_3^2) + \eta_3(2Q_3^2 - Q_1^2 - Q_2^2)]$$
$$+ e_3(Q_1Q_2\eta_6 + Q_1Q_3\eta_5 + Q_2Q_3\eta_4) + \tfrac{1}{2}\sum_{\alpha\beta} C_{\alpha\beta}\eta_\alpha\eta_\beta,$$

Can minimize over the eta’s, obtain a “renormalized” energy for the Q’s

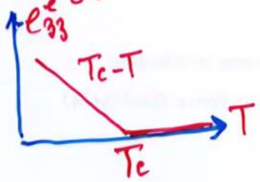
$$G = \bar{G}_0 + \tfrac{1}{2}\bar{a}(T - T_{\mathrm{C}})\boldsymbol{Q}^2 + \bar{u}(\boldsymbol{Q}^2)^2 + \bar{v}(Q_1^4 + Q_2^4 + Q_3^4)$$

$$F(T,P,p_z,e_{zz}) = a_0(T,P) + a_{02}(T-T_c)\,p_z^2 + a_4\,p_z^4 + \delta\, e_{zz}\, p_z^2 + \frac{C_{33}}{2} e_{zz}^2$$

Coupling between e_{zz} and OP — Elastic energy

$$\frac{\partial F}{\partial e_{zz}} = \delta\, p_z^2 + C_{33}\, e_{zz} = 0$$

$$e_{zz}^e = -\frac{\delta}{C_{33}} p_z^2 = -\frac{\delta a_{02}(T_c - T)}{2 a_4 C_{33}}$$



$$F(T,P,p_z) = a_0(T,P) + a_{02}(T-T_c)\,p_z^2 + \left(a_4 - \frac{\delta^2}{2C_{33}}\right) p_z^4$$

$\Rightarrow e_{zz}$ renormalizes a_4 coefficient $\Rightarrow$ { change of limit of stability of the phase; 2nd order $\rightarrow$ 1st order ($a_4 < 0$) }

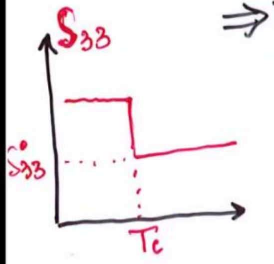
Elastic susceptibility $\chi_{\sigma_{zz}} = \lim_{\sigma_{zz}\to 0} \frac{\partial e_{zz}}{\partial \sigma_{zz}}$

$$G = F - \sigma_{zz} e_{zz} \Rightarrow \begin{cases} \frac{\partial G}{\partial e_{zz}} = \delta\, p_z^2 + C_{33} e_{zz} - \sigma_{zz} = 0 \\ 2 p_z \left[a_{02}(T-T_c) + 2 a_4 p_z^2 + \delta e_{zz} \right] = 0 \end{cases}$$

Second minimization with respect to σ_{zz}:

$$\Rightarrow T \geq T_c\ (p_z = 0) \Rightarrow \chi^0_{\sigma_{zz}} = \frac{1}{C_{33}} = S^0_{33}$$

$$T < T_c\ (p_z \neq 0) \Rightarrow \chi_{zz} = \frac{1}{C_{33} - \frac{\delta^2}{2a_4}}$$



No divergence at T_c but small anomaly

$\Rightarrow$ Secondary OP can be distinguished by critical anomalies (Also by domain pattern)

Experiments:

- **X-ray and neutron scattering** give lattice structure

- In ordered phase, rotation generates **strain**:

$$e_1(\eta_1 + \eta_2 + \eta_3)\boldsymbol{Q}^2 \qquad e_2[\eta_1(2Q_1^2 - Q_2^2 - Q_3^2)\ ]$$

- **EPR spectra** (below)

- **Specific heat**, …

Static Critical Exponents at Structural Phase Transitions

K. A. Müller and W. Berlinger
IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland

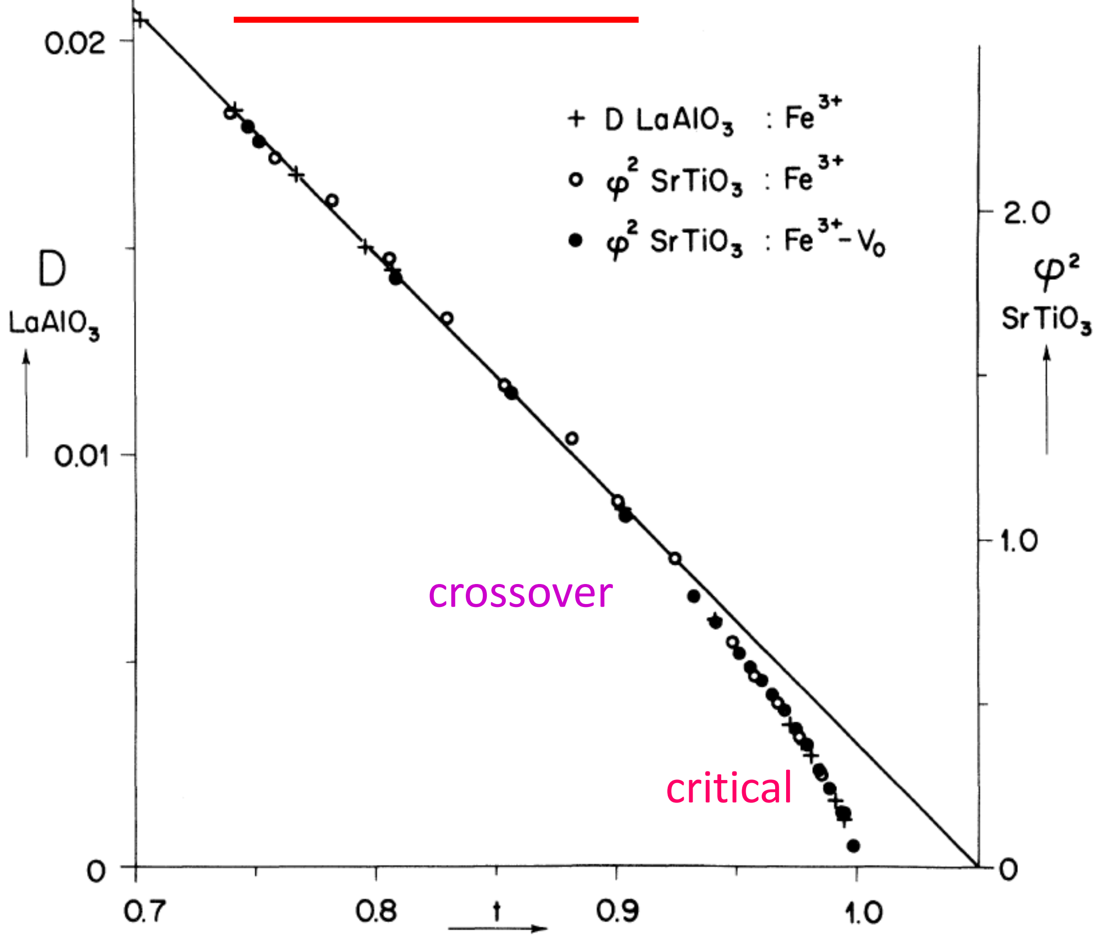
(Received 27 October 1970)

The temperature dependence of the rotational displacement parameters below the second-order phase transitions in $SrTiO_3$ and $LaAlO_3$ at $T_a = 105.5$ and $797°K$ is described by an exponent $\beta = 0.33 \pm 0.02$ down to $t = T/T_a = 0.95$. For smaller t's there occurs a change to Landau behavior approximately followed between $t = 0.9$ and 0.7. The observation of static critical exponents near displacive phase transitions confirms now the notion of universality in this field.

Fe^{3+} *replaces* Al^{3+}, *EPR line shifts with rotation angle*

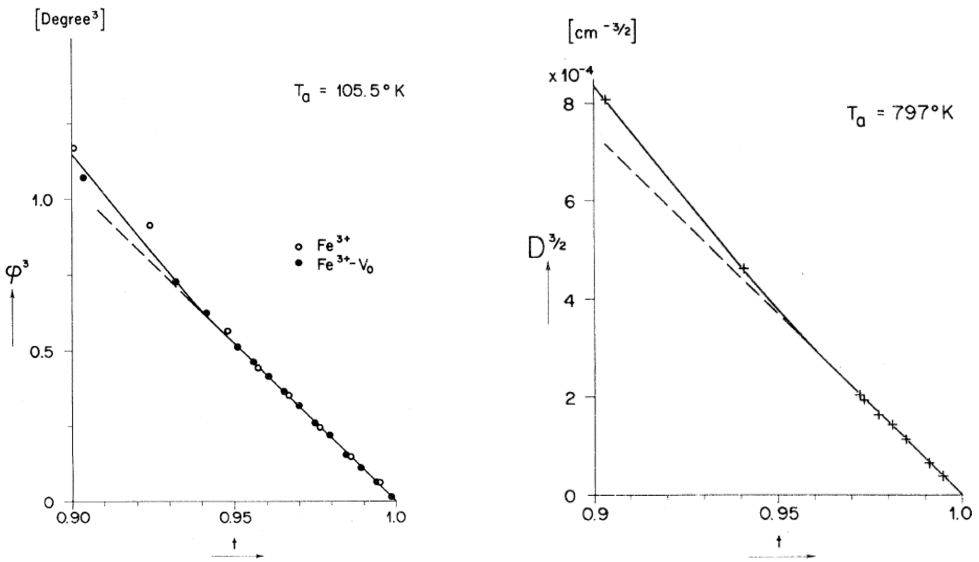
MFT

$|\mathbf{Q}|^2 \propto (T_c - T)$



T_c is lowered, power law changes

$\varphi \propto \epsilon^{\beta}, \quad \epsilon = (T_a - T)/T_a,$



$\beta = 0.33 \pm 0.02$

Which **universality class** is this??

Critical phenomena

Reduced Temperature, t

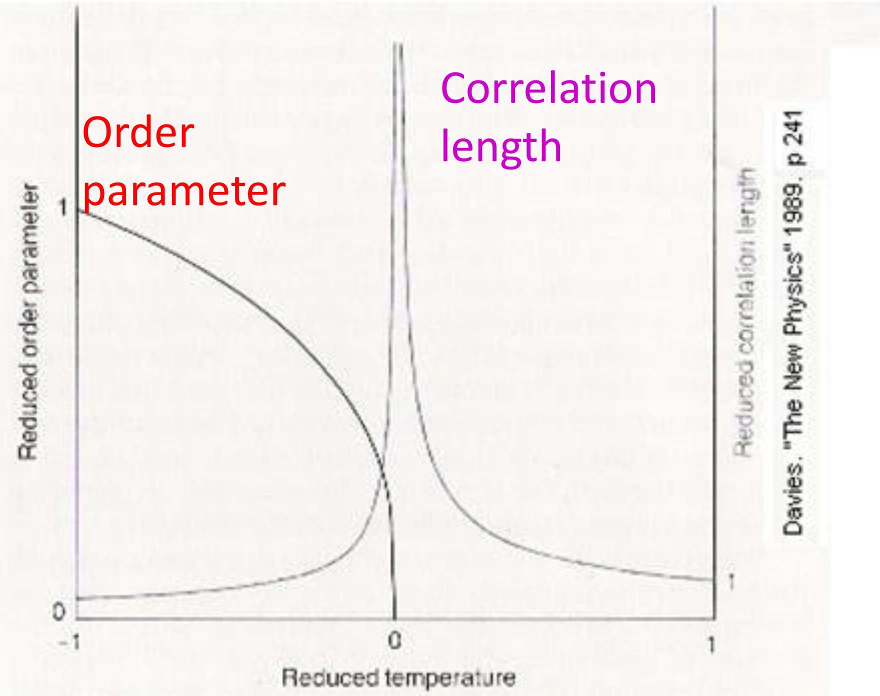
$$t \equiv \frac{T - T_C}{T_C}$$

Specific heat $C \propto |t|^{-\alpha}$

Magnetization $M \propto |t|^{\beta}$

Magnetic susceptibility $\chi \propto |t|^{-\gamma}$

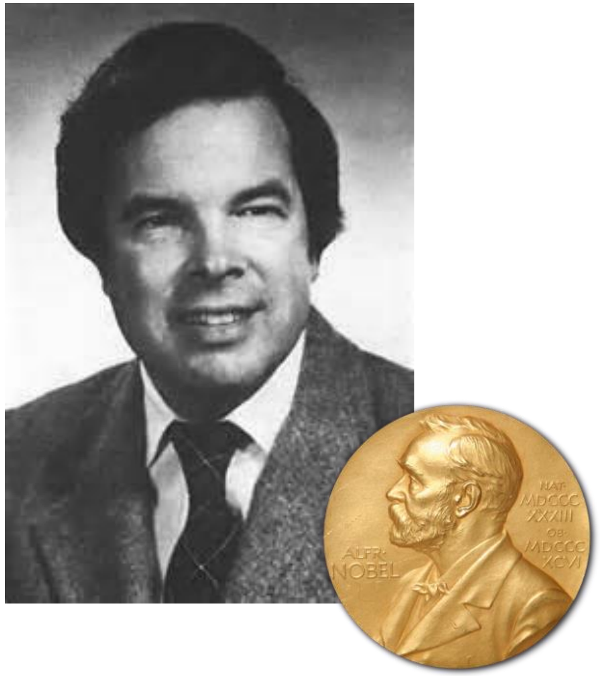
Correlation length $\xi \propto |t|^{-\nu}$



Critical behavior of the order parameter an the correlation length. The order parameter vanishes with the power β of the reduced temperature t as the critical point is approached along the line of phase coexistance. The correlation length diverges with the power ν of the reduced temperature.

Need introduction to RG

I arrived at Cornell in July 1972,
Took the RG course from Wilson.
Also there: Pfeuty, Bruce, Kosterlitz, Nelson



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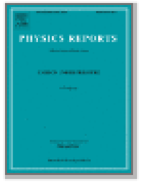
Physics Reports
Volume 12, Issue 2, August 1974, Pages 75-199



The renormalization group and the ϵ expansion ☆

Kenneth G. Wilson [a, b, †], J. Kogut [‡]

PRL 28, 240 (1972)

Critical Exponents in 3.99 Dimensions*

Kenneth G. Wilson and Michael E. Fisher
Laboratory of Nuclear Studies and Baker Laboratory, Cornell University, Ithaca, New York 14850

(Received 11 October 1971)

Critical exponents are calculated for dimension $d = 4 - \epsilon$ with ϵ small, using renormalization-group techniques. To order ϵ the exponent γ is $1 + \frac{1}{6}\epsilon$ for an Ising-like model and $1 + \frac{1}{5}\epsilon$ for an XY model.

RG

A. Aharony, *Dependence of universal critical behavior on symmetry and range of interaction* in **Phase Transitions and Critical Phenomena**, C. Domb and M. S. Green, eds., Vol. 6 (Academic Press, NY, 1976), p. 357

Partition function: $Z = \exp[-F/(k_B T)]$, $Z = \int_{\mathbf{Q}(\mathbf{x})} \left(\Pi_{\mathbf{x}} d^n \mathbf{Q}(\mathbf{x})\right) \exp[\bar{\mathcal{H}}\{\mathbf{Q}(\mathbf{x})\}]$

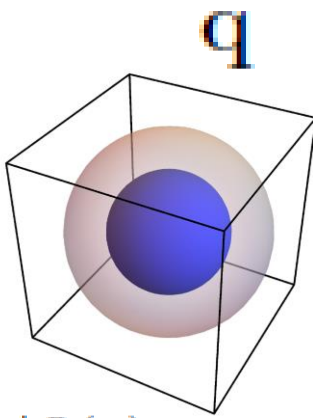
Local Hamiltonian includes gradients, $\bar{\mathcal{H}} = \int d^n\mathbf{x}[-\frac{1}{2}(\nabla\mathbf{Q}(\mathbf{x})\cdot\nabla\mathbf{Q}(\mathbf{x}) - G[\mathbf{Q}(\mathbf{x})]/(k_B T)]$ Landau-Ginzburg

Fourier transform: $\mathbf{Q}(\mathbf{q}) = C\sum_{\mathbf{x}} \exp[i\mathbf{q}\cdot\mathbf{x}]\mathbf{Q}(\mathbf{x})$, $\mathbf{q}$ in 1st Brillouin zone (BZ), $|q_\alpha| < \pi/a$

Approximate BZ by a sphere, $|\mathbf{q}| < \Lambda$

$$\bar{\mathcal{H}} = -\frac{1}{2}\int_{\vec{q}} (r+q^2)\sum_{\alpha} S^{\alpha}_{\vec{q}}\cdot S^{\alpha}_{-\vec{q}} - \sum_{\alpha\beta}(u+v\delta_{\alpha\beta}) \times \int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''} S^{\alpha}_{\vec{q}} S^{\alpha}_{\vec{q}'} S^{\beta}_{\vec{q}''} S^{\beta}_{-\vec{q}-\vec{q}'-\vec{q}''},$$

Q ⇔ ***S*** $\mathbf{q}$ near (π,π,π)



Renormalization group: integrate in Z over short length scales $\Lambda/b < |\mathbf{q}| < \Lambda$

Then rescale length, $\mathbf{q} \Rightarrow \mathbf{q}' = b\mathbf{q}$ and order-parameter scale, $\mathbf{Q}(\mathbf{q}) \Rightarrow \mathbf{Q}'(b\mathbf{q}) = \mathbf{Q}'(\mathbf{q}') = \zeta^{-1}\mathbf{Q}(\mathbf{q})$

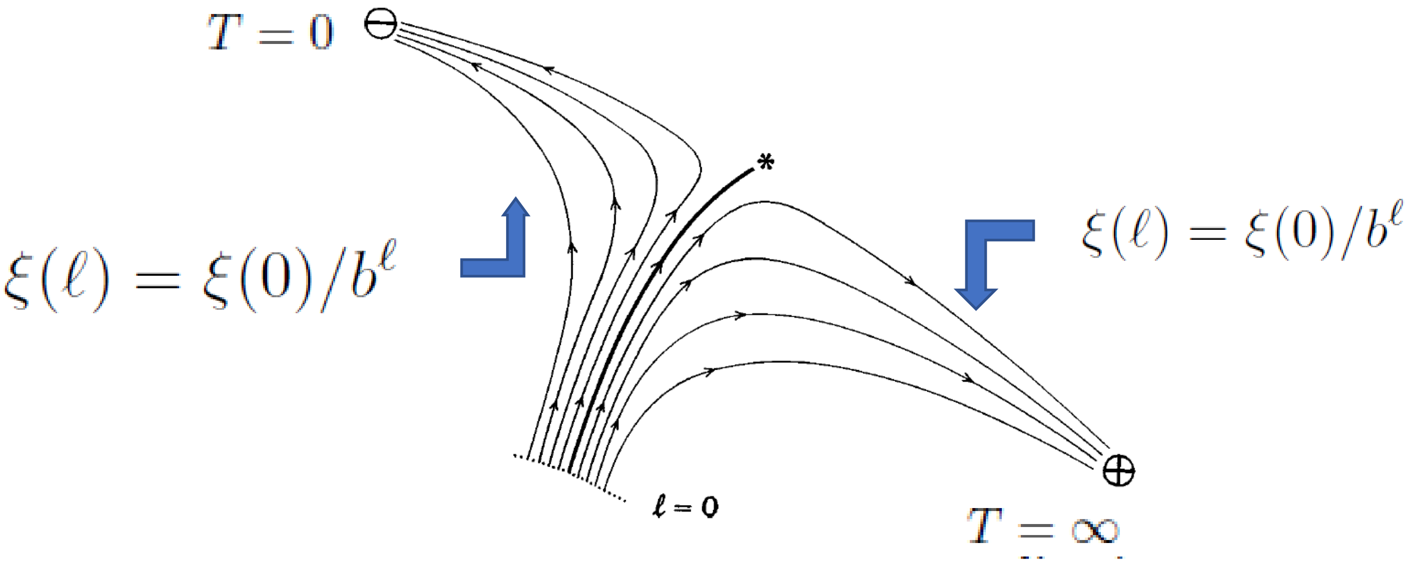
⇒ $\bar{\mathcal{H}}\{\mathbf{Q}'(\mathbf{q}')\} \equiv \bar{\mathcal{H}}_1\{\mathbf{Q}(\mathbf{q})\}$

New Hamiltonian similar to previous one, but may contain new terms

$$\text{Under renormalization step,} \quad \mathbf{x} \Rightarrow \mathbf{x}/b, \quad \mathbf{q} \Rightarrow b\mathbf{q}, \quad \xi \Rightarrow \xi/b$$

$$\text{At fixed point,} \quad \xi = \infty \quad (\text{or } 0).$$

Critical point flows to the Fixed point!



$$\bar{\mathcal{H}}_1 \equiv \Re\bar{\mathcal{H}}, \quad \ldots \quad \bar{\mathcal{H}}_\ell \equiv \Re^\ell \bar{\mathcal{H}}$$

Search for fixed points,

$$\bar{\mathcal{H}}^* \equiv \Re\bar{\mathcal{H}}^*$$

If $\xi(0) < \infty$, iterate

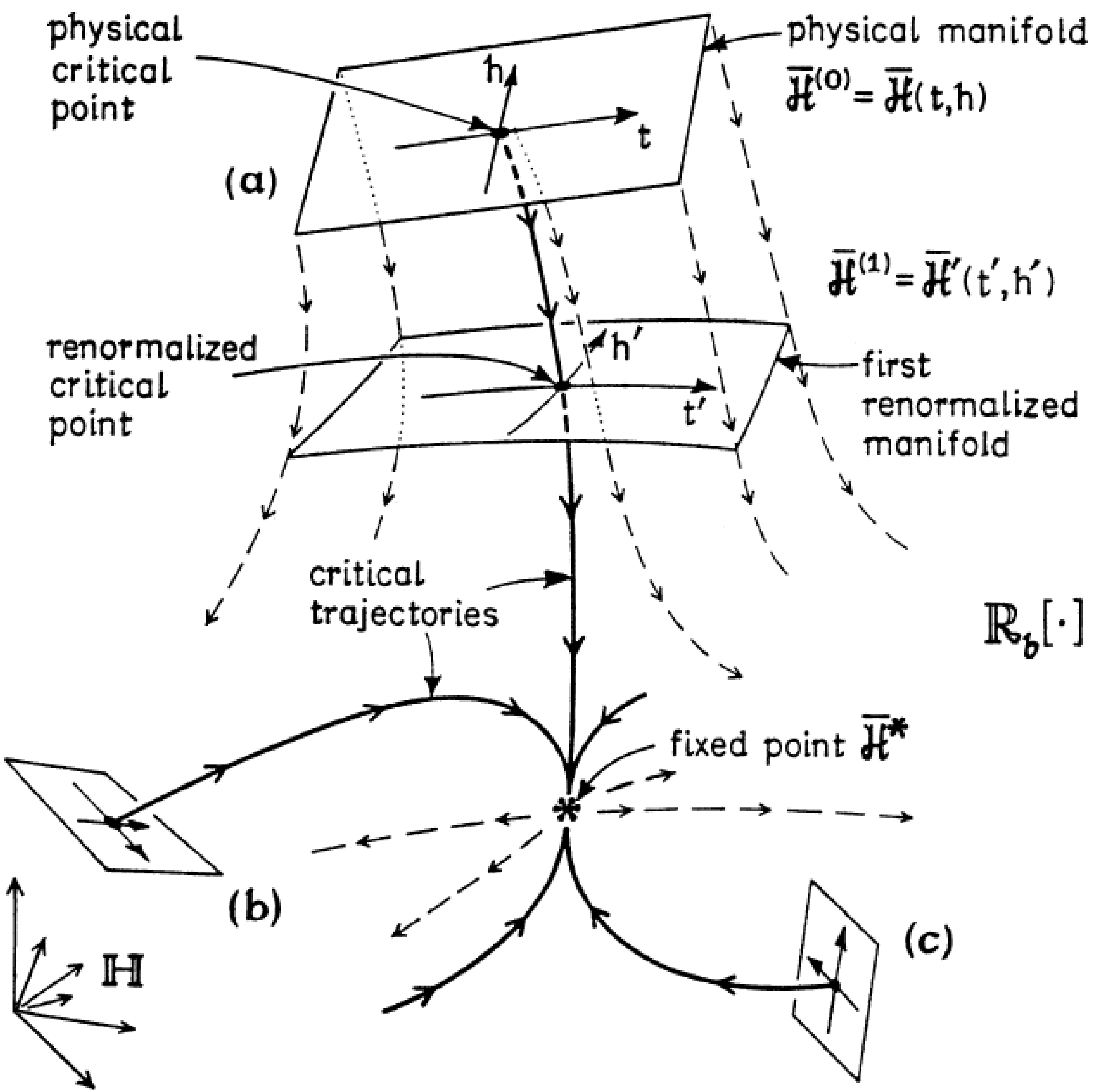
until $\ell = \ell^*$, where

$\xi(\ell^*) = \xi(0)/b^{\ell^*} \sim 1$

then solve Landau theory

with renormalized parameters

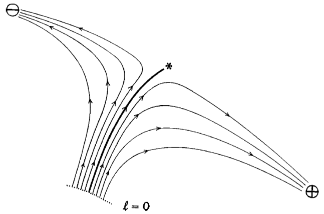
Michael E. Fisher: Renormalization group theory



Near fixed point, $\bar{\mathcal{H}} = \bar{\mathcal{H}}^* + \sum_i \mu_i \mathcal{O}_i + \text{ higher order in the } \mu'\text{s}$

$$\Rightarrow \quad \bar{\mathcal{H}}' = \bar{\mathcal{H}}^* + \sum_i [\mu_i]' \mathcal{O}_i + \text{higher order in the } \mu'\text{s}$$

Linearize, diagonalize the matrix of derivatives, $[\mu_i]' = \frac{\partial \bar{\mathcal{H}}'}{\partial \mu_i} \Big|_* \equiv b^{\lambda_i} \mu_i$



$\mathcal{O}_i$ are **scaling operators**, and the couplings μ_i epend on original Hamiltonian parameters

Singular part of the **free energy density**:

$$f(\mu_1,\ \mu_2,\ \mu_3,\ \ldots) = b^{-\ell d} f(b^{\ell\lambda_1}\mu_1,\ b^{\ell\lambda_2}\mu_2,\ b^{\ell\lambda_3}\mu_3,\ \ldots)$$

Scaling, Homogeneous functions – Widom, Kadanoff

$$f(\mu_1,\ \mu_2,\ \mu_3,\ \ldots) = b^{-\ell d} f(b^{\ell\lambda_1}\mu_1,\ b^{\ell\lambda_2}\mu_2,\ b^{\ell\lambda_3}\mu_3,\ \ldots)$$

$\lambda_i > 0 \Rightarrow \mathcal{O}_i$ is **relevant** $\qquad \lambda_i < 0 \Rightarrow \mathcal{O}_i$ is **irrelevant**

$\lambda_i = 0 \Rightarrow \mathcal{O}_i$ is **marginal** --- Log corrections

---

$$\mu_1 = t \propto (T - T_c) \qquad \xi(0) \approx \xi_0 |t|^{-\nu} = \xi(\ell)/b^\ell$$

$$b^\ell \propto \xi(0)/\xi(\ell) = \xi(0) \approx \xi_0|t|^{-\nu} \qquad b^{\ell\lambda_1}(T - T_c) = (T - T_c)^{1-\nu\lambda_1} \qquad \lambda_1 = 1/\nu$$

$$f(\mu_1,\ \mu_2,\ \mu_3,\ \ldots) = |t|^{d\nu} \tilde{f}(\mu_2|t|^{-\phi_2}, \mu_3|t|^{-\phi_3},\ \ldots) \qquad \phi_\ell = \nu\lambda_\ell$$

$$\mu_2 = H$$ Ordering field $$\phi_2 = \Delta = \beta + \gamma > 0 \;\Rightarrow\; H \text{ is relevant}$$

If there are no other relevant operators then the fixed point represents a **regular critical point,**

$$f(t,\, H,\, \mu_3,\, \ldots) = |t|^{d\nu} \tilde{f}(H/t^{\Delta}, \mu_3 |t|^{-\phi_3},\, \ldots) \approx |t|^{d\nu} \tilde{f}_0(H/t^{\Delta})[1 + a_1 \mu_3 |t|^{\omega} + \ldots]$$

Irrelevant

Correction to scaling

$$Q(H=0) = -\partial f/\partial H \propto |t|^{d\nu - \Delta} f' \propto |t|^{\beta}[1 + a_Q |t|^{\omega} + \ldots]$$

$$\omega = -\phi_3 = -\lambda_3 \nu > 0$$

$$\beta = d\nu - \Delta = 2 - \alpha - (\beta - \gamma) = 2\beta + \gamma) - (\beta + \gamma)$$

Scaling relations among exponents

$$\chi = \partial Q/\partial H \propto |t|^{-\gamma}[1 + a_\chi |t|^{\omega} + \ldots]$$

$$\beta_{\text{eff}} = \frac{\partial \log Q}{\partial \log |t|} = \beta + \omega a_Q |t|^{\omega}$$

effective exponents

If there is another relevant operator → **crossover** to another behavior

→ **multicritical point**

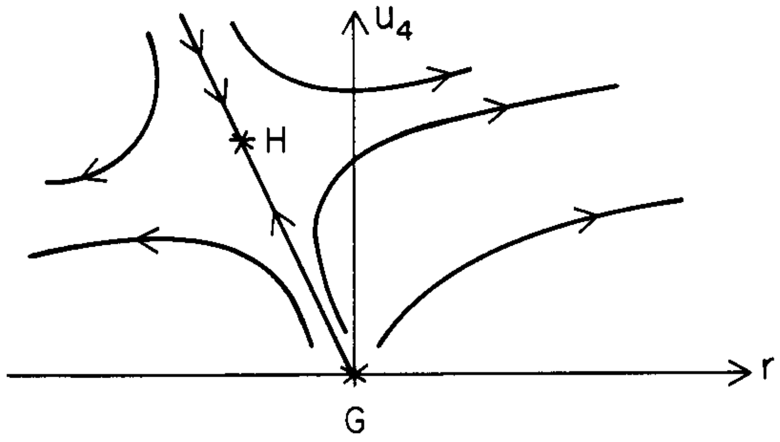
$$f(t,\ H,\ \mu_3,\ \ldots) = |t|^{d\nu} \tilde{f}(H/t^{\Delta}, \mu_3|t|^{-\phi_3},\ \ldots)$$

relevant

Example: Isotropic Heisenberg model

$$\bar{\mathcal{H}} \equiv -\frac{\mathcal{H}}{k_B T} = -\int d\vec{R}\left[\tfrac{1}{2}(\vec{\nabla}\vec{S})^2 + \tfrac{1}{2}r\,|\vec{S}|^2 + u\,|\vec{S}|^4 - \vec{h}\cdot\vec{S}\right]$$

$u = -Ar$



$u > Ar \ \Rightarrow\ r$ increases, $T \to \infty$

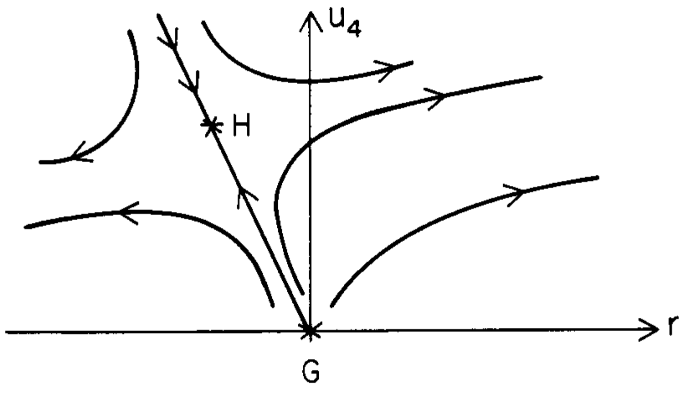
$u < Ar \ \Rightarrow\ u$ becomes negative and large

→ first order transition

Gaussian fixed point represents a **tricritical point**

Crossover from Gaussian to Heisenberg

$u = -Ar$



$$dr/dl = 2r + A(r)u + O(u^2)$$

$$du/dl = \epsilon u - B(r)u^2 + O(u^3)$$

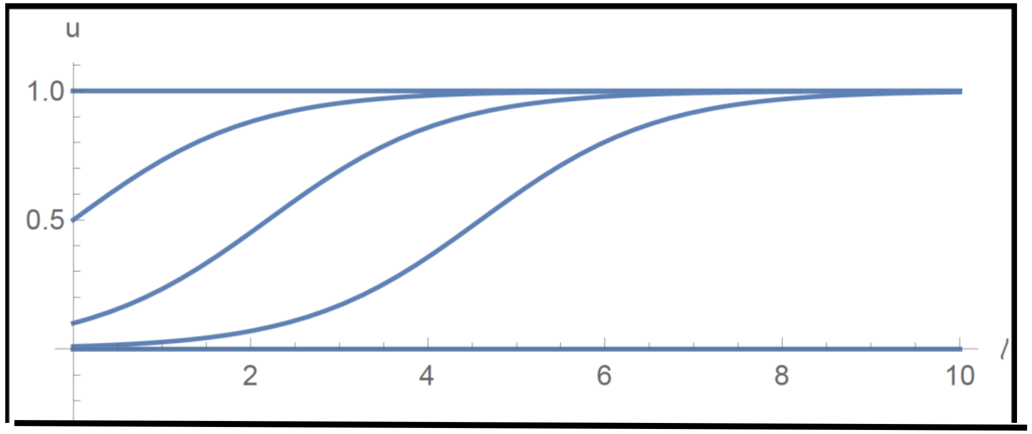
$$u_l = e^{\epsilon l}/[(1/u_0) + B(e^{\epsilon l} - 1)/\epsilon]$$

$$F(t,u,v) = -\frac{nt^2}{16u(4-n)}\left[R^{(4-n)/(n+8)} - 1\right]$$

$$+ \min\left(\frac{1}{2}t_R M^2 + u_R M^4\right)$$

$$R = 1 + (n+8)u\left(e^{\epsilon l^*} - 1\right)2\pi^2\epsilon$$

$$t_R = tR^{-(n+2)/(n+8)}, \quad u_R = u/R$$



PHYSICAL REVIEW B VOLUME 13, NUMBER 5 1 MARCH 1976

Equations of state and renormalization-group recursion relations

Joseph Rudnick
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106

David R. Nelson*
Baker Laboratory and Materials Science Center, Cornell University, Ithaca, New York 14853

UNIVERSALITY

$$f(t,\ H,\ \mu_3,\ \ldots) = |t|^{d\nu}\tilde{f}(H/t^{\Delta}, \mu_3|t|^{-\phi_3},\ \ldots)$$

Scaling function comes from fixed point – independent of initial Hamiltonian – **UNIVERSAL**

Also implies **UNIVERSAL AMPLITUDE RATIOS, equation of state, correlation functions**

V. Privman, P. C. Hohenberg, and A. Aharony
Universal critical-point amplitude relations
in **Phase Transitions and Critical Phenomena**,
C. Domb and J. L. Lebowitz, eds., Vol. 14
(Academic, NY, 1991), pp. 1-134, 364-367

$$\xi_+ = \xi_0 t^{-\nu} \quad (t>0,\ H=0)$$

$$C_+^s = (A/\alpha)t^{-\alpha} \quad (H=0,\ t>0)$$

$$R_\xi^+ = A^{1/d}\xi_0 \iff d\nu = 2-\alpha$$

$$(R_\xi^+)^d = \tfrac{1}{4}nK_d\left[1+\epsilon\left(1-\frac{9}{n+8}\right)+O(\epsilon^2)\right]$$

$$C_-^s = (A'/\alpha')|t|^{-\alpha'} \quad (H=0,\ t<0)$$

$$A/A' = 2^{\alpha-2}(1+\epsilon)n + O(\epsilon^2)$$

Also finite size scaling

$$L \to L/b^\ell \Rightarrow L(\ell)=1 \Rightarrow b^\ell = L$$

$$f(\mu_1,\ \mu_2,\ \mu_3,\ \ldots) = L^{-d}\tilde{f}(\mu_1 L^{\lambda_1}, \mu_2 L^{\lambda_2}, \mu_3 L^{\lambda_3},\ \ldots)$$

2nd example: cubic problem

PHYSICAL REVIEW B VOLUME 8, NUMBER 9 1 NOVEMBER 1973

Critical Behavior of Anisotropic Cubic Systems

Amnon Aharony

Wilson-Fisher
Renormalization Group

$$\bar{\mathcal{H}} = -\frac{1}{2}\int_{\vec{q}} (r+q^2) \sum_{\alpha} S^{\alpha}_{\vec{q}} S^{\alpha}_{-\vec{q}} - \sum_{\alpha\beta} (u + v\delta_{\alpha\beta}) \times \int_{\vec{q}} \int_{\vec{q}\,'} \int_{\vec{q}\,''} S^{\alpha}_{\vec{q}} S^{\alpha}_{\vec{q}\,'} S^{\beta}_{\vec{q}\,''} S^{\beta}_{-\vec{q}-\vec{q}\,'-\vec{q}\,''} ,$$

$$u' = b^{\epsilon-2\eta}\{u - 4K_d \ln b(1+\tfrac{1}{2}\epsilon \ln b)[(n+8)u^2 + 6uv] + 16K_4^2 \ln^2 b[(n^2+6n+20)u^3 + 9(n+4)u^2 v + 27uv^2] + 32K_4^2 \ln b(1+\ln b)[(5n+22)u^3 + 36u^2 v + 9uv^2]\},$$

$$v' = b^{\epsilon-2\eta}\{v - 4K_d \ln b(1+\tfrac{1}{2}\epsilon \ln b)(12uv + 9v^2) + 16K_4^2 \ln^2 b(36u^2 v + 54uv^2 + 27v^3) + 32K_4^2 \ln b(1+\ln b)[3(n+14)u^2 v + 72uv^2 + 27v^3]\},$$

Gaussian:

$$u^G = v^G = 0;$$

Ising:

$$u^I = 0,\ \ v^I = \frac{\epsilon}{36K_d}\left(1 + \tfrac{17}{27}\epsilon\right) + O(\epsilon^3);$$

Heisenberg:

$$v^H = 0,\quad u^H = \frac{\epsilon}{4K_d(n+8)}\left(1 + \frac{3(3n+14)}{(n+8)^2}\epsilon\right) + O(\epsilon^3);$$

Cubic:

$$u^C = \frac{\epsilon}{12K_d n}\left(1 + \frac{(n-1)(106-19n)}{27n^2}\epsilon\right) + O(\epsilon^3),$$

$$v^C = \frac{\epsilon}{12K_d n}\left(\frac{n-4}{3} + \frac{(n-1)(17n^2+110n-424)}{8\,\mathrm{l}n^2}\epsilon\right)$$

$$+ O(\epsilon^3).$$

Gaussian:

$$\lambda_u^G = \lambda_v^G = \epsilon;$$

Ising:

$$\lambda_u^I = \tfrac{1}{3}\epsilon - \tfrac{19}{81}\epsilon^2 + O(\epsilon^3),\quad \lambda_v^I = -\epsilon + \tfrac{17}{27}\epsilon^2 + O(\epsilon^3);$$

Heisenberg:

$$\lambda_u^H = -\epsilon + \frac{9n+42}{(n+8)^2}\epsilon^2 + O(\epsilon^3),$$

$$\lambda_v^H = \frac{n-4}{n+8}\epsilon + \frac{5n^2+14n+152}{(n+8)^3}\epsilon^2 + O(\epsilon^3).$$

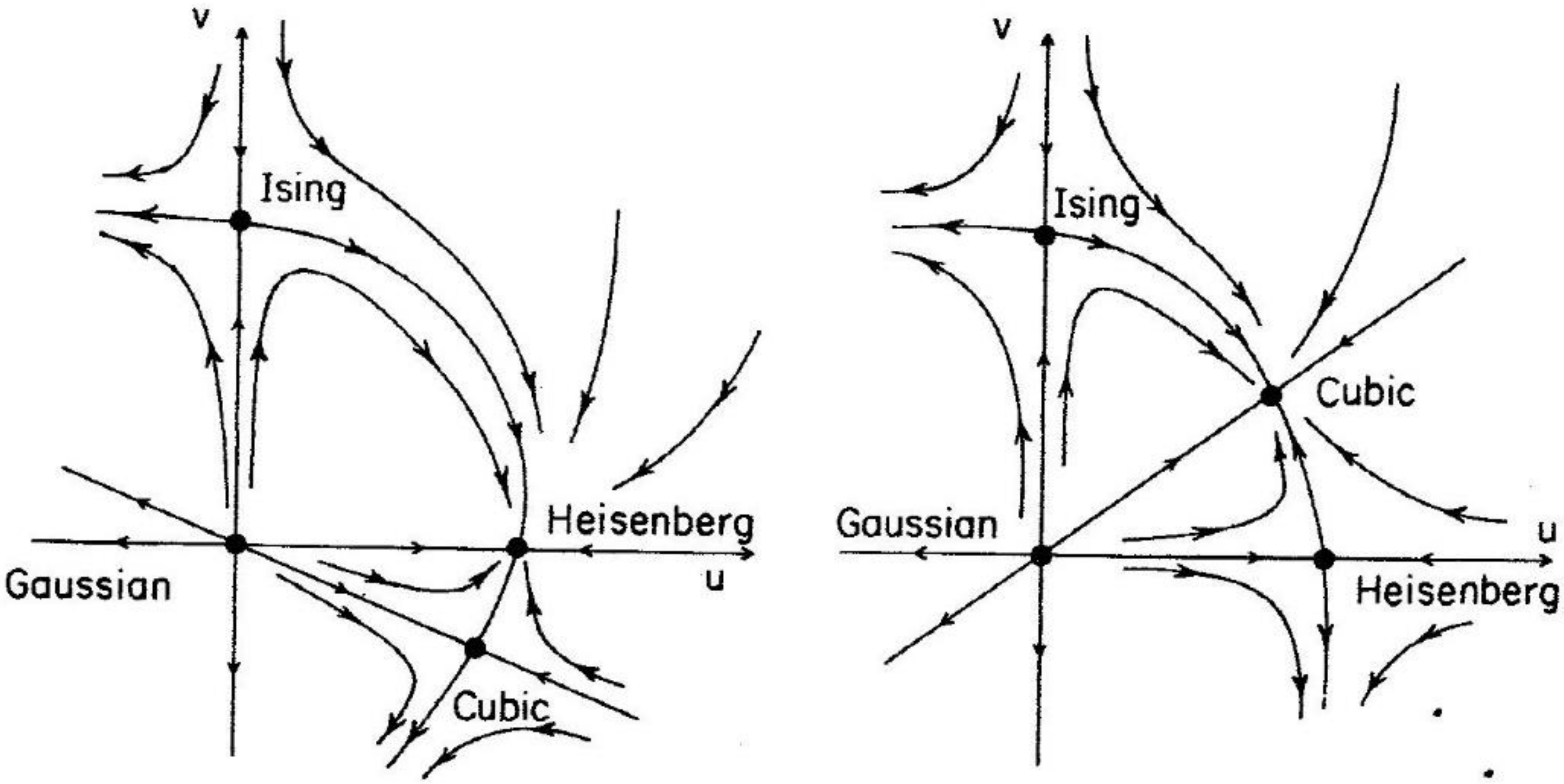
Cubic:

$$\lambda_1^C = -\epsilon + \frac{(n-1)(17n^2-4n+212)}{27n^2(n+2)}\epsilon^2 + O(\epsilon^3)$$

$$\lambda_2^C = \frac{4-n}{3n}\epsilon$$

$$+ \frac{(n-1)(19n^3-72n^2-660n+848)}{81n^3(n+2)}\epsilon^2 + O(\epsilon^3).$$

A. Aharony, *Dependence of universal critical behavior on symmetry and range of interaction* in **Phase Transitions and Critical Phenomena**, C. Domb and M. S. Green, eds., Vol. 6 (Academic Press, NY, 1976), p. 357



$n < n_c$ $n > n_c$

crossover

Critical Behavior of Anisotropic Cubic Systems

Amnon Aharony

borderline of stability of the Heisenberg fixed point depends on the order of the truncated series used for λ_v^H. Using the order-ϵ result, the Heisenberg fixed point is stable for $n < 4$. This explains Cowley and Bruce's[6] result for $n = 3$. Using the order-ϵ^2 expression, given above, one finds that $\lambda_v^H > 0$ for $n > 2$, and $\lambda_v^H = 0$ at $n = 2$. Recently, Ketley and Wallace[15] calculated the order-ϵ^3 term in λ_v^H, using a Feynman-graph expansion near the Heisenberg fixed point. To that order, they find the borderline goes up and lies close to $n = 4$. Ketley and Wallace conclude that for $n = 3$ the radius of convergence of the series for λ_v^H is smaller than 1, and that no conclusions may be drawn. We conclude tentatively that λ_v^H is either positive or very small in magnitude, so that it is important to study other possible fixed points and other types of critical behavior.[16]

A. Aharony, *Dependence of universal critical behavior on symmetry and range of interaction*
in **Phase Transitions and Critical Phenomena**, C. Domb and M. S. Green, eds., Vol. 6 (Academic Press, NY, 1976), p. 357

$$v^C = 0, \quad \lambda_v^H = \lambda_2^C = 0$$

$$\Rightarrow \quad n_c = 4 - 2\epsilon + \frac{5}{12}[6\zeta(3) - 1]\epsilon^2 + \mathcal{O}[\epsilon^3] \approx \frac{4 + 3.176\epsilon}{1 + 1.294\epsilon} \approx 3.128 \text{ at } \epsilon = 1.$$

$$\Downarrow$$

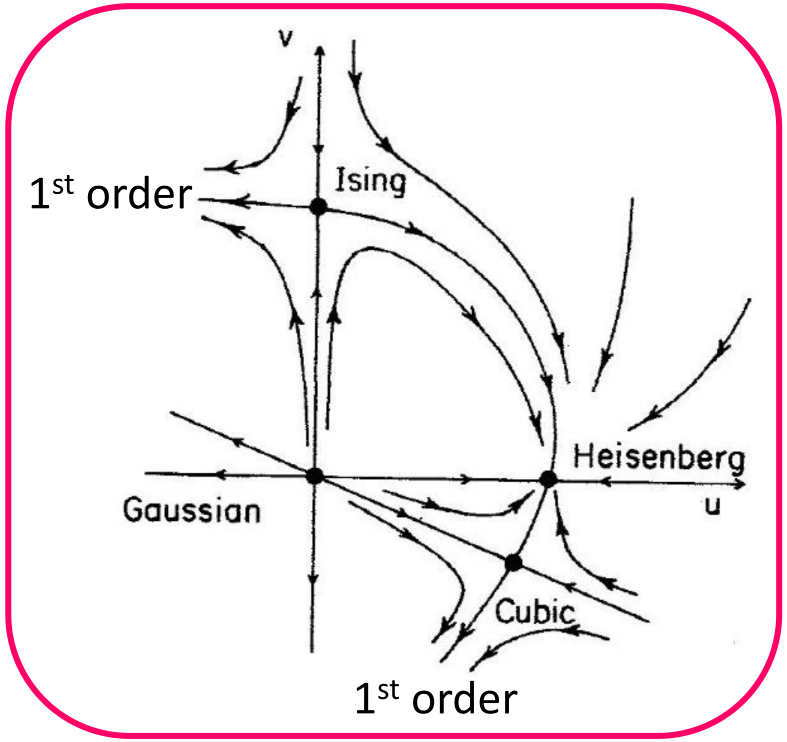
$$n_c > 3?$$

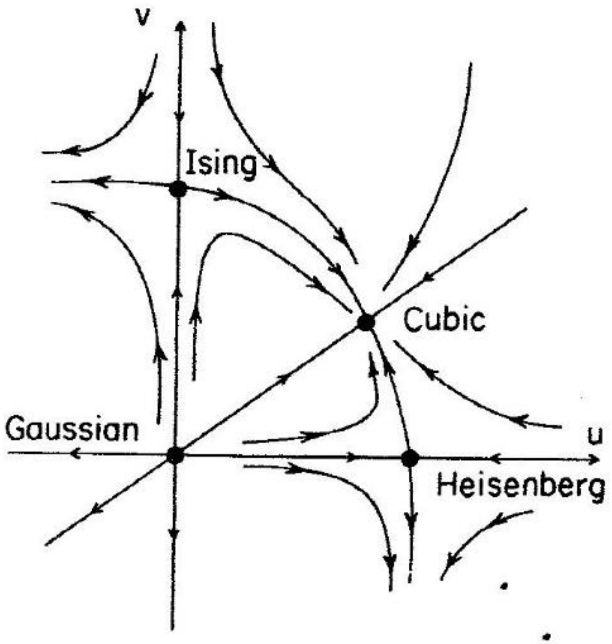
Heisenberg is stable

$$\beta_H = 0.37 \pm 0.01$$

Larger than experiments!

$$\beta = 0.33 \pm 0.02$$

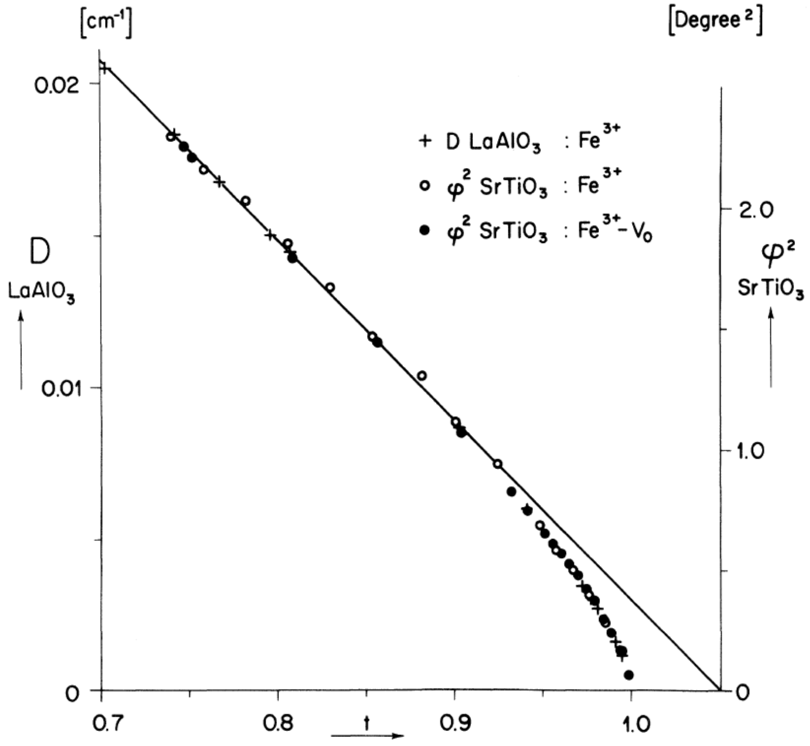




Alex Müller visited Cornell in 1973

Static Critical Exponents at Structural Phase Transitions

K. A. Müller and W. Berlinger



cancy (Fe^{3+}-V_O) is observed.[8] Very recently proper shaping of $SrTiO_3$ crystals led to samples which became nearly monodomain below the phase transitions.[10] These samples allowed us a heretofore unattained accuracy in the determination of φ. The c axis of the monodomain was aligned parallel to the rotation axis of the magnet. Under this geometry there are essentially

(Usually, samples have many domains)

Coupling to strain

$$G = G_0 + \frac{1}{2}[\alpha(T - T_c)(Q_1^2 + Q_2^2 + Q_3^2) + \underline{2p_1 Q_1^2}$$

$$\underline{+ p_2(Q_2^2 + Q_3^2)]} + u(Q_1^2 + Q_2^2 + Q_3^2)^2$$

$$+ v(Q_1^4 + Q_2^4 + Q_3^4),$$

Polycritical Points and Floplike Displacive Transitions in Perovskites

Amnon Aharony and Alastair D. Bruce*
Baker Laboratory and Materials Science Center and Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14850
(Received 29 April 1974)

$$\bar{\mathcal{H}} = \int d^3x \left\{ \tfrac{1}{2}\left[r_0 \vec{Q}^2 + (\nabla\cdot\vec{Q})^2\right] + \bar{u}_0 \vec{Q}^4 + \bar{v}_0 \sum_{\alpha=1}^{3} Q_\alpha{}^4 - \sum_{\alpha=1}^{3} T_\alpha\left[(L_1 - L_2)Q_\alpha{}^2 + L_2\vec{Q}^2\right] - L_3(Q_1Q_2T_6 + Q_2Q_3T_4 + Q_3Q_1T_5)\right\}$$

stress along the $[100]$ *axis,* $T_i = -p\delta_{i1}$

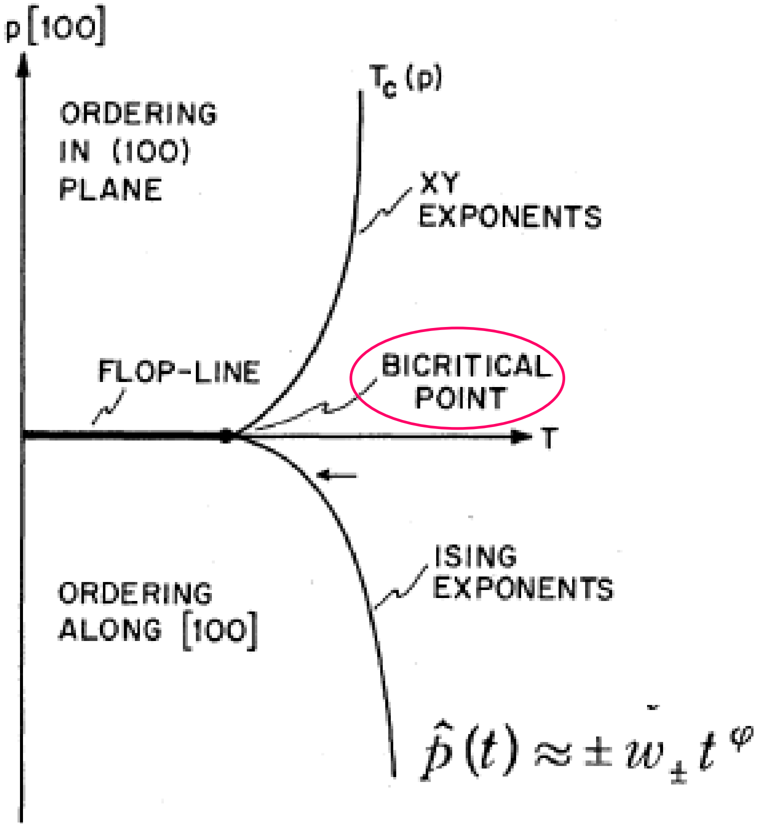
$$\bar{\mathcal{H}}^{[100]} = \int d^3x \left\{ \tfrac{1}{2}\left[r_1Q_1{}^2 + r_2(Q_2{}^2 + Q_3{}^2) + (\nabla\cdot\vec{Q})^2\right] + \bar{u}_0\vec{Q}^4 + \bar{v}_0 \sum_{\alpha=1}^{3} Q_\alpha{}^4\right\}$$

$r_1 = r_0 + 2pL_1$ and $r_2 = r_0 + 2pL_2$

$$G(T,p) \approx |t|^{2-\alpha_H} f(\hat{p}/|t|^{\varphi}).$$

φ is the crossover exponent for p

$$\mathcal{O} = Q_1^2 - \frac{1}{n-1}\sum_{i>1} Q_i^2$$



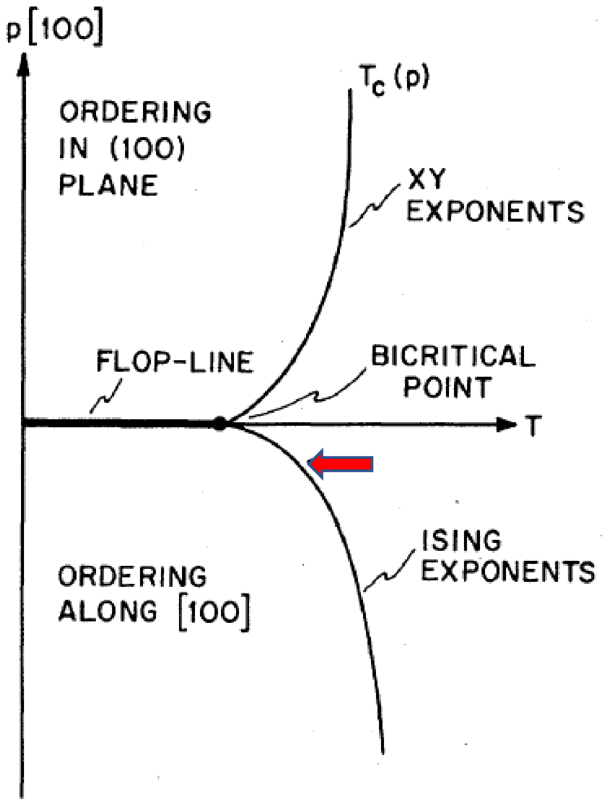


FIG. 1. Schematic phase diagram for a perovskite with a [100] stress, assuming $v_c < \bar{v}_0 < 0$ ($SrTiO_3$). The arrow locates qualitatively what we conjecture to be the position of the observed transition in monodomain $SrTiO_3$.

In the light of the above analysis we conjecture that this mechanism is simply a systematic strain field which breaks the cubic symmetry, as discussed above, and leads to a *crossover* from the purely Heisenberg critical behavior expected for a strain-free sample to an *Ising-like behavior*, as indicated in Fig. 1.[22] We note that the experimental value[8] of $\beta = 0.33 \pm 0.02$ is quite consistent with this suggestion since we must presume that the experiments were performed in the crossover region, in which the effective value of β changes from its Heisenberg value, $\beta_H \simeq 0.37$, to that appropriate for the Ising system, $\beta_I \simeq 0.31$.[19]

PHYSICAL REVIEW B VOLUME 11, NUMBER 1 1 JANUARY 1975

Coupled order parameters, symmetry-breaking irrelevant scaling fields, and tetracritical points

Alastair D. Bruce*
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14850

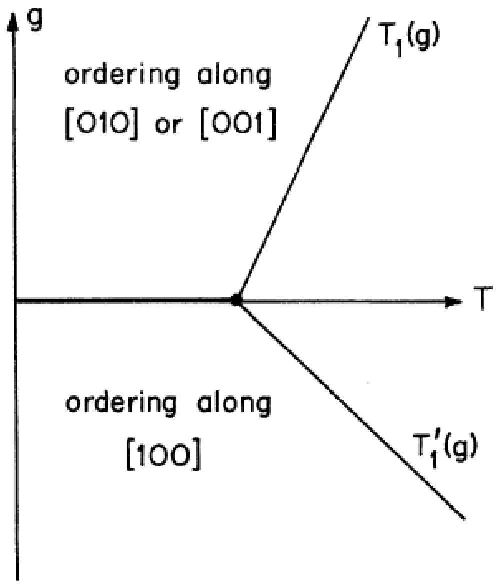
Amnon Aharony†

$$t_1 = [T_1(g) - T_c]/T_c \sim g^{1/\psi_1}$$

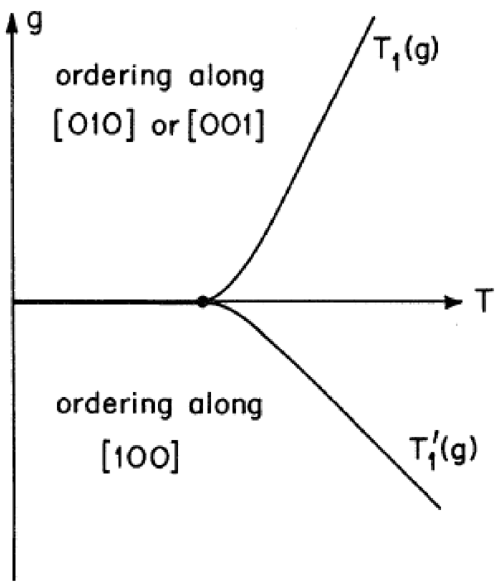
$$A(T, g, v, \vec{M}) = \tfrac{1}{2} r_0 \vec{M}^2 + u \vec{M}^4 + \tfrac{1}{2} g \sum_\alpha c_\alpha M_\alpha^2 + v \sum_\alpha M_\alpha^4$$

$$v < 0,$$

$SrTiO_3$



Landau theory



$$\psi_1 = \phi_g$$

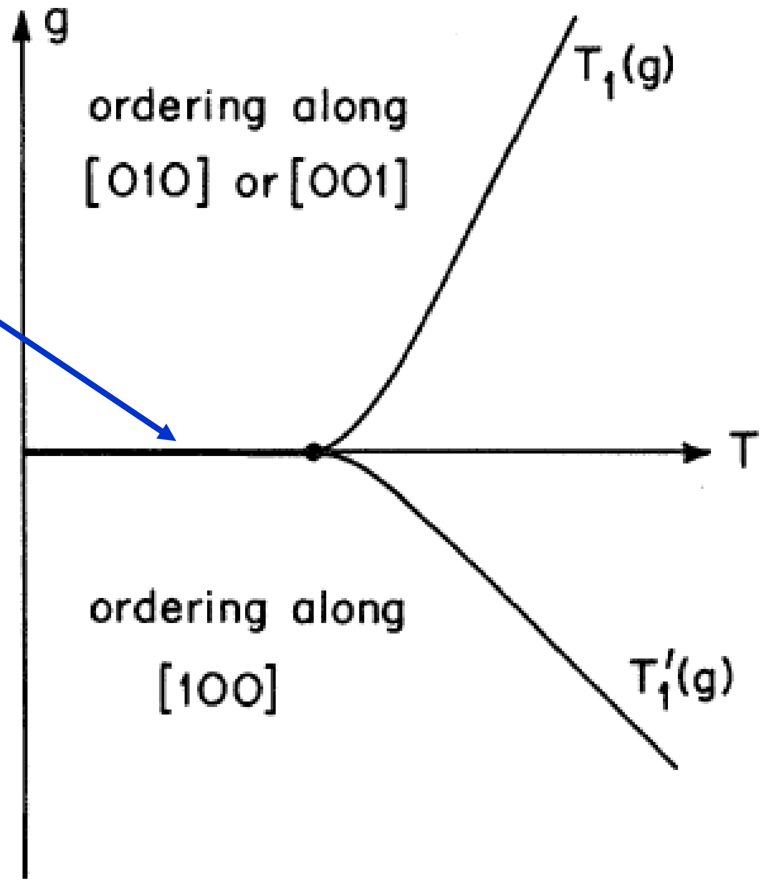
RG, near H or C fixed point

In ordered phase of the symmetric structure,

Goldstone modes now have a "mass"

$$\chi_\perp \propto \frac{1}{\frac{E}{Q} + v(\ell)Q(\ell)^2}$$

$$\propto |t|^{\phi_v}|t|^{2\beta}$$



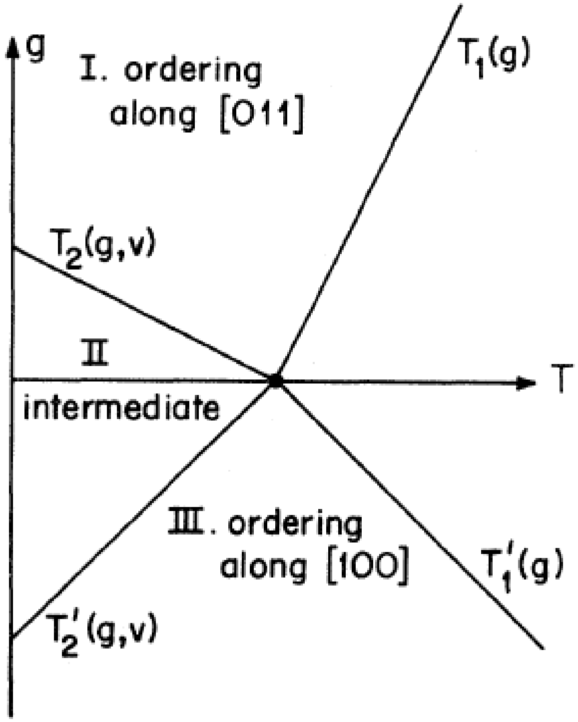
Different for cubic and Heisenberg fixed points!

$t_1 = [T_1(g) - T_c]/T_c \sim g^{1/\psi_1}$

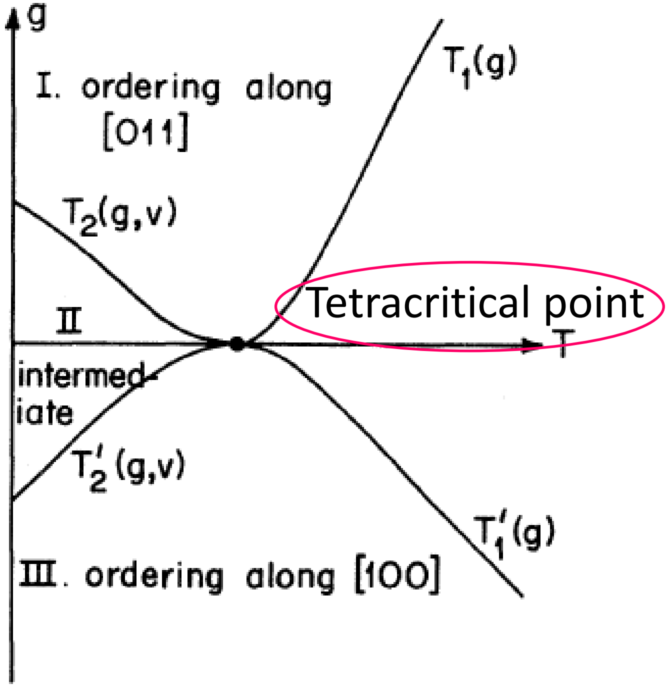
$\psi_1 = \phi_g$

$v > 0$

$LaAlO_3$



Landau theory



RG, near H or C fixed point

$$F(t,g,v) \approx t^{2-\alpha}\,\mathfrak{F}(g t^{-\phi_g},\ v t^{-\phi_v})$$

$$[T_c - T_2(g,\nu)] \sim (g/\nu)^{1/\psi_2}$$

$$\psi_2 = \phi_g - \phi_\nu \qquad \text{H}$$

$$\psi_2 = \phi_g \qquad \text{C}$$

For H fixed point, v is a “dangerous irrelevant variable”

$\psi_2 > \phi_g$

Volume 59A, number 2 PHYSICS LETTERS 15 November 1976

AXIAL AND DIAGONAL ANISOTROPY CROSSOVER EXPONENTS FOR CUBIC SYSTEMS

A. AHARONY
IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland
and Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel *

$$\mathcal{E}_{\alpha\beta} = -g\Sigma_{\langle ij\rangle}[S^{\alpha}(i)S^{\beta}(j) - \delta_{\alpha\beta}\boldsymbol{S}(i)\cdot\boldsymbol{S}(j)/n],$$

Near the cubic fixed point,

$$\phi_{\text{diag}} = 1 + \frac{n-2}{3n}\epsilon + \frac{(n-2)(7n^2+196n-212)}{162n^3}\epsilon^2 + O(\epsilon^3)$$

$$\mathcal{O}_{\text{axial}} = Q_1^2 - \frac{1}{n-1}\sum_{i>1} Q_i^2$$

$$\phi_{\text{axis}} = 1 + \tfrac{1}{6}\epsilon - \frac{5n^2 - 49n + 53}{81n^2}\epsilon^2 + O(\epsilon^3)\,.$$

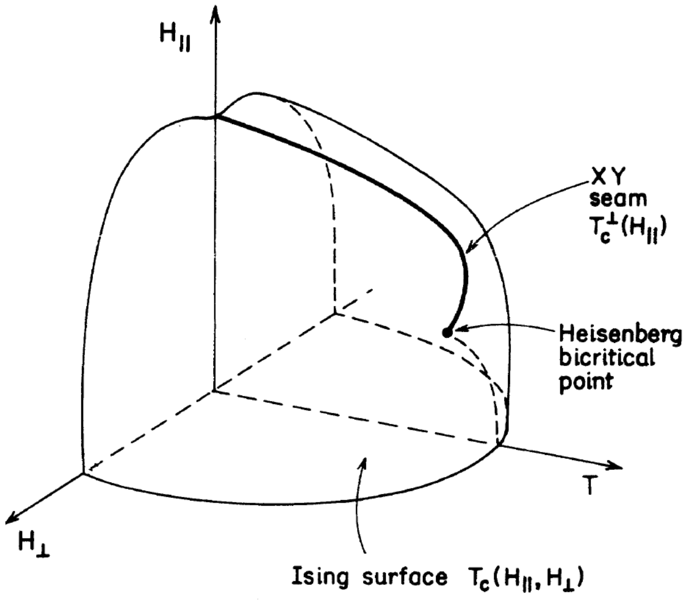
$$\mathcal{O}_{\text{diag}} = Q_1 Q_2$$

Implies different shapes of phase diagram, different Goldstone modes, etc

Note: all the multicritical points discussed here

were simultaneously also discussed in the context

of magnetic systems:



P. Pfeuty, J. M. Kosterlitz, D. R. Nelson, E. Domany, D. Mukamel,

D. J. Wallace, E. Brézin, S. Hikami, R. Abe, T. Nattermann, J. Rudnick,

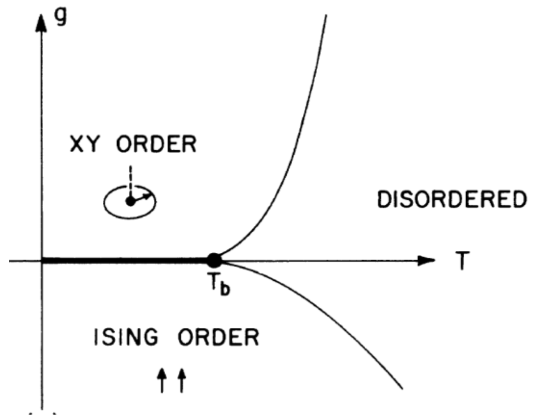
E. Riedel, F. Wegner, A. J. Larkin, D. E. Khmel’nitzkii,….

PHYSICAL REVIEW B VOLUME 13, NUMBER 1 1 JANUARY 1976

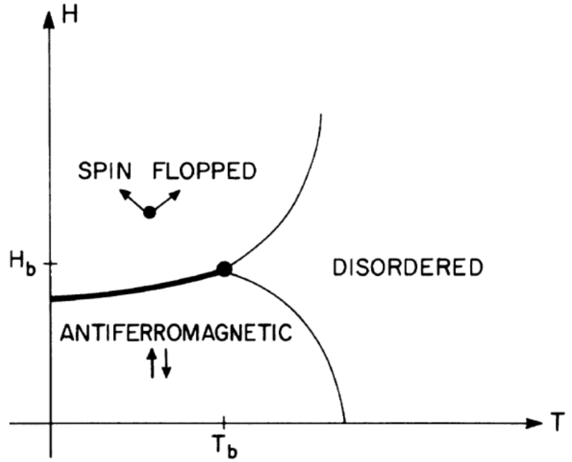
Equations of state for bicritical points. I. Calculations in the disordered phase

David R. Nelson*
Baker Laboratory and Materials Science Center, Cornell University, Ithaca, New York 14853

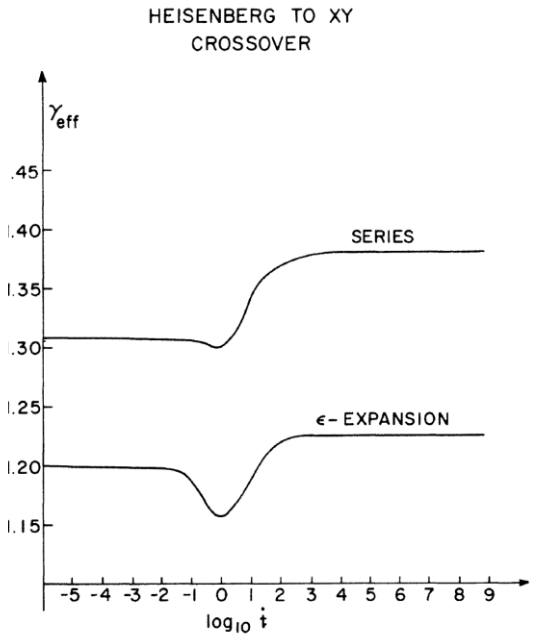
Eytan Domany
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853



$$\chi(t,g) \approx t^{-\gamma}\Psi(g/t^{\phi}) \ .$$



$$\gamma_{\text{eff}} = \frac{-d\ln\chi}{d\ln\dot{t}}$$



$$\dot{t} = t/t_c(g) - 1$$

Polycritical Points and Floplike Displacive Transitions in Perovskites

Amnon Aharony and Alastair D. Bruce*
Baker Laboratory and Materials Science Center and Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14850
(Received 29 April 1974)

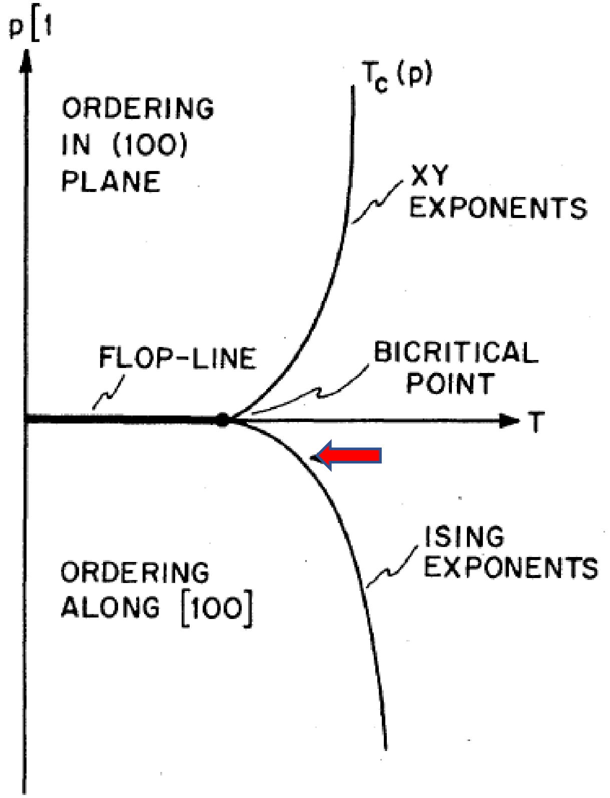
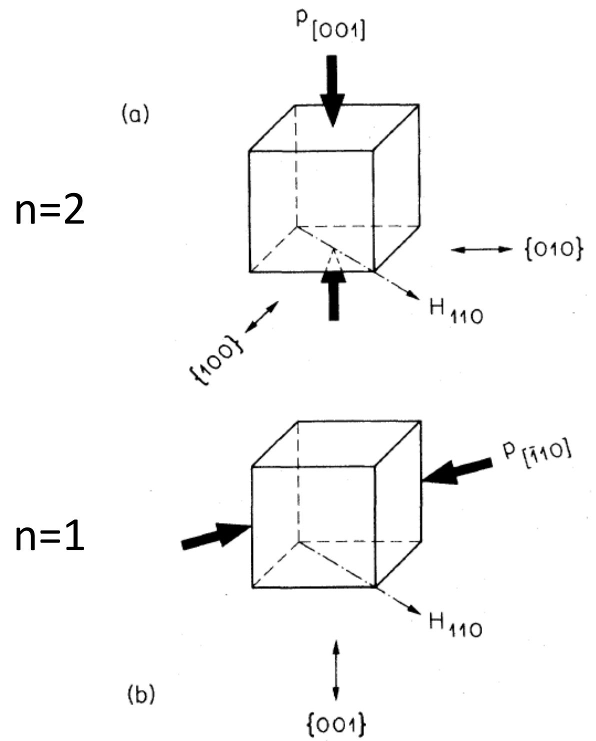


FIG. 1. Schematic phase diagram for a perovskite with a [100] stress, assuming $v_c < \overline{v}_0 < 0$ ($SrTiO_3$). The arrow locates qualitatively what we conjecture to be the position of the observed transition in monodomain $SrTiO_3$.

Behavior of $SrTiO_3$ near the [100]-Stress–Temperature Bicritical Point

K. A. Müller and W. Berlinger
IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland

(Received 2 September 1975)



The earlier measurements of the temperature dependence of the order parameter have been carried out on monodomain samples[8] to achieve a better accuracy near T_c. The experimental value obtained gave $\beta = 0.33 \pm 0.02$.[1] Because of the uniaxial character of the sample, the Ising value $\beta_I \equiv \beta(1) = 0.315$ rather than the Heisenberg β_H

$$\varphi(T) = \varphi_0(1 - T/T_c)^{\beta(1)} \times [1 + b_1(1 - T/T_c)^x + \ldots]$$

Order parameter

correction

$$\beta(1) = 0.315$$

Ising, *n=1*

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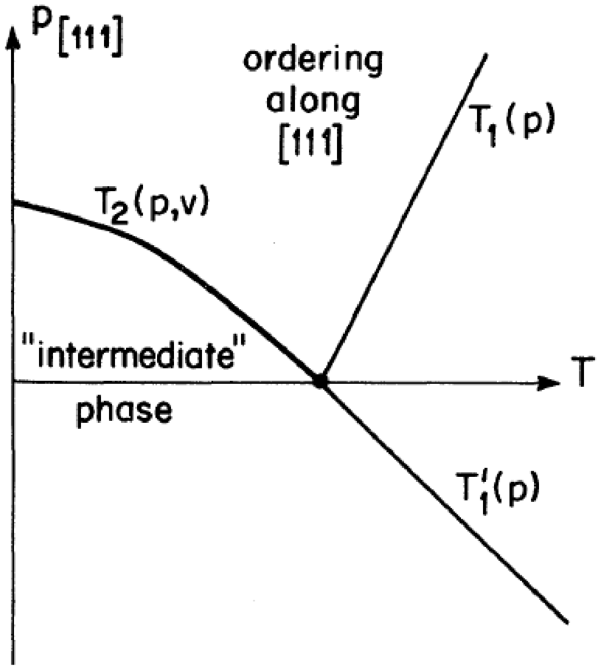
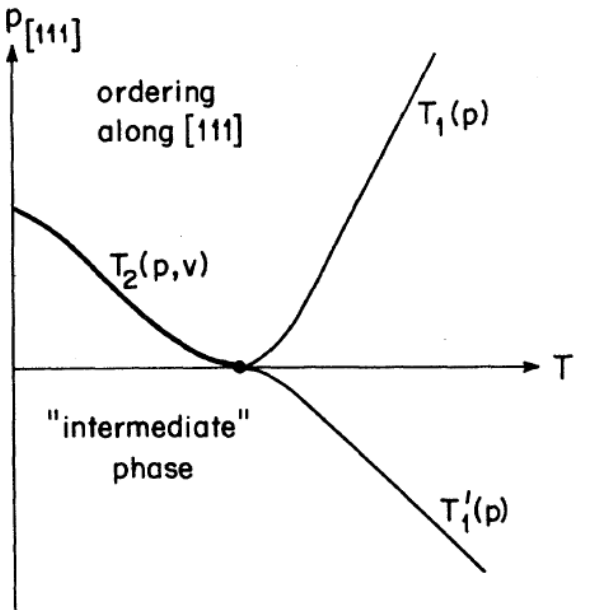
Coupled order parameters, symmetry-breaking irrelevant scaling fields, and tetracritical points

Alastair D. Bruce*
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14850

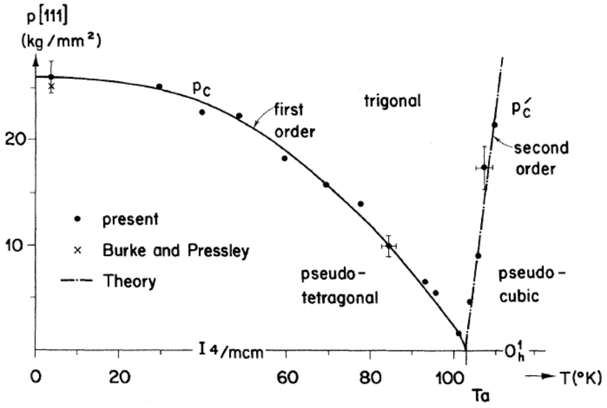
Amnon Aharony†



Landau theory



RG, near H or C fixed point



Stress along [111]

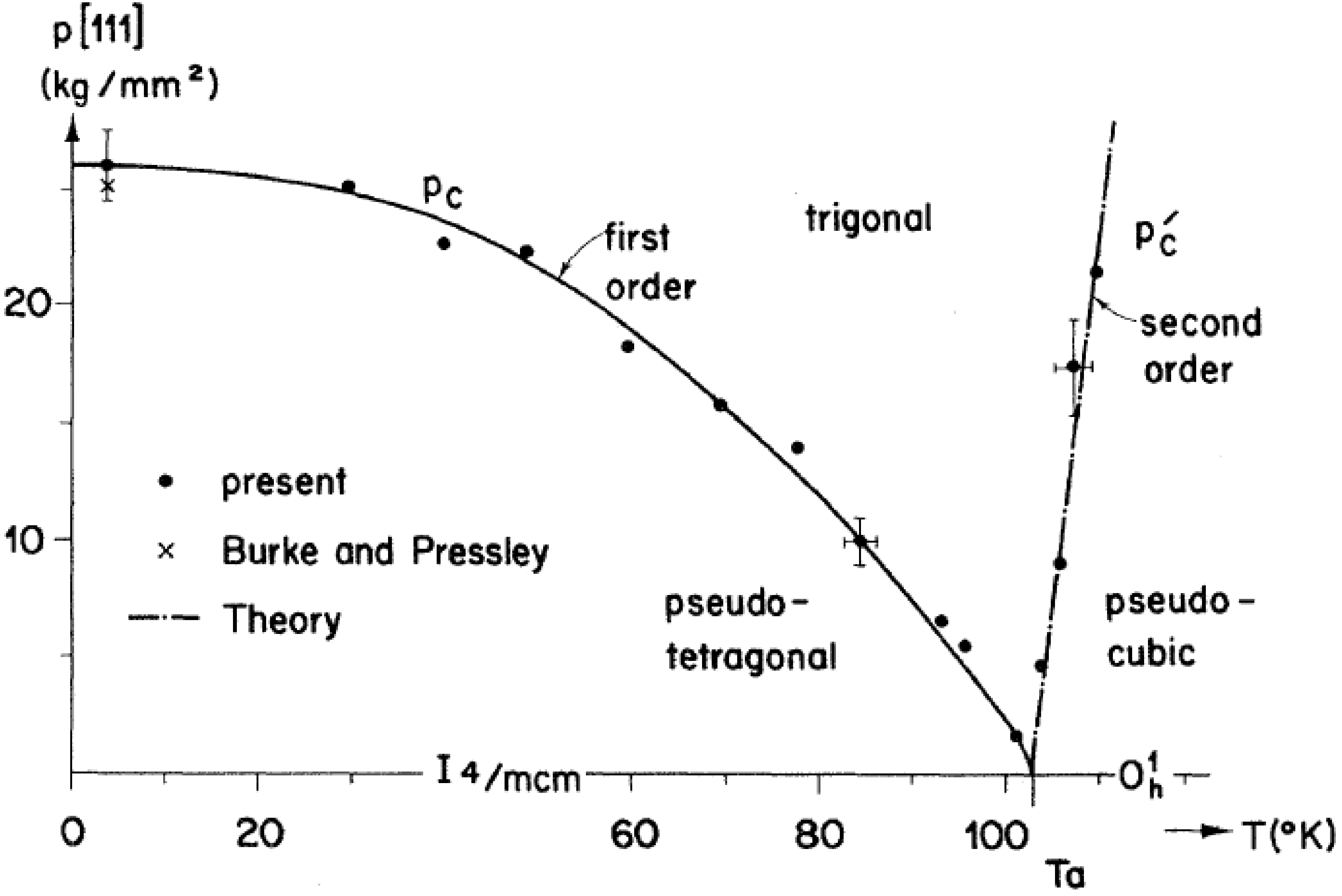
ORDER PARAMETER AND PHASE TRANSITIONS OF STRESSED $SrTiO_3$

K. A. Müller, W. Berlinger, and J. C. Slonczewski*
IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland

(Received 6 July 1970)

EPR spectra of Fe-V_O pairs were used to study how the order parameter of $SrTiO_3$ varies with uniaxial stress applied to a (111) face. A second-order cubic-trigonal phase boundary appears above the stress-free transition temperature T_a. A first-order tetragonal-trigonal phase boundary is found below T_a. An independently determined Landau potential describes the results.

$$\tilde{U} = \tfrac{1}{2}KQ^2 + A'Q^4 + A_n' \sum_{i<j} Q_i^2 Q_j^2$$
$$- b_e \sum_i T_{ii}(3Q_i^2 - Q^2) - b_t \sum_{i<j} T_{ij} Q_i Q_j$$



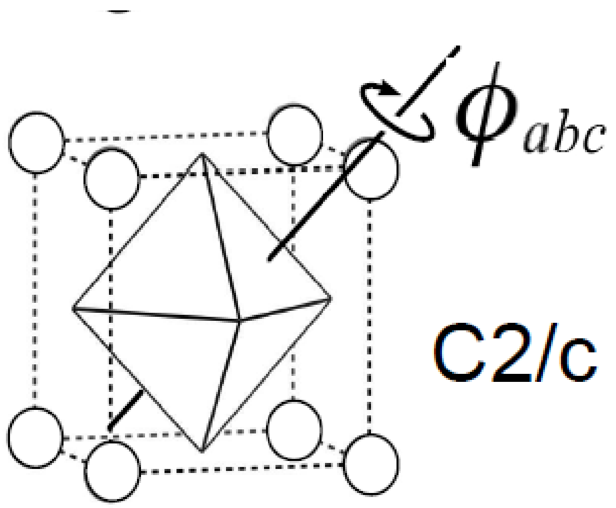
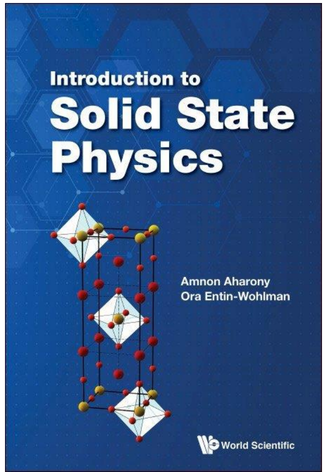


FIG. 3. Phase diagram of $SrTiO_3$ for stress-induced $R\bar{3}c$ phase. For $T < T_a$ the transition is first order, for $T > T_a$ second order.

In the mountains near IBM Zürich, circa 1975

… Many mutual visits in Tel Aviv and Zürich:
Phase transitions in perovskites or high-Tc Superconductivity?



April 1987: Conference in Zürich, celebrating Müller's 60th birthday

My talk: "My life with Alex Müller and the perovskites"

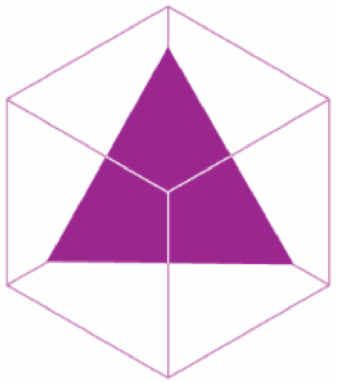
October 1987: Announcing the Nobel Prize, …….

Trigonal-to-Tetragonal Transition in Stressed $SrTiO_3$: A Realization of the Three-State Potts Model

Amnon Aharony
*IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland, and Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel**

and

K. A. Müller and W. Berlinger
IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland
(Received 18 October 1976)



$$S_1 = (Q_1 + Q_2 + Q_3)/\sqrt{3}, \quad S_2 = (Q_1 - Q_2)/\sqrt{2}, \text{ and } S_3 = (Q_1 + Q_2 - 2Q_3)/\sqrt{6},$$

$$\Delta\mathcal{H} = \int d^dx\{[r_1 + 4(u + \tfrac{1}{3}v)M^2]MS_1 + 4(u+v)MS_1(S_2^2 + S_3^2) + 4(u + \tfrac{1}{3}v)MS_1^3 + 2\sqrt{2}VMS_3(S_2^2 - \tfrac{1}{3}S_3^2)\}$$

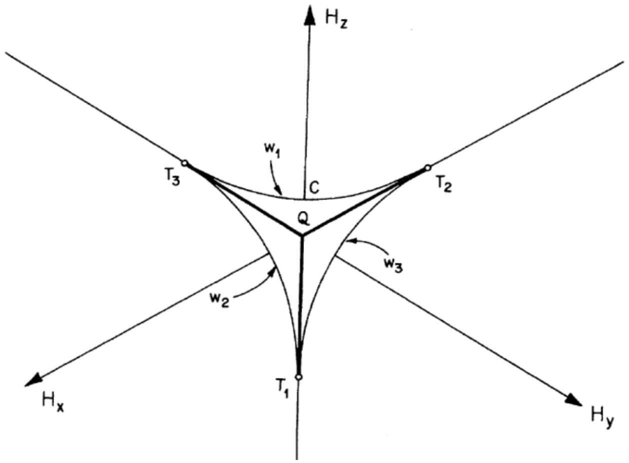
$$\mathcal{H}_{\text{eff}} = \int d^dx\{\tfrac{1}{2}[\tilde{r}_2(S_2^2 + S_3^2) + (\nabla S_2)^2 + (\nabla S_3)^2] + \tilde{u}_2(S_2^2 + S_3^2)^2 + wS_3(S_2^2 - \tfrac{1}{3}S_3^2)\}$$

PRL **37**,565 (1976)

Magnetization of Cubic Ferromagnets and the Three-Component Potts Model

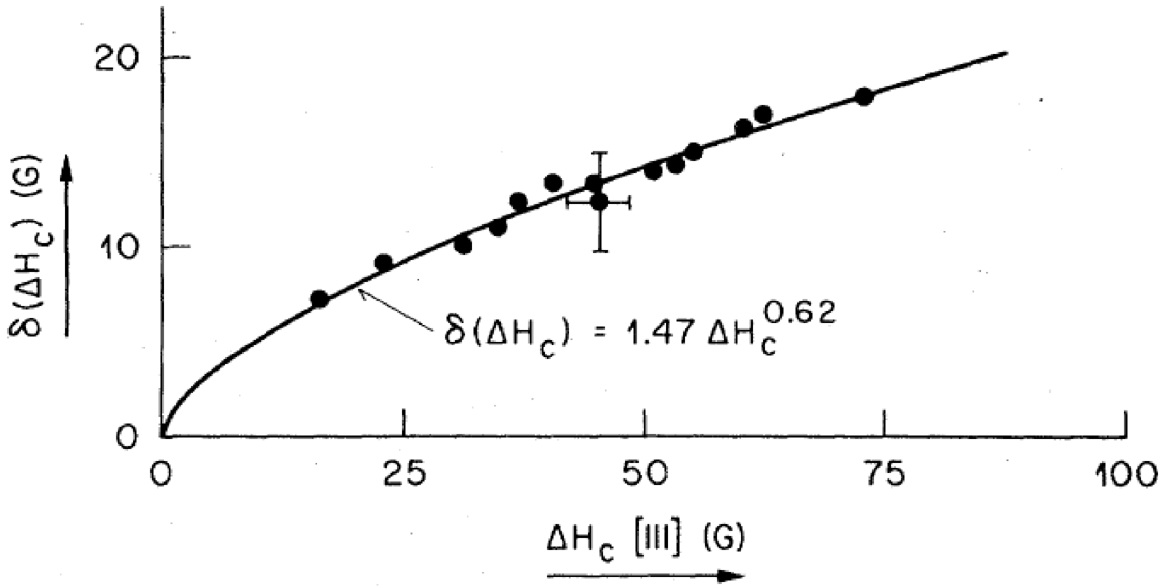
David Mukamel, Michael E. Fisher, and Eytan Domany
Baker Laboratory and Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853
(Received 19 April 1976)

The (H_x, H_y, H_z) phase diagram of a cubic ferromagnet with three easy axes, in a field $\vec{H}=(H_x, H_y, H_z)$, is studied by mean-field, scaling, and renormalization-group theories. For $T < T_c(H=0)$ and $\vec{H} \parallel [111]$ there is a phase transition at fields $\pm H_0(T)$, described by the three-component Potts model. By varying $\vec{H}$ the full phase diagram of the three-dimensional Potts model is experimentally accessible and competing predictions of the multicritical behavior can be tested.



$$\tfrac{1}{3}v\left[6\sigma_0^{\,2}(\sigma_1^{\,2}+\sigma_2^{\,2}) - 2\sqrt{2}\,\sigma_0(\sigma_2^{\,3} - 3\sigma_1^{\,2}\sigma_2) + \tfrac{3}{2}(\sigma_1^{\,2}+\sigma_2^{\,2})^2 + \sigma_0^{\,4}\right].$$

For $w=0$, the Hamiltonian (4) represents a second-order XY-like phase transition. However, w is a relevant variable, with exponent[13] $\lambda_w = 1 - \epsilon/10 - 3\epsilon^2/100 + O(\epsilon^3)$. Simple scaling arguments[13] now yield for $w \neq 0$ a first-order transition, with an order-parameter discontinuity $|\langle\Delta\vec{S}\rangle| \propto |w|^{\delta^*}$, where $\delta^* = (d-x)/\lambda_w = \frac{1}{2}(d-2+\eta)/\lambda \approx 0.62$



Side comment

The p–state Potts model has a 2nd order transition for $p < p_c$, due to fluctuations, and a 1st order transition for $p > p_c$.

$p_c = 4$ at $d = 2$. What is p_c at $d = 3$?

PHYSICAL REVIEW B VOLUME 23, NUMBER 1 1 JANUARY 1981

First- and second-order transitions in the Potts model near four dimensions

Amnon Aharony* and E. Pytte
IBM T. J. Watson Research Center, P. O. Box 218, Yorktown Heights, New York 10598
(Received 28 July 1980)

The continuum generalization of the p-state Potts model is analyzed in the ordered phase. Renormalization-group iterations in $d = 4 - \epsilon$ dimensions are followed by an elimination of the transverse modes and a mapping onto an effective Ising model. This model is then used to show that the transition is first order for $p > p_c(d)$ and continuous for $p < p_c(d)$. We find that $p_c(d) = 2$ for $d > 4$ and $p_c(4 - \epsilon) = 2 + \epsilon + O(\epsilon^2)$.

Plan for 3rd lecture

Critical behavior of compressible systems

What is $n_c = ?$ What are the implications?

Fluctuation driven 1st order transitions

PHYSICAL REVIEW B VOLUME 13, NUMBER 5 1 MARCH 1976

Critical behavior of an Ising model on a cubic compressible lattice

D. J. Bergman* and B. I. Halperin
Bell Laboratories, Murray Hill, New Jersey 07974
(Received 22 July 1975)

Renormalization-group methods are applied to the critical behavior of an Ising-like system on an elastic solid of either cubic or isotropic symmetry. Except in the special case where $dT_c/dV = 0$, the bulk modulus is found to be negative very close to T_c, so that the phase transition at constant pressure must be at least weakly first order. In the isotropic case the solid may be stabilized by pinned boundary conditions, if crystal fracture can be avoided. A "Fisher-renormalized" critical point can then be observed. By contrast, the anisotropic cubic solid will develop a microscopic instability so that T_c cannot be reached, regardless of boundary conditions. Estimates of the size of these effects are given, and contact is made with the Baker-Essam model and a liquid, as limiting cases with a vanishing shear modulus.

PHYSICAL REVIEW B VOLUME 13, NUMBER 5 1 MARCH 1976

Coupling to anisotropic elastic media: Magnetic and liquid-crystal phase transitions*

Marco A. de Moura
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174
and Departamento de Fisica,[†] *Universidade Federal de Pernambuco, Recife, 50,000, Brasil*

T. C. Lubensky[‡]
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174

Yoseph Imry[§]
Department of Physics and Institute of Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037
and Department of Physics, Brookhaven National Laboratory, Upton, New York 11973

Amnon Aharony[||]
Department of Physics and Institute of Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037
and Bell Laboratories, Murray Hill, New Jersey 09794

PHYSICAL REVIEW B VOLUME 13, NUMBER 5 1 MARCH 1976

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As was pointed out by Rice,[1] Domb,[2] and Pippard,[3] and later by many others, this question is of particular importance if the specific heat of the ideal system diverges at T_c, as it does for the Ising model.

$$\frac{H^0}{T} = \int d^dx \left(\tfrac{1}{2}\tilde{r}_0\psi^2 + \tfrac{1}{2}(\nabla\psi)^2 + \tilde{u}_0\psi^4 + \frac{g_0}{T^{1/2}}\psi^2(\nabla\cdot u)\right) + \frac{H_e^0}{T},$$

$$H_e^0 = \int d^dx \left(\tfrac{1}{2}C_{11}^0 \sum_{\alpha=1}^{d} e_{\alpha\alpha}^2 + C_{12}^0 \sum_{\alpha<\beta} e_{\alpha\alpha}e_{\beta\beta} + \tfrac{1}{2}C_{44}^0 \sum_{\alpha<\beta} e_{\alpha\beta}^2 \right),$$

$$e_{\alpha\beta}(x) = \tfrac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)$$

$$\frac{H^0}{T} = \sum_q \tfrac{1}{2}(\tilde{r}_0 + q^2)\psi_q\psi_{-q} + \frac{\tilde{u}_0}{V}\sum_{q_1q_2q_3}\psi_{q_1}\psi_{q_2}\psi_{q_3}\psi_{-q_1-q_2-q_3} + \sum_q \frac{1}{2T} u_q^* \cdot D^0 \cdot u_q + \frac{g_0}{(TV)^{1/2}}\sum_{qq_1}(iq\cdot u_q)\psi_{q_1}\psi_{-q-q_1},$$

Martin Zirnbauer

PHYSICAL REVIEW B VOLUME 13, NUMBER 5 1 MARCH 1976

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Renormalization-group methods are applied to the critical behavior of an Ising-like system on an elastic solid of either cubic or isotropic symmetry. Except in the special case where $dT_c/dV = 0$, the bulk modulus is found to be negative very close to T_c, so that the phase transition at constant pressure must be at least weakly first order. In the isotropic case the solid may be stabilized by pinned boundary conditions, if crystal fracture can be avoided. A "Fisher-renormalized" critical point can then be observed. By contrast, the anisotropic cubic solid will develop a microscopic instability so that T_c cannot be reached, regardless of boundary conditions. Estimates of the size of these effects are given, and contact is made with the Baker-Essam model and a liquid, as limiting cases with a vanishing shear modulus.

As was pointed out by Rice,[1] Domb,[2] and Pippard,[3] and later by many others, this question is of particular importance if the specific heat of the ideal system diverges at T_c, as it does for the Ising model.

$$\frac{H^0}{T} = \int d^dx \left(\tfrac{1}{2}\tilde{r}_0\psi^2 + \tfrac{1}{2}(\nabla\psi)^2 + \tilde{u}_0\psi^4 + \frac{g_0}{T^{1/2}}\psi^2(\nabla\cdot u)\right) + \frac{H_e^0}{T},$$

$$H_e^0 = \int d^dx \left(\tfrac{1}{2}C_{11}^0 \sum_{\alpha=1}^{d} e_{\alpha\alpha}^2 + C_{12}^0 \sum_{\alpha<\beta} e_{\alpha\alpha}e_{\beta\beta} + \tfrac{1}{2}C_{44}^0 \sum_{\alpha<\beta} e_{\alpha\beta}^2 \right),$$

$$e_{\alpha\beta}(x) = \tfrac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)$$

$$\frac{H^0}{T} = \sum_q \tfrac{1}{2}(\tilde{r}_0 + q^2)\psi_q\psi_{-q} + \frac{\tilde{u}_0}{V}\sum_{q_1q_2q_3}\psi_{q_1}\psi_{q_2}\psi_{q_3}\psi_{-q_1-q_2-q_3} + \sum_q \frac{1}{2T} u_q^* \cdot D^0 \cdot u_q + \frac{g_0}{(TV)^{1/2}}\sum_{qq_1}(iq\cdot u_q)\psi_{q_1}\psi_{-q-q_1},$$

Martin Zirnbauer, Slava Rychkov

PHYSICAL REVIEW B VOLUME 13, NUMBER 5 1 MARCH 1976

Coupling to anisotropic elastic media: Magnetic and liquid-crystal phase transitions*

Marco A. de Moura
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174
and Departamento de Fisica,[†] *Universidade Federal de Pernambuco, Recife, 50,000, Brasil*

T. C. Lubensky[‡]
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174

Yoseph Imry[§]
Department of Physics and Institute of Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037
and Department of Physics, Brookhaven National Laboratory, Upton, New York 11973

Amnon Aharony[||]
Department of Physics and Institute of Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037
and Bell Laboratories, Murray Hill, New Jersey 09794

$$\beta\mathcal{H}_{el} = \tfrac{1}{2}V \sum_{\alpha,\beta,\gamma,\delta} \lambda_{\alpha\beta\gamma\delta} e^0_{\alpha\beta} e^0_{\gamma\delta} + \frac{1}{2V}\sum_{\vec{k}\neq 0}\sum_{\alpha,\beta} A_{\alpha\beta}(\vec{k}) u_\alpha(\vec{k}) u_\beta(-\vec{k})$$

$$A_{\alpha\beta}(\vec{k}) = \sum_{\gamma,\delta} \lambda_{\alpha\gamma\beta\delta} k_\gamma k_\delta$$

$$\beta\mathcal{H}_{eff} = \frac{1}{V^3}\sum_{\vec{k}\neq 0}\sum_{\vec{q},\vec{p}} v(\hat{k})(\vec{S}_{\vec{q}}\cdot\vec{S}_{\vec{k}-\vec{q}})(\vec{S}_{\vec{p}}\cdot\vec{S}_{\vec{k}-\vec{p}}),$$

with $\hat{k} = \vec{k}/|\vec{k}|$, and

$$v(\hat{k}) = -\frac{1}{2}\sum_{\alpha,\beta;\gamma,\delta} [A^{-1}(\hat{k})]_{\alpha\beta} g_{\alpha\gamma} g_{\beta\delta} k_\gamma k_\delta .$$

$$\beta\mathcal{H}_{int} = \sum_{\alpha,\beta} g_{\alpha\beta} e^0_{\alpha\beta} \int d^dx\, |\vec{S}(\vec{x})|^2 + \frac{1}{V}\sum_{\vec{k}\neq 0}\sum_{\alpha} B_\alpha(\vec{k}) u_\alpha(-\vec{k}),$$

$$B_\alpha(\vec{k}) = -\frac{i}{V}\sum_{\vec{q}} (\vec{S}_{\vec{q}}\cdot\vec{S}_{\vec{k}-\vec{q}}) \sum_\beta g_{\alpha\beta} k_\beta,$$

We close with the observation that in practice it may be very difficult to see any evidence of the anisotropic runaway discussed here. This is because the crossover exponent is equal to α, which is always quite small.

So why is n_c important?

- What are the theoretical exponents for the fully cubic system?
- How do the bicritical and tetracritical phase diagrams look like?
- What is the universality class of the preovskites?

$$n_c = ?$$

The Phase Transition of Strontium Titanate

Author(s): R. A. Cowley

Source: *Philosophical Transactions: Mathematical, Physical and Engineering Sciences* , Dec. 15, 1996, Vol. 354, No. 1720, Phase Transitions with Elastic Interactions: Equilibrium and Kinetics (Dec. 15, 1996), pp. 2799-2814

Published by: Royal Society

The theoretical situation for the cubic $n = 3$, $d = 3$ system is also somewhat uncertain. If the cubic terms, v and f_0, are small, Aharony (1976) showed that the system evolved to the isotropic fixed point at which $v = 0$ and $f = 0$, when the exponents would be expected to be $\beta = 0.365$, $\gamma = 1.386$ and $\upsilon = 0.705$ (le Guillou & Zinn-Justin 1980). These results are similar to the experimental values although the differences between the β and υ are at the limit of what might be expected. This success is, however, modified by the calculations of Nattermann (1976) who showed that the isotropic $n = 3$, $d = 3$ fixed point would only describe the behaviour very close to T_C and that, in the presence of the cubic antisotropies, all of the exponents might be larger than those calculated for the isotropic model. Second, if f_0 is close to -1, Bruce (1974) and Nattermann (1976) concluded that the system probably had a first-order transition. Clearly, although critical fluctuations are important in $SrTiO_3$, there is still the need for more theoretical and experimental work to clarify the results. After developing the mean field theory of a phase transition, the next step is the understanding of the critical phenomena, although this has not been accomplished for $SrTiO_3$.

N-component Ginzburg-Landau Hamiltonian with cubic anisotropy: A six-loop study

José Manuel Carmona*
Dipartimento di Fisica dell'Università and I.N.F.N., Via Buonarroti 2, I-56127 Pisa, Italy

Andrea Pelissetto†
Dipartimento di Fisica dell'Università di Roma I and I.N.F.N., I-00185 Roma, Italy

Ettore Vicari‡
Dipartimento di Fisica dell'Università and I.N.F.N., Via Buonarroti 2, I-56127 Pisa, Italy

(Received 9 December 1999)

	Method	Results
Ref. 26, 1974	ϵ expansion: $O(\epsilon^3)$	$N_c \simeq 3.128$
Ref. 28, 1977	approximate RG	$\nu_s\, \omega_{2,s} = -0.11$, $N_c \simeq 2.3$
Ref. 30, 1981	H.T. expansion: $O(\beta^{10})$	$\nu_s\, \omega_{2,s} = -0.63(10)$, $N_c < 3$
Ref. 23, 1982	scaling-field	$N_c \simeq 3.38$
Ref. 33, 1989	$d=3$ expansion: $O(g^4)$	$\omega_{2,c} \simeq 0.008$, $N_c \simeq 2.91$
Ref. 34, 1995	ϵ expansion: $O(\epsilon^5)$	$N_c \simeq 2.958$
Ref. 36, 1997	ϵ expansion: $O(\epsilon^5)$	$\omega_{2,s} = -0.00214$, $\omega_{2,c} = 0.00213$, $N_c < 3$
Ref. 37, 1997	ϵ expansion: $O(\epsilon^5)$	$N_c \simeq 2.86$
Ref. 38, 1998	Monte Carlo	$\omega_{2,s} = 0.0007(29)$, $N_c \approx 3$
Ref. 40, 1999	$d=3$ expansion: $O(g^4)$	$\omega_{2,s} = -0.0081$, $\omega_{2,c} = 0.0077$, $N_c = 2.89(2)$
This work	ϵ expansion: $O(\epsilon^5)$	$\omega_{2,s} = -0.003(4)$, $\omega_{2,c} = 0.006(4)$, $N_c = 2.87(5)$
This work	$d=3$ expansion: $O(g^6)$	$\omega_{2,s} = -0.013(6)$, $\omega_{2,c} = 0.010(4)$, $N_c = 2.89(4)$

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Physics Reports 368 (2002) 549–727

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Critical phenomena and renormalization-group theory

Andrea Pelissetto[a, *], Ettore Vicari[b]

Ref.	Method	$\omega_{2,s}$	$\omega_{2,c}$	N_c
[437] 2000	$d = 3$ exp: $O(g^6)$		0.015(2)	2.862(5)
[270] 2000	$d = 3$ exp: $O(g^6)$	−0.013(6)	0.010(4)	2.89(4)
[270] 2000	ϵ exp: $O(\epsilon^5)$	−0.003(4)	0.006(4)	2.87(5)
[1078] 2000	$d = 3$ exp: $O(g^4)$	−0.0081	0.0077	2.89(2)
[1001] 1997	ϵ exp: $O(\epsilon^5)$			2.86
[660,661] 1995	ϵ exp: $O(\epsilon^5)$	−0.00214	0.00213	$N_c < 3$
[658] 1995	ϵ exp: $O(\epsilon^5)$			2.958
[775] 1989	$d = 3$ exp: $O(g^4)$		0.008	2.91
[831] 1974	ϵ exp: $O(\epsilon^3)$			3.128
[1064] 2002	CRG			3.1
[835] 1982	Scaling-field			3.38
[1083] 1977	CRG	−0.16		2.3
[273] 1998	MC	0.0007(29)		$N_c \approx 3$
[386] 1981	HT exp: $O(\beta^{10})$	−0.89(14)		$N_c < 3$

Bootstrapping Heisenberg Magnets and their Cubic Instability

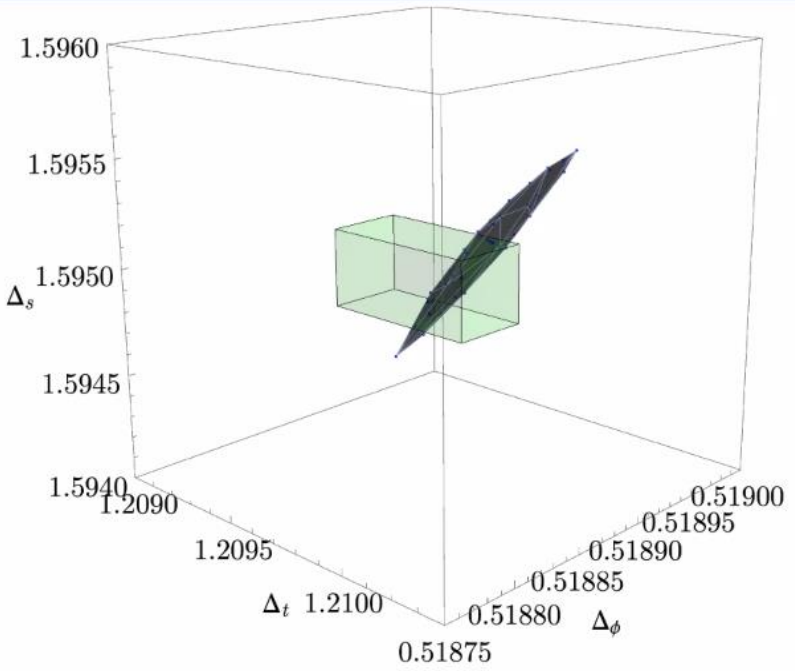
Shai M. Chester[a], Walter Landry[b,c], Junyu Liu[c,d], David Poland[e], David Simmons-Duffin[c], Ning Su[f], Alessandro Vichi[f,g]

[a] *Department of Particle Physics and Astrophysics, Weizmann Institute of Science, Rehovot, Israel*
[b] *Simons Collaboration on the Nonperturbative Bootstrap*
[c] *Walter Burke Institute for Theoretical Physics, Caltech, Pasadena, CA 91125, USA*
[d] *Institute for Quantum Information and Matter, Caltech, Pasadena, CA 91125, USA*
[e] *Department of Physics, Yale University, New Haven, CT 06520, USA*
[f] *Institute of Physics, École Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland*
[g] *Department of Physics, University of Pisa, I-56127 Pisa, Italy*

Abstract

We study the critical $O(3)$ model using the numerical conformal bootstrap. In particular, we use a recently developed cutting-surface algorithm to efficiently map out the allowed space of CFT data from correlators involving the leading $O(3)$ singlet s, vector ϕ, and rank-2 symmetric tensor t. We determine their scaling dimensions to be $(\Delta_s, \Delta_\phi, \Delta_t) = (0.518942(51), 1.59489(59), 1.20954(23))$, and also bound various OPE coefficients. We additionally introduce a new "tip-finding" algorithm to compute an upper bound on the leading rank-4 symmetric tensor t_4, which we find to be relevant with $\Delta_{t_4} < 2.99056$. The conformal bootstrap thus provides a numerical proof that systems described by the critical $O(3)$ model, such as classical Heisenberg ferromagnets at the Curie transition, are unstable to cubic anisotropy.

$O(3)$ from $\{\phi_i, s, t_{ij}\}$ System



Black (Bootstrap): [Chester, Landry, Liu, DP, Simmons-Duffin, Vichi, '20]

Green (Monte Carlo): [Hasenbusch, Vicari '11; Hasenbusch '20]

- Using `tiptop` search, find it is relevant: $\Delta_{\phi^{\{i}\phi^j\phi^k\phi^{l\}}} < 2.99056$!
- Proof that critical Heisenberg magnets are unstable to cubic anisotropy, should flow to fixed point with cubic symmetry C_3 rather than $O(3)$

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Physics Letters B 342 (1995) 284–296

Exact five-loop renormalization group functions of ϕ^4-theory with $O(N)$-symmetric and cubic interactions. Critical exponents up to ϵ^5

H. Kleinert, V. Schulte-Frohlinde

$$N_c = 4 - 2\,\epsilon + \epsilon^2\left(-\frac{5}{12} + \frac{5\,\zeta(3)}{2}\right) + \epsilon^3\left(-\frac{1}{72} + \frac{5\,\zeta(3)}{8} + \frac{15\,\zeta(4)}{8} - \frac{25\,\zeta(5)}{3}\right) + \epsilon^4\left(-\frac{1}{384} + \frac{93\,\zeta(3)}{128} - \frac{229\,\zeta^2(3)}{144} + \frac{15\,\zeta(4)}{32} - \frac{3155\,\zeta(5)}{1728} - \frac{125\,\zeta(6)}{12} + \frac{11515\,\zeta(7)}{384}\right) + O[\epsilon^5]\,. \tag{19}$$

Padé [1, 1] :	$N_c = 3.128$	Padé [2, 2] :	$N_c = 2.958$
Padé [2, 1] :	$N_c = 2.792$	Padé [1, 2] :	$N_c = 2.893$
Padé [3, 1] :	$N_c = 3.068$	Padé [1, 3] :	$N_c = 2.972$.

Nuclear Physics B 940 (2019) 332–350

Six-loop ε expansion study of three-dimensional n-vector model with cubic anisotropy

Loran Ts. Adzhemyan, Ella V. Ivanova, Mikhail V. Kompaniets, Andrey Kudlis *, Aleksandr I. Sokolov

Detailed study of the n-vector cubic model including evaluation of critical exponents and n_c was carried out by many groups [2–32] having used both field-theoretical methods and lattice calculations. Early numerical estimates of n_c obtained in the lower-order approximations within the ε expansion approach [2,4–6] and in the frame of 3D RG machinery [12,15,16] turned out to be in favor of the conclusion that $n_c > 3$, while lattice calculations implied n_c is practically equal to 3 [13]. This made the study of the cubic class of universality less interesting from the physical point of view. Later, however, the higher-order analysis including resummation of RG perturbative series was performed and shown that numerical value of n_c falls below 3 [17–22, 24–26,28,32]. To date, the most advanced estimates of n_c obtained within the ε expansion, 3D RG and pseudo-ε expansion approaches are $n_c = 2.855, 2.87$ [21,26], $n_c = 2.89, 2.91$ [24,26] and $n_c = 2.86$ [28,32], respectively.

$$n_c = 4 - 2\varepsilon + 2.588476\varepsilon^2 - 5.874312\varepsilon^3 + 16.82704\varepsilon^4 - 56.62195\varepsilon^5 + \mathcal{O}\left(\varepsilon^6\right).$$

Table 1
Padé triangle for the ε expansion of n_c. Here Padé estimate of k-th order (lower line, RoC) is the number given by corresponding diagonal approximant [L/L] or by a half of the sum of the values given by approximants [L/L−1] and [L−1/L] when a diagonal approximant does not exist. Three estimates are absent because corresponding Padé approximants have poles close to the physical value $\varepsilon = 1$.

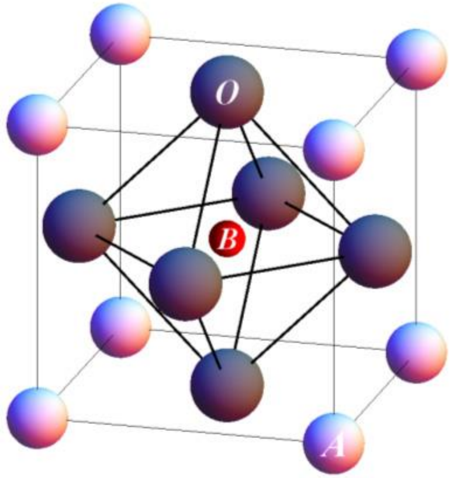
$M \setminus L$	0	1	2	3	4	5
0	4	2	4.5885	−1.2858	15.5412	−41.0807
1	2.6667	3.1283	2.7917	3.0684	2.5692	
2	–	2.8930	2.9576	2.8828		
3	1.9518	–	2.9138			
4	–	2.7887				
5	0.4549					
RoC	4	2.3333	3.1283	2.8424	2.9576	2.8983

Table 2
Padé–Borel–Leroy estimates of n_c obtained from ε expansion (25) under the optimal value of the shift parameter $b_{opt} = 1.845$. The estimate of k-th order (lower line, RoC) is the number given by corresponding diagonal approximant [L/L] or by a half of the sum of the values given by approximants [L/L−1] and [L−1/L] when a diagonal approximant does not exist. Two estimates are absent because corresponding Padé approximants turn out to be spoiled by dangerous poles.

$M \setminus L$	0	1	2	3	4	5
0	4	2	4.58848	−1.28584	15.5412	−41.0807
1	2.75996	3.05988	2.87042	2.92283	2.91341	
2	–	2.93394	2.91132	2.91499		
3	2.57775	2.91419	2.91416			
4	–	2.91416				
5	2.39138					
RoC	4	2.3800	3.0599	2.9022	2.9113	2.9146

$$n_c = n_c^{(6)} = 2.915 \pm 0.003$$

“New” approach:



Physically, *n* **must** equal *d.*

Later, we also consider $\mathscr{H}_0^f = \frac{f_0}{2} \sum_\alpha \int_{\mathbf{k}}^{\Lambda} k_\alpha^2 |Q_\alpha(\mathbf{k})|^2$

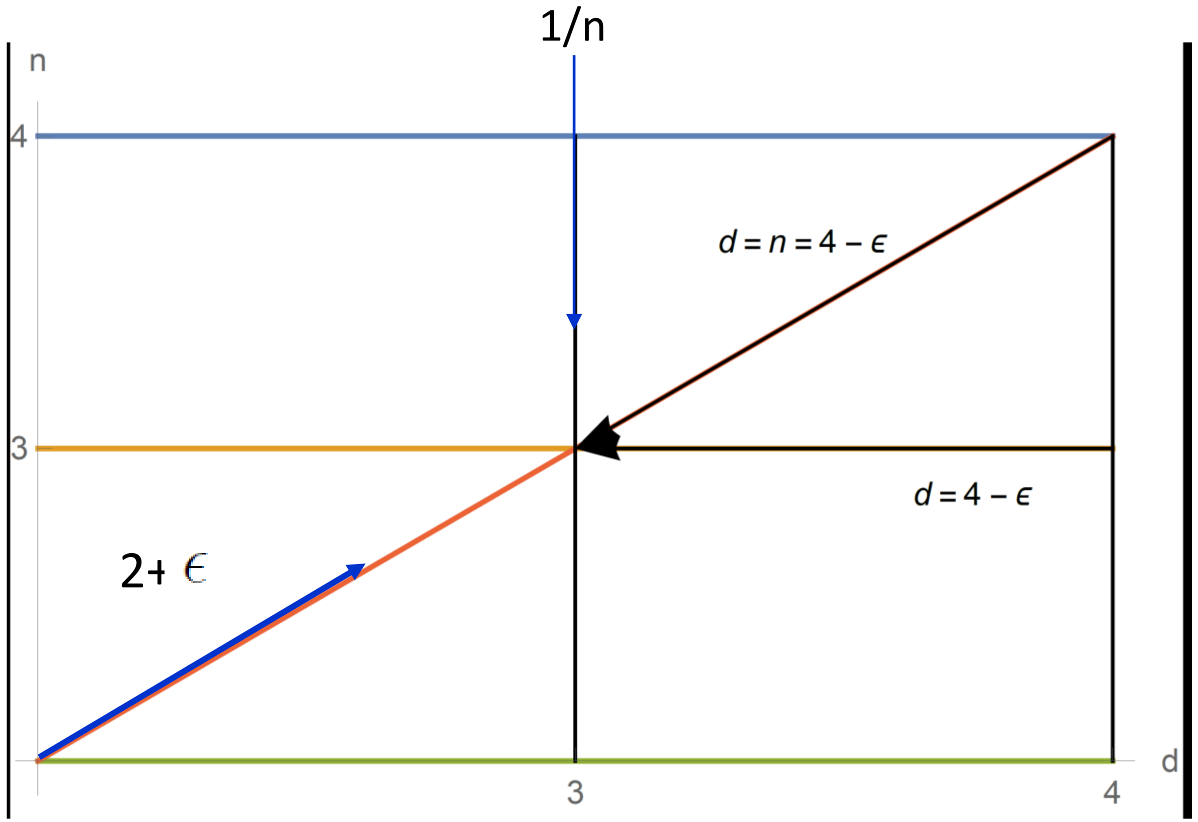
Historically, started in 1973,

M. E. Fisher and A. Aharony
Dipolar interactions at ferromagnetic critical points
Phys. Rev. Lett. **30**, 559-562 (1973)

$$\frac{1}{R(ij)^d}\left[\mathbf{S})(i)\cdot\mathbf{S}(j) - d\frac{\mathbf{R}(ij)\cdot\mathbf{S}(i)\mathbf{R}(ij)\cdot\mathbf{S}(j)}{R(ij)^2}\right]$$

$$\Rightarrow \quad n = d = 4 - \epsilon$$

$$n = d = 4 - \epsilon$$



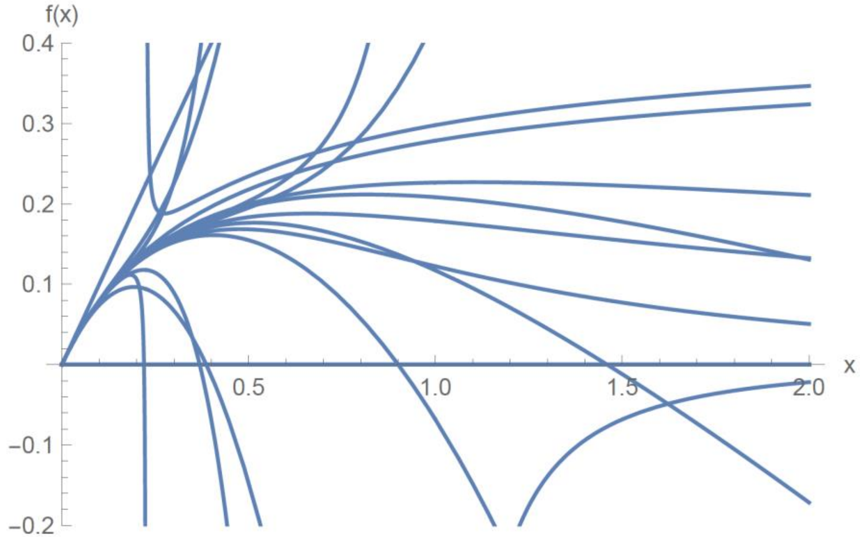
$$n = n_c(\epsilon) = 4 - 2\epsilon + 2.58848\epsilon^2 - 5.87431\epsilon^3 + 16.827\epsilon^4 - 56.62196\epsilon^5 + \mathcal{O}[\epsilon^6].$$

$$4 - \epsilon - n_c(\epsilon) = \epsilon - 2.58848\epsilon^2 + 5.87431\epsilon^3 - 16.827\epsilon^4 + 56.62196\epsilon^5 + \mathcal{O}[\epsilon^6] = 0$$

$$4 - \epsilon - n_c(\epsilon) = \epsilon - 2.58848\epsilon^2 + 5.87431\epsilon^3 - 16.827\epsilon^4 + 56.62196\epsilon^5 + \mathcal{O}[\epsilon^6].$$

$$\begin{pmatrix} \frac{1-0.31906578597938484\,\epsilon}{1+2.269409805252934\,\epsilon} & \frac{1-4.551834202156295\,\epsilon}{1-1.9633586109239758\,\epsilon-10.956417728613014\,\epsilon^2} & \frac{1+2.177564360014011\,\epsilon}{1+4.76603995124633\,\epsilon+6.462466193238685\,\epsilon^2+5.557769889409281\,\epsilon^3} \\ \frac{1+0.2760367726022195\,\epsilon-1.5404084471683843\,\epsilon^2}{1+2.8645123638345384\,\epsilon} & \frac{1+2.684825951080864\,\epsilon-2.308970659658309\,\epsilon^2}{1+5.273301542313183\,\epsilon+5.466529780426597\,\epsilon^2} & 0 \\ \frac{1.+0.7764628211312679\,\epsilon-2.8357490590026666\,\epsilon^2+2.9396586856390527\,\epsilon^3}{1+3.364938412363587\,\epsilon} & 0 & 0 \end{pmatrix}$$

(with Ora Entin-Wohlman)



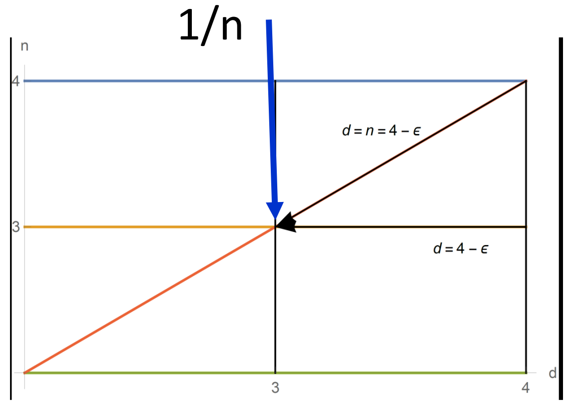
$$\epsilon_c \;=\; .220,\;\; .369,\;\; 0.386,\;\; .900,\;\; 1.460$$

Small, so more reliable?

Large n

Critical Behavior of Anisotropic Cubic Systems in the Limit of Infinite Spin Dimensionality

Amnon Aharony
Baker Laboratory, Cornell University, Ithaca, New York 14850
(Received 29 October 1973)

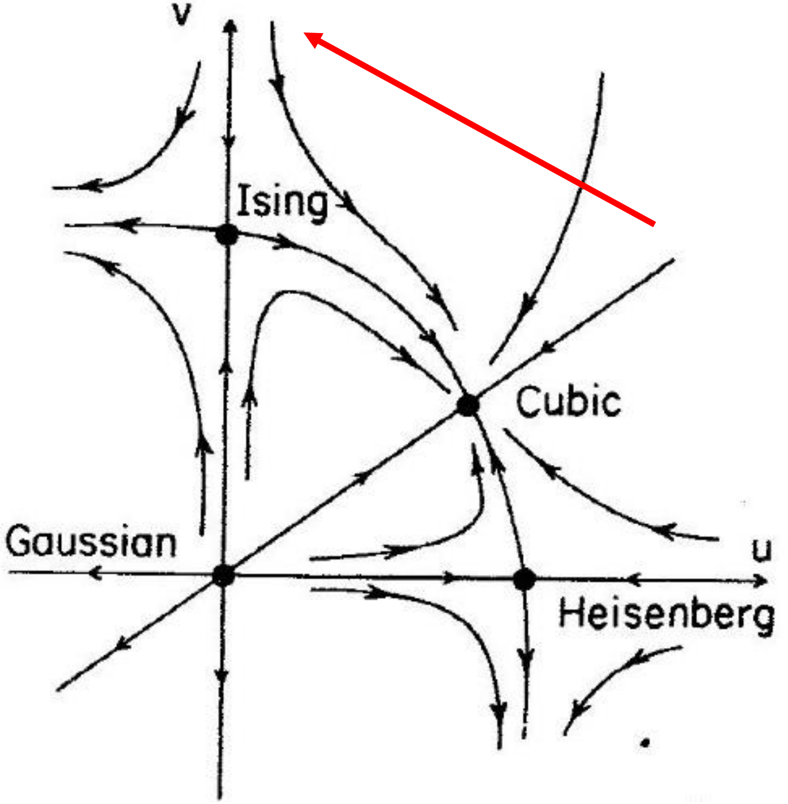


$$\mathcal{H}_0=\sum_\alpha\{-\tfrac{1}{2}\int(r_0+q^2)\sigma_{\vec{q}}^{\alpha}\sigma_{-\vec{q}}^{\alpha}-v_0\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}'}^{\alpha}\sigma_{\vec{q}''}^{\alpha}\sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^{\alpha}\}$$

$$\mathcal{H}_1=-u_0\sum_{\alpha\beta}\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}'}^{\alpha}\sigma_{\vec{q}''}^{\beta}\sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^{\beta}$$

$$\langle\sigma_{\vec{q}}^{\alpha}\sigma_{-\vec{q}}^{\alpha}\rangle_0=G_0(r_1,\vec{q}).$$

$$\gamma^c=\gamma^I/(1-\alpha^I)+O(n^{-1}).$$



C

H

$$\eta^c\simeq0.056+O(n^{-2})\text{ and }\gamma^c\simeq1.43+O(n^{-1})$$

$$\eta^s=0,\quad\gamma^s=2.$$

$d = 2 + \epsilon$

Volume 57A, number 1 PHYSICS LETTERS 17 May 1976

BICRITICAL POINTS IN 2 + ϵ DIMENSIONS

R.A. PELCOVITS and D.R. NELSON *

$$\bar{H} = \int \mathrm{d}\boldsymbol{R} \left\{ \frac{1}{2f} [\partial_\mu \boldsymbol{S}(\boldsymbol{R})]^2 + \frac{g_1}{f} [\partial_\mu S_n(\boldsymbol{R})]^2 + \frac{g_2}{f} [S_n(\boldsymbol{R})]^2 \right\}$$

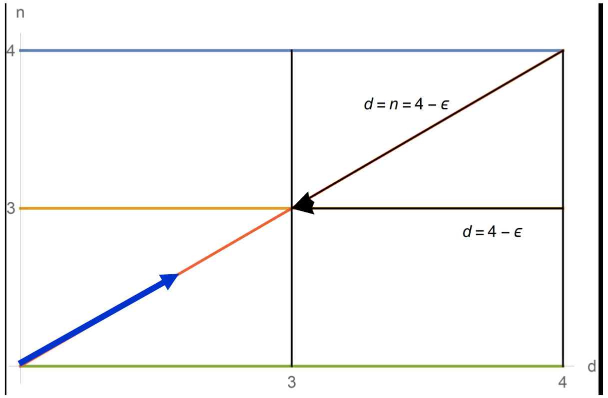
$$\lambda_{g_2} = 2 - 2\epsilon/(n-2) + O(\epsilon^2).$$

$$\bar{H}_v = (v/f) \int \sum_{a=1}^{n} [S_n(\boldsymbol{R})]^4$$

$$\lambda_v = 2 - \epsilon(n+6)/(n-2) + O(\epsilon^2),$$

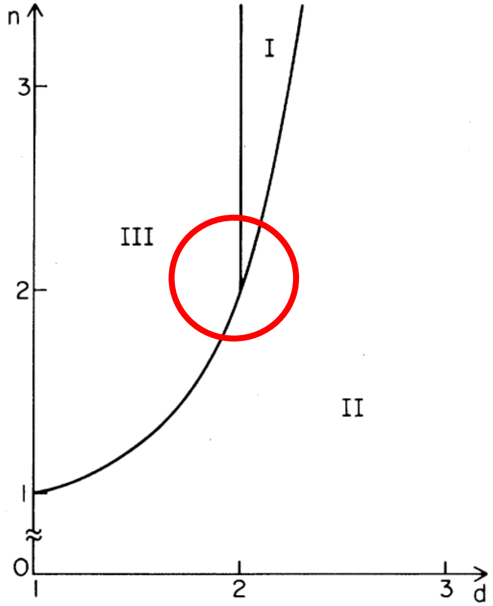
$$n = d = 2 + \epsilon$$

???



$O(n)$ Heisenberg Model Close to $n=d=2$

John L. Cardy and Herbert W. Hamber
Department of Physics, University of California, Santa Barbara, California
(Received 19 May 1980)



10 May 2021

SciPost Physics
Submission

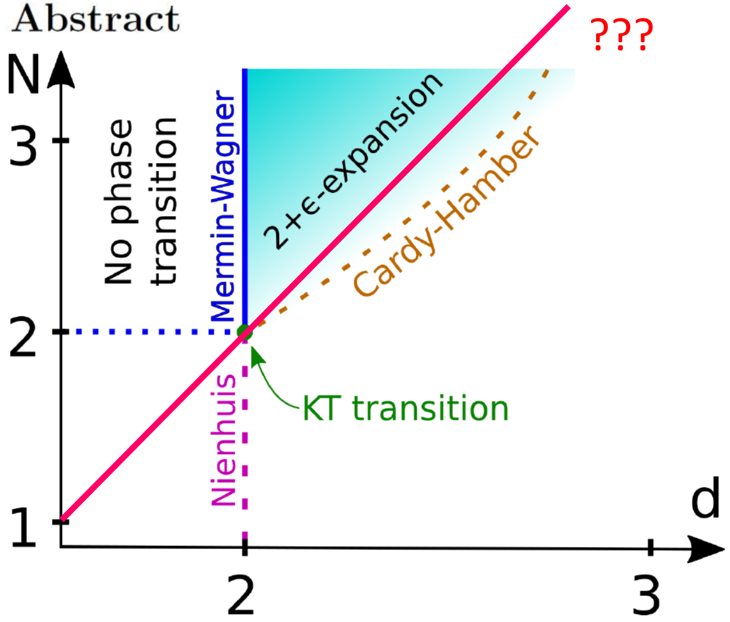
Analyticity of critical exponents of the $O(N)$ models from nonperturbative renormalization

A. Z. Chlebicki[1], P. M. Jakubczyk[1*]

1 Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland
* pawel.jakubczyk@fuw.edu.pl

May 11, 2021

Abstract

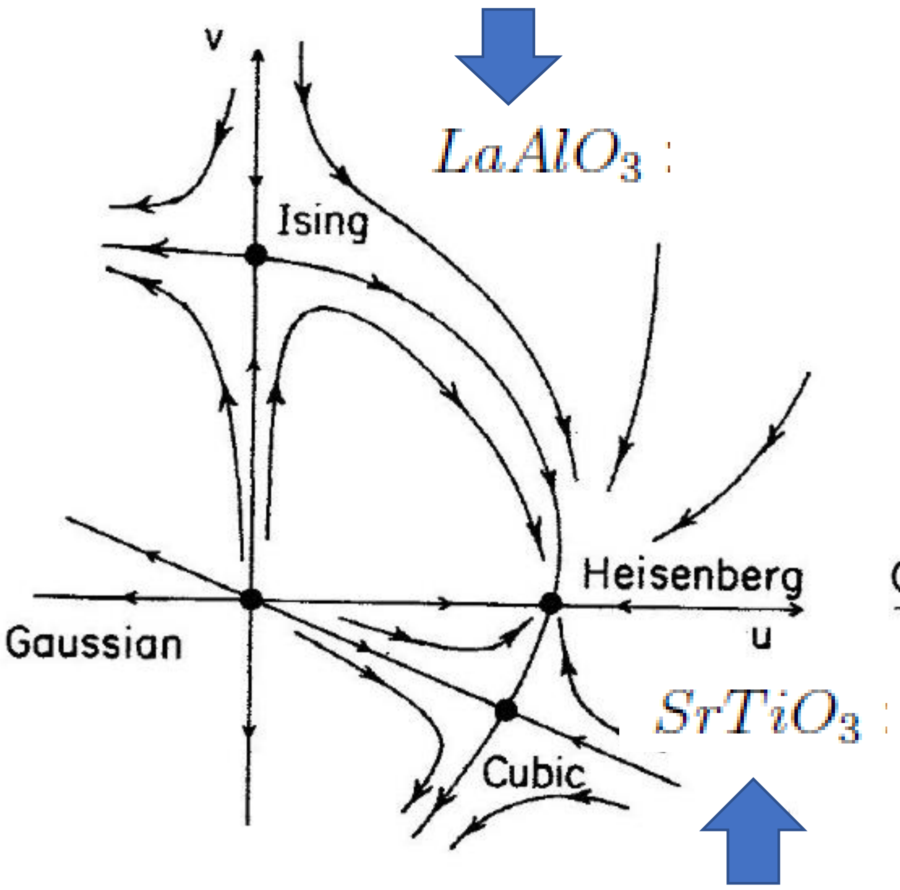


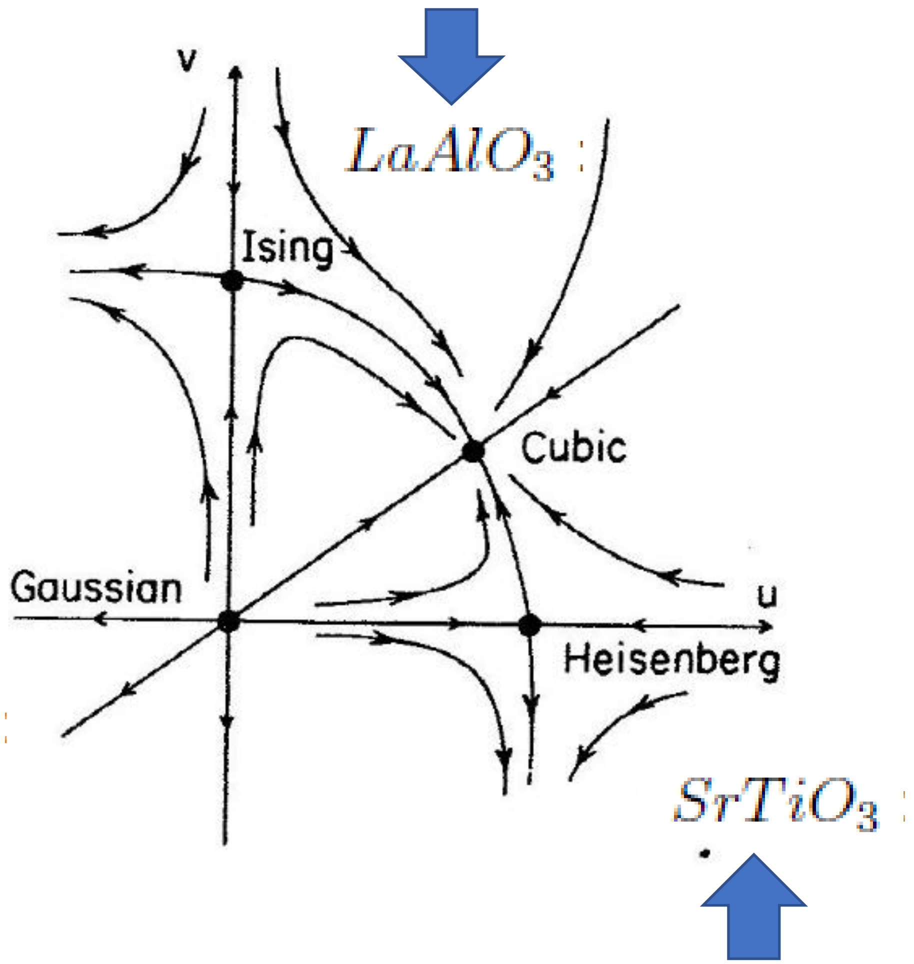
December 2018-
Slava visits Barak Kol at HUJI
and Ofer Aharony at WIS

If the cubic fixed point wins, what are the implications for

The displacive phase transitions in the perovskites??

Back to the beginning

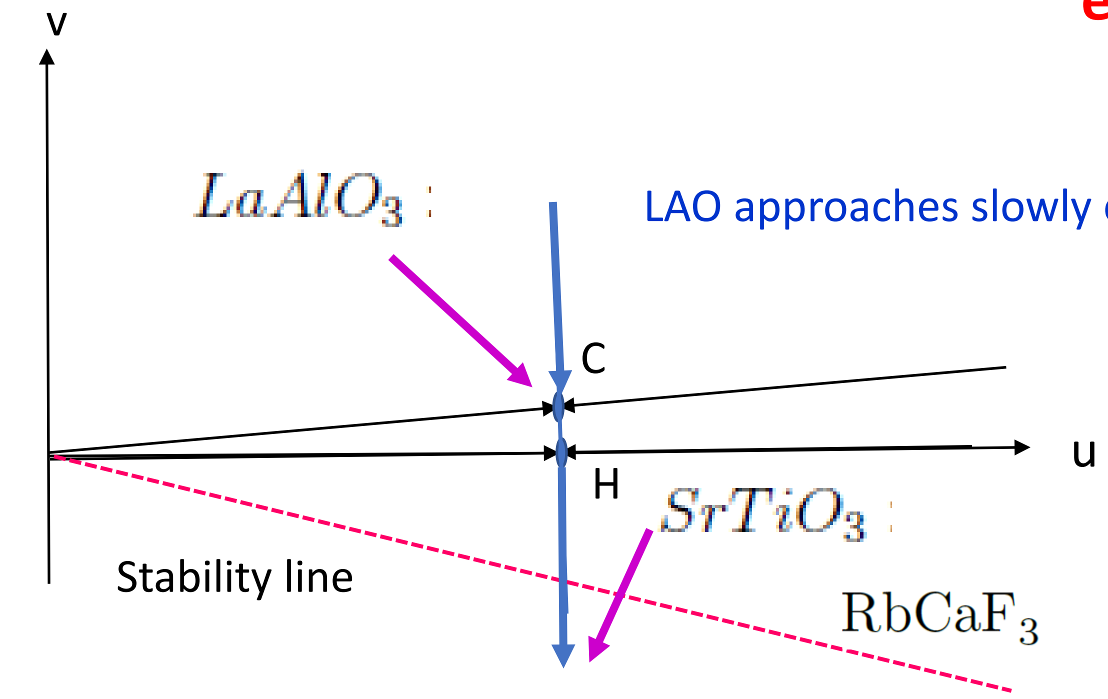




What is predicted if $n_c \preceq 3$?

Exponents for the flow of v are very small (almost logarithmic) – always see v-dependent **effective exponents**

LAO approaches slowly **cubic** behavior



1st order

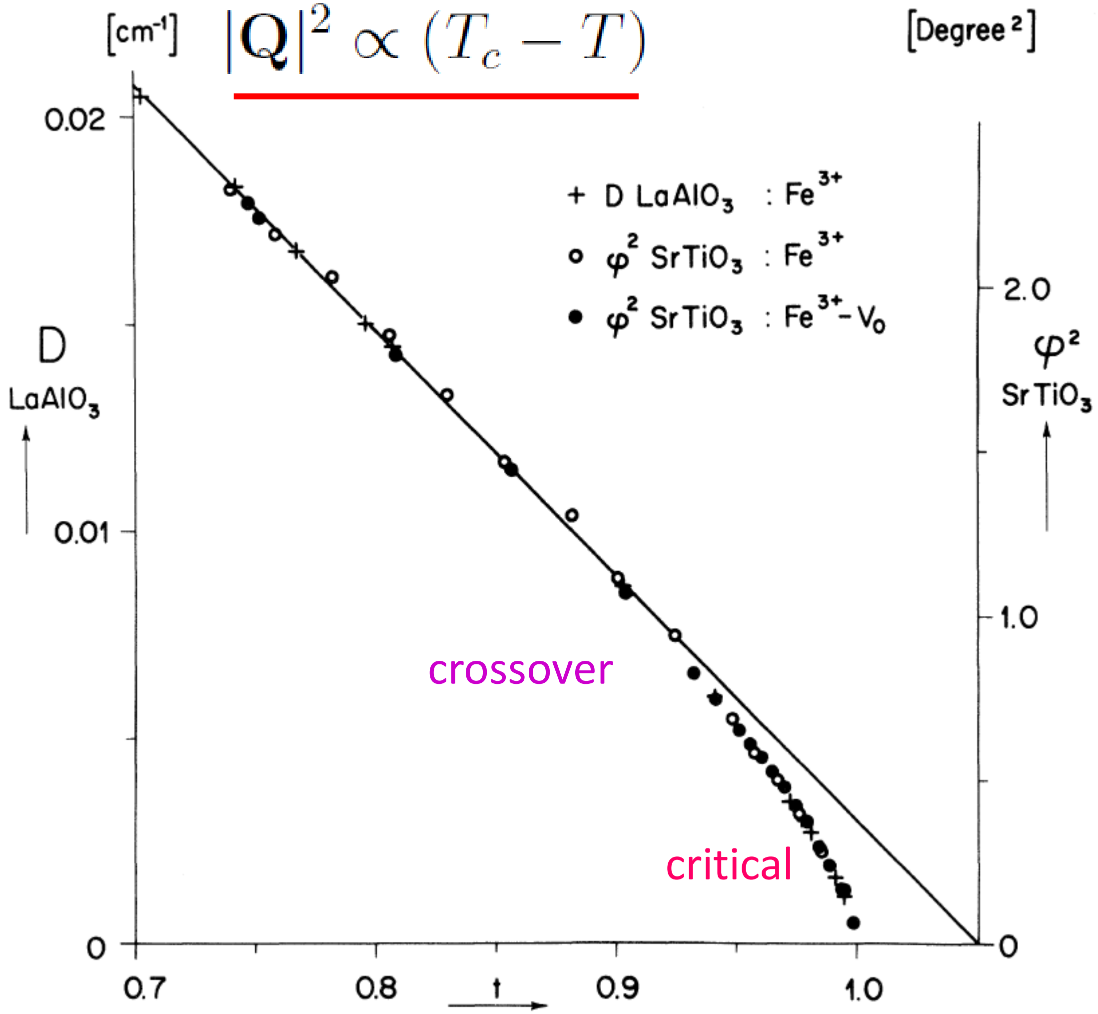
No Universality!

STO starts with **isotropic** exponents, but goes **very slowly** to 1st order

What about universality??

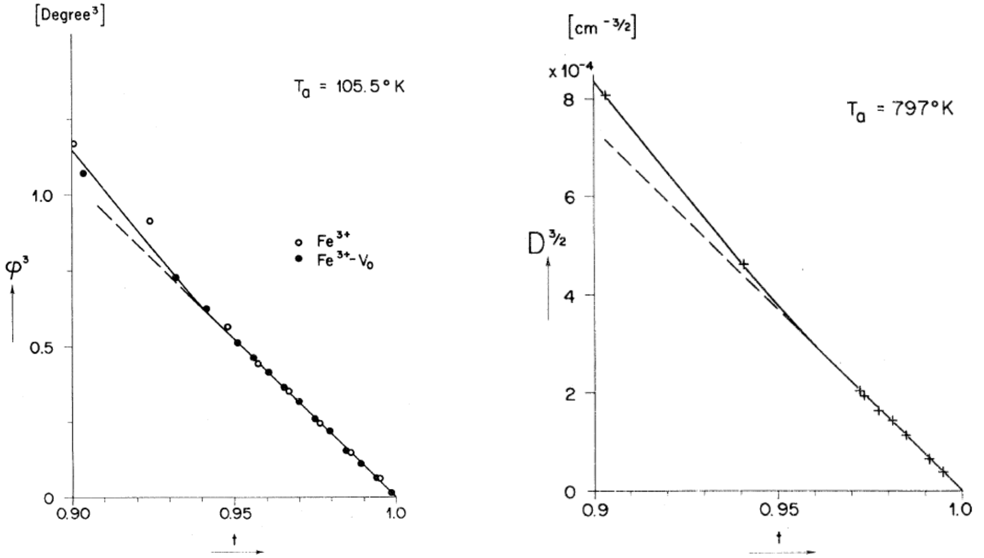
MFT

$$|\mathbf{Q}|^2 \propto (T_c - T)$$



T_c is lowered, power law changes

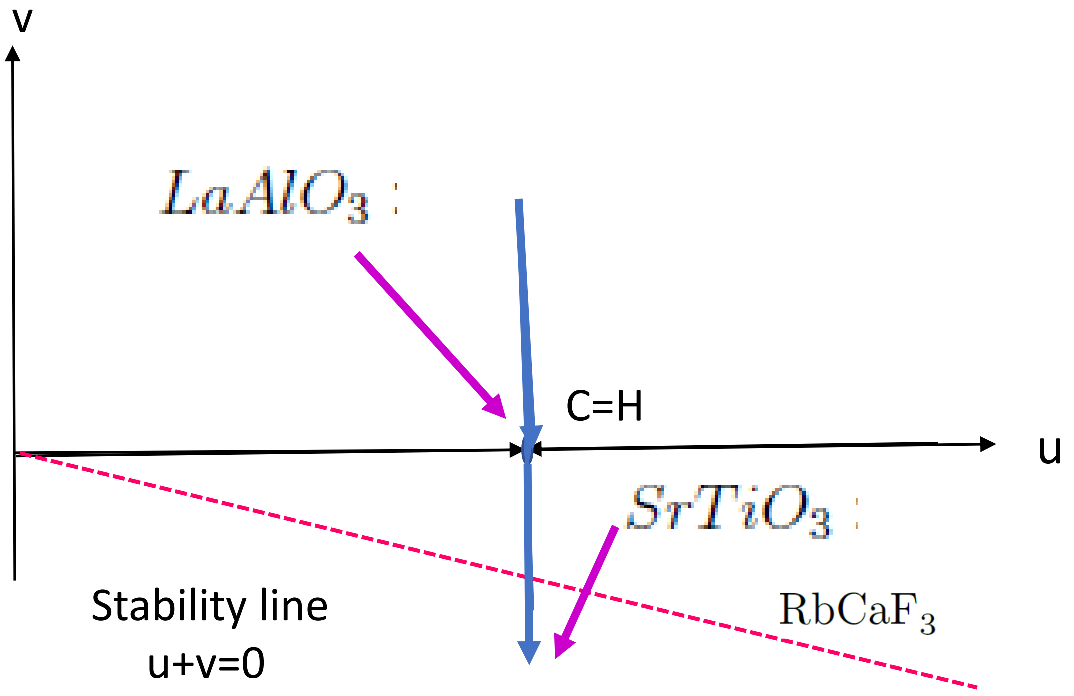
$$\varphi \propto \epsilon^{\beta}, \quad \epsilon = (T_a - T)/T_a,$$



$$\beta = 0.33 \pm 0.02$$

Which **universality class** is this??

$$n_c = 3$$



©

$$\frac{dv}{d\ell} = \cancel{\lambda_v v} + A(u)v^2 + \dots$$

$$\frac{dv}{v^2} = A\ d\ell$$

$$\frac{1}{v(0)} - \frac{1}{v(\ell)} = A\ \ell$$

$$v(\ell) = \frac{v(0)}{1 - Av(0)\ell}$$

$$e^{\ell} = \xi/\xi(0) \propto t^{-\nu}$$

$$\ell \sim -\nu \log t + const.$$

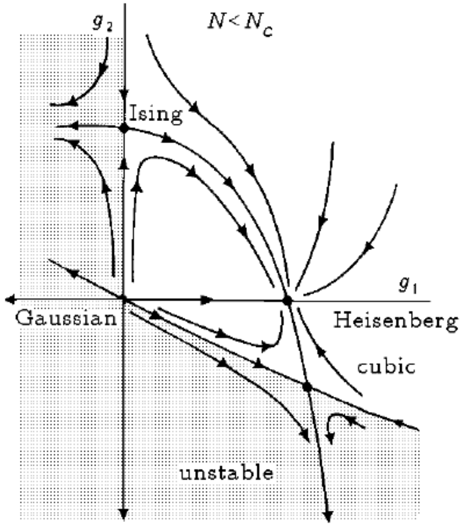
A<0 ??

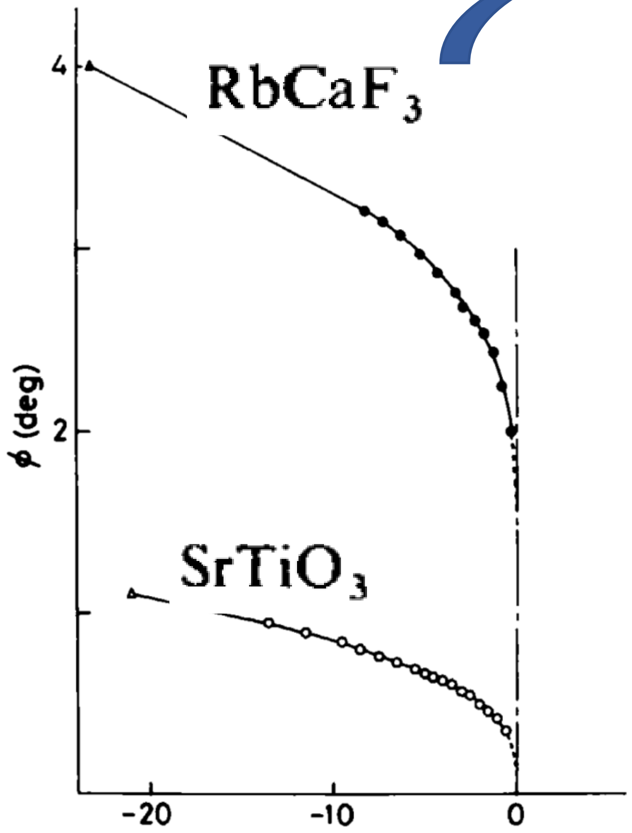
First order transitions, tricritical points

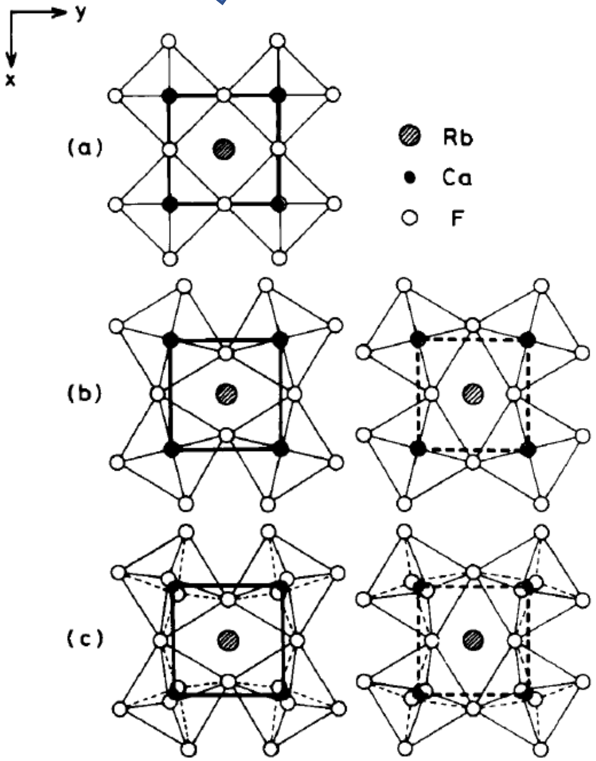
Phase Transitions, 1985, Vol. 5, pp. 219–232
0141-1594/85/0503-0219$18.50/0

Structural Phase Transitions of $RbCaF_3$

M. HIDAKA, S. MAEDA
Department of Physics, Kyushu University 33, Fukuoka 812, Japan
and
J. S. STOREY
Clarendon Laboratory, Department of Physics, University of Oxford, Oxford, OX1 3PU U.K.







$KMnF_3$,[2] $RbCaF_3$,[3] $KCaF_3$,[4] . . .

Need scenarios that turn 2nd order transitions into 1st order ones

Fluctuation driven 1st order transitions

Scenarios for **fluctuation driven 1st order transitions**: The role of **irrelevant** fields

PHYSICAL REVIEW B VOLUME 28, NUMBER 1 1 JULY 1983

Fluctuation-induced tricritical points

Daniel Blankschtein*
Department of Electronics, The Weizmann Institute of Science, Rehovot 76100, Israel and Department of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel

Amnon Aharony

$$\mathscr{H}=\int d^dx[\tfrac{1}{2}|\vec{\nabla}\vec{S}|^2+\tfrac{1}{2}r|\vec{S}|^2+u_4|\vec{S}|^4+u_6|\vec{S}|^6+O(|\vec{S}|^8)],$$

$$\frac{dr}{dl}=2r+4(n+2)K_d u_4(1-r),$$

$$\frac{du_4}{dl}=(4-d)u_4+3(n+4)K_d u_6-4(n+8)K_d u_4^2,$$

$$\frac{du_6}{dl}=(6-2d)u_6-12(n+14)K_d u_6 u_4,$$

$$t(l)=t(0)e^{2l}/Q(l)^{(n+2)/(n+8)},$$

$$t(0)=r+2(n+2)K_d\times[u_4+3(n+4)K_d u_6/2(d-2)]$$

$$\tilde{u}_4(l)=\tilde{u}_4(0)e^{(4-d)l}/Q(l),$$

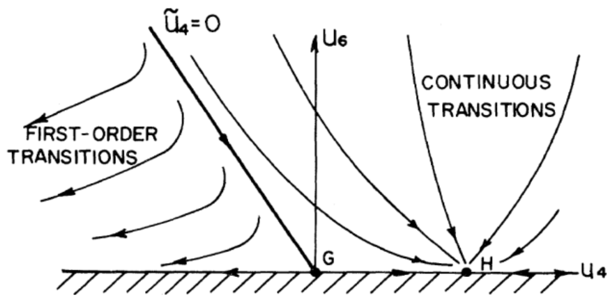
$$\tilde{u}_4(0)=u_4+C(n)u_6$$

$$u_6(l)=u_6 e^{(6-2d)l}/Q(l)^{3(n+14)/(n+8)},$$

$$u_6>0$$

where

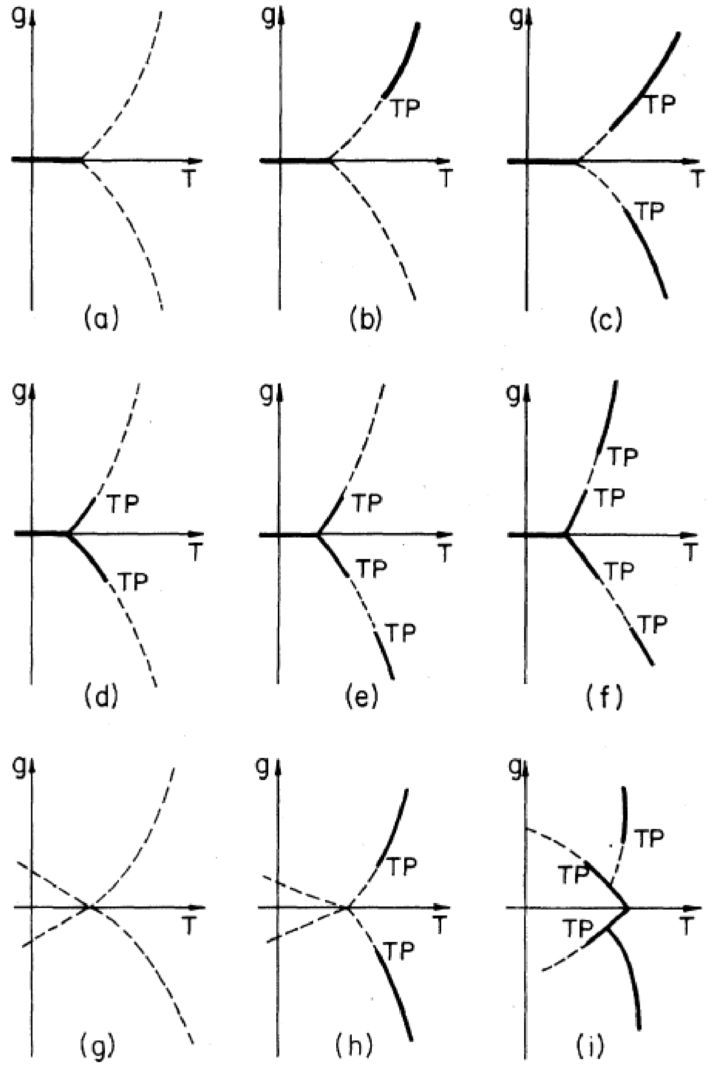
$$Q(l)=1+[\tilde{u}_4(0)/\tilde{u}_4^H](e^{\epsilon l}-1).$$



VOLUME 47, NUMBER 6 PHYSICAL REVIEW LETTERS 10 AUGUST 1981

Crossover from Fluctuation-Driven Continuous Transitions to First-Order Transitions

Daniel Blankschtein and Amnon Aharony



isotropic

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2}|\nabla\vec{S}|^2 + \frac{1}{2} r|\vec{S}|^2 + u_4|\vec{S}|^4 + u_6|\vec{S}|^6 + O(|\vec{S}|^8)\right\}$$

Cubic
V<0

$$\tilde{u}_4 = u_4 + C(n)u_6, \quad C(n) = \frac{3}{2}K_4(n+4)$$

$$\mathcal{H}_v = v \sum_{\alpha=1}^{n} (S^\alpha)^4$$

Cubic
V>0

PHYSICAL REVIEW B VOLUME 28, NUMBER 1 1 JULY 1983

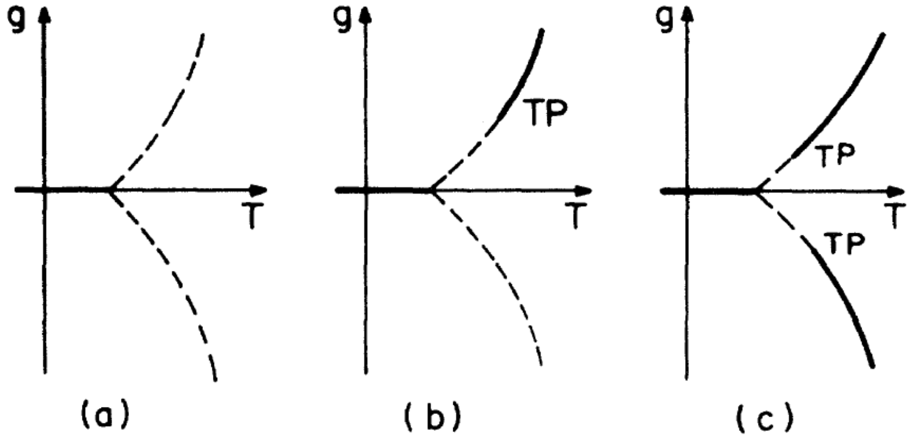
Fluctuation-induced tricritical points

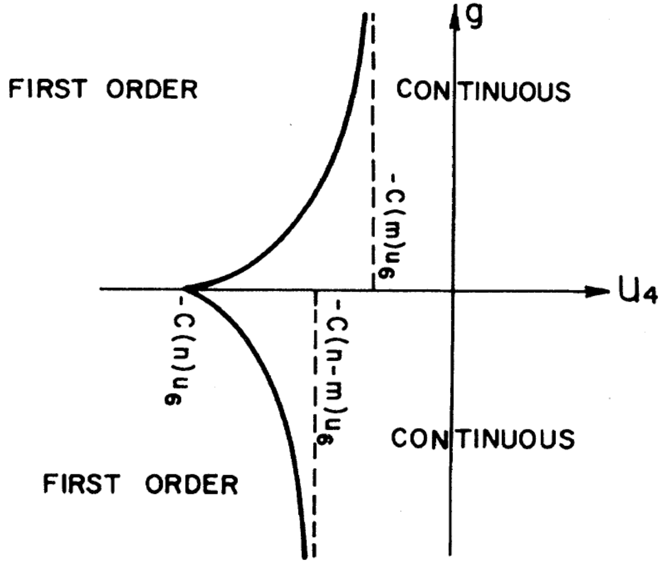
Daniel Blankschtein*
Department of Electronics, The Weizmann Institute of Science, Rehovot 76100, Israel
and Department of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel

Amnon Aharony

We now add the *quadratic anisotropy*,

$$\mathscr{H}_g = \tfrac{1}{2} g \int d^d x \left[m \,|\vec{S}_{n-m}|^2 - (n-m)\,|\vec{S}_m|^2 \right]/n$$





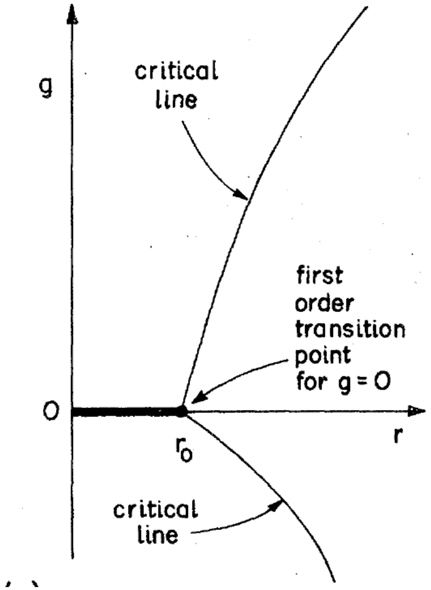
Scenarios for **fluctuation driven 1st order transitions**: The role of **relevant** fields

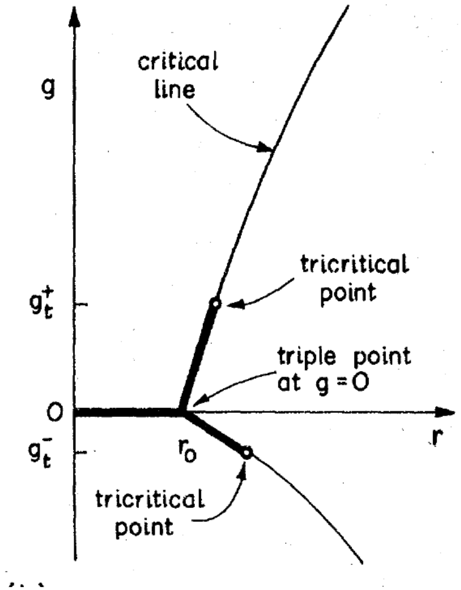
PHYSICAL REVIEW B VOLUME 15, NUMBER 11 1 JUNE 1977

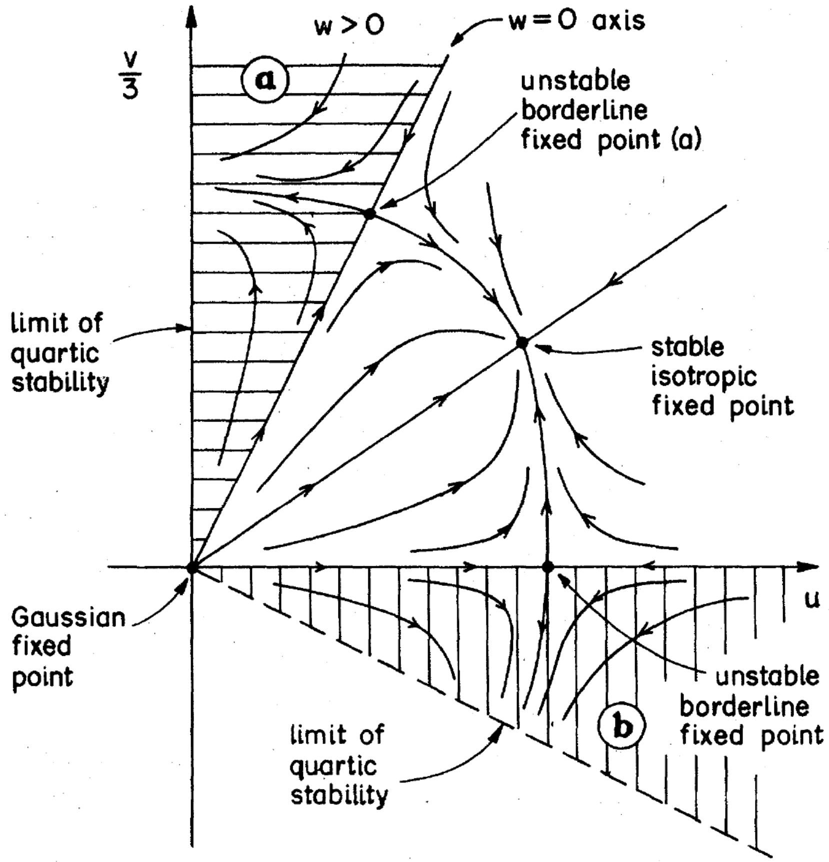
Destruction of first-order transitions by symmetry-breaking fields

Eytan Domany, David Mukamel, and Michael E. Fisher

$$w = \begin{cases} v - 6u \text{ for } n=2 , \\ v - 3u + O(\epsilon^2) \text{ for } n=3 \\ v - 2u \text{ for } n \geq 4 , \end{cases}$$



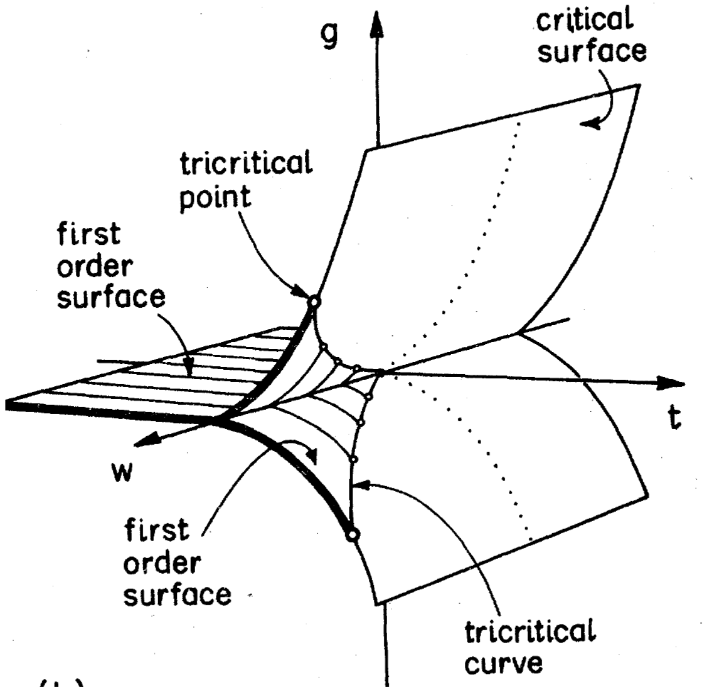




$$\mathcal{H} = \int d\vec{R} \left(-\tfrac{1}{2} |\nabla \vec{\psi}|^2 - \tfrac{1}{2} r_1 \sum_{\alpha=1}^{m} \psi_\alpha^2 - \tfrac{1}{2} r_2 \sum_{\alpha=m+1}^{n} \psi_\alpha^2 - u \sum_{\alpha=1}^{n} \psi_\alpha^4 - v \sum_{\alpha<\beta=1}^{n} \psi_\alpha^2 \psi_\beta^2 \right) , \quad (2.$$

h

$r_1 = r - [1 - (m/n)] g$ and $r_2 = r + (m/n) g$.



$$f(t,g,w) \approx t^{2-\alpha} W\left(g/t^{\phi_g}, \omega/t^{\phi_w}\right)$$

Unstable fixed point:
Ising of cubic for
$n < n_c$

[5](a) D. Mukamel, Phys. Rev. Lett. 34, 481 (1975); (b) D. Mukamel and S. Krinsky, J. Phys. C 8, L496 (1975); (c) P. Bak, S. Krinsky, and D. Mukamel, Phys. Rev. Lett. 36, 52 (1976); (d) D. Mukamel and S. Krinsky, Phys. Rev. B 13, 5065, 5078 (1976); (e) P. Bak and D. Mukamel, Phys. Rev. B 13, 5086 (1976).

Weakly First-Order Phase Transitions: The ϵ Expansion vs Numerical Simulations in the Cubic Anisotropy Model

Peter Arnold, Stephen R. Sharpe, Laurence G. Yaffe, and Yan Zhang
Department of Physics, University of Washington, Seattle, Washington 98195–1560

In recent years, renewed interest in quantitative predictions for weakly first-order transitions has arisen in cosmology, specifically the genesis of matter. Current

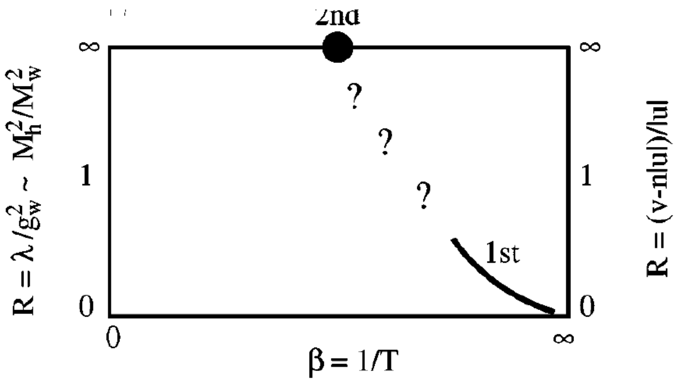


FIG. 1. Phase diagram of electroweak theory, or the cubic anisotropy model (for $u \leq 0, v \geq 0$), based purely on analytic arguments. The left and right axes show the relevant ratio R of couplings for electroweak theory, or for the cubic anisotropy model, respectively.

$$S = \int d^d x \left(\frac{1}{2} |\partial \vec{\phi}|^2 + \frac{t}{2} |\vec{\phi}|^2 + \frac{u}{4!} |\vec{\phi}|^4 + \frac{v}{4!} \sum_{i=1}^{n} \phi_i^4 \right)$$

We shall focus on the simplest case, $n = 2$. This model is analogous to electroweak theory when $u \leq 0$ and $v \geq 0$.

PHYSICAL REVIEW B VOLUME 25, NUMBER 11 1 JUNE 1982

Fluctuation-induced first-order transitions and symmetry-breaking fields: The $n=3$-component cubic model

Daniel Blankschtein
Department of Physics and Astronomy, Tel-Aviv University, Ramat Aviv, Israel

David Mukamel*

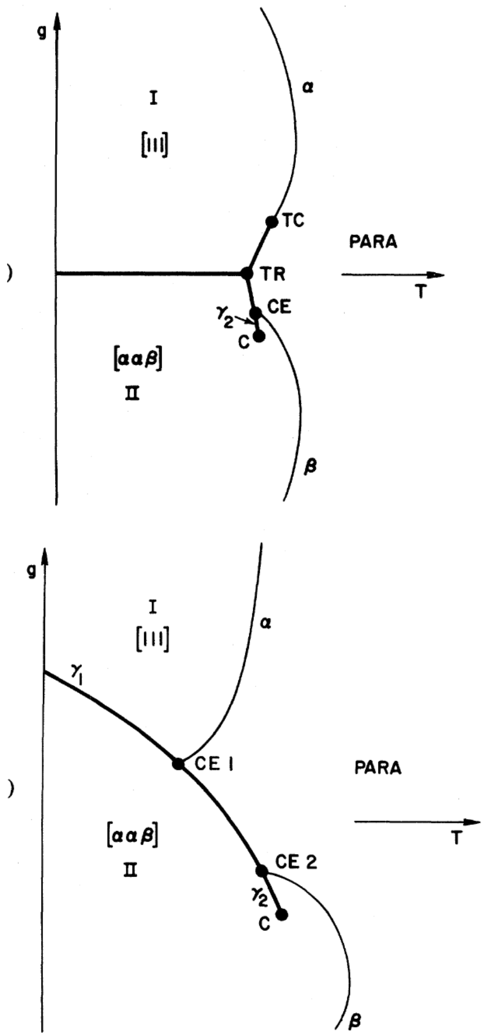


FIG. 3. (a) Schematic (g,T) phase diagram associated with the $n=3$ cubic model with easy axes along the cube diagonals $(v-2u<0)$ when the stable isotropic fixed point is not accessible $(-u<v<0)$ [region (b) of Fig. 1]. Thin lines represent continuous transitions and thick lines represent first-order transitions. The point TC is tricritical, C is a critical point and CE a critical end point and TR is a triple point. (b) Schematic (g,T) phase diagram associated with the $n=3$ cubic model with easy axes along the cube edges $(v-2u>0)$ when the stable isotropic fixed point is not accessible $(0<u<v/3)$ [region (a) of Fig. 1]. The points CE1 and CE2 are critical end points, C is a critical point.

PHYSICAL REVIEW B VOLUME 12, NUMBER 3 1 AUGUST 1975

Critical behavior of amorphous magnets*

Amnon Aharony†

$$\bar{\mathcal{H}}_{\text{eff}} = -\int d^dR \left(\tfrac{1}{2}\left[r|\vec{\sigma}|^2 + |\vec{\nabla}\vec{\sigma}|^2 \right] + u|\vec{\sigma}|^4 + v\sum_{\alpha=1}^{n} |\vec{S}_\alpha|^4 + w\sum_{i=1}^{m}\sum_{\alpha,\beta=1}^{n} S_{\alpha i}^2 S_{\beta i}^2 + y\sum_{i=1}^{m}\sum_{\alpha=1}^{n} S_{\alpha i}^4 + \cdots \right).$$

Random cubic case, n=0, m=d=3

TABLE II. Fixed points and exponents (to order ϵ) for random cubic case, $n=0$.

	Fixed point	$4K_4u^*$	$4K_4v^*$	$4K_4w^*$	$4K_4y^*$	Exponents
I.	Gaussian	0	0	0	0	$\lambda_u=\lambda_v=\lambda_w=\lambda_y=\epsilon$
II.	Decoupled m-component	0	$\frac{\epsilon}{m+8}$	0	0	$\lambda_v=-\epsilon,\ \lambda_u=\frac{4-m}{m+8}\epsilon,\ \lambda_w=\frac{m+4}{m+8}\epsilon,\ \lambda_y=\frac{m-4}{m+8}\epsilon$
III.	Isotropic $n=0$	$\epsilon/8$	0	0	0	$\lambda_u=-\epsilon,\ \lambda_v=\lambda_w=\lambda_y=-\epsilon/2$
IV.	Decoupled $n=0$	0	0	$\epsilon/8$	0	$\lambda_u=\lambda_v=\epsilon/2,\ \lambda_w=-\epsilon,\ \lambda_y=-\epsilon/2$
V.	Decoupled Ising	0	0	0	$\epsilon/9$	$\lambda_u=\lambda_v=\lambda_w=\epsilon/3,\ \lambda_y=-\epsilon$
VI.	Mixed $(0,m)$	$\frac{(m-4)\epsilon}{16(m-1)}$	$\frac{\epsilon}{4(m-1)}$	0	0	$\lambda_1=-\epsilon,\ \lambda_2=\lambda_y=\frac{m-4}{4(m-1)}\epsilon,\ \lambda_w=\frac{m+4}{4(m-1)}\epsilon$
VII.	Mixed $(m,0)$	$\epsilon/4$	0	$-\epsilon/4$	0	$\lambda_1=\lambda_v=-\epsilon,\ \lambda_3=\lambda_y=\epsilon$
VIII.	Decoupled m-component cubic	0	$\frac{\epsilon}{3m}$	0	$\frac{m-4}{9m}\epsilon$	$\lambda_u=\lambda_2=\frac{4-m}{3m}\epsilon,\ \lambda_w=\frac{m+4}{3m}\epsilon,\ \lambda_4=-\epsilon$
IX.		$\frac{m-4}{24(m-2)}\epsilon$	$\frac{\epsilon}{6(m-2)}$	0	$\frac{m-4}{18(m-2)}\epsilon$	$\lambda_1=-\epsilon,\ \lambda_2=\frac{m-4}{6(m-2)}\epsilon,\ \lambda_w=\frac{m+4}{6(m-2)}\epsilon,\ \lambda_4=\frac{4-m}{6(m-2)}\epsilon$
X.		$\epsilon/12$	0	$-\epsilon/12$	$\epsilon/9$	$\lambda_1=-\epsilon,\ \lambda_v=\lambda_3=-\epsilon/3,\ \lambda_4=\epsilon/3$
XI. XII. VIII. XIV.	(α_+,β_+) (α_+,β_-) (α_-,β_+) (α_-,β_-)	$\frac{\alpha_\pm\epsilon}{2A_{\pm\pm}}$ [a]	$\frac{\epsilon}{2A_{\pm\pm}}$	$\frac{m+4}{8A_{\pm\pm}}\epsilon$	$\frac{\beta_\pm\epsilon}{2A_{\pm\pm}}$	$\lambda_1=-\epsilon$; for other exponents see text [Eq. (58)]

[a] $\alpha_\pm=[m-4\pm(m^2+48)^{1/2}]/8,\ \beta_\pm=-[m+12\pm(m^2+48)^{1/2}]/6,\ A_{\pm\pm}=6\alpha_\pm+3\beta_\pm+m+6.$

Cubic symmetry in the quadratic terms

$$\mathscr{H}_0^f = \frac{f_0}{2}\sum_{\alpha}\int_{\mathbf{k}}^{\Lambda} k_\alpha^2 |Q_\alpha(\mathbf{k})|^2$$

$$R_{ij}(\mathbf{k}) = [a(T-T_c) + \lambda(k^2 + f k_i^2)]\delta_{ij} + \lambda h k_i k_j (1-\delta_{ij}) \qquad \text{(I.4.40)}$$

and the coefficients λ, f and h have been measured for $SrTiO_3$ by Stirling (1972) from the anisotropy in the phonon dispersion relations about the R point, with the results

$$\lambda = 216 \pm 20 (\mathrm{THz\,Å})^2, \quad f = -0{\cdot}97 \pm 0{\cdot}01, \quad h = 0{\cdot}19 \pm 0{\cdot}04$$

showing a very large degree of anisotropy.

Negative *f* reflects the small fluctuations along the axis of rotation

PHYSICAL REVIEW B VOLUME 8, NUMBER 7 1 OCTOBER 1973

Critical Behavior of Magnets with Dipolar Interactions. III. Antiferromagnets

Amnon Aharony

$$n = d = 4 - \epsilon$$

$$\mathcal{H}_0 = -\frac{1}{2}\sum_{\alpha\beta}\int_{\vec{q}} U_2^{0,\alpha\beta}(\vec{q})\sigma_{\vec{q}}^{\alpha}\sigma_{-\vec{q}}^{\beta} - \sum_{\alpha\beta}(u_0 + \delta_{\alpha\beta}v_0) \times \int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}_1}^{\alpha}\sigma_{\vec{q}_2}^{\beta}\sigma_{-\vec{q}-\vec{q}_1-\vec{q}_2}^{\beta},$$

$$U_2^{0,\alpha\beta}(\vec{q}) = [r_0 + q^2 - f_0(q^\alpha)^2]\delta_{\alpha\beta} + h_0 q^\alpha q^\beta,$$

$$r_0 = \tilde{r}kT/\tilde{J}\pi^2 = k(T - T_0)/\tilde{J}\pi^2, \qquad kT_0 = c|J|$$

$$u_{l+1} = b^{\epsilon}\{u_l - 4K_4 \ln b[(12 + 6h_l)u_l^2 + (6 + 3h_l)u_l v_l] + 4K_4 \ln b \times f_l[\tfrac{57}{10}u_l^2 + 3u_l v_l] + O(u_l^3, u_l^2 h_l^2, u_l^2 f_l^2, \ldots)\}$$

$$v_{l+1} = b^{\epsilon}\{v_l - 4K_4 \ln b[(12 + 5h_l)u_l v_l + (9 + \tfrac{9}{2}h_l)v_l^2] + 4K_4 \ln b \times f_l[6u_l v_l + \tfrac{9}{2}v_l^2] + O(v_l^3, v_l^2 f_l^2, \ldots)\}$$

$$f_{l+1} = b^{-\eta_l}\{[1 + K_4^2(\tfrac{80}{3}u_l^2 - \tfrac{256}{9}u_l v_l + 24v_l^2)\ln b]f_l + K_4^2(\tfrac{128}{3}u_l v_l - 24v_l^2)\ln b\, h_l\}$$
$$= b^{-\eta_l^f}f_l + b^{-\eta_l}K_4^2(\tfrac{128}{3}u_l v_l - 24v_l^2)\ln b\, h_l,$$

$$\eta_l^f = K_4^2(\tfrac{64}{3}u_l^2 + \tfrac{688}{9}u_l v_l)$$

Isotropic: $\eta^f = \frac{1}{108}\epsilon^2 + O(\epsilon^3)$

$f \Leftrightarrow v$

f is irrelevant near zero, highly relevant near 1

J. Phys. C: Solid State Phys., Vol. 9, 1976. Printed in Great Britain. © 1976

On the decay of anisotropy approaching the critical point

T Nattermann
Sektion Physik, Karl-Marx-Universität, Karl-Marx-Platz, 701 Leipzig, DDR

$$\frac{H}{k_B T} = \frac{1}{2}\sum_{\alpha}\int_q (r_{0\alpha} + q^2 - f q_\alpha^2)\, Q_q^\alpha Q_{-q}^\alpha + \sum_{\alpha,\beta}\int_{q_1}\int_{q_2}\int_{q_3} (u_0 + v_0\delta_{\alpha\beta})\, Q_{q_1}^\alpha Q_{q_2}^\alpha Q_{q_3}^\beta Q_{-q_1-q_2-q_3}^\beta$$

f pushes *v* to more negative values before it decays → 1st order transition

Should not expand in *f* if close to *1* (*f=1* means 2D behavior?)

Tricritical Behavior in Uniaxially Stressed $RbCaF_3$

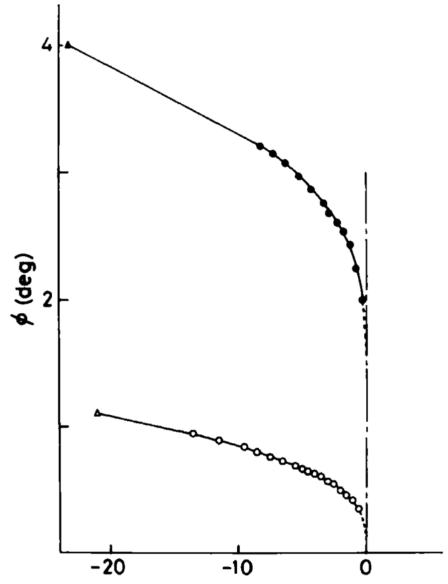
J. Y. Buzaré
IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland, and Laboratoire de Spectroscopie du Solide, 72017 Le Mans Cedex, France

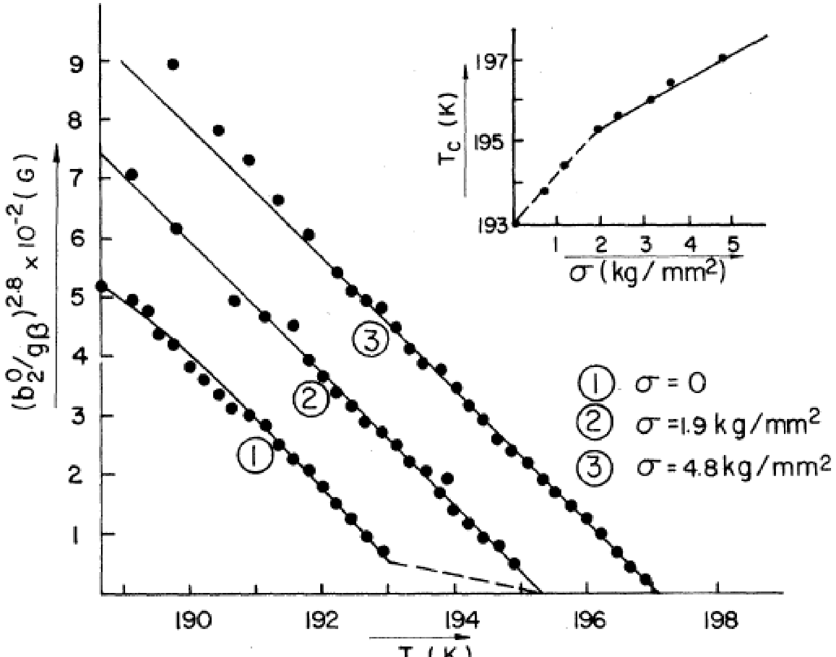
and

J. C. Fayet
Laboratoire de Spectroscopie du Solide, 72017 Le Mans Cedex, France

and

W. Berlinger and K. A. Müller
IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland
(Received 27 November 1978)





$$\beta_c \approx 0.32 \quad (Ising?)$$

$$\beta_t \approx 0.18 \quad (Lifshitz\ Ising\ TCP?$$

$$Normal\ Ising\ TCP?)$$

VOLUME 42, NUMBER 7 PHYSICAL REVIEW LETTERS 12 FEBRUARY 1979

Lifshitz-Point Critical and Tricritical Behavior in Anisotropically Stressed Perovskites

Amnon Aharony
Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel

and

Alastair D. Bruce[a]
IBM Zurich Research Laboratory, 8803 Rüschlikon, Switzerland
(Received 27 November 1978)

Back o back with experimental paper

$$\mathcal{H}=\frac{1}{2}\sum_{\alpha=1}^{n}\int_{\vec{q}}U_{2,\alpha}(\vec{q})Q_{\alpha}(\vec{q})Q_{\alpha}(-\vec{q})+\sum_{\alpha,\beta=1}^{n}(u+v\delta_{\alpha\beta})\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3}Q_{\alpha}(\vec{q}_1)Q_{\alpha}(\vec{q}_2)Q_{\beta}(\vec{q}_3)Q_{\beta}(-\vec{q}_1-\vec{q}_2-\vec{q}_3)$$

$$\mathcal{H} = \frac{1}{2} \sum_{\alpha=1}^{n} \int_{\vec{q}} U_{2,\alpha}(\vec{q}) Q_\alpha(\vec{q}) Q_\alpha(-\vec{q}) + \sum_{\alpha,\beta=1}^{n} (u + v\delta_{\alpha\beta}) \int_{\vec{q}_1} \int_{\vec{q}_2} \int_{\vec{q}_3} Q_\alpha(\vec{q}_1) Q_\alpha(\vec{q}_2) Q_\beta(\vec{q}_3) Q_\beta(-\vec{q}_1 - \vec{q}_2 - \vec{q}_3)$$

Cubic: $U_{2,\alpha}(\vec{q}) = r_\alpha + q_{\perp,\alpha}{}^2 + \underline{a q_\alpha{}^{2L}}$ $\quad r_\alpha = A(T - T_0)$ $\quad q_{\perp,\alpha}{}^2 \equiv q^2 - q_\alpha{}^2$

a close to 0.01 for both $KMnF_3$ and $RbCaF_3$ for $L = 1$

If *L* is infinite – two dimensional fluctuations only. Otherwise –**Lifshitz** behavior

critical Lifshitz point.

$$d_c = 5 - 1/L \qquad d = d_c - \epsilon_c$$

$$\beta_c = \tfrac{1}{2} - \tfrac{1}{6}\epsilon_c + O(\epsilon_c{}^2) = \frac{1}{2[1 + \frac{1}{3}\epsilon_c]} + O(\epsilon_c{}^2).$$

tricritical Lifshitz point.

$$d_t = 4 - 1/L \qquad \epsilon_t \equiv d_t - d$$

$$\beta_t = \tfrac{1}{4} - \tfrac{1}{4}\epsilon_t + O(\epsilon_t{}^2) = \frac{1}{4(1 + \epsilon_t)} + O(\epsilon_t{}^2)$$

More perovskites

- H. Rohrer, A. Aharony and S. Fishman
Critical and multicritical properties of random antiferromagnets $GdAlO_3 : La$
JMMM **15-18**, 396 (1980)

BCP under random fields

A. Aharony, R. J. Birgeneau, A. Coniglio, M. A. Kastner, and H. E. Stanley
Magnetic phases and magnetic pairing in doped $La_2 CuO_4$
Phys. Rev. Lett. **60**, 1330-1333 (1988)

High-T superconductors parent

- R. Sachidanandam, T. Yildirim, A. B. Harris, A. Aharony, and O. Entin-Wohlman
Single ion anisotropy, crystal field effects, spin reorientation transitions and spin waves in $R_2 CuO_4$ *(R=Nd, Pr, and Sm)*
Phys. Rev. **B56**, 260-286 (1997)

Same family

- Y. J. Kim, A. Aharony, R. J. Birgeneau, F. C. Chou, O. Entin-Wohlman, R. W. Erwin, M. Greven, A. B. Harris, M. A. Kastner, I. Ya. Korenblit, Y. S. Lee and G. Shirane
Ordering due to Quantum fluctuations in $Sr_2 Cu_3 O_4 Cl_2$
Phys. Rev. Lett. **83**, 852-855 (1999).

Same family

- R. Schmitz, O. Entin-Wohlman, A. Aharony, A. B. Harris, and E. Mueller-Hartmann
The magnetic structure of the Jahn-Teller system $LaTiO_3$
Phys. Rev. B **71**, 144412 (2005).

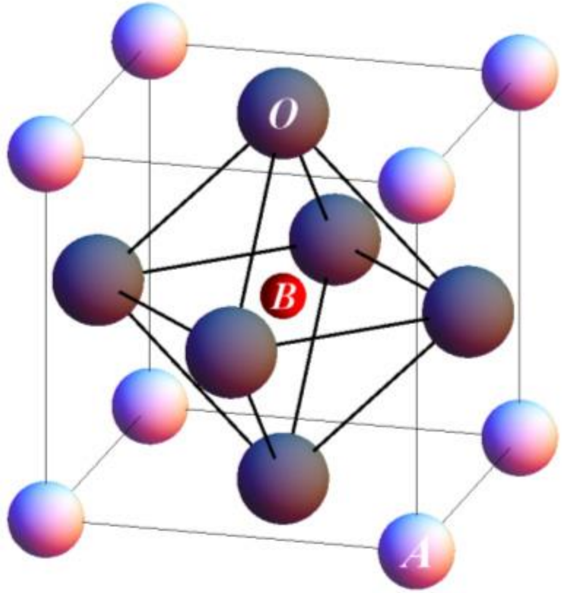
Orbital order

- S. Matityahu, O. Entin-Wohlman and A. Aharony
Landau theory for the phase diagram of the multiferroic $Mn_{1-x}(Fe,Zn,Mg)_x WO_4$
Phys. Rev. B **85**, 174408 (2012).

multiferroic

Remaining issues

- There is much more than exponents: scaling functions, amplitudes, multicritical points, crossover exponents, effective exponents – the importance of irrelevant operators
- What happens if $n_c = 3$? Log corrections?
- More work needed on Lifshitz critical and tricritical points
- The line n=d should be followed for many physical problems
- More work is needed on predicting and promoting calculations and experiments near the multicritical points of cubic systems
- Communications between CFT, conformal bootstrap and condensed matter physics are useful; let's continue!



Thank you

The end