From complementations on lattices to locality (Or from renormalisation to quantum logic)

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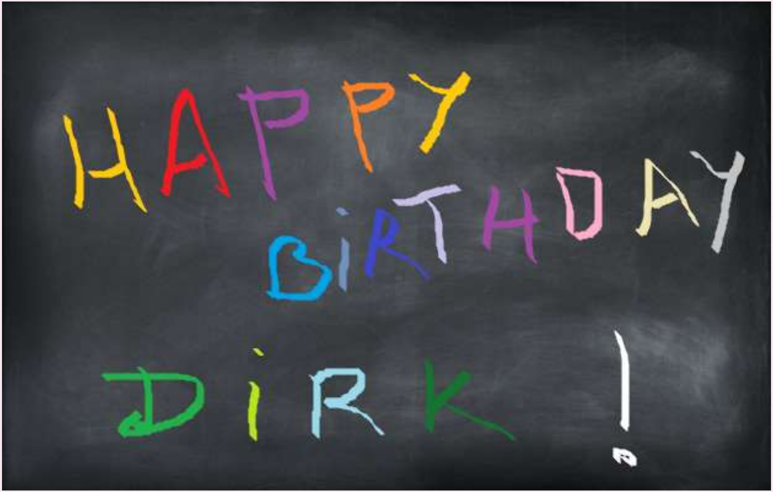
joint work with Pierre Clavier, Li Guo and Bin Zhang

Bures sur Yvette, November 17th 2020


HAPPY
BiRTHDAY
DiRK !

Motivations

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Our aim

We want to generalise Laurent expansions to meromorphic germs in **several variables**, so on $\mathcal{M}(\mathbb{C}^\infty)$, we need a **separating device** on the underlying spaces $V = \mathbb{C}^k$ to distinguish the **polar part** from the **holomorphic part**.

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- to prove Euler-Maclaurin formulae on convex polytopes [Garoufalidis, Pommersheim (2010)], [Berline, Vergne (2007)].

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to a 1-1 correspondence on a class of locality lattices $(L, \top)$

$$\top \longleftrightarrow \Psi^\top$$

with ”orthocomplementations” $\Psi^\top$.

We expect that:

$$U \top W \Leftrightarrow W \in \downarrow U^\top \quad \text{and} \quad \Psi^\top(U) = \max U^\top.$$

Orthogonality in Laurent expansions

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Meromorphic germs with linear poles

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- $(z_1 - z_2) \perp^Q (z_1 + z_2)$ with Q: canonical inner product on $\mathbb{R}^2$.

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(L. Guo, S.P., B. Zhang PJM 2020)

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Warning: The holomorphic germ h is unique yet the decomposition is not unique: $\frac{1}{L_1 L_2} = \frac{1}{L_1(L_1+L_2)} + \frac{1}{L_2(L_1+L_2)}$.

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- $(z_1 - z_2) \perp^Q (z_1 + z_2)$ with Q: canonical inner product on $\mathbb{R}^2$.

The lattice $G(V)$

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- Given a finite dimensional vector space V, $(G(V), \leq)$ is a non distributive lattice equipped with the sum $\vee = +$ and the intersection $\wedge = \cap$ as lattice operations. It is bounded by $0 = \{0\}$ and $1 = V$.
- In a lattice $(L, \leq)$, the set $\downarrow a := \{b \leq a,\ b \in L\}$ is a sub-lattice (even a lattice ideal) of L.

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- (i) ($a \mapsto a^\top$ is antitone) $a \leq b \Longrightarrow b^\top \subseteq a^\top$ (called a Galois connection on $P \times \mathcal{P}(P)$)
- (ii) (absorbing) if $a \leq b$ then $c \top b \Longrightarrow c \top a \quad \forall c \in P$ (i.e. $c^\top$ is a poset ideal),
- $\downarrow a \subset (a^\top)^\top$.

We call $(P, \leq, \top)$ a (or weak degenerate orthogonal) **locality poset**.

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The power set $(\mathcal{P}(X), \subseteq)$ equipped with $A \top B \Longleftrightarrow A \cup B = X$ is not a locality poset. Indeed, let $X := \{1, 2, 3\}$, $A = \{2\}$, $B = \{2, 3\}$ and $C = \{1\}$. Then $A \subseteq B$ and $C \top B$, yet C is not independent of A.

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Example

Given a Hilbert (finite or infinite dimensional) vector space (V, Q), the locality relation $U \perp^{Q} W$ defines a lattice locality relation.

$\perp^Q$ is a separating locality relation on $G(V)$

A locality relation $\top$ on a lattice $(L, \leq, 0)$ with a bottom element 0, is called **separating** if for any $a \in L$ we have

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3. (completeness) the set $a^\top$ admits a maximal element $\max(a^\top)$ for any $a \in L$.

In this case, we say that $(L, \leq, 0, \top)$ is a **separated locality** (or complete orthogonality poset) lattice. Recall that $\downarrow a \subset (a^\top)^\top$ since $\top$ is a locality relation on the poset $(L, \leq)$. If moreover,

- $\downarrow a = (a^\top)^\top$ or equivalently, if $\max\left((a^\top)^\top\right) = a$ for any $a \in L$,

we call the relation **strongly separating** and the lattice strongly separated.

Example

Given a Hilbert (finite or infinite dimensional) vector space $(V, \langle\cdot,\cdot\rangle)$, the poset $G(V)$ is a strongly separated locality lattice for $W_1 \top W_1 \Longleftrightarrow W_1 \perp^Q W_2$. For three subspaces W, U_1, U_2 in V we have $\left(\forall W \subseteq V,\ W \perp^Q U_1 \Rightarrow W \perp^Q U_2\right) \Longrightarrow U_2 \leq U_1$.

Locality versus complements

From locality to complements and back

Main result (P. Clavier, L.Guo, S.P., B. Zhang (2020), G. Cattaneo, A. Mania (74!))

Let L be a bounded lattice.There is a one-to-one correspondence

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Corollary

If the lattice is $\oplus$-modular, this yields a one-to-one correspondence

$$\text{orthomodular orthocomplementations} \quad \longleftrightarrow \quad \text{strongly separating locality relations.}$$

Example: $L = G(V)$

This generalises the correspondence orthogonality $\longleftrightarrow$ orthogonal complement on vector spaces.

Locality on lattices versus locality on vector spaces

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Corollary

A locality vector space $(V, \top)$ *is* **strongly non-degenerate** *if, and only if* $(G(V), \top)$ *is* **orthocomplemented**.

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Example

On a Hilbert space $(V, \langle \cdot, \cdot \rangle)$ this amounts to the correspondence we started from

$$\perp \quad \longleftrightarrow \quad (\Psi^\perp : U \mapsto U^\perp)\,.$$

Example beyond orthogonality

Take $V := \mathbb{R}^2$,

$$G(\mathbb{R}^2) = \{\{0\}, \mathbb{R}^2\} \cup \{U_\theta := \mathbb{R}\, e^{i\theta} \,|\, \theta \in [0, \pi)\}.$$

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$$\psi(\theta) = \pi - \theta, \quad \theta \in [0,\pi)$$

yields back $\Psi^\perp$ for the canonical inner product.

Openings

- generalise to orthocomplements beyond $\perp^Q$ the Laurent expansions for multi-variable meromorphic germs with linear poles built in [L. Guo, S.P., B. Zhang, to appear in PJM].

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- generalise to orthocomplements beyond $\perp^Q$ the Laurent expansions for multi-variable meromorphic germs with linear poles built in [L. Guo, S.P., B. Zhang, to appear in PJM].
- study the Galois group of transformations of multi-variable meromorphic germs with linear poles which stablise holomorphic germs at zero.

THANK YOU FOR YOUR ATTENTION!

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Modularity

The poset $(G(V), \leq)$ is a ($\oplus$-) modular lattice

Modular lattices (conditional distributivity)

A lattice $(L, \leq, \wedge, \vee)$ is **modular** if $a \geq c \Rightarrow (a \wedge b) \vee c = a \wedge (b \vee c)$, for any a, b, c in L

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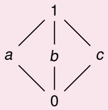
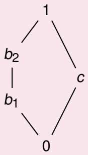
Remark: $\oplus$-modularity (resp. $\oplus$-cancellation) combined with sectional completeness implies modularity.

Special lattices

Distributivity and modularity are hereditary properties.

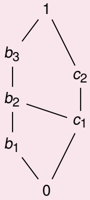


diamond lattice



pentagon lattice

The diamond lattice is modular and the pentagon lattice is not modular. They are both non distributive, non $\oplus$ distributive, non $\oplus$-modular and have no orthocomplementation.



extended pentagon lattice

The extended pentagon lattice $\oplus$-modular but not modular.