SEMINAR ON GEOMETRIC REPRESENTATION THEORY OF WEYL GROUPS

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1. Talks

1.1. **Semisimple Lie-Algebras and Flag Varieties.** (April 13, Jens Eberhardt) This talk should recall some of the prerequisites for this seminar.

- Basic structure theory of complex semisimple groups and Lie algebras.
  - Recall the definitions and important subgroups/algebras, the Weyl group.
  - Consider the example SL_n and maybe SO_n .
- Conjugacy classes of semisimple and nilpotent elements.
  - Define semisimple and nilpotent elements. Prove that regular semisimple elements form a Zariski-open subset.
  - Explain their conjugacy classes in the example of $\mathfrak{sl}_n(\mathbb{C})$.
- Recall the Jacobson–Morozov theorem.
- Flag variety, torus fixed points and Bruhat decomposition.
  - Define the flag variety, give the different definitions. Quickly explain why it is a variety.
  - Introduce the example $\mathbb{P}^1$.
  - Explain the torus fixed points and how, in general, torus actions define stratifications.
  - Discuss the Bruhat stratification. Do the example $\mathbb{P}^1$ and SL_3/B.

Please coordinate with the next speaker.

References: [CG10, Pages: 127-133]

1.2. **The Universal Resolution.** (April 20, Nico Wolf) This talk should provide a general approach to conjugacy classes of semisimple and nilpotent elements in semisimple Lie algebras.

- Explain how the coefficients of the characteristic polynomial and symmetric polynomials are related. Discuss how this yields a map $\mathfrak{sl}_n(\mathbb{C}) \to \mathbb{C}^{n-1}/S_n$.
- Explain how one order the eigenvalues by considering $\tilde{\mathfrak{sl}}_n \to \mathfrak{sl}_n$. Describe the fibers of this map and how much they vary between, for example, semisimple and nilpotent elements.
- Explain the diagram [CG10, (3.1.21)].
- Introduce the abstract Weyl group and root systems (very short).
- Define the univeral resolution for general semisimple Lie algebras. Discuss the main properties.
- Maybe try to discuss the example $\mathfrak{so}_n$.
- State and sketch a proof of the Chevalley restriction theorem. Explain how this leads to die diagram [CG10, (3.1.41)].

Please coordinate with the previous speaker.

References: [CG10, Pages: 133-144]

1.3. **Nilpotent Cone.** (April 27, Rein Janssen Groesbeek) The goal of this talk is to make us familiar with the nilpotent cone.

- Discuss the different definitions of nilpotent elements , either using ad or invariant polynomials.
- Provide different definitions of the nilpotent cone in $\mathfrak{g}$ and $\mathfrak{g}^*$.
- Define the Springer resolution. Explain that the cotangent bundle of the flag variety can be understood as balanced product $G \times_B (\mathfrak{g}/\mathfrak{b})^*$.
- Discuss diagram [CG10, (3.2.6)].
- Discuss basic properties of the resolution.
- Sketch the proof that there are finitely many conjugacy classes of nilpotent elements.
- Discuss regular nilpotent elements.
- Proof that the Springer resolution *is* a resolution of singularities.
- Try to give an extremely explicit description for SL_2 and SL_3 .

References: [CG10, Pages: 144-153]

1.4. **Steinberg Variety.** (May 4, Fernando Pena)

- Definition and explicit computation for SL_2
- Generalities on the relative cotangent space
- Description of the irreducible components
- Orbits and Langrangian slices

References: [CG10, Pages: 154-161]

1.5. **Borel–Moore homology.** (May 11, Junyan Liu) In this talk, we want to learn about Borel–Moore homology. Borel–Moore homology coincides with singular homology on compact spaces. On non-compact spaces it has several advantages for us. Most importantly, the existence of fundamental classes.

- Give different definition of of Borel–Moore homology
  1. Via relative singular homology of the one-point compactification.
  2. Via relative singular homology of any compactification.
  3. Via the complex of locally finite infinite chains.
- Do not prove that they are equivalent. Rather sketch how to use definition (1) and (3) to compute the Borel–Moore homology of the affine space $\mathbb{R}^1$ and state what the Borel–Moore homology of $\mathbb{R}^n$ is in general.
- Emphasize the difference between Borel–Moore and singular homology in these examples.
- Show that Borel–Moore homology coincides with singular homology on compact spaces.
- Introduce proper pushforward.
- Very quickly recall Poincaré duality between for compact manifolds. Show that Poincaré duality fails without the compactness condition on the example $\mathbb{R}^1$.
- Explain without proof that Poincaré duality for non-compact manifolds still holds when one considers Borel–Moore homology. Illustrate this on the example $\mathbb{R}^1$.

- Quickly state the long exact sequence in Borel–Moore homology from an open/closed subset decomposition of a space.
- Introduce fundamental classes. State without proof that any smooth manifold has a fundamental class. Show that also non-smooth irreducible complex varieties have fundamental classes. State Proposition 2.6.14.
- Define the intersection pairing for Borel–Moore homology using Poincaré duality. Explain very briefly in a sketch that the intersection pairing can be interpreted as an intersection, when considering nice enough representatives of the homology classes and counting multiplicities.
- Introduce restriction with supports. Give an example of $i : \mathbb{R}^1 \to \mathbb{R}^2$.
- Quickly state the Künneth formula.
- Explain smooth pullback. Consider the example of a trivial fibration. Illustrate this on the case $p : \mathbb{R}^2 \to \mathbb{R}^1$. State the projection formula.
- Collect all the functorialities: Proper pushforward, restriction with supports and smooth pullback.

References: [CG10, Pages: 93-110]

1.6. **Convolution.** (May 18, Tanja Helme)

- Convolution for functions
- Correspondences as generalization of functions
- Composition of correspondences
- Definitions and basic properties of convolution
- Example: Steinberg variety for SL_2

References: [CG10, Pages: 110-126]

1.7. **Lagrangian Construction of the Weyl group.** (June 1, David Cueto Noval)

- Correspondences associated to Weyl group elements
- Proof: Top Borel–Moore homology of the Steinberg variety is the group algebra

References: [CG10, Pages: 161-168]

1.8. **General Springer Theory.** (June 8, Aaron Wild)

- Generalisation of setup: Steinberg variety and Springer fibers
- Action on Borel–Moore homology of Springer fibers
- Component groups and Springer correspondence

References: [CG10, Pages: 168-183]

1.9. **Springer Fibers for SL_n, I.** (June 15, Xiaoxiang Zhou)

- Quick recap: Young tableaux and representations of the symmetric group S_n.
  - Introduce all the terminology on Young diagrams/tableaux from [Ful96] which are necessary for the next points.
  - Explain the results in [Ful96, Section 7.1, Section 7.2].
  - Provide examples: Reduced permutation representations. Maybe two-row case.
- Recall the definition of Springer fibers for SL_n .
- Discuss the example of the partition $(n-1 \geq 1)$, see [JN04, Section 10.2].

- Give Spaltensteins description of components of Springer fibers for SL_n , see [Spa76].
- Illustrate this in the example $(n-1 \geq 1)$ or the general two-row case.
- Show that Springer fibers for SL_n admit an affine paving, see [JN04, Section 11.2].
- Again, illustrate on the examples.
- As a bonus, fully compute the actions in the SL_2 example.

References: For a standard reference on Young tableaux, see [Ful96]. Spaltenstein describes the irreducible components of the Springer fiber for SL_n in [Spa76], see also [Spa82, Section 5]. Jantzen gives a description of an affine paving of Springer fibers of SL_n in [JN04, Section 11.2]. Also see [JN04, Section 10.2] for examples. For a reference on the two-row case, see [Kho02].

1.10. **Springer Fibers for SL_n, II.** (June 22, Daniel Bermudez)

- Explain the Robinson-Schensted correspondence combinatorially, see [Ful96, Section 4]. Do example calculations.
- Recall the Bruhat stratification and relative position of two flags for SL_n .
- Recall the parametrizations of irreducible components of the Steinberg variety from Talk 4 in the example of SL_n using the description of the components of the Springer fiber from Talk 9.
- Explain how this yields a geometric Robinson-Schensted correspondence.
- Show that this agrees with the combinatorial description.

References: See references of last talk. This paper [Ste88] by Steinberg proves that the combinatorial and geometric Robinson-Schensted correspondence coincide.

1.11. **Basic properties of (equivariant) K-theory.** (June 29, Jiaxi Mo) The goal of this talk is to introduce the basics of (equivariant) K-theory in analogy to Borel–Moore homology.

- Define the category of equivariant coherent sheaves. Explain the geometric intuition behind equivariant vector bundles.
- Define the equivariant algebraic K-theory. Focus on K_0 and just quickly mention that there are higher K-groups.
- Quickly sketch the relation to the representation ring of a group and do the examples $G = \mathbb{G}_m$ and $G = \mathbb{G}_a$.
- Sketch pullback, pushforward, tensor product, restriction with support, induction/restriction and reduction to the reductive part of the group. Explain, how non-exact functors can be handled. Explain to which constructions in Borel–Moore homology the functorialities correspond.
- Explain that this can be used to define convolution.
- State the projective bundle formula.
- Instead of proving the projective bundle formula, compute the (non-equivariant) K-theory of $\mathbb{P}_1$. Try to compute the $\mathbb{G}_m$-equivariant K-theory $\mathbb{P}_1$ directly. As as starting point, explain which $\mathbb{G}_m$ equivariant line bundles there are.
- Explain what the K-theory of $\mathbb{P}_n$ is.
- State the cellular fibration lemma.
- Introduce the Chern character map. Very briefly sketch its construction (only for smooth varieties) and five an overview over the most important properties.
- State that the Chern character map is an isomorphism for cellular varieties.

- Try, as briefly as possible, to motivate and explain formula [CG10, (5.11.9)].

References: The main reference is [CG10, Chapter 5].

1.12. **Equivariant K-theory of Flag varieties.** (July 6, Liao Wang)

- Computation of G- and T-equivariant K-theory of flag varieties
- Connection to Weyl's character formula

1.13. **Affine Weyl Groups and Equivariant K-theory.** (July 13, Noah Schaumburg)

- Recall the (co)-weight and (co)-root lattices and that they are being acted upon by the Weyl group. Give and example where weight and root lattice to not agree.
- Recall the generators and relations of Weyl groups and then give the definition of the Hecke algebra.
- Explain, without proof, the standard basis of the Hecke algebra.
- Quickly recall how the group of affine linear transformations can be understood as a semi-direct product of linear transformations and translations.
- Use this as a motivation to introduce the affine Weyl group as a semi-direct product. Explain that, if you use the root lattice, the resulting group is also a Coxeter group.
- Recall that the group algebra of the weight lattice is the representation ring of the torus. Hence, the representation ring of the torus sits inside the group algebra of the affine Weyl group.
- Illustrate everything above in the example of SL_2 and SL_3 .
- Introduce the affine Hecke algebra.
- To get familiar with calculations in the affine Hecke algebra, prove [CG10, Lemma 7.1.10].
- State, without proof, what the center of the affine Hecke algebra is [CG10, Lemma 7.1.14].
- Quickly recall the definition of the Springer resolution and Steinberg variety. Recall that the top Borel-Moore homology of the Steinberg variety, equipped with the convolution product, is the group algebra of the Weyl group.
- Explain that the same definition for the convolution product also work for equivariant K-theory.
- State [CG10, Theorem 7.2.2].
- Introduce the $\mathbb{C}^\times$ action and motivate [CG10, Theorem 7.2.5].
- Explain the the diagram in [CG10, Equation (7.2.12)].
- Your next goal is a sketch of the proof in [CG10, Section 7.3].
- Recall the deformation setup and the definition of the subsets Λ^h_w.
- Explain that the goal is to let $h \to 0$ in order to obtain elements in the K-theory of the Steinberg variety.
- Explain that we can also define equivariant line bundles L_λ on the Λ^h_w.
- Explain how this allows to state [CG10, Lemma 7.3.4]. Do not prove the Lemma.
- Now, we are ready for the specialisation argument. Recall, that we choose a line along which we want to take the limit $h \to 0$. Explain how this allows to consider [CG10, Equation (7.3.5)]. Explain the [CG10, Claim 7.3.6].
- State, without proof, [CG10, Lemma 7.3.9, 11 and 13].

- Explain how all this yields a proof of [CG10, Theorem 7.2.2].

References: The main reference is [CG10, Chapter 7].

1.14. **Representations of p-adic groups.** (July 20, Mingyu Ni)

- First, we start with the Hecke-Algebra for $\mathrm{GL}_n(\mathbb{F}_q)$.
- Show that for finite groups $H \subset G$ one can identify $\mathrm{End}_G(\mathbb{C}[G/H])$ with a convolution algebra on the double cosets $H(G,H) = \mathbb{C}[H\backslash G/H]$. This is called the *Hecke algebra* of G with respect to H.
- Explain that this induces an equivalence of categories between representations of G that are generated by H-invariant vectors and representations of the Hecke algebra.
- Give a set of generators and relations for $H(\mathrm{GL}_n(\mathbb{F}_q), B(\mathbb{F}_q))$ using the Bruhat decomposition. Do the example $n = 2$ in great detail.
- Explain how to define the generic Hecke alegebra $H_v(S_n)$ and how one can specialize it to $H(\mathrm{GL}_n(\mathbb{F}_q), B(\mathbb{F}_q))$.
- Our next goal is to see how the affine Hecke algebra arises when studying representations of p-adic groups.
- Quickly recall the setup $\mathbb{F}_p \leftarrow \mathbb{Z}_p \to \mathbb{Q}_p$.
- Explain the Iwahori subgroup $I \subset \mathrm{GL}_n(\mathbb{Z}_p)$ and the Iwahori decomposition of $\mathrm{GL}_n(\mathbb{Q}_p)$ and its relation to the affine Weyl group.
- Give generators and relations for the Hecke algebra $H(\mathrm{GL}_n(\mathbb{Q}_p), I)$ and explain the representation theoretic significance.
- Match this to the definition of the affine Hecke algebra from the previous talk.

References: For a nice quick reference for $H(\mathrm{GL}_n(\mathbb{F}_q), B(\mathbb{F}_q))$, see here and [Bum04, Section 48]. For the Iwahori-Matsumoto story see [IM65]. For the affine Hecke algebra, see [How02].

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