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Linear incongruences for generalized eta-quotients

Steffen Löbrich*

*Correspondence:
steffen.loebrich@uni-koeln.de
University of Cologne, Cologne, Germany

Abstract

For a given generalized eta-quotient, we show that linear progressions whose residues fulfill certain quadratic equations do not give rise to a linear congruence modulo any prime. This recovers known results for classical eta-quotients, especially the partition function, but also yields linear incongruences for more general weakly holomorphic modular forms like the Rogers-Ramanujan functions.

1 Introduction and statement of results

Ever since Ramanujan established his famous linear congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+6) &\equiv 0 \pmod{7}, \\ p(11n+7) &\equiv 0 \pmod{11} \end{aligned} \tag{1}$$

for the partition function $p(n)$, the challenge of proving and generalizing them triggered a vast amount of research. For instance, Ono [8] found analogues of (1) for every modulus coprime to 6. See also [3] and the sources contained therein for further results. However, recent work of Radu [9] proved that there are no linear congruences of $p(n)$ modulo 2 and 3, affirming a famous conjecture of Subbarao. The main ingredients of his proof are skillful computations and the *q-expansion principle* due to Deligne and Rapoport [4]. Adapting the methods of Radu's proof, Ahlgren and Kim [2] showed analogous results for the mock theta functions $f(q)$ and $\omega(q)$, as well as for certain classes of weakly holomorphic modular forms, including (classical) eta-quotients. In this paper, we extend their approach to generalized eta-quotients.

These functions are defined as follows: For $\delta \in \mathbb{Z}^+$ and a residue class $g \pmod{\delta}$, we set

$$\eta_{\delta,g}(z) := q^{\frac{\delta}{2}P_2\left(\frac{g}{\delta}\right)} \prod_{\substack{m>0 \\ m\equiv g \pmod{\delta}}} (1-q^m) \prod_{\substack{m>0 \\ m\equiv -g \pmod{\delta}}} (1-q^m),$$

where $z \in \mathbb{H}$ and $q := e^{2\pi i z}$ throughout. Here, for $x \in \mathbb{R}$ and $\{x\} := x - \lfloor x \rfloor$, we let

$$P_2(x) := \{x\}^2 - \{x\} + \frac{1}{6}$$

be the *second Bernoulli function*.

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Note that if

$$\eta(z) := q^{1/24} \prod_{m>0} (1-q^m)$$

denotes the usual Dedekind eta-function, then

$$\eta_{\delta,0}(z) = \eta(\delta z)^2 \qquad \text{and} \qquad \eta_{\delta,\frac{\delta}{2}}(z) = \frac{\eta\left(\frac{\delta}{2}z\right)^2}{\eta(\delta z)^2}.$$

Furthermore, for $g \notin \left\{0, \frac{\delta}{2}\right\}$ we have

$$\eta_{\delta,g}(z)^{-1} = q^{\frac{\delta}{2}P_2\left(\frac{g}{\delta}\right)} \sum_{n\geq 0} p_{\delta,g}(n) q^n,$$

where $p_{\delta,g}(n)$ denotes the number of partitions of n with all parts congruent to $\pm g$ $\pmod{\delta}$.

For $\delta = 5$, these functions occur in the well-known *Rogers-Ramanujan identities*, which state that

$$q^{\frac{1}{60}} \eta_{5,1}^{-1}(z) = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots$$

and

$$q^{-\frac{11}{60}} \eta_{5,2}^{-1}(z) = \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \cdots,$$

where $(q;q)_n := \prod_{j=1}^{n} (1-q^j)$.

For $N \in \mathbb{Z}^+$, a residue class $a \pmod{N}$, let $r := (r_{\delta,g})_{\delta|N, g \pmod{\delta}}$ be a tuple of half-integers, indexed by the divisors of N and their residue classes, with $r_{\delta,g} \in \mathbb{Z}$ unless $g = 0$ or $g = \frac{\delta}{2}$. In this paper, we study *generalized eta-quotients* of the form

$$H_r(z) := \prod_{\substack{\delta|N \\ g \pmod{\delta}}} \eta_{\delta,g}(z)^{r_{\delta,g}} =: q^{P(r)} \sum_{n\geq 0} c_r(n) q^n,$$

where

$$P(r) := \frac{1}{2} \sum_{\substack{\delta|N \\ g \pmod{\delta}}} \delta r_{\delta,g} P_2\left(\frac{g}{\delta}\right).$$

Note that the denominator of $P(r)$ divides $12N$.

For every modulus $m \in \mathbb{Z}^+$ and residue class $t \pmod{m}$, we give conditions on prime numbers p that guarantee that the linear progression $t \pmod{m}$ does not satisfy a linear congruence mod p for the generalized eta-quotient H_r. Here, for any residue class a $\pmod{N}$, we denote by r_a the tuple $(r_{\delta,ag})_{\delta|N, g \pmod{\delta}}$.

Theorem 1 *Let $m \in \mathbb{Z}^+$ and $t \in \{0, \ldots, m-1\}$. For $a, d \in \mathbb{Z}$ with $ad \equiv 1 \pmod{24Nm}$, let n be the smallest nonnegative integer for which*

$$d^2(n + P(r_a)) - P(r) \equiv t \pmod{m}.$$

Then for every prime p not dividing $c_{r_a}(n)$, we have

$$\sum_{n\geq 0} c_r(mn+t) q^n \not\equiv 0 \pmod{p}.$$

Remark 1 Since we always have $c_r(0) = 1$, the linear incongruence is satisfied for any prime p if

$$d^2 P(r_a) - P(r) \equiv t \pmod{m}.$$

Remark 2 By work of Ahlgren and Boylan [1], if the conditions of Theorem 1 are satisfied, we even have that

$$\#\left\{n \leq X : c_r(mn+t) \not\equiv 0 \pmod{p}\right\} \gg_{p,r,m,t,K} \frac{\sqrt{X}}{\log X} (\log\log X)^K$$

for every positive integer K.

Theorem 1 has several immediate applications.

Example 1 Let $N = a = 1$ and $r = -\frac{1}{2}$. Then

$$H_{-\frac{1}{2}}(z) = \eta^{-1}(z) = q^{-\frac{1}{24}} \sum_{n\geq 0} p(n) q^n.$$

Since $p(0) = p(1) = 1$, Theorem 1 then implies that

$$\sum_{n\geq 0} p(mn+t) q^n \not\equiv 0 \pmod{\ell}$$

for every prime ℓ if there is a d coprime to $6m$ with

$$t \equiv \frac{1-d^2}{24} \pmod{m} \quad \text{or} \quad t \equiv \frac{1+23d^2}{24} \pmod{m}.$$

Now assume that $\ell \geq 5$ is prime with $\left(\frac{-23}{\ell}\right) = -1$. Then the classes $d^2 \pmod{\ell}$ and $-23d^2 \pmod{\ell}$ together run over all residue classes except for 0 as d runs over residue classes coprime to ℓ. Since $(\ell, 24) = 1$, the classes $\frac{1-d^2}{24}$ and $\frac{1+23d^2}{24}$ cover every residue class modulo ℓ except for $\frac{1-\ell^2}{24}$. It follows that we can only have a linear congruence

$$\sum_{n\geq 0} p(\ell n+t) q^n \equiv 0 \pmod{\ell}$$

if $t \equiv \frac{1-\ell^2}{24} \pmod{\ell}$. This result was shown by Kiming and Olsson for every prime ℓ [7]. In particular, for $\ell \in \{5, 7, 11\}$, this implies that the residues in (1) are the only ones for which such a congruence can hold.

Example 2 More generally, Theorem 1 specializes to classical eta-quotients if $r_{\delta,g} = 0$ for $g \neq 0$. Then we have $P(r_a) = \frac{1}{12}\sum_{\delta|N} \delta r_\delta$ for all a. Since we always have $c_a(0) = 1$, we obtain that for every prime p, we have

$$\sum_{n\geq 0} c_r(mn+t) q^n \not\equiv 0 \pmod{p} \qquad \text{if} \qquad t \equiv \frac{d^2-1}{12} \sum_{\delta|N} \delta r_\delta \pmod{m}$$

for some d coprime to $6Nm$.

Example 3 Another interesting example are partitions occurring in *Schur's Theorem* [12]. These are given by

$$q^{\frac{1}{12}} \eta_{6,1}^{-1}(z) = \sum_{n\geq 0} p_{6,1}(n) q^n = 1 + q + q^2 + q^3 + q^4 + 2q^5 + 2q^6 + \cdots$$

with $N = 6$, $r_{6,1} = -1$ and $r_{\delta,g} = 0$ otherwise, and $P(r_a) = -\frac{1}{12}$ for every a coprime to 6. Thus Theorem 1 implies that

$$\sum_{n\geq 0} p_{6,1}(mn+t)q^n \not\equiv 0 \pmod{p}$$

for any prime p if there is a d coprime to $6m$ and $j \in \{-1, 11, 23, 35, 47\}$ with

$$t \equiv \frac{1+jd^2}{12} \pmod{m}.$$

As in Example 1, if $\ell \geq 5$ is a prime with at least one of $\left(\frac{-11}{\ell}\right)$, $\left(\frac{-23}{\ell}\right)$, $\left(\frac{-35}{\ell}\right)$, or $\left(\frac{-47}{\ell}\right)$ equal to -1, then there can only be a linear congruence if $t \equiv \frac{1-\ell^2}{12} \pmod{\ell}$.

Example 4 Now we take a closer look at the Rogers-Ramanujan functions $\eta_{5,1}^{-1}$ and $\eta_{5,2}^{-1}$. If $H_{r_1} = H_{r_4} = \eta_{5,1}^{-1}$, then we have $N = 5$, $r_{5,1} = -1$, $r_{5,2} = 0$, $H_{r_2} = H_{r_3} = \eta_{5,2}^{-1}$, and

$$P(r_a) = \begin{cases} -\frac{1}{60} & \text{if} \quad a \equiv 1 \text{ or } \quad 4 \pmod{5}, \\ \frac{11}{60} & \text{if} \quad a \equiv 2 \text{ or } \quad 3 \pmod{5}. \end{cases}$$

Hence Theorem 1 states that $\sum_{n\geq 0} p_{5,1}(mn+t)q^n \not\equiv 0 \pmod{p}$ for every prime p, if

$$t \equiv nd^2 + \frac{1-d^2}{60} \pmod{m} \text{ for } n \in \{0,1,2,3\} \text{ and}$$
$$d \equiv 1, 4 \pmod{5} \text{ coprime to } 6m$$

or

$$t \equiv nd^2 + \frac{11d^2+1}{60} \pmod{m} \text{ for } n \in \{0,2,3,4,5\} \text{ and}$$
$$d \equiv 2, 3 \pmod{5} \text{coprime to } 6m.$$

If we switch the roles of $\eta_{5,1}^{-1}$ and $\eta_{5,2}^{-1}$, we obtain that $\sum_{n\geq 0} p_{5,2}(mn+t)q^n \not\equiv 0 \pmod{p}$ for every prime p, if

$$t \equiv nd^2 - \frac{d^2+11}{60} \pmod{m} \text{ for } n \in \{0,1,2,3\} \text{ and}$$
$$d \equiv 2, 3 \pmod{5} \text{ coprime to } 6m$$

or

$$t \equiv nd^2 + \frac{11(d^2-1)}{60} \pmod{m} \text{ for } n \in \{0,2,3,4,5\} \text{ and}$$
$$d \equiv 1, 4 \pmod{5} \text{ coprime to } 6m.$$

In contrast, applying work of Gordon [5], Hirschhorn [6] found linear congruences (mod 2) for $p_{5,1}$ and $p_{5,2}$. For example, Theorem 3 of [6] states that

$$p_{5,1}(98n+t) \equiv 0 \pmod{2}$$

for $t \in \{23, 37, 51, 65, 79, 93\}$ and

$$p_{5,2}(98n+t) \equiv 0 \pmod{2}$$

for $t \in \{6, 20, 34, 62, 76, 90\}$. The above discussion precludes all the other residues (mod 98) except for $t \in \{9, 16, 58, 72, 86\}$ resp. $t \in \{13, 27, 48, 55, 97\}$ from satisfying these congruences.

The paper is organized as follows: In Sect. 2 we define generalized eta-quotients and study their transformation behavior under $\Gamma_0(12N)$, slightly adapting a result of Robins [10]. This will lead to modularity properties for the functions $H_{m,r,t}$ whose Fourier coefficients are given by those of H_r on the arithmetic progression $t \pmod m$. In Sect. 3 we prove Theorem 1 using the q-expansion principle.

2 Transformation properties of eta-quotients

We begin by studying modularity properites of $\eta_{\delta,g}$. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\delta)$ we define $\mu_{A,g,\delta}$ by

$$\eta_{\delta,g}(Az) = e\left(\mu_{A,g,\delta}\right) j(A,z)^{\delta_{g,0}} \eta_{\delta,ag}(z),$$

where $j(A,z) := cz + d$ and $e(w) := e^{2\pi i w}$ throughout. An analogue of the following proposition for the subgroup $\Gamma_1(\delta)$ was shown in Theorem 2 of [10].

Proposition 1 *For* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(12\delta)$ *we have*

$$\mu_{A,g,\delta} \equiv \frac{1}{2} db\delta P_2\left(\frac{ag}{\delta}\right) - \frac{a-1}{4} + \frac{1}{2}\left\lfloor \frac{ag}{\delta} \right\rfloor \pmod 1.$$

Proof An equation on p. 122 of [10] states that (note the different normalization of $\mu_{A,g,\delta}$)

$$\mu_{A,g,\delta} = \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a}\right)\right)\left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) + \frac{\delta b}{2a} P_2\left(\frac{ag}{\delta}\right) - \frac{c}{12\delta a}$$

with

$$((x)) := \begin{cases} \{x\} - \frac{1}{2} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

By Eqn. (34) of [11], Ch. VIII §4, the denominator of $\mu_{A,g,\delta}$ divides 12δ. This implies that for $A \in \Gamma_0(12\delta)$ we have, using that $ad \equiv 1 \pmod{12\delta}$,

$$\mu_{A,g,\delta} \equiv ad \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a}\right)\right)\left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) + \frac{\delta db}{2} P_2\left(\frac{ag}{\delta}\right) \pmod 1.$$

We compute

$$\begin{aligned} ad \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a}\right)\right)\left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) &= d \sum_{\mu=1}^{a-1} \left(\mu - \frac{a}{2}\right)\left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) \\ &= d \sum_{\mu=1}^{a-1} \mu\left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) - \frac{ad}{2} \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a} + \frac{g}{\delta}\right)\right). \end{aligned}$$

Now

$$d \sum_{\mu=1}^{a-1} \mu\left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) \equiv d \sum_{\mu=1}^{a-1} \mu\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta} - \frac{1}{2}\right) \equiv \frac{a-1}{2}\left(\frac{g}{\delta} - \frac{1}{2}\right) \pmod 1$$

and

$$\frac{ad}{2}\sum_{\mu=1}^{a-1}\left(\left(\frac{\mu}{a}+\frac{g}{\delta}\right)\right) \equiv \frac{1}{2}\sum_{\mu=1}^{a-1}\left(\frac{\mu}{a}+\frac{g}{\delta}-\left\lfloor\frac{\mu}{a}+\frac{g}{\delta}\right\rfloor-\frac{1}{2}\right)$$
$$\equiv \frac{a-1}{4}+\frac{a-1}{2}\left(\frac{g}{\delta}-\frac{1}{2}\right)-\frac{1}{2}\left\lfloor\frac{ag}{\delta}\right\rfloor \pmod 1,$$

using that $\sum_{\mu=0}^{a-1}\left\lfloor\frac{\mu}{a}+x\right\rfloor = \lfloor ax\rfloor$. □

Let

$$H_{r,m,t}(z) := \frac{1}{m}\sum_{\lambda \pmod m} e\left(-\frac{\lambda}{m}(t+P(r))\right) H_r\left(\begin{pmatrix}1 & \lambda\\ & m\end{pmatrix} z\right), \tag{2}$$

so that

$$H_{r,m,t}(z) = q^{\frac{t+P(r)}{m}}\sum_{n\geq 0} c_r(mn+t)q^n$$

and for every $\lambda \pmod m$ and $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(m)$ choose λ' with

$$a\lambda' \equiv b + d\lambda \pmod m.$$

Note that λ' runs over all residue classes modulo m with λ.

Moreover, let $k := \sum_{\delta|N} r_{\delta,0}$, so that k is the weight of H_r.

Proposition 2 *For $A \in \Gamma_0(24Nm)$, we have*

$$H_{r,m,t}(Az) = j(A,z)^k\frac{\zeta}{m}\sum_{\lambda \pmod m} e\left(\frac{\lambda}{m}\left(d^2P(r_a)-P(r)-t\right)-\frac{\lambda'}{m}P(r_a)\right)$$
$$\times H_{r_a}\left(\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix} z\right),$$

where ζ is a $24Nm$-th root of unity depending on r, m, and A. In particular, $H_{r,m,t}^{24Nm}$ is a weakly holomorphic modular form of weight $24Nmk$ for $\Gamma_1(24Nm)$, i.e. a meromorphic modular form whose poles are supported at the cusps.

Proof Let

$$A_\lambda := \begin{pmatrix} a+c\lambda & \frac{1}{m}\left(b+d\lambda-\lambda'(a+c\lambda)\right)\\ mc & d-c\lambda'\end{pmatrix},$$

so that

$$\begin{pmatrix}1 & \lambda\\ & m\end{pmatrix} A = A_\lambda\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}$$

Then for $A \in \Gamma_0(24Nm)$, we have by Proposition 1

$$\begin{aligned}\eta_{\delta,g}\left(\begin{pmatrix}1 & \lambda\\ & m\end{pmatrix}Az\right)^{r_{\delta,g}} &= \eta_{\delta,g}\left(A_\lambda\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right)^{r_{\delta,g}}\\ &= e\left(r_{\delta,g}\mu_{A_\lambda,g,\delta}\right) j\left(A_\lambda,\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right)^{\delta_{g,0}r_{\delta,g}}\eta_{\delta,ag}\left(\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right)^{r_{\delta,g}}\\ &= \zeta_0 e\left(\frac{r_{\delta,g}\delta}{2}P_2\left(\frac{ag}{\delta}\right)(d-c\lambda')\frac{1}{m}\left(b+d\lambda-\lambda'(a+c\lambda)\right)\right)\\ &\quad\times j(A,z)^{\delta_{g,0}r_{\delta,g}}\eta_{\delta,ag}\left(\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right)^{r_{\delta,g}}\\ &= \zeta_0 e\left(\frac{r_{\delta,g}\delta}{2m}P_2\left(\frac{ag}{\delta}\right)\left(db+d^2\lambda-\lambda'\right)\right)\\ &\quad\times j(A,z)^{\delta_{g,0}r_{\delta,g}}\eta_{\delta,ag}\left(\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right)^{r_{\delta,g}},\end{aligned}$$

where ζ_0 is a fourth root of unity depending on $r_{\delta,g}$ and A. Thus,

$$H_r\left(\begin{pmatrix}1 & \lambda\\ & m\end{pmatrix}Az\right) = \zeta j(A,z)^k e\left(\frac{P(r_a)}{m}\left(d^2\lambda-\lambda'\right)\right)H_{r_a}\left(\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right).$$

Together with (2), this yields the formula in the proposition.

Moreover, note that for $A \in \Gamma_1(24Nm)$, we have $\lambda' \equiv \lambda + b \pmod m$ and

$$\begin{aligned}&H_{r,m,t}(Az)\\ &= j(A,z)^k\frac{\zeta}{m}\sum_{\lambda' \pmod m} e\left(\frac{\lambda'-b}{m}\left((d^2-1)P(r)-t\right)-\frac{\lambda'}{m}P(r)\right)H_r\left(\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right)\\ &= j(A,z)^k\frac{\zeta_1}{m}\sum_{\lambda' \pmod m} e\left(-\frac{\lambda'}{m}(t+P(r))\right)H_r\left(\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right) = \zeta_1 j(A,z)^k H_{r,m,t}(z)\end{aligned}$$

with $\zeta_1 := e\left(\frac{bt}{m}\right)\zeta$. Since ζ_1 is a $24Nm$-th root of unity, we conclude that $H_{r,m,t}^{24Nm}$ is a weakly holomorphic modular form of weight $24Nmk$ for $\Gamma_1(24Nm)$. □

3 Proof of Theorem 1

Proof For j large enough, we have that $H_{r,m,t}^{24Nm}\Delta^j$ is a holomorphic modular form for $\Gamma_1(24Nm)$. Let $A = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(24Nm)$ and let $|_k$ denote the Petersson slash-operator, i.e. $(f|_kA)(z) := j(A,z)^{-k}f(Az)$. Then since

$$H_{r_a}\left(\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right) = e\left(\frac{\lambda'}{m}P(r_a)\right)q^{\frac{P(r_a)}{m}}\sum_{n\geq 0}c_{r_a}(n)e\left(\frac{\lambda'}{m}n\right)q^{\frac{n}{m}},$$

we obtain by Proposition 2

$$\begin{aligned}\left(H_{r,m,t}|_kA\right)(z) &= \frac{\zeta}{m}\sum_{\lambda \pmod m} e\left(\frac{\lambda}{m}\left(d^2P(r_a)-P(r)-t\right)-\frac{\lambda'}{m}P(r_a)\right)H_{r_a}\left(\begin{pmatrix}1 & \lambda'\\ & m\end{pmatrix}z\right)\\ &= \frac{\zeta}{m}\sum_{\lambda \pmod m} e\left(\frac{\lambda}{m}\left(d^2P(r_a)-P(r)-t\right)\right)q^{\frac{P(r_a)}{m}}\sum_{n\geq 0}c_{r_a}(n)e\left(\frac{\lambda'}{m}n\right)q^{\frac{n}{m}}\\ &= \frac{\zeta_2}{m}q^{\frac{P(r_a)}{m}}\sum_{n\geq 0}c_{r_a}(n)\sum_{\lambda \pmod m} e\left(\frac{\lambda}{m}\left(d^2(P(r_a)+n)-P(r)-t\right)\right)q^{\frac{n}{m}}\end{aligned}$$

with $\zeta_2 := e\left(\frac{db}{m}\right)\zeta$, since $\lambda' \equiv db + d^2\lambda \pmod m$.

Now assume that n is the smallest nonnegative integer with $d^2(P(r_a)+n)-P(r)\equiv t \pmod{m}$. Then we have

$$\left(H_{r,m,t}|_k A\right)(z)=\zeta_2 c_{r_a}(n) q^{\frac{P(r_a)+n}{m}}\left(1+O\left(q^{\frac{1}{m}}\right)\right)$$

Suppose that $H_{r,m,t}\equiv 0 \pmod{p}$. Then

$$(p^{-1}H_{r,m,t})^{24Nm}\Delta^j\in M_{24Nmk+12j}(\Gamma_1(24Nm))\cap\mathbb{Z}[\zeta_{24Nm}][q].$$

The q-expansion principle from Corollaire 3.12 of [4], Ch. VII states that if f is a modular form of weight κ for $\Gamma_1(N)$ whose Fourier coefficients at $i\infty$ lie in $\mathbb{Z}[\zeta_N]$, then for any $A\in\Gamma_0(N)$, also $f|_\kappa A$ has Fourier coefficients in $\mathbb{Z}[\zeta_N]$ (see also Corollary 5.3 of [9]). Thus it follows that

$$\left((p^{-1}H_{r,m,t})^{24Nm}\Delta^j\right)|_{24Nmk+12j}A\in\mathbb{Z}[\zeta_{24Nm}][q]$$

for every $A\in\Gamma_0(24Nm)$. By the above computation we have

$$\begin{aligned}&\left(\left((p^{-1}H_{r,m,t})^{24Nm}\Delta^j\right)|_{24Nmk+12j}A\right)\\&=p^{-24Nm}\left(\left(H_{r,m,t}|A\right)(z)\right)^{24Nm}\Delta(z)^j\\&=\left(\frac{c_{r_a}(n)}{p}\right)^{24Nm}q^{24N(P(r_a)+n)+j}\left(1+O(q)\right)\in\mathbb{Z}[\zeta_{24Nm}][q].\end{aligned}$$

This can only hold if p divides $c_{r_a}(n)$. □

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