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Applications of the generalised Dirichlet integral inequality to the Neumann problem with fast-growing continuous data

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Abstract

By using the generalised Dirichlet integral inequality with continuous functions on the boundary of the upper half-space, we prove new types of solutions for the Neumann problem with fast-growing continuous data on the boundary. Given any harmonic function with its negative part satisfying similarly fast-growing conditions, we obtain weaker boundary integral condition.

Keywords: Neumann problem; Neumann integral; upper half-space

1 Introduction

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space, where $n \geq 3$. We denote two points L and N in $\mathbf{R}^n$ by $L = (x', x_n)$ and $N = (y', y_n)$, respectively, where $x' = (x_1, x_2, \ldots, x_{n-1})$, $y' = (y_1, y_2, \ldots, y_{n-1})$, $x_n \in \mathbf{R}$ and $y_n \in \mathbf{R}$. The Euclidean distance of them is denoted by $|L - N|$. Let E be a subset of $\mathbf{R}^n$, we denote the boundary and closure of it by ∂E and $\overline{E}$, respectively.

The set

$$\{L = (x', x_n) \in \mathbf{R}^n; x_n > 0\},$$

is denoted by $\mathcal{T}_n$, which is called the upper half-space. Let F be a subset of $\mathbf{R}_+ \cup \{0\}$. Then two sets

$$\{L = (x', x_n) \in \mathcal{T}_n; |L| \in F\} \quad \text{and} \quad \{N = (y', 0) \in \partial\mathcal{T}_n; |N| \in F\}$$

are denoted by $\mathcal{T}_n E$ and $\partial\mathcal{T}_n E$, respectively.

Let $B_n(r)$ denote the open ball with center at the origin and radius r, where $r > 0$. By $S_n(r)$ we denote $\mathcal{T}_n \cap \partial B_n(r)$. When g is a function defined by $\sigma_n(r) = \mathcal{T}_n \cap B_n(r)$, the mean of g is defined by

$$\mathrm{M}(g)(r) = \frac{2s_n}{r^{n-1}} \int_{\sigma_n(r)} g(L)\, d\sigma_L,$$

where s_n is the surface area of $B_n(1)$ and $d\sigma_L$ is the surface element on $B_n(r)$ at $L \in \sigma_n(r)$.

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Let $h(L)$ be a function on $\mathcal{T}_n$. In this paper we denote $h^+ = \max\{h, 0\}$, $h^- = -\min\{h, 0\}$ and $[c]$ is the integer part of c, where $c \in \mathbf{R}$. Let $\partial/\partial n$ denote differentiation along the inward normal into $\mathcal{T}_n$. We use the Lebesgue measure $dL = dx'\,dx_n$, where $dx' = dx_1 \cdots dx_{n-1}$.

Let f be a continuous function on $\partial\mathcal{T}_n$. If h is a harmonic function on $\mathcal{T}_n$ and

$$\lim_{L\to N\in\partial\mathcal{T}_n, L\in\mathcal{T}_n(\Omega)} \frac{\partial h(L)}{\partial x_n} = f(N),$$

then we say that h is a solution of the Neumann problem on $\mathcal{T}_n$ with respect to f.

The uniqueness and the existence of solutions of the Neumann problem on $\mathcal{T}_n$ with a continuous function on $\partial\mathcal{T}_n$ were given by Su (see [1, 2]).

Theorem A (see [3], Theorem 1) *Let $f(N)$ ($N = (y', 0)$) be a function continuous on $\partial\mathcal{T}_n$ such that*

$$\int_{\partial\mathcal{T}_n} \left|f(y')\right|\left(\left|1 + |y'|\right|\right)^{2-n} dy' < +\infty. \tag{1.1}$$

Then the Neumann integral

$$\mathbb{H}_{0,n}[f](L) = -\rho_n \int_{\partial\mathcal{T}_n} f(N)|L - N|^{2-n}\,dN$$

is a solution of the Neumann problem on $\mathcal{T}_n$ with respect to f satisfying

$$\mathrm{M}\bigl(\mathbb{H}_{0,n}[f]\bigr)(r) = O(1)$$

as $r \to +\infty$, where $\rho_n = 2\{(n-2)s_n\}^{-1}$.

Theorem B (see [3], Theorem 3) *Let k be a positive integer, f be a continuous function on $\partial\mathcal{T}_n$ such that* (1.1) *holds and $h(L)$ be a solution of the Neumann problem on $\mathcal{T}_n$ with respect to f satisfying*

$$\mathrm{M}\bigl(h^+\bigr)(r) = o\bigl(r^k\bigr)$$

as $r \to +\infty$. Then

$$h(L) = \mathbb{H}_{0,n}(f)(L) + \begin{cases} d & \textit{when } k = 1, \\ \Pi(x') + \sum_{j=1}^{[\frac{k}{2}]} \frac{(-1)^j}{(2j)!} x_n^{2j} \Delta^j \Pi(x') & \textit{when } k \geq 2, \end{cases}$$

for any $L = (x', x_n)$, where d is a constant, $\Pi(x')$ is a polynomial of degree less than k on $\partial\mathcal{T}_n$ and

$$\Delta^j = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\right) \quad (j = 1, 2 \ldots).$$

Recently, Ren and Yang (see [4]) extended Theorems A and B by defining generalised Neumann integrals with continuous functions under less restricted conditions than (1.1).

Meanwhile, they also proved that for any continuous function f on $\partial\mathcal{T}_n$ there exists a solution of Neumann problem on $\mathcal{T}_n$. To state them, we need some preliminaries.

Let L and N be two points on $\mathcal{T}_n$ and $\partial\mathcal{T}_n$, respectively. By $\langle L, N\rangle$ we denote the usual inner product in $\mathbf{R}^n$. We denote

$$|L-N|^{2-n} = \sum_{k=0}^{\infty} d_{k,n}|N|^{-k-n+2}|L|^k G_{k,n}(t),$$

where $|N| > |L|$,

$$t = |L|^{-1}|N|^{-1}\langle L, N\rangle, \quad d_{k,n} = \binom{k+n-3}{k}$$

and $G_{k,n}$ is the n-dimensional Legendre polynomial of degree k.

As in [2], we shall use the following generalised Dirichlet kernel. For a non-negative integer l, two points $L \in \mathcal{T}_n$ and $N \in \partial\mathcal{T}_n$, we put

$$\mathbb{V}_{l,n}(L,N) = \begin{cases} -\rho_{n+1}\sum_{k=0}^{l-1} d_{k,n}|N|^{-n-k+2}|L|^k G_{k,n}(t) & \text{when } |N| \geq 1 \text{ and } l \geq 1, \\ 0 & \text{when } |N| < 1 \text{ and } l \geq 1, \\ 0 & \text{when } l = 0. \end{cases} \tag{1.2}$$

The generalised Neumann kernel $\mathbb{K}_{l,n}(L,N)$ on $\mathcal{T}_n$ is defined by (see [2])

$$\mathbb{K}_{l,n}(L,N) = \mathbb{K}_{0,n}(L,N) - \mathbb{V}_{l,n}(L,N),$$

where $L \in \mathcal{T}_n$, $N \in \partial\mathcal{T}_n$ and

$$\mathbb{K}_{0,n}(L,N) = -\alpha_n|L-N|^{2-n}.$$

As for similar generalised Dirichlet kernel in a half plane and smooth cone, we refer the reader to the papers by Yang and Ren (see [5]), Zhao and Yamada (see [6]) and Su (see [1]).

Let $f(N)$ be a continuous function on $\partial\mathcal{T}_n$. Then the generalised Neumann integral on $\mathcal{T}_n$ can be defined by

$$\mathbb{H}_{l,n}[f](L) = \int_{\partial\mathcal{T}_n} f(N)\mathbb{K}_{l,n}(L,N)\,dN.$$

Ren and Yang proved the following results.

Theorem C (see [4], Corollary 1) *Let* $1 < p < \infty$, $n + \beta - 2 > -(n-1)(p-1)$ *and*

$$1 - \frac{1-\beta}{p} < m < 2 - \frac{1-\beta}{p}.$$

Let $f(N)$ $(N = (y', 0))$ *be a continuous function on* $\partial\mathcal{T}_n$ *such that*

$$\int_{\partial H} \left|f(y')\right|^p \left(1 + |y'|\right)^{2-\beta-n} dy' < \infty. \tag{1.3}$$

Then the generalised Neumann integral $\mathbb{H}_{l,n}[f](L)$ *is a solution of the Neumann problem on* $\mathcal{T}_n$ *with respect to* f *satisfying*

$$\mathrm{M}\big(\big|\mathbb{H}_{l,n}[f]\big|\big)(r) = O\big(|x|^{1+\frac{\beta-1}{p}}\sec^{n-2}\theta\big)$$

as $r \to +\infty$.

Theorem D (see [4], Theorem 3) *Let* $1 \leq p < \infty$, $\beta > 1 - p$, l *be a positive integer and*

$$1 - \frac{1-\beta}{p} < m < 2 - \frac{1-\beta}{p} \quad \textit{when } p > 1,$$

$$\beta \leq m < \beta + 1 \quad \textit{when } p = 1.$$

Let $f(N)$ *be a continuous function on* $\partial\mathcal{T}_n$ *satisfying* (1.3). *If* $h(L)$ *is a solution of the Neumann problem on* $\mathcal{T}_n$ *with respect to* f *such that*

$$\lim_{r\to\infty, L=(r,\Theta)\in H} h^+(L) = o\big(r^{l+[1+\frac{\beta-1}{p}]}\big),$$

then

$$h(L) = N_m[f](L) + \Pi\big(x'\big) + \sum_{j=1}^{[\frac{l+[1+\frac{\beta-1}{p}]}{2}]} \frac{(-1)^j}{(2j)!} x_n^{2j} \Delta^j \Pi\big(x'\big)$$

for any $L = (x', x_n)$, *where* d *is a constant,* $\Pi(x')$ *is a polynomial of degree less than* $l + [1 + \frac{\beta-1}{p}]$ *on* $\partial\mathcal{T}_n$.

From Theorems A, B, C and D, it is easy to see that the continuous boundary function f grows slowly on $\partial\mathcal{T}_n$. It is natural to ask what will happen if f is replaced by a fast-growing continuous function on $\partial\mathcal{T}_n$. In this paper, we shall solve this problem and explicitly give a new solution of the Neumann problem on $\partial\mathcal{T}_n$.

Define

$$\varepsilon_0 = \limsup_{r\to\infty} \tau^{-1}(r) r \tau'(r) \log r < 1,$$

where $\tau(r)$ is a nondecreasing and continuously differentiable function satisfying $\tau(r) \geq 1$ for any $r \in \mathbf{R}^+ \cup \{0\}$.

From these we see that there is a sufficiently large positive number r such that for any $t > r$

$$\tau(e)(\ln t)^{\epsilon_0+\epsilon} > \tau(t), \tag{1.4}$$

where ϵ is a sufficiently small positive number satisfying $\epsilon_0 + \epsilon < 1$.

Let $\mathfrak{A}_\varpi$ be the set of continuous functions $f(N)$ ($N = (y', 0)$) on $\partial\mathcal{T}_n$ satisfying

$$\int_{\partial\mathcal{T}_n} \big|f\big(y'\big)\big|\big(1 + \big|y'\big|\big)^{3-n-\varpi-\tau(|y'|)} dy' < +\infty, \tag{1.5}$$

where ϖ is a real number such that $\varpi > 2$.

2 Results

Now we state our results.

Theorem 1 *If* $f \in \mathfrak{A}_{\varpi}$, *then generalised Neumann integral* $\mathbb{H}_{[\tau(|y'|)+\varpi],n}[f](L)$ *is a solution of the Neumann problem on* $\mathcal{T}_n$ *with respect to* f.

Then we shall prove that if the negative part of a harmonic function satisfies a fast-growing condition, then its positive part satisfies the similar condition. That is to say, the condition of Theorem 1 may be replaced by a weaker integral condition. To state this result, we also need some notations.

Let $\mathfrak{B}_{\varpi}$ be the set of continuous functions $f(N)$ $(N = (y', y_n))$ on $\mathcal{T}_n$ satisfying

$$\int_{\mathcal{T}_n} |f(N)| \big(1 + |N|\big)^{1-n-\varpi-\tau(|N|)} y_n \, dN < +\infty. \tag{2.1}$$

By $\mathfrak{C}_{\varpi}$ we denote the set of all continuous functions $h(N)$ on $\overline{\mathcal{T}_n}$, harmonic on $\mathcal{T}_n$ with $h^-(N) \in \mathfrak{B}_{\varpi}$ and $h^-(y') \in \mathfrak{A}_{\varpi}$.

Theorem 2 *The conclusion of Theorem* 1 *remains valid if its condition is replaced by* $h \in \mathfrak{C}_{\varpi}$.

Theorem 3 *If* $h \in \mathfrak{C}_{\varpi}$, *then there exists a harmonic function* $\Lambda(L)$ *with normal derivative vanishes on* $\partial\mathcal{T}_n$ *such that*

$$h(L) = \Lambda(L) + \mathbb{H}_{[\tau(|y'|)+\varpi],n}[h](L),$$

where $L \in \overline{\mathcal{T}}_n$.

3 Lemmas

Lemma 1 *Let* $L \in \mathcal{T}_n$ *and* $N \in \partial\mathcal{T}_n$ *such that* $|N| \geq \max\{1, 2|L|\}$. *Then* (*see* [7])

$$\big|\mathbb{K}_{l,n}(L,N)\big| \leq M|N|^{-l-n+2}|L|^l,$$

where M *is a positive constant.*

Lemma 2 *Let* $\mathbb{W}(L,N)$ $(N \in \partial\mathcal{T}_n)$ *be a locally integrable function for any fixed point* $L \in \mathcal{T}_n$, $g(N)$ *be a upper semicontinuous and locally integrable function on* $\partial\mathcal{T}_n$. *Set*

$$\mathbb{K}(L,N) = \mathbb{K}_{0,n}(L,N) - \mathbb{W}(L,N)$$

for any $N \in \partial\mathcal{T}_n$ *and* $L \in \mathcal{T}_n$.

Suppose that the following two conditions hold:

(I) *There are a positive number* R *and a neighborhood* $B(N^*)$ *of* N^* $(\in \partial\mathcal{T}_n)$ *satisfying*

$$\int_{\partial\mathcal{T}_n[R,+\infty)\cup\partial\mathcal{T}_n(-\infty,-R]} |g(N)| \left|\frac{\partial}{\partial x_n}\mathbb{K}(L,N)\right| dN < \epsilon,$$

where $\epsilon > 0$.

(II) *There exists a positive number R satisfying*

$$\limsup_{L\to N^*, L\in\mathcal{T}_n} \int_{\partial\mathcal{T}_n(-R,R)} \bigl|g(N)\bigr| \left|\frac{\partial}{\partial x_n}\mathbb{W}(L,N)\right| dN = 0$$

for any $N^* \in \partial\mathcal{T}_n$.

Then

$$\limsup_{L\to N^*\in\partial\mathcal{T}_n, L\in\mathcal{T}_n} \int_{\partial\mathcal{T}_n} g(N)\frac{\partial}{\partial x_n}\mathbb{W}(L,N)\,dN \le g\bigl(N^*\bigr). \tag{3.1}$$

Proof Let N^* be any point of $\partial\mathcal{T}_n$ and ϵ be any positive number. There exists a positive number R^* satisfying

$$\int_{\partial\mathcal{T}_n[R^*,+\infty)\cup\partial\mathcal{T}_n(-\infty,-R^*]} \bigl|g(N)\bigr| \left|\frac{\partial}{\partial x_n}\mathbb{K}(L,N)\right| dN \le \frac{\epsilon}{2} \tag{3.2}$$

for any $L = (x', x_n) \in \mathcal{T}_n \cap B(N^*)$ from (I).

Let ϕ be a continuous function on $\partial\mathcal{T}_n$ such that $0 \le \phi \le 1$ and

$$\phi = \begin{cases} 1 & \text{if } \partial\mathcal{T}_n[-R^*,R^*], \\ 0 & \text{if } \partial\mathcal{T}_n(-\infty,-2R^*) \cup \partial\mathcal{T}_n(2R^*,+\infty). \end{cases}$$

Let $\mathbb{K}^j_{0,n}(L,N)$ be the Neumann function of $\mathcal{T}_n(-j,j)$, where j is a positive integer. Since

$$\Gamma_j(L,N) = \mathbb{K}_{0,n}(L,N) - \mathbb{K}^j_{0,n}(L,N)$$

on $\mathcal{T}_n(-j,j)$ converges monotonically to 0 as $j \to \infty$, we can find an integer j^* satisfying $j^* > 2R^*$ such that

$$\int_{\partial\mathcal{T}_n(-2R^*,2R^*)} \bigl|\phi(N)g(N)\bigr| \left|\frac{\partial}{\partial x_n}\Gamma_{j^*}(L,N)\right| d\sigma < \frac{\epsilon}{4} \tag{3.3}$$

for any $L = (x', x_n) \in B(N^*) \cap \mathcal{T}_n$.

Then we have from (3.2) and (3.3) that

$$\begin{aligned}
\int_{\partial\mathcal{T}_n} g(N)\frac{\partial}{\partial x_n}\mathbb{K}(L,N)\,dN &\le \int_{\partial\mathcal{T}_n(-2R^*,2R^*)} g(N)\frac{\partial\mathbb{K}^{j^*}_{0,n}(L,N)}{\partial x_n}\phi(N)\,dN \\
&\quad + \int_{\partial\mathcal{T}_n(-2R^*,2R^*)} \bigl|g(N)\bigr| \left|\frac{\partial\Gamma_{j^*}(L,N)}{\partial x_n}\right| \bigl|\phi(N)\bigr|\,dN \\
&\quad + \int_{\partial\mathcal{T}_n(-2R^*,2R^*)} \bigl|g(N)\bigr| \left|\frac{\partial\mathbb{W}(L,N)}{\partial x_n}\right| dN \\
&\quad + 2\int_{\partial\mathcal{T}_n[R^*,+\infty)\cup\partial\mathcal{T}_n(-\infty,-R^*]} \bigl|g(N)\bigr| \left|\frac{\partial\mathbb{K}(L,N)}{\partial x_n}\right| dN \\
&\le \int_{S_n(\Gamma;(-2R^*,2R^*))} g(N)\frac{\partial\mathbb{K}^{j^*}_{0,n}(L,N)}{\partial x_n}\phi(N)\,dN \\
&\quad + \int_{\partial\mathcal{T}_n(-2R^*,2R^*)} \bigl|g(N)\bigr| \left|\frac{\partial\mathbb{W}(L,N)}{\partial x_n}\right| dN + \frac{5}{4}\epsilon
\end{aligned} \tag{3.4}$$

for any $L = (x', x_n) \in \mathcal{T}_n \cap B(N^*)$.

Consider an upper semicontinuous function

$$\psi(N)=\begin{cases}\phi(N)g(N) & \text{if } \partial\mathcal{T}_n[-2R^*,2R^*],\\ 0 & \text{if } \partial\mathcal{T}_n[-j^*,j^*]-\partial\mathcal{T}_n[-2R^*,2R^*]\end{cases}$$

on $\partial\mathcal{T}_n(-j^*,j^*)$ and denote the Perron-Wiener-Brelot solution of the Neumann problem on $\mathcal{T}_n(-j^*,j^*)$ by $\mathbb{H}_\psi(L;\mathcal{T}_n(-j^*,j^*))$. We know that

$$\int_{\partial\mathcal{T}_n(-2R^*,2R^*)} g(N)\frac{\partial\mathbb{K}_{0,n}^{j^*}(L,N)}{\partial x_n}\phi(N)\,dN=\mathbb{H}_\psi\bigl(L;\mathcal{T}_n\bigl(-j^*,j^*\bigr)\bigr).$$

We also have

$$\limsup_{L\to N^*,L\in\mathcal{T}_n}\mathbb{H}_\psi\bigl(L;\mathcal{T}_n\bigl(-j^*,j^*\bigr)\bigr)\le\limsup_{N\in\partial\mathcal{T}_n,N\to N^*}\psi(N)=g\bigl(N^*\bigr).$$

Hence we obtain

$$\limsup_{L\to N^*,L\in\mathcal{T}_n}\int_{\partial\mathcal{T}_n(-2R^*,2R^*)} g(N)\frac{\partial\mathbb{K}_{0,n}^{j^*}(L,N)}{\partial x_n}\phi(N)\,dN\le g\bigl(N^*\bigr),$$

which together with (II) and (3.4) gives (3.1). □

Lemma 3 *Let $r>1$ and $h(N)$ $(N=(y',y_n))$ be a function harmonic on $\mathcal{T}_n$. Then*

$$\int_{S_n(r)} r^{-1-n}h(N)ny_n\,dN+\int_{\partial\mathcal{T}_n(1,r)} h\bigl(y'\bigr)\bigl(\bigl|y'\bigr|^{-n}-r^{-n}\bigr)\,dy'=d_1+d_2r^{-n},$$

where

$$d_1=\int_{S_n(1)} y_n\left((n-1)h(N)+\frac{\partial h(N)}{\partial n}\right)dN$$

and

$$d_2=\int_{S_n(1)} y_n\left(h(N)-\frac{\partial h(N)}{\partial n}\right)dN.$$

4 Proof of Theorem 1

We have from (1.4)

$$M_1(r)\ge(2r)^{\tau(k+1)+\varpi+1}k^{\frac{2-\varpi}{2}} \tag{4.1}$$

for any $k>k_r=[2r]+1$, where $M_1(r)$ is a positive constant dependent only on r.

We have for any $L\in\mathcal{T}_n$ and $|L|\le R$

$$\begin{aligned}&\sum_{k=k_r}^{\infty}\int_{\partial\mathcal{T}_n[k,k+1)}\bigl|f\bigl(y'\bigr)\bigr|\bigl(2|L|\bigr)^{[\tau(|y'|)+\varpi]}\bigl|y'\bigr|^{2-n-[\tau(|y'|)+\varpi]}\,dy'\\&\quad\le\sum_{k=k_r}^{\infty}k^{\frac{2-\varpi}{2}}(2r)^{1+\varpi+\tau(k+1)}\int_{\partial\mathcal{T}_n[k,k+1)}2\bigl|f\bigl(y'\bigr)\bigr|\bigl(1+\bigl|y'\bigr|\bigr)^{1-n-\frac{\varpi-2}{2}-\tau(|y'|)}\,dy'\end{aligned}$$

$$\leq 2M_1(r)\int_{\partial\mathcal{T}_n[k_r,+\infty)}\left|f(y')\right|\left(1+|y'|\right)^{1-n-\frac{\varpi-2}{2}-\tau(|y'|)}\,dy'$$
$$< +\infty \tag{4.2}$$

from Lemma 1 and (1.5). So $\mathbb{H}_{[\tau(|y'|)+\varpi],n}(L)$ is absolutely convergent.

Next we shall prove that

$$\lim_{L\to N',L=(x',x_n)\in\mathcal{T}_n}\frac{\partial\mathbb{H}_{[\tau(|y'|)+\varpi],n}(L)}{\partial x_n}=h(N')$$

for any $N'=(y',0)\in\partial\mathcal{T}_n$. By applying Lemma 2 to $-g(y')$ and $g(y')$ by setting

$$\mathbb{W}(L,N)=\mathbb{V}_{[\tau(|y'|)+\varpi],n}(L,N),$$

then we shall see that (I) and (II) hold. Take any $N'=(y',0)\in\partial\mathcal{T}_n$ and any $\epsilon>0$. There exists a number R ($>\max\{2(\delta+y'),1\}$) satisfying

$$\int_{\partial\mathcal{T}_n[R,+\infty)\cup\partial\mathcal{T}_n(-\infty,-R]}\left|f(N)\right|\left|\frac{\partial}{\partial x_n}\mathbb{K}_{[\tau(|y'|)+\varpi],n}(L,N)\right|\,dN<\epsilon$$

for any $L\in\mathcal{T}_n\cap U(N',\delta)$ from (1.5) and (4.2), which is (I) in Lemma 2. To see (II), we only need to observe from (1.2) that for any $N'\in\partial\mathcal{T}_n$

$$\limsup_{L=(x',x_n)\to N^*,L\in\mathcal{T}_n}\frac{\partial}{\partial x_n}\mathbb{V}_{[\tau(|y'|)+\varpi],n}(L,N)=0.$$

So Theorem 1 is proved.

5 Proof of Theorem 2

Lemma 2 gives

$$P_-(r)+\int_{\partial\mathcal{T}_n(1,r)}h^-(y')\left(|y'|^{-n}-r^{-n}\right)dy'$$
$$=P_+(r)+\int_{\partial\mathcal{T}_n(1,r)}h^+(y')\left(|y'|^{-n}-r^{-n}\right)dy'-d_1-d_2r^{-n},$$

where

$$P_\pm(r)=\int_{\sigma_n(r)}nh^\pm(y)r^{-n-1}y_n\,dN.$$

Since $h\in\mathfrak{C}_\varpi$, we obtain by (2.1)

$$\int_1^{+\infty}P_-(r)r^{2-\varpi-\tau(r)}\,dr=n\int_{\mathcal{T}_n(1,+\infty)}h^-(N)y_n|N|^{1-\varpi-n-\tau(|N|)}\,dN<+\infty. \tag{5.1}$$

We have by (1.5)

$$\int_1^{+\infty}r^{2-\varpi-\tau(r)}\left(\int_{\partial\mathcal{T}_n(1,r)}h^-(y')\left(|y'|^{-n}-r^{-n}\right)dy'\right)dr$$
$$=\int_{\partial\mathcal{T}_n(1,+\infty)}h^-(y')\left(\int_{|y'|}^{\infty}r^{2-\varpi-\tau(r)}\left(|y'|^{-n}-r^{-n}\right)dr\right)dy'$$

$$\begin{aligned}&\leq \frac{n}{n+1}\int_{\partial\mathcal{T}_n(1,+\infty)} h^-(y')|y'|^{3-\varpi-n-\tau(|y'|)}\,dy'\\&<+\infty.\end{aligned}\tag{5.2}$$

From (5.1), (5.2) and Lemma 2, we see that

$$\begin{aligned}&\int_1^{+\infty} r^{\frac{2-\varpi}{2}-\tau(r)}\left(\int_{\partial\mathcal{T}_n(1,r)} h^+(y')\left(|y'|^{-n}-r^{-n}\right)dy'\right)dr\\&=\int_{\partial\mathcal{T}_n[1,+\infty)} h^+(y')\left(\int_{|y'|}^{\infty} r^{\frac{2-\varpi}{2}-\tau(r)}\left(|y'|^{-n}-r^{-n}\right)dr\right)dy'\\&\leq \int_1^{+\infty} P_-(r)r^{\frac{2-\varpi}{2}-\tau(r)}\,dr-\int_1^{+\infty} r^{\frac{2-\varpi}{2}-\tau(r)}\left(d_1+d_2r^{-n}\right)dr\\&\quad+\int_1^{+\infty} r^{\frac{2-\varpi}{2}-\tau(r)}\left(\int_{\partial\mathcal{T}_n(1,r)} h^-(y')\left(|y'|^{-n}-r^{-n}\right)dy'\right)dr\\&<+\infty.\end{aligned}\tag{5.3}$$

Set

$$\mathbb{Q}(\varpi)=\lim_{|y'|\to\infty}\int_{|y'|}^{\infty} r^{\frac{2-\varpi}{2}-\tau(r)}\left(|y'|^{-n}-r^{-n}\right)dr|y'|^{-3+\varpi+n+\tau(|y'|)}.$$

It is easy to see that

$$\mathbb{Q}(\varpi)=+\infty,$$

from (1.4), which shows that

$$M_2|y'|^{3-\varpi-n-\tau(|y'|)}\leq \int_{|y'|}^{\infty} r^{\frac{2-\varpi}{2}-\tau(r)}\left(|y'|^{-n}-r^{-n}\right)dr$$

for any $|y'|\geq 1$, where M_2 is a positive constant.

It follows that

$$\begin{aligned}&M_2\int_{\partial\mathcal{T}_n[1,+\infty)} h^+(y')|y'|^{3-\varpi-n-\tau(|y'|)}\,dx'\\&\quad\leq \int_{\partial\mathcal{T}_n[1,+\infty)} h^+(y')\int_{|y'|}^{\infty} r^{\frac{2-\varpi}{2}-\tau(r)}\left(|y'|^{-n}-r^{-n}\right)dr\,dy'\\&\quad<+\infty\end{aligned}$$

from (5.3).

Then Theorem 2 is proved from $|h|=h^++h^-$.

6 Proof of Theorem 3

Put $h'(L)=h(L)-\mathbb{H}_{[\tau(|y'|)+\varpi],n}(L)$. Then it is easy to see that $h'(L)$ is harmonic on $\mathcal{T}_n$ with normal derivative vanishes on $\partial\mathcal{T}_n$ and $h'(L)$ can be continuously extended to $\overline{\mathcal{T}_n}$. By applying the Schwarz reflection principle [8], p.68, to $h'(L)$, it follows that there is a function harmonic on $\mathcal{T}_n$ satisfying $h(L^*)=-h'(L)=-(h(L)-\mathbb{H}_{[\tau(|y'|)+\varpi],n}(L))$ for $L\in\overline{T}_n$, where $*$

denotes reflection in $\partial\mathcal{T}_n$ just as $L^* = (x', -x_n)$. Thus $h(L) = \Lambda(L) + \mathbb{H}_{[\tau(|y'|)+\varpi],n}(L)$ for all $L \in \overline{\mathcal{T}}_n$, where $\Lambda(L)$ is a harmonic function on $\mathcal{T}_n$ with normal derivative which vanishes continuously on $\partial\mathcal{T}_n$. Theorem 3 is proved.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References
1. Su, B: Dirichlet problem for the Schrödinger operator in a half space. Abstr. Appl. Anal. **2012**, Article ID 578197 (2012)
2. Su, B: Growth properties of harmonic functions in the upper half-space. Acta Math. Sin. **55**(6), 1095-1100 (2012)
3. Armitage, D: The Neumann problem for a function harmonic in $R^n \times (0,\infty)$. Arch. Ration. Mech. Anal. **63**(1), 89-105 (1976)
4. Ren, Y, Yang, P: Growth estimates for modified Neumann integrals in a half space. J. Inequal. Appl. **2013**, Article ID 572 (2013)
5. Yang, D, Ren, Y: Dirichlet problem on the upper half space. Proc. Indian Acad. Sci. Math. Sci. **124**(2), 175-178 (2014)
6. Zhao, T, Yamada, A Jr.: A remark on the Dirichlet problem in a half-plane. J. Inequal. Appl. **2014**, Article ID 497 (2014)
7. Hayman, W, Kennedy, P: Subharmonic Functions. Vol. I. London Mathematical Society Monographs, vol. 9. Academic Press, London (1976)
8. Axler, S, Bourdon, P, Ramey, W: Harmonic Function Theory, 2nd edn. Springer, New York (1992)

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