On isotropicity with respect to a measure

Liran Rotem

Abstract A body C is said to be isoptropic with respect to a measure μ if the function

$$\theta \to \int_C \langle x, \theta \rangle^2 \, d\mu(x)$$

is constant on the unit sphere. In this note, we extend a result of Bobkov, and prove that every body can be put in isotropic position with respect to any rotation invariant measure.
When the body C is convex, and the measure μ is log-concave, we relate the isotropic position with respect to μ to the famous M-position, and give bounds on the isotropic constant.

1 Introduction

Let μ be a finite Borel measure on $\mathbb{R}^n$ with finite second moments. For simplicity, we will always assume our measures are *even*, i.e. measures which satisfy $\mu(A) = \mu(-A)$ for every Borel set A. We will say that such a measure is isotropic if the function

$$\theta \mapsto \int \langle x, \theta \rangle^2 \, d\mu(x)$$

is constant on the unit sphere $S^{n-1} = \{x : \ |x| = 1\}$.

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In particular, let C be an origin-symmetric and compact set in $\mathbb{R}^n$ with non-empty interior. From now on, such sets will simply be called *bodies*. Let λ_C be the restriction of the Lebesgue measure λ to the set C:

$$\lambda_C(A) = \lambda\left(A \cap C\right).$$

We say that C is isotropic if the measure λ_C is isotropic, i.e. if the integrals

$$\int_C \langle x, \theta\rangle^2 \, dx$$

are independent of $\theta \in S^{n-1}$.

For a discussion of isotropic bodies and measures see, for example, [9] and [1]. Notice that at the moment we do not assume our measures and bodies satisfy any convexity properties, nor do we assume any normalization condition. This will change in section 3.

In this note we will study the following notion:

Definition 1. Let μ be an even locally finite Borel measure on $\mathbb{R}^n$. Let C be a body with $\mu(C) > 0$. We say that C is isotropic with respect to μ, or that the pair (C, μ) is isotropic, if

$$\int_C \langle x, \theta\rangle^2 \, d\mu(x)$$

is independent of $\theta \in S^{n-1}$.

From a formal point of view, this is not a new definition. Isotropicity of the pair (C, μ) is nothing more than isotropicity of the measure μ_C, where μ_C is the restriction of μ to C. In particular, C is isotropic with respect to the Lebesgue measure if and only if it is isotropic. However, this new notation is better suited for our needs, as we want to separate the roles of μ and C.

Let us demonstrate this point by discussing the notion of an *isotropic position*. It is well known that for any measure μ one can find a linear map $T \in \mathrm{SL}(n)$ such that the push-forward $T_\sharp\mu$ is isotropic. The proof may be written in several ways (again, see for example the proofs in [9] and [1]), but in any case this is little more than an exercise in linear algebra. Since $T_\sharp\left(\mu_C\right) = \left(T_\sharp\mu\right)_{T(C)}$ we see that for every pair (C, μ) one can find a map $T \in \mathrm{SL}(n)$ such that $(TC, T_\sharp\mu)$ is isotropic.

However, we are interested in a different problem. For us, the measure μ is a fixed "universal" measure, which we are unwilling to change. Given a body C, we want to put it in an isotropic position with respect to this given μ. In other words, we want to find a map $T \in \mathrm{SL}(n)$ such that (TC, μ) is isotropic. This is already a non-linear problem, and it is far less obvious that such a T actually exists.

Of course, there is one choice of μ for which the problem is trivial. For the Lebesgue measure λ we know that $T_\sharp\lambda = \lambda$ for any $T \in \mathrm{SL}(n)$. Hence the two problems coincide, and there is nothing new to prove.

There is one non-trivial case where the problem was previously solved. Let γ be the Gaussian measure on $\mathbb{R}^n$, defined by

$$\gamma(A) = (2\pi)^{-\frac{n}{2}} \int_A e^{-|x|^2/2} dx.$$

In [2], Bobkov proved the following result:

Theorem 1. *Let C be a body in $\mathbb{R}^n$. If $\gamma(C) \geq \gamma(TC)$ for all $T \in \mathrm{SL}(n)$, then C is isotropic with respect to γ.*

If C is assumed to be convex, then the converse is also true.

A simple compactness argument shows that the map $T \mapsto \gamma(TC)$ attains a maximum on $\mathrm{SL}(n)$ for some map T_0. Theorem 1 implies that T_0C is isotropic with respect to γ.

The main goal of this note is to discuss isotropicity with respect any rotation invariant measure. In the next section we will extend Bobkov's argument, and prove that C can be put in isotropic position with respect to any rotation invariant measure μ. Then in section 3 we will restrict our attention to the case where C is convex and μ is log-concave (all relevant definitions will be given there). We will relate the isotropicity of (C, μ) to the M-position, and give an upper bound on the isotropic constant of C with respect to μ.

2 Existence of isotropic position

In this section we will assume that μ is rotation invariant:

Definition 2. We say that μ is rotation invariant if there exists a bounded function $f : [0, \infty) \to [0, \infty)$ such that

$$\frac{d\mu}{dx} = f\left(|x|\right).$$

We will always assume that f has a finite first moment, i.e. $\int_0^\infty tf(t)dt < \infty$.

Our goal is to prove that if μ is rotation invariant, then for every C one can find a map $T \in \mathrm{SL}(n)$ such that (TC, μ) is isotropic. To do so we will need the following definitions:

Definition 3. Let $\mu = f\left(|x|\right) dx$ be a rotation invariant measure on $\mathbb{R}^n$. Then:

1. The associated measure $\hat{\mu}$ is the measure on $\mathbb{R}^n$ with density $g\left(|x|\right)$, where $g : [0, \infty) \to [0, \infty)$ is defined by

$$g(t) = \int_t^\infty sf(s)ds.$$

2. Given a body C we define the associated functional $J_{\mu,C} : \mathrm{SL}(n) \to \mathbb{R}$ by

$$J_{\mu,C}(T) = \hat{\mu}(TC) = \int_C g(|Tx|)\,dx.$$

When the measure μ is obvious from the context, we will write J_C instead of $J_{\mu,C}$.

As one example of the definitions, notice that for the Gaussian measure γ we have $\hat{\gamma} = \gamma$. Hence the functional $J_{\gamma,C}$ is exactly the one being maximized in Bobkov's Theorem 1.

In the general case, we have the following Proposition:

Proposition 1. *Fix a rotation invariant measure μ, a body C, and $T \in \mathrm{SL}(n)$. Then (TC, μ) is isotropic if and only if T is a critical point of $J_{\mu,C}$.*

Proof. We will first show that the identity matrix I is a critical point for J_C if and only if (C, μ) is isotropic. Indeed, I is a critical point if and only if

$$\left.\frac{d}{dt}\right|_{t=0} J_C\left(e^{tA}\right) = 0$$

for all maps $A \in \mathfrak{sl}(n)$, i.e. all linear maps A with $\operatorname{Tr} A = 0$.

An explicit calculation of the derivative gives

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} J_C\left(e^{tA}\right) &= \left.\frac{d}{dt}\right|_{t=0}\left(\int_C g\left(\left|e^{tA}x\right|\right)dx\right) = \int_C\left(\left.\frac{d}{dt}\right|_{t=0} g\left(\left|e^{tA}x\right|\right)\right)dx \\
&= \int_C g'(|x|)\cdot\left\langle \frac{x}{|x|}, Ax\right\rangle dx = -\int_C |x|\, f(|x|)\left\langle \frac{x}{|x|}, Ax\right\rangle dx \\
&= -\int_C \langle x, Ax\rangle\, d\mu(x).
\end{aligned}$$

Hence we see that I is a critical value for J_C if and only if

$$\int_C \langle x, Ax\rangle\, d\mu(x) = 0$$

for all maps A with $\operatorname{Tr} A = 0$. This condition is known and easily seen to be equivalent to isotropicity of (C, μ).

So far we have proved the result only for $T = I$. For general case notice that $J_{TC}(S) = J_C(ST)$ for every $S, T \in \mathrm{SL}(n)$. Hence T is a critical point for J_C if and only if I is a critical point for J_{TC}, which holds if and only if (TC, μ) is isotropic.

From here it is easy to deduce the main result:

Theorem 2. *Let μ be a rotation invariant measure on $\mathbb{R}^n$, and let C be a body in $\mathbb{R}^n$. Then there exists a map $T \in \mathrm{SL}(n)$ such that TC is isotropic with respect to μ.*

Proof. Write $s_n(T)$ for the smallest singular value of a map $T \in \mathrm{SL}(n)$. It is not hard to see that $J_{\mu,C}(T) \to 0$ as $s_n(T) \to 0$. Since

$$\{T \in \mathrm{SL}(n) : \ s_n(T) \geq \varepsilon\}$$

is compact, it follows that $J_{\mu,C}$ attains a global maximum at some point T. In particular, T is a critical point for $J_{\mu,C}$, so (TC, μ) is isotropic.

Remark 1. When $\mu = \gamma$ and the body C is convex, the functional $J_{\mu,C}$ has a unique positive definite critical point. In other words, there exists a positive definite matrix $S \in \mathrm{SL}(n)$ such that the set of critical points of $J_{\mu,C}$ is exactly $\{US : \ U \in O(n)\}$. Moreover, every such critical point is a global maximum. The proof of these facts, which appears in [2], is based on the so-called (B) conjecture, proved by Cordero-Erausquin, Fradelizi and Maurey ([5]). This fact explains the second half of Theorem 1. It also implies that the isotropic position with respect to γ is unique, up to rotations and reflections.

For general measures μ, we have no analog of the (B) conjecture, and so $J_{\mu,C}$ may have critical points which are not the global maximum. Hence we define:

Definition 4. We say that C is in principle isotropic position with respect to μ if $J_{\mu,C}$ is maximized at the identity matrix I.

Proposition 1 shows that if C is in principle isotropic position with respect to μ, then it is also isotropic with respect to μ in the sense of Definition 1. If the (B) conjecture happens to hold for the measure μ, then these two notions coincide. However, we currently know the (B) conjecture for very few measures: the original result concerns the Gaussian measure, and Livne Bar-On has recently proved the conjecture when μ is a uniform measure in the plane ([7]).

Let us conclude this section with one application of Theorem 2 for isotropicity of bodies:

Proposition 2. *Let B be a Euclidean ball of some radius $r > 0$. Then for every body C one can find a map $T \in \mathrm{SL}(n)$ such that $TC \cap B$ is isotropic.*

Proof. Let $\mu = \lambda_B$ be the uniform measure on B. μ is rotation invariant, so we can apply Theorem 2 and find a map $T \in \mathrm{SL}(n)$ such that TC is isotropic with respect to μ. This just means that $\mu_{TC} = \lambda_{TC \cap B}$ is an isotropic measure, or that $TC \cap B$ is an isotropic body.

Following Proposition 2, Prof. Bobkov asked about an interesting variant concerning Minkowski addition. Recall that the Minkowski sum of sets $A, B \subseteq \mathbb{R}^n$ is defined by

$$A+B=\{a+b:\ a\in A,\ b\in B\}.$$

Bobkov then posed the following question:

Problem 1. Let B be a Euclidean ball of some radius $r>0$. Given a convex body C, is it always possible to find $T\in \mathrm{SL}(n)$ such that $TC+B$ is isotropic?

Unfortunately, we do not know the answer to this question.

3 properties of isotropic pairs

Let μ be an (even) isotropic measure with density f. The isotropic constant of μ is defined as

$$L_\mu=\frac{f(0)^{\frac{1}{n}}}{\mu\left(\mathbb{R}^n\right)^{\frac{1}{n}+\frac{1}{2}}}\left(\int \langle x,\theta\rangle^2\,d\mu(x)\right)^{\frac{1}{2}}.$$

We define the isotropic constant of the pair (C,μ) to be the isotropic constant of μ_C, so

$$L\left(C,\mu\right)=\frac{f(0)^{\frac{1}{n}}}{\mu\left(C\right)^{\frac{1}{n}+\frac{1}{2}}}\left(\int_C \langle x,\theta\rangle^2\,d\mu(x)\right)^{\frac{1}{2}}.$$

A major open question, known as the slicing problem, asks if $L_K=L\left(K,\lambda\right)$ is bounded from above by a universal constant for every dimension n and every isotropic convex body K in $\mathbb{R}^n$ (see [9] for a much more information). It turns out that for certain rotation invariant measures μ it is possible to give an upper bound on $L\left(K,\mu\right)$ whenever K is a convex body in principle isotropic position with respect to μ. In the Gaussian case, this was done by Bobkov in [2]. We will demonstrate how his methods can be extended, starting with the case where μ is a uniform measure on the Euclidean ball.

In order to prove our result, we will need the notion of M-position. Let B be the Euclidean ball of volume (Lebesgue measure) 1. A convex body K of volume 1 is said to be in M-position with constant $C>0$ if

$$|K\cap B|\geq C^{-n}.$$

There are many other equivalent ways to state this definition, but this definition will be the most convenient for us. A remarkable theorem of Milman shows that for every convex body K of volume 1 there exists a map $T\in\mathrm{SL}(n)$ such that TK is in M-position, with some universal constant C (see [8]).

After giving all the definitions, we are ready to prove the following:

Theorem 3. *Let K be a convex body of volume 1 such that $K\cap B$ is in principle isotropic position (i.e. K is in principle isotropic position with respect to μ, when μ is the uniform measure on B). Then*

1. K is in M-position. In fact
$$|K\cap B| \geq \left(\frac{1}{2}\right)^{n+1} \sup_{T\in SL(n)} |TK\cap B| \geq C^{-n}$$
for some universal $C>0$.
2. The isotropic constant of $K\cap B$ is bounded by an absolute constant.

Proof. We should understand how $J_{K,\mu}$ looks in this special case. If we denote the radius of B by r, then $d\mu = f(|x|)dx$, where $f = \mathbf{1}_{[0,r]}$. Hence
$$g(t) = \int_t^\infty s\cdot \mathbf{1}_{[0,r]}(s)ds = \begin{cases} \frac{r^2-t^2}{2} & t\leq r \\ 0 & t>r, \end{cases}$$
and
$$J_K(T) = \int_{TK} g\left(|x|\right)dx = \int_{TK\cap B} \frac{r^2-|x|^2}{2}dx.$$

Notice that $J_K(T)$ is almost the same as the volume $|TK\cap B|$ (properly normalized) for every $T\in \mathrm{SL}(n)$. Indeed, on the one hand we have
$$J_K(T) = \int_{TK\cap B} \frac{r^2-|x|^2}{2}dx \leq \int_{TK\cap B}\frac{r^2}{2}dx = \frac{r^2}{2}\left|TK\cap B\right|,$$
and on the other hand we have
$$\begin{aligned} J_K(T) &= \int_{TK\cap B} \frac{r^2-|x|^2}{2}dx \geq \int_{\frac{TK\cap B}{2}} \frac{r^2-|x|^2}{2}dx \\ &\geq \int_{\frac{TK\cap B}{2}} \frac{r^2-\left(\frac{r}{2}\right)^2}{2}dx = \frac{3}{8}r^2\left|\frac{TK\cap B}{2}\right| \geq \left(\frac{1}{2}\right)^{n+1}\frac{r^2}{2}\left|TK\cap B\right|. \end{aligned}$$

Since K is in principle isotropic position we know that for every $T\in\mathrm{SL}(n)$ we have $J_K(T)\leq J_K(I)$, and then
$$|TK\cap B| \leq 2^{n+1}\cdot\frac{2}{r^2}J_K\left(T\right) \leq 2^{n+1}\frac{2}{r^2}J_K(I) \leq 2^{n+1}\left|K\cap B\right|,$$
so the first inequality of (1) is proven. In particular, by Milman's theorem we get that $|K\cap B|\geq C^{-n}$ for some universal constant $C>0$.

Finally, in order to prove (2), Notice that it follows from the definition that
$$L_{K\cap B} = \frac{1}{|K\cap B|^{\frac{1}{n}+\frac{1}{2}}}\left(\int_{K\cap B}\frac{|x|^2}{n}dx\right)^{\frac{1}{2}}.$$
Since B has radius $\leq C\sqrt{n}$ we get that

$$L_{K\cap B} \leq \frac{1}{|K\cap B|^{\frac{1}{n}+\frac{1}{2}}} \cdot \left(\int_{K\cap B} \frac{C^2 n}{n} dx\right)^{\frac{1}{2}} = \frac{C}{|K\cap B|^{\frac{1}{n}}} \leq C',$$

and the theorem is proven.

What happens for general rotation invariant measures μ? In order to prove a similar estimate, we will need to assume that the measure μ is log-concave:

Definition 5. A Borel measure μ on $\mathbb{R}^n$ is log-concave if for every Borel sets A and B and every $0 < \lambda < 1$ we have

$$\mu\left(\lambda A + (1-\lambda) B\right) \geq \mu(A)^{\lambda} \mu(B)^{1-\lambda}.$$

Borel ([3], [4]) gave a simple and useful characterization of log-concave measures: assume μ is not supported on any affine hyperplane. Then μ is log-concave if and only if μ has a density f, which is log-concave. Log-concavity of f just means that $(-\log f)$ is a convex function.

For log-concave measures we have the following bound on the isotropic constant of (K, μ):

Proposition 3. *Let μ be a log-concave, rotation invariant measure on $\mathbb{R}^n$, and let $K \subseteq \mathbb{R}^n$ be a convex body. Then*

$$L(K,\mu) \leq C \cdot \mu\left(\mathbb{R}^n\right)^{\frac{1}{n}} \cdot \mu(K)^{-\frac{1}{n}}$$

for some universal constant $C > 0$.

Proof. Write $d\mu(x) = f\left(|x|\right) dx$. Since both sides of the inequality are invariant to a scaling of f, we may assume without loss of generality that $f(0) = 1$.

Recall the following construction of Ball ([1]): If μ is a log-concave measure with density f, and $p \geq 1$, we define

$$K_p(\mu) = \left\{ x \in \mathbb{R}^n : \int_0^\infty f(rx) r^{p-1} dr \geq \frac{f(0)}{p} \right\}.$$

Ball proved that $K_p(\mu)$ is a convex body, but we won't need this fact in our proof. We will need that fact that if $p = n+1$ and $f(0) = 1$, then $L_{K_{n+1}(\mu)} \asymp L_\mu$ and $|K_{n+1}(\mu)| \asymp \mu\left(\mathbb{R}^n\right)$. This is proven, for example, in Lemma 2.7 of [6]. Here the notation $A \asymp B$ means that $\frac{A}{B}$ is bounded from above and from below by universal constants.

Since our μ is rotation invariant, $K_{n+1}(\mu)$ is just a Euclidean ball. If we denote its radius by R, then

$$\mu\left(\mathbb{R}^n\right)^{\frac{1}{n}} \asymp |K_{n+1}(\mu)|^{\frac{1}{n}} \asymp \frac{R}{\sqrt{n}}.$$

Now we turn our attention to the body $\widetilde{K} = K_{n+1}(\mu_K)$. It is obvious that $\widetilde{K} \subseteq K_{n+1}(\mu)$. Hence we get

$$L(K,\mu) \asymp L_{\widetilde{K}} = \frac{1}{\left|\widetilde{K}\right|^{\frac{1}{n}+\frac{1}{2}}} \left(\int_{\widetilde{K}} \frac{|x|^2}{n} dx \right)^{\frac{1}{2}} \leq \frac{1}{\left|\widetilde{K}\right|^{\frac{1}{n}+\frac{1}{2}}} \left(\int_{\widetilde{K}} \frac{R^2}{n} dx \right)^{\frac{1}{2}}$$
$$= \frac{R}{\sqrt{n}} \cdot \left|\widetilde{K}\right|^{-\frac{1}{n}} \asymp \mu(\mathbb{R}^n)^{\frac{1}{n}} \cdot \mu_K(\mathbb{R}^n)^{-\frac{1}{n}} = \mu(\mathbb{R}^n)^{\frac{1}{n}} \cdot \mu(K)^{-\frac{1}{n}},$$

and the proof is complete.

Therefore in order to bound $L(K,\mu)$ from above we need to bound $\mu(K)$ from below. Notice that we have three distinct functionals on $\mathrm{SL}(n)$:

$$\begin{aligned} &T \mapsto \mu(TK) \\ &T \mapsto J_{K,\mu}(T) = \hat{\mu}(TK) \\ &T \mapsto |TK \cap B|. \end{aligned}$$

The estimates of Theorem 3 only depend on these functionals being close to each other. More concretely, we have the following:

Theorem 4. *Let μ be a log-concave rotation invariant measure on $\mathbb{R}^n$, and let $K \subseteq \mathbb{R}^n$ be a convex body of volume 1, which is in principle isotropic position with respect to μ. Assume that*

$$\sup_{T\in \mathrm{SL}(n)} \frac{J_{K,\mu}(T)}{|TK\cap B|} \leq a^n \cdot \inf_{T\in \mathrm{SL}(n)} \frac{J_{K,\mu}(T)}{|TK\cap B|},$$

and that

$$\mu(K) \geq b^{-n} \cdot |K \cap B|.$$

Then:

1. K is in M-position with constant $C \cdot a$ for some universal $C > 0$.
2. $L(K,\mu) \leq C\mu(\mathbb{R}^n)^{\frac{1}{n}} \cdot ab$ for some universal constant $C > 0$.

Proof. There is very little to prove here. For (1), define for simplicity

$$m = \inf_{T\in \mathrm{SL}(n)} \frac{J_{K,\mu}(T)}{|TK\cap B|}.$$

Then for every $T \in \mathrm{SL}(n)$ we have

$$|TK\cap B| \leq \frac{1}{m} \cdot J_{K,\mu}(T) \leq \frac{1}{m} \cdot J_{K,\mu}(I) \leq \frac{a^n m}{m} \cdot |K\cap B| = a^n |K\cap B|.$$

By this estimate and Milman's theorem, it follows that K is in M-position with constant $C \cdot a$.

Now for (2) we use Proposition 3 together with part (1) and immediately obtain

$$L(K,\mu) \le C\mu\left(\mathbb{R}^n\right)^{\frac{1}{n}}\mu(K)^{-\frac{1}{n}} \le C\mu\left(\mathbb{R}^n\right)^{\frac{1}{n}} \cdot b\left|K \cap B\right|^{-\frac{1}{n}}$$
$$\le C\mu\left(\mathbb{R}^n\right)^{\frac{1}{n}} \cdot b \cdot (Ca) = C'\mu\left(\mathbb{R}^n\right)^{\frac{1}{n}} \cdot ab$$

Of course, in general there is no reason for a and b to be small. However, for any specific μ, one may try and compute explicit values for these constants. Both Theorem 3 and Bobkov's theorem in the Gaussian case follow from this general scheme.

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