CHAP 4

FEA for Elastoplastic Problems

Nam-Ho Kim

Introduction

- Elastic material: a strain energy is differentiated by strain to obtain stress
  - History-independent, potential exists, reversible, no permanent deformation
- Elatoplastic material:
  - Permanent deformation for a force larger than elastic limit
  - No one-to-one relationship between stress and strain
  - Constitutive relation is given in terms of the rates of stress and strain (**Hypo-elasticity**)
  - Stress can only be calculated by integrating the stress rate over the past load history (**History-dependent**)
- Important to separate elastic and plastic strain
  - Only elastic strain generates stress

Table of Contents

4.2

1D Elastoplasticity

Goals

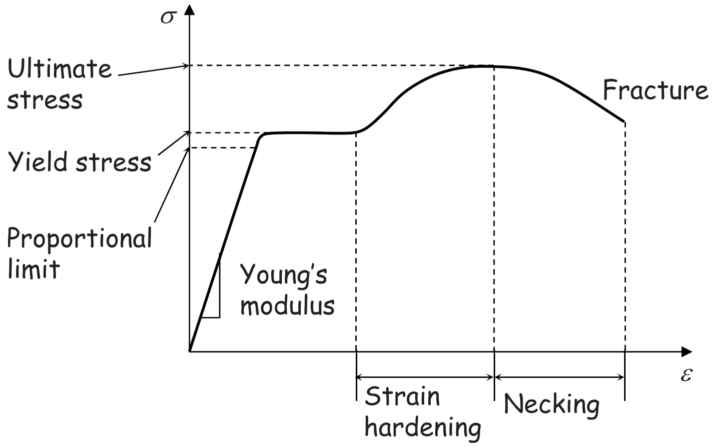
- Understand difference between elasticity and plasticity
- Learn basic elastoplastic model
- Learn different hardening models
- Understand different moduli used in 1D elastoplasticity
- Learn how to calculate plastic strain when total strain increment is given
- Learn state determination for elastoplastic material

Plasticity

- Elasticity – A material deforms under stress, but then returns to its original shape when the stress is removed
- Plasticity - deformation of a material undergoing **non-reversible changes of shape** in response to applied forces
  - Plasticity in metals is usually a consequence of **dislocations**
  - Rough nonlinearity
- Found in most metals, and in general is a good description for a large class of materials
- **Perfect plasticity** – a property of materials to undergo irreversible deformation without any increase in stresses or loads
- **Hardening** - need increasingly higher stresses to result in further plastic deformation

Behavior of a Ductile Material

Terms	Explanation
Proportional limit	The greatest stress for which the stress is still proportional to the strain
Elastic limit	The greatest stress without resulting in any permanent strain on release of stress
Young's Modulus	Slope of the linear portion of the stress-strain curve
Yield stress	The stress required to produce 0.2% plastic strain
Strain hardening	A region where more stress is required to deform the material
Ultimate stress	The maximum stress the material can resist
Necking	Cross section of the specimen reduces during deformation



Elastoplasticity

- Most metals have both elastic and plastic properties
  - Initially, the material shows elastic behavior
  - After yielding, the material becomes plastic
  - By removing loading, the material becomes elastic again
- We will assume **small (infinitesimal) deformation** case
  - Elastic and plastic strain can be **additively decomposed** by

$$\boxed{\varepsilon = \varepsilon_e + \varepsilon_p}$$

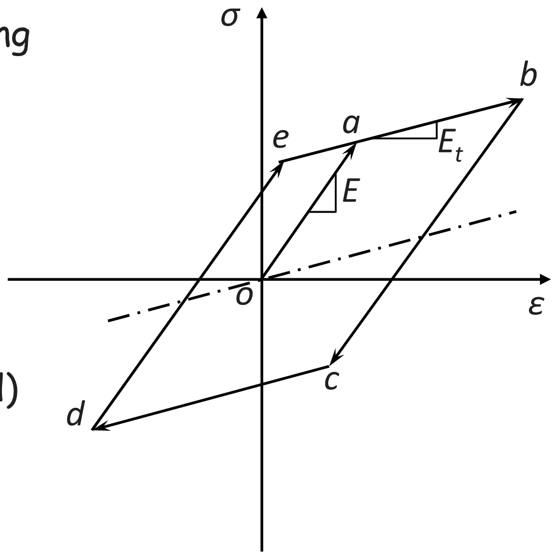
  - Strain energy density exists in terms of elastic strain

$$U_0 = \tfrac{1}{2}E(\varepsilon_e)^2$$

  - Stress is related to the elastic strain, not the plastic strain
- The plastic strain will be considered as an internal variable, which evolves according to plastic deformation

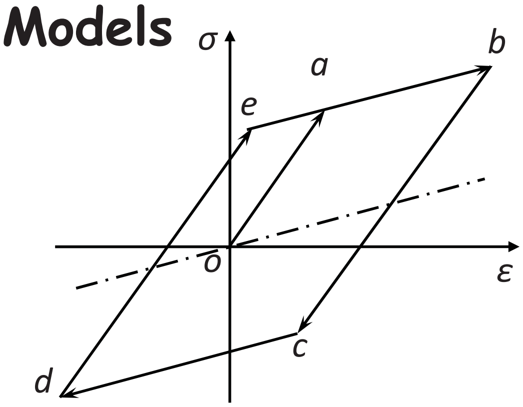
1D Elastoplasticity

- Idealized elastoplastic stress-strain behavior
  - Initial elastic behavior with slope E (**elastic modulus**) until yield stress σ_Y (line o–a)
  - After yielding, the plastic phase with slope E_t (**tangent modulus**) (line a–b).
  - Upon removing load, elastic unloading with slope E (line b-c)
  - Loading in the opposite direction, the material will eventually yield in that direction (point c)
  - **Work hardening** – more force is required to continuously deform in the plastic region (line a-b or c-d)

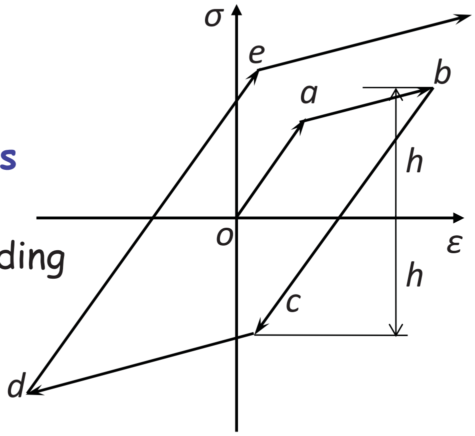


Work Hardening Models

- Kinematic hardening
  - **Elastic range remains constant**
  - Center of the elastic region moves parallel to the work hardening line
  - bc = de = 2oa
  - Use the center of elastic domain as an evolution variable



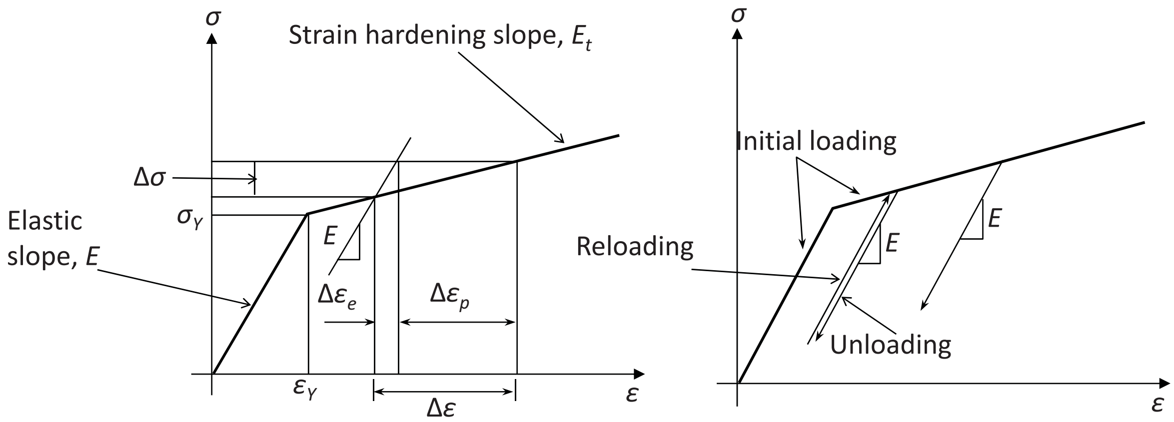
- Isotropic hardening
  - **Elastic range (yield stress) increases proportional to plastic strain**
  - The yield stress for the reversed loading is equal to the previous yield stress
  - Use plastic strain as an evolution variable



- No difference in proportional loading (line o-a-b)

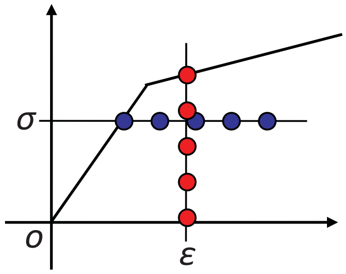
Elastoplastic Analysis

- Additive decomposition
  - Only elastic strain contributes to stress (but we don't know how much of the total strain corresponds to the elastic strain)
  - Let's consider an increment of strain: $\boxed{\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p}$
  - Elastic strain increases stress by $\boxed{\Delta\sigma = E\Delta\varepsilon_e}$
  - Elastic strain disappears upon removing loads or changing direction



Elastoplastic Analysis cont.

- Additive decomposition (continue)
  - Plastic strain remains constant during unloading
  - The effect of load-history is stored in the plastic strain
  - The yield stress is determined by the magnitude of plastic strain
  - **Decomposing elastic and plastic part of strain is an important part of elastoplastic analysis**
- For given stress σ, strain cannot be determined.
  - Complete history is required (path- or history-dependent)
  - History is stored in evolution variable (plastic strain)



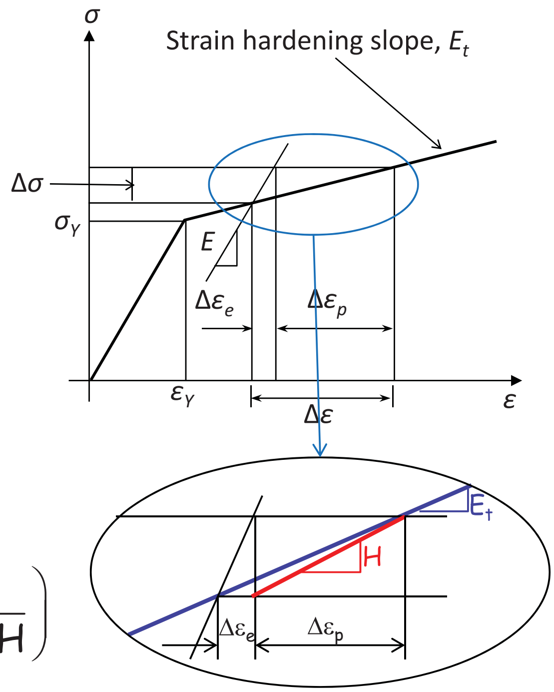
Plastic Modulus

- Strain increment $\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$
- Stress increment $\Delta\sigma = E\Delta\varepsilon_e$
- **Plastic modulus** $H = \dfrac{\Delta\sigma}{\Delta\varepsilon_p}$
- Relation between moduli

$$\boxed{\Delta\sigma = E\Delta\varepsilon_e = H\Delta\varepsilon_p = E_t\Delta\varepsilon}$$

$$\frac{\Delta\sigma}{E_t} = \frac{\Delta\sigma}{E} + \frac{\Delta\sigma}{H} \quad \Rightarrow \quad \frac{1}{E_t} = \frac{1}{E} + \frac{1}{H}$$

$$H = \frac{EE_t}{E - E_t} \qquad E_t = \frac{EH}{E + H} = E\left(1 - \frac{E}{E + H}\right)$$



- Both kinematic and isotropic hardenings have the same plastic modulus

---

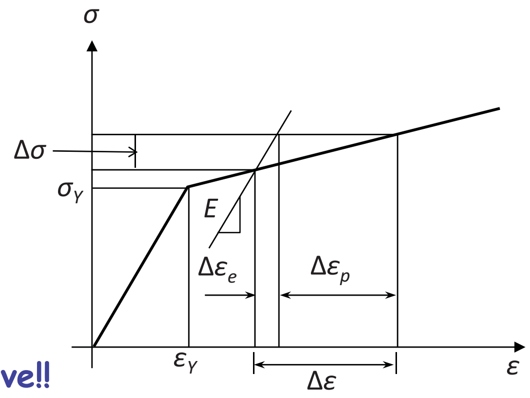
Analysis Procedure

- Analysis is performed with a given incremental strain
  - N-R iteration will provide $\Delta\mathbf{u} \Rightarrow \Delta\varepsilon$
  - But, we don't know $\Delta\varepsilon_e$ or $\Delta\varepsilon_p$
- When the material is in the initial elastic range, regular elastic analysis procedure can be used
- When the material is in the plastic range, we have to **determine incremental plastic strain**

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p = \frac{\Delta\sigma}{E} + \Delta\varepsilon_p = \frac{H\Delta\varepsilon_p}{E} + \Delta\varepsilon_p$$

$$= \Delta\varepsilon_p\left(\frac{H}{E} + 1\right)$$

$$\Rightarrow \quad \boxed{\Delta\varepsilon_p = \frac{\Delta\varepsilon}{1 + H/E}}$$



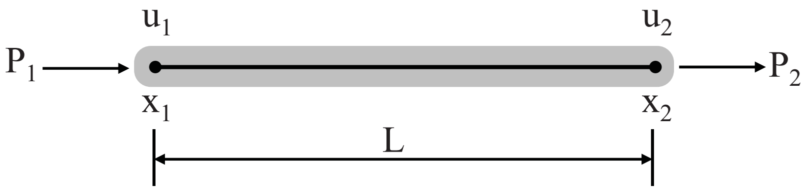
Only when the material is on the plastic curve!!

1D Finite Element Formulation

- Load increment
  - applied load is divided by N increments: $[t_1, t_2, \ldots, t_N]$
  - analysis procedure has been completed up to load increment t_n
  - a new solution at t_{n+1} is sought using the Newton-Raphson method
  - iteration k has been finished and the current iteration is k+1

- Displacement increments
  - From last increment t_n: $\Delta\mathbf{d}^k = {}^{n+1}\mathbf{d}^k - {}^{n}\mathbf{d}$
  - From previous iteration: $\delta\mathbf{d}^k = {}^{n+1}\mathbf{d}^{k+1} - {}^{n+1}\mathbf{d}^k$

$$\mathbf{d} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



1D FE Formulation cont.

- Interpolation

$$\Delta u(x) = [N_1\ N_2]\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix} = \mathbf{N}\cdot\Delta\mathbf{d}$$

$$\Delta\varepsilon = \frac{d}{dx}(\Delta u) = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix} = \mathbf{B}\cdot\Delta\mathbf{d}$$

$$\delta u = \mathbf{N}\cdot\delta\mathbf{d}$$
$$\delta\varepsilon = \mathbf{B}\cdot\delta\mathbf{d}$$

$$\bar{u} = \mathbf{N}\cdot\bar{\mathbf{d}}$$
$$\bar{\varepsilon} = \mathbf{B}\cdot\bar{\mathbf{d}}$$

- Weak form (1 element)
  - Internal force = external force

$$\bar{\mathbf{d}}^T \int_0^L \mathbf{B}^T\, {}^{n+1}\sigma^{k+1} A\, dx = \bar{\mathbf{d}}^T\, {}^{n+1}\mathbf{F}, \quad \forall \bar{\mathbf{d}} \in R^2$$

$$\bar{\mathbf{d}} = \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}$$

1D FE Formulation cont.

- Stress-strain relationship (Incremental)

$$ {}^{n+1}\sigma^{k+1} \approx {}^{n+1}\sigma^{k} + \frac{\partial\sigma}{\partial\varepsilon}\delta\varepsilon = {}^{n+1}\sigma^{k} + D^{ep}\delta\varepsilon $$

  - Elastoplastic tangent modulus

$$ \boxed{D^{ep} = \begin{cases} E & \text{if elastic} \\ E_t & \text{if plastic} \end{cases}} $$

- Linearization of weak form

$$ \bar{\mathbf{d}}^T \underbrace{\left[\int_0^L \mathbf{B}^T D^{ep} \mathbf{B} A\,dx \right]}_{\text{Tangent stiffness}} \delta\mathbf{d} = \underbrace{\bar{\mathbf{d}}^T\,{}^{n+1}\mathbf{F} - \bar{\mathbf{d}}^T \int_0^L \mathbf{B}^T\,{}^{n+1}\sigma^k A\,dx}_{\text{Residual}} $$

1D FE Formulation cont.

- Tangent Stiffness

$$ \boxed{\mathbf{k}_T = \frac{AD^{ep}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} $$

- Residual

$$ {}^{n+1}\mathbf{R}^k = {}^{n+1}\mathbf{F} - \int_0^L \mathbf{B}^T\,{}^{n+1}\sigma^k A\,dx = \begin{Bmatrix} {}^{n+1}F_1 + {}^{n+1}\sigma^k A \\ {}^{n+1}F_2 - {}^{n+1}\sigma^k A \end{Bmatrix} $$

- State Determination: ${}^{n+1}\sigma^k = f({}^n\sigma, {}^n\varepsilon_p, \Delta\varepsilon^k, \ldots)$

Will talk about next slides

- Incremental Finite Element Equation
  - N-R iteration until the residual vanishes

$$ \boxed{\mathbf{k}_T \cdot \delta\mathbf{d}^k = {}^{n+1}\mathbf{R}^k} $$

Isotropic Hardening Model

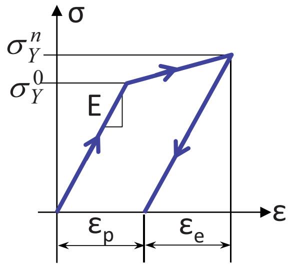
- Yield strength gradually increases proportional to the plastic strain
  - Yield strength is always positive for both tension or compression

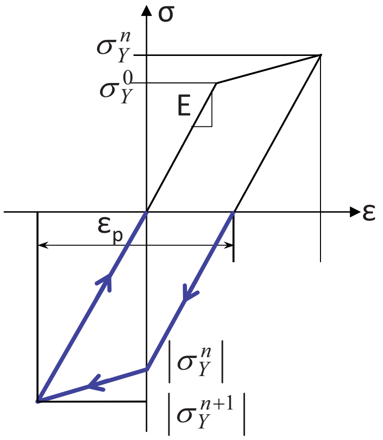
$$\sigma_Y^n = \sigma_Y^0 + H\varepsilon_p^n$$

Total plastic strain

Initial yield stress

  - Plastic strain is always positive and continuously **accumulated** even in cycling loadings





State Determination (Isotropic Hardening)

- How to determine stress
  - Given: strain increment ($\Delta\varepsilon$) and all variables in load step n

$$(E, H, \sigma_Y^0, \sigma^n, \varepsilon_p^n)$$

1. Computer current yield stress (pont d)

$$\sigma_Y^n = \sigma_Y^0 + H\varepsilon_p^n$$

2. Elastic predictor (point c)

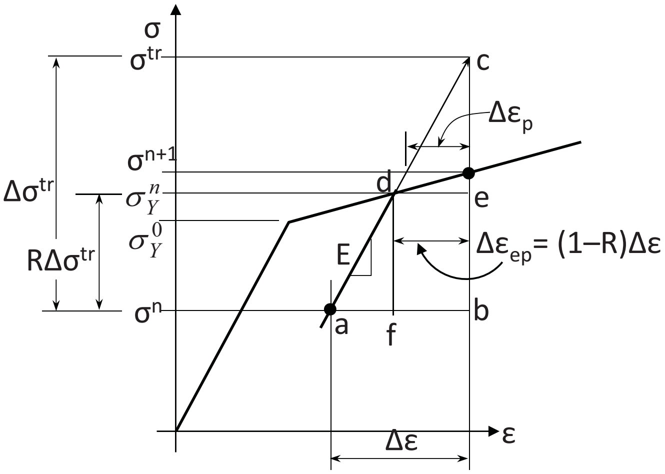
$$\Delta\sigma^{tr} = E\Delta\varepsilon \qquad \sigma^{tr} = \sigma^n + \Delta\sigma^{tr}$$

3. Check yield status

Trial yield function (c – e)

$$f^{tr} = |\sigma^{tr}| - \sigma_Y^n$$

$$f^{tr} = (1 - R)E\Delta\varepsilon$$



R: Fraction of $\Delta\sigma^{tr}$ to the yield stress

State Determination (Isotropic Hardening) cont.

- If $f^{tr} \leq 0$, material is elastic

$$\boxed{\sigma^{n+1} = \sigma^{tr}}$$

Either initial elastic region or unloading

- If $f^{tr} > 0$, material is plastic (yielding)

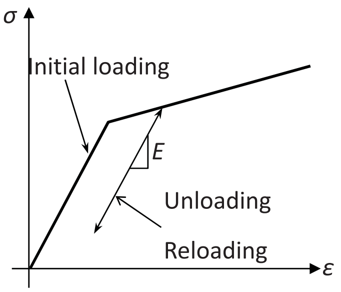
Either transition from elastic to plastic or continuous yielding

- Stress update (return to the yield surface)

$$\boxed{\sigma^{n+1} = \sigma^{tr} - sgn(\sigma^{tr})E\Delta\varepsilon_p}$$

- Update plastic strain

$$\boxed{\varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p}$$



Plastic strain increment is unknown

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$$

For a given strain increment, how much is elastic and plastic?

State Determination (Isotropic Hardening) cont.

- Plastic consistency condition
  - to determine plastic strain increment

$$\boxed{f^{n+1} = \left|\sigma^{n+1}\right| - \sigma_y^{n+1} = 0}$$

  - Stress must be on the yield surface after plastic deformation

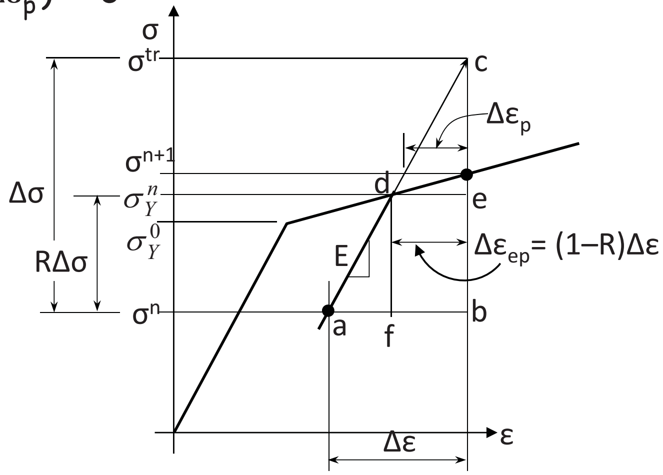
$$\Rightarrow \left|\sigma^{tr} - sgn(\sigma^{tr})E\Delta\varepsilon_p\right| - (\sigma_y^n + H\Delta\varepsilon_p) = 0$$

$$\Rightarrow \left|\sigma^{tr}\right| - \sigma_y^n - (E+H)\Delta\varepsilon_p = 0$$

$$\boxed{\Delta\varepsilon_p = \frac{\left|\sigma^{tr}\right| - \sigma_y^n}{E+H} = \frac{f^{tr}}{E+H}}$$

$$\Delta\varepsilon_p = (1-R)\frac{E}{E+H}\Delta\varepsilon$$

$$R = 1 - \frac{f^{tr}}{\left|\Delta\sigma^{tr}\right|}$$



%Note: $\Delta\varepsilon_p$ is always positive!!

State Determination (Isotropic Hardening) cont.

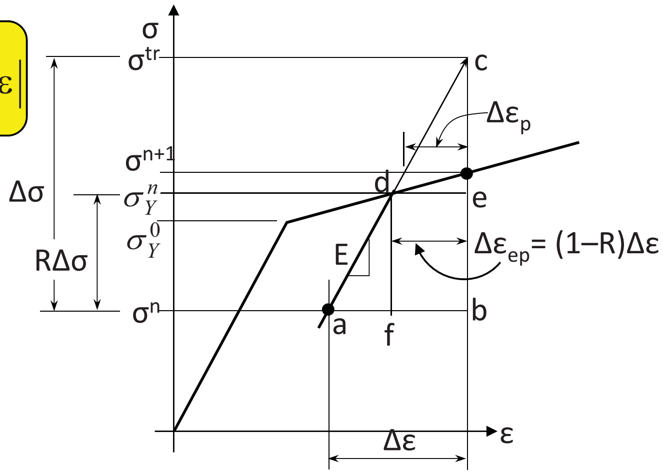
- Update stress

$$\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\sigma^{tr}) E \Delta\varepsilon_p$$

$$\boxed{\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\sigma^{tr}) \frac{(1-R)E^2}{E+H} |\Delta\varepsilon|}$$

Elastic trial

Plastic compensation (return mapping)



- Algorithm
  1) Elastic trial
  2) Plastic return mapping
  - No iteration is required in linear hardening models

Algorithmic Tangent Stiffness

- Continuum tangent modulus
  - The slope of stress-strain curve $D^{ep} = \begin{cases} E & \text{if elastic} \\ E_t & \text{if plastic} \end{cases}$
- Algorithmic tangent modulus
  - Differentiation of the state determination algorithm

$$D^{alg} = \frac{\partial \Delta\sigma}{\partial \Delta\varepsilon} = \frac{\partial\, {}^{tr}\sigma}{\partial \Delta\varepsilon} - \mathrm{sgn}({}^{tr}\sigma) E \frac{\partial \Delta\varepsilon_p}{\partial \Delta\varepsilon}$$

$$\frac{\partial \Delta\varepsilon_p}{\partial \Delta\varepsilon} = \frac{1}{E+H} \frac{\partial\, {}^{tr}f}{\partial \Delta\varepsilon} = \mathrm{sgn}({}^{tr}\sigma) \frac{E}{E+H}$$

$$\boxed{D^{alg} = \begin{cases} E & \text{if elastic} \\ E_t & \text{if plastic} \end{cases}}$$

- $D^{alg} = D^{ep}$ for 1D plasticity!!
  - We will show that they are different for multi-dimension

Algorithm for Isotropic Hardening

- Given: $\Delta\varepsilon, \mathrm{E}, \mathrm{H}, \sigma_\mathrm{Y}^0, \sigma^n, \varepsilon_p^n$

1. Trial state $\sigma^{tr} = \sigma^n + E\Delta\varepsilon$

$$\sigma_Y^n = \sigma_Y^0 + H\varepsilon_p^n$$

$$f^{tr} = \left|\sigma^{tr}\right| - \sigma_Y^n$$

1. If $f^{tr} \leq 0$ (elastic)

   – Remain elastic: $\boxed{\sigma^{n+1} = \sigma^{tr}, \quad \varepsilon_p^{n+1} = \varepsilon_p^n}$; exit

2. If $f^{tr} > 0$ (plastic)

   a. Calculate plastic strain: $\boxed{\Delta\varepsilon_p = \dfrac{f^{tr}}{E+H}}$

   b. Update stress and plastic strain (store them for next increment)

$$\boxed{\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\sigma^{tr})E\Delta\varepsilon_p} \qquad \boxed{\varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p}$$

Ex) Elastoplastic Bar (Isotropic Hardening)

- E = 200GPa, H = 25GPa, ${}^0\sigma_Y$ = 250MPa
- ${}^n\sigma$ = 150MPa, ${}^n\varepsilon_p$ = 0.0001, $\Delta\varepsilon$ = 0.002
- Yield stress: ${}^n\sigma_Y = {}^0\sigma_Y + H\,{}^n\varepsilon_p = 252.5\text{MPa}$
  - Material is elastic at t_n
- Trial stress: $\Delta\,{}^{tr}\sigma = E\Delta\varepsilon = 400\text{MPa}$

$${}^{tr}\sigma = {}^n\sigma + \Delta\,{}^{tr}\sigma = 550\text{MPa}$$

Now material is plastic

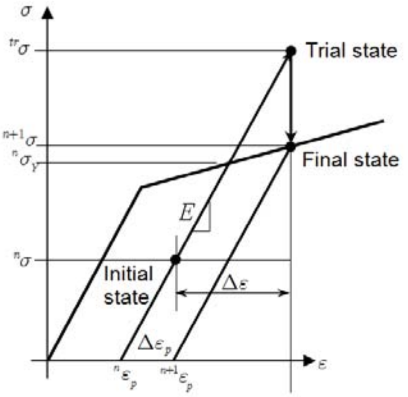
- Plastic consistency condition

$$\Delta\varepsilon_p = \frac{{}^{tr}f}{E+H} = 1.322 \times 10^{-3}$$

- State update

$${}^{n+1}\sigma = {}^{tr}\sigma - \mathrm{sgn}({}^{tr}\sigma)E\Delta\varepsilon_p = 285.6\text{MPa}$$

$${}^{n+1}\varepsilon_p = {}^n\varepsilon_p + \Delta\varepsilon_p = 1.422 \times 10^{-3}$$



Kinematic Hardening Model

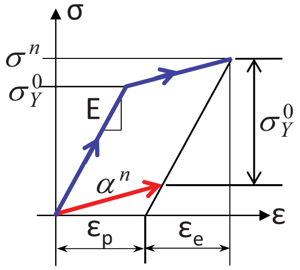
- Yield strength remains constant, but the center of elastic region moves parallel to the hardening curve
- Effective stress is defined using the **shifted stress**

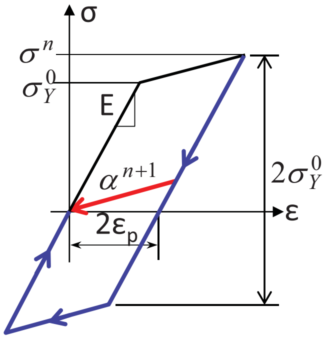
$$\boxed{\eta = \sigma - \alpha}$$

- Use the center of elastic domain as an evolution variable

$$\boxed{\alpha^{n+1} = \alpha^n + \mathrm{sgn}(\eta) H \Delta\varepsilon_p}$$

Back stress





State Determination (Kinematic Hardening)

- Given: Material properties and state at increment n:

$$(\Delta\varepsilon, E, H, \sigma_Y^0, \sigma^n, \alpha^n, \varepsilon_p^n)$$

- Elastic predictor

$$\boxed{\sigma^{tr} = \sigma^n + E\Delta\varepsilon, \quad \alpha^{tr} = \alpha^n, \quad \eta^{tr} = \sigma^{tr} - \alpha^{tr}}$$

- Check yield status

Trial yield function

$$\boxed{f^{tr} = \left|\eta^{tr}\right| - \sigma_Y^0}$$

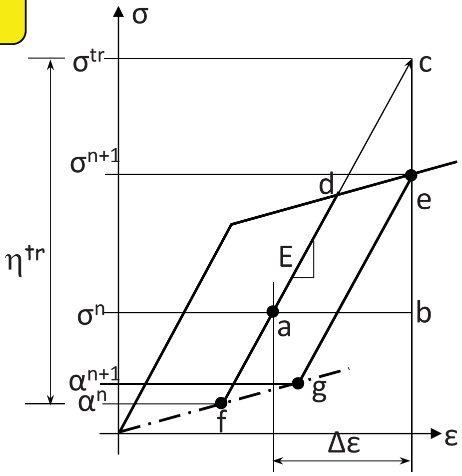
- If $f^{tr} \le 0$, material is elastic

$$\boxed{\sigma^{n+1} = \sigma^{tr}}$$

Either initial elastic region or unloading

- If $f^{tr} > 0$, material is plastic (yielding)

Either transition from elastic to plastic or continuous yielding



State Determination (Kinematic Hardening) cont.

- Updating formulas for stress, back stress & plastic strain

$$\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\eta^{tr})E\Delta\varepsilon_p \qquad \alpha^{n+1} = \alpha^{tr} + \mathrm{sgn}(\eta^{tr})H\Delta\varepsilon_p \qquad \varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p$$

- Plastic consistency condition
  - To determine unknown plastic strain increment
  - Stress must be on the yield surface during plastic loading

$$\boxed{f^{n+1} = \left|\eta^{n+1}\right| - \sigma_y^0 = 0}$$

$$\Rightarrow \left|\sigma^{tr} - \mathrm{sgn}(\eta^{tr})E\Delta\varepsilon_p - \alpha^{tr} - \mathrm{sgn}(\eta^{tr})H\Delta\varepsilon_p\right| - \sigma_y^0 = 0$$

$$\Rightarrow \left|\sigma^{tr} - \alpha^{tr}\right| - \sigma_y^0 - (E+H)\Delta\varepsilon_p = 0$$

$$\boxed{\Delta\varepsilon_p = \frac{\left|\eta^{tr}\right| - \sigma_y^n}{E+H} = \frac{f^{tr}}{E+H}}$$

%Note: the same formula with isotropic hardening model!!

Algorithm for Kinematic Hardening

- Given: $\Delta\varepsilon, E, H, \sigma_y^0, \sigma^n, \alpha^n, \varepsilon_p^n$

1. Trial state $\sigma^{tr} = \sigma^n + E\Delta\varepsilon$

$$\alpha^{tr} = \alpha^n$$

$$\eta^{tr} = \sigma^{tr} - \alpha^{tr}$$

$$f^{tr} = \left|\eta^{tr}\right| - \sigma_y^0$$

2. If $f^{tr} \le 0$ (elastic)
   - Remain elastic: $\boxed{\sigma^{n+1} = \sigma^{tr}, \quad \alpha^{n+1} = \alpha^n, \quad \varepsilon_p^{n+1} = \varepsilon_p^n}$; exit
3. If $f^{tr} > 0$ (plastic)
   a. Calculate plastic strain: $\boxed{\Delta\varepsilon_p = \dfrac{f^{tr}}{E+H}}$

   b. Update stress and plastic strain (store them for next increment)

$$\boxed{\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\eta^{tr})E\Delta\varepsilon_p} \qquad \boxed{\alpha^{n+1} = \alpha^n + \mathrm{sgn}(\eta^{tr})H\Delta\varepsilon_p}$$

$$\boxed{\varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p}$$

Ex) Elastoplastic Bar (Kinematic Hardening)

- $E = 200\text{GPa}$, $H = 25\text{GPa}$, ${}^{0}\sigma_y = 200\text{MPa}$
- ${}^{n}\sigma = 150\text{MPa}$, ${}^{n}\alpha = 50\text{MPa}$, $\Delta\varepsilon = -0.002$
- Since ${}^{n}\eta = {}^{n}\sigma - {}^{n}\alpha = 100 < {}^{0}\sigma_y$, elastic state at t_n
- Trial stress:

$$\Delta\, {}^{tr}\sigma = E\Delta\varepsilon = -400\text{MPa}, \quad {}^{tr}\sigma = {}^{n}\sigma + \Delta\, {}^{tr}\sigma = -250\text{MPa}$$

$$ {}^{tr}\alpha = {}^{n}\alpha = 50\text{MPa}, \quad {}^{tr}\eta = {}^{tr}\sigma - {}^{tr}\alpha = -300\text{MPa}$$

- Since ${}^{tr}f = \left|{}^{tr}\eta\right| - {}^{0}\sigma_y > 0$, material yields in compression
- Plastic strain

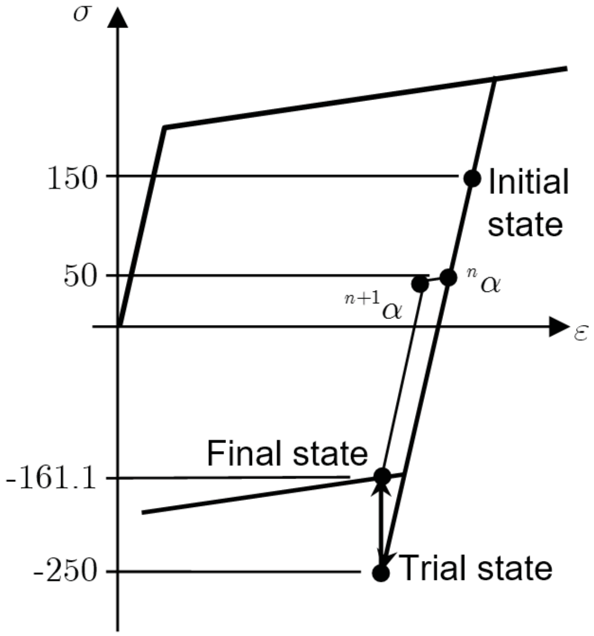
$$\Delta\varepsilon_p = \frac{{}^{tr}f}{E+H} = 0.444 \times 10^{-3}$$

- State update

$$ {}^{n+1}\sigma = {}^{tr}\sigma - \text{sgn}({}^{tr}\eta)E\Delta\varepsilon_p = -161.1\text{MPa}$$

$$ {}^{n+1}\alpha = {}^{tr}\alpha - \text{sgn}({}^{tr}\eta)H\Delta\varepsilon_p = 38.9\text{MPa}$$

Ex) Elastoplastic Bar (Kinematic Hardening)



Combined Hardening Model

- Baushinger effect
  - conditions where the yield strength of a metal decreases when the direction of strain is changed
  - Common for most polycrystalline metals
  - Related to the dislocation structure in the cold worked metal. As deformation occurs, the dislocations will accumulate at barriers and produce dislocation pile-ups and tangles.
- Numerical modeling of Baushinger effect
  - Modeled as a combined kinematic and isotropic hardening

$$\boxed{\sigma_y^{n+1} = \sigma_y^n + (1-\beta)H\Delta\varepsilon_p} \qquad \boxed{\alpha^{n+1} = \alpha^n + \mathrm{sgn}(\eta)\beta H\Delta\varepsilon_p}$$

$$0 \le \beta \le 1$$

β = 0: isotropic hardening

β = 1: kinematic hardening

Combined Hardening Model cont.

- Trial state

$$\sigma^{tr} = \sigma^n + E\Delta\varepsilon$$
$$\alpha^{tr} = \alpha^n$$
$$\eta^{tr} = \sigma^{tr} - \alpha^{tr}$$
$$f^{tr} = \left|\eta^{tr}\right| - \sigma_y^n$$

- Stress update

$$\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\eta^{tr})E\Delta\varepsilon_p$$
$$\alpha^{n+1} = \alpha^{tr} + \mathrm{sgn}(\eta^{tr})\beta H\Delta\varepsilon_p$$
$$\sigma_y^{n+1} = \sigma_y^n + (1-\beta)H\Delta\varepsilon_p$$

- Show that the plastic increment is the same

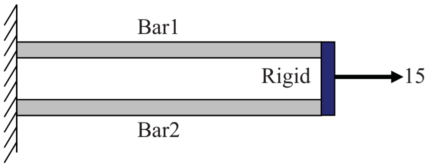
$$\boxed{\Delta\varepsilon_p = \frac{f^{tr}}{E+H}}$$

MATLAB Program combHard1D

```
%
% 1D Linear combined isotropic/kinematic hardening model
%
function [stress, alpha, ep]=combHard1D(mp, deps, stressN, alphaN, epN)
% Inputs:
% mp = [E, beta, H, Y0];
% deps = strain increment
% stressN = stress at load step N
% alphaN = back stress at load step N
% epN = plastic strain at load step N
%
E=mp(1); beta=mp(2); H=mp(3); Y0=mp(4);          %material properties
ftol = Y0*1E-6;                                  %tolerance for yield
stresstr = stressN + E*deps;                     %trial stress
etatr = stresstr - alphaN;                       %trial shifted stress
fyld = abs(etatr) - (Y0+(1-beta)*H*epN);         %trial yield function
if fyld < ftol                                   %yield test
    stress = stresstr; alpha = alphaN; ep = epN;%trial states are final
    return;
else
    dep = fyld/(E+H);                            %plastic strain increment
end
stress = stresstr - sign(etatr)*E*dep;           %updated stress
alpha = alphaN + sign(etatr)*beta*H*dep;         %updated back stress
ep = epN + dep;                                  %updated plastic strain
return;
```

Ex) Two bars in parallel

- Bar 1: A = 0.75, E = 10000, E_t = 1000, ${}^0\sigma_Y$ = 5, kinematic
- Bar 2: A = 1.25, E = 5000, E_t = 500, ${}^0\sigma_Y$ = 7.5, isotropic
- MATLAB program



```
%
% Example 4.5 Two elastoplastic bars in parallel
%
E1=10000; Et1=1000; sYield1=5;
E2=5000;  Et2=500;  sYield2=7.5;
mp1 = [E1, 1, E1*Et1/(E1-Et1), sYield1];
mp2 = [E2, 0, E2*Et2/(E2-Et2), sYield2];
nS1 = 0; nA1 = 0; nep1 = 0;
nS2 = 0; nA2 = 0; nep2 = 0;
A1 = 0.75; L1 = 100;
A2 = 1.25; L2 = 100;
tol = 1.0E-5; u = 0; P = 15; iter = 0;
Res = P - nS1*A1 - nS2*A2;
Dep1 = E1; Dep2 = E2;
conv = Res^2/(1+P^2);
fprintf('\niter        u       S1       S2       A1       A2');
fprintf('      ep1       ep2    Residual');
fprintf('\n %3d  %7.4f %7.3f %7.3f %7.3f %7.3f %8.6f %8.6f %10.3e',...
      iter,u,nS1,nS2,nA1,nA2,nep1,nep2,Res);
```

Ex) Two bars in parallel cont.

```
while conv > tol && iter < 20
  delu = Res / (Dep1*A1/L1 + Dep2*A2/L2);
  u = u + delu;
  delE = delu / L1;
  [Snew1, Anew1, epnew1]=combHard1D(mp1,delE,nS1,nA1,nep1);
  [Snew2, Anew2, epnew2]=combHard1D(mp2,delE,nS2,nA2,nep2);
  Res = P - Snew1*A1 - Snew2*A2;
  conv = Res^2/(1+P^2);
  iter = iter + 1;
  Dep1 = E1; if epnew1 > nep1; Dep1 = Et1; end
  Dep2 = E2; if epnew2 > nep2; Dep2 = Et2; end
  nS1 = Snew1; nA1 = Anew1; nep1 = epnew1;
  nS2 = Snew2; nA2 = Anew2; nep2 = epnew2;
  fprintf('\n %3d  %7.4f %7.3f %7.3f %7.3f %7.3f %8.6f %8.6f %10.3e',...
      iter,u,nS1,nS2,nA1,nA2,nep1,nep2,Res);
end
```

Iteration	u	s_1	s_2	e_{p1}	e_{p2}	Residual
0	0.0000	0.000	0.000	0.000000	0.000000	1.50E+1
1	0.1091	5.591	5.455	0.000532	0.000000	3.99E+0
2	0.1661	6.161	7.580	0.001045	0.000145	9.04E−1
3	0.2318	6.818	7.909	0.001636	0.000736	0.00E+0

Summary

- Plastic deformation depends on load-history and its information is stored in plastic strain
- Stress only depends on elastic strain
- Isotropic hardening increases the elastic domain, while kinematic hardening maintains the size of elastic domain but moves the center of it
- Major issue in elastoplastic analysis is to decompose the strain into elastic and plastic parts
- Algorithmic tangent stiffness is consistent with the state determination algorithm
- State determination is composed of (a) elastic trial and (b) plastic return mapping

1D Elastoplastic Analysis Using ABAQUS

- Material Card

```
*MATERIAL,NAME=ALLE
*ELASTIC
200.E3,.3
*PLASTIC
200.,0.
220.,.0009
220.,.0029
```

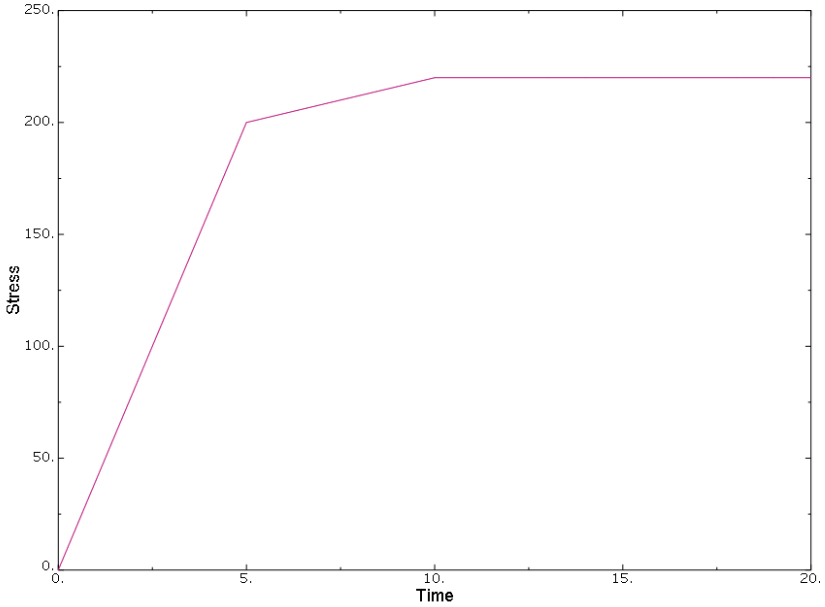
Plastic strain

Yield stress

1D Elastoplastic Analysis Using ABAQUS

```
*HEADING
UniaxialPlasticity
*NODE,NSET=ALLN
1,0.,0.,0.
2,1.,0.,0.
3,1.,1.,0.
4,0.,1.,0.
5,0.,0.,1.
6,1.,0.,1.
7,1.,1.,1.
8,0.,1.,1.
*ELEMENT,TYPE=C3D8,ELSET=ALLE
1,1,2,3,4,5,6,7,8
*SOLID SECTION,ELSET=ALLE,MATERIAL=ALLE
*MATERIAL,NAME=ALLE
*ELASTIC
200.E3,.3
*PLASTIC
200.,0.
220.,.0009
220.,.0029
*BOUNDARY
1,PINNED
2,2
5,2
6,2
4,1
5,1
8,1
2,3
3,3
4,3
*STEP,INC=20
*STATIC,DIRECT
1.,20.
*BOUNDARY
7,3,,.004
5,3,,.004
6,3,,.004
8,3,,.004
*EL PRINT,FREQ=1
S,
E,
EP,
*NODE PRINT
U,RF
*END STEP
```

- Stress Curve



4.3

Multi-Dimensional Elastoplastic Analysis

Goals

- Understand failure criteria, equivalent stress, and effective strain
- Understand how 1D tension test data can be used for determining failure of 3D stress state
- Understand deviatoric stress and strain
- Understand the concept of elastic domain and yield surface
- Understand hardening models
- Understand evolution of plastic variables along with that of the yield surface

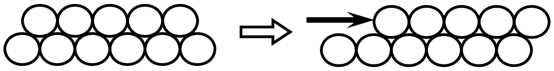
---

Multi-Dimensional Elastoplasticity

- How can we generalize 1D stress state (σ_{11}) to 3D state (6 components)?
  - Need scalar measures of stress and strain to compare with 1D test
  - **Equivalent stress & effective strain**
  - Key ingredients: **yield criteria**, **hardening model**, stress-strain relation
- We will assume small (infinitesimal) strains
- **Rate independent elastoplasticity**- independent of strain rate
- Von Mises yield criterion with associated hardening model is the most popular

Failure Criteria

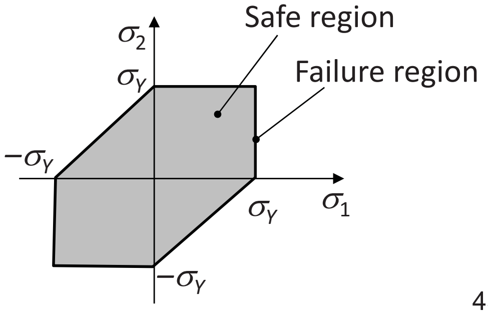
- Material yields due to relative sliding in lattice structures



- Sliding preserves volume ⇒ plastic deformation is related to shear or deviatoric part
- **Tresca (1864, max. shear stress)**
  - Material fails when max. shear stress reaches that of tension test
  - Tension test: yield at $\sigma_1 = \sigma_Y$, $\sigma_2 = \sigma_3 = 0$

$$\boxed{\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \le \tau_Y = \frac{\sigma_Y}{2}}$$

  - Yielding occurs when $\tau_{max} = \tau_Y$



Failure Criteria cont.

- Distortion Energy Theory (von Mises)
  - Material fails when distortion energy reaches that of tension test

$$\boxed{U_d \le U_d(\text{tension test})}$$

  - We need preliminaries before deriving U_d
- Volumetric stress and mean strain

$$\sigma_m = \tfrac{1}{3}\mathrm{tr}(\sigma) = \tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$$\varepsilon_m = \tfrac{1}{3}\mathrm{tr}(\varepsilon) = \tfrac{1}{3}\varepsilon_v = \tfrac{1}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

- Deviatoric stress and strain

$$\mathbf{s} = \sigma - \sigma_m \mathbf{1} = \mathbf{I}_{dev} : \sigma \qquad I_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$$

$$\mathbf{e} = \varepsilon - \varepsilon_m \mathbf{1} = \mathbf{I}_{dev} : \varepsilon \qquad \mathbf{I}_{dev} = \mathbf{I} - \tfrac{1}{3}\mathbf{1} \otimes \mathbf{1}$$

Failure Criteria cont.

- Example: Linear elastic material

$$\sigma = [\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}] : \varepsilon \equiv \mathbf{D} : \varepsilon$$

$$\begin{aligned} \sigma &= \lambda(3\varepsilon_m)\mathbf{1} + 2\mu(\mathbf{e} + \varepsilon_m \mathbf{1}) \\ &= \underbrace{(3\lambda + 2\mu)\varepsilon_m \mathbf{1}}_{\text{volumetric}} + \underbrace{2\mu \mathbf{e}}_{\text{deviatoric}} \end{aligned}$$

$$\boxed{\begin{aligned} \sigma_m &= (3\lambda + 2\mu)\varepsilon_m \\ \mathbf{s} &= 2\mu \mathbf{e} \end{aligned}}$$

Bulk modulus

$$K = \frac{3\lambda + 2\mu}{3}$$

- Distortion energy density

$$U = \tfrac{1}{2}\sigma : \varepsilon = \tfrac{1}{2}(\sigma_m \mathbf{1} + \mathbf{s}) : (\varepsilon_m \mathbf{1} + \mathbf{e}) = \tfrac{3}{2}\sigma_m \varepsilon_m + \tfrac{1}{2}\mathbf{s} : \mathbf{e}$$

$$\boxed{U_d = \tfrac{1}{2}\mathbf{s} : \mathbf{e} = \tfrac{1}{4\mu}\mathbf{s} : \mathbf{s}}$$

Failure Criteria cont.

- 1D Case

$$\sigma_{11} = \sigma \qquad \sigma_m = \tfrac{1}{3}\sigma \qquad \mathbf{s} = \sigma \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

$$U_d\big|_{1D} = \frac{1}{4\mu}\mathbf{s} : \mathbf{s} = \frac{1}{4\mu}\frac{2}{3}\sigma^2 = \frac{1}{6\mu}\sigma^2$$

- Material yields when

$$U_d = \frac{1}{4\mu}\mathbf{s} : \mathbf{s} = \frac{1}{6\mu}\sigma_Y^2 = U_d\big|_{1D}$$

- Let's define an **equivalent stress**

$$\boxed{\sigma_e = \sqrt{\tfrac{3}{2}\mathbf{s} : \mathbf{s}}}$$

von Mises stress

- Then, material yields when

$$\boxed{\sigma_e = \sigma_Y}$$

- **Stress can increase from zero to σ_Y, but cannot increase beyond that**

Equivalent Stress and Effective Strain

- Equivalent stress is the scalar measure of 3D stress state that can be compared with 1D stress from tension test
- Effective strain is the scalar measure of 3D strain state that makes conjugate with equivalent stress

$$U_d = \tfrac{1}{2}\mathbf{s} : \mathbf{e} = \tfrac{1}{2}\sigma_e e_e$$

$$U_d = \tfrac{1}{4\mu}\mathbf{s} : \mathbf{s} = \tfrac{1}{6\mu}\sigma_e^2 = \tfrac{1}{2}\sigma_e e_e$$

Effective strain

$$\boxed{e_e = \tfrac{1}{3\mu}\sigma_e = \tfrac{1}{3\mu}\sqrt{\tfrac{3}{2}\mathbf{s} : \mathbf{s}} = \tfrac{1}{3\mu}\sqrt{\tfrac{3}{2}2\mu\mathbf{e} : 2\mu\mathbf{e}} = \sqrt{\tfrac{2}{3}\mathbf{e} : \mathbf{e}}}$$

Equivalent Stress and Effective Strain cont.

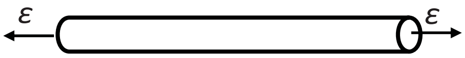
- 1D Case cont.

$$\varepsilon_{11} = \varepsilon \qquad \varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon \qquad \varepsilon_m = \frac{1-2\nu}{3}\varepsilon$$

$$\mathbf{e} = \frac{(1+\nu)\varepsilon}{3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{e} : \mathbf{e} = 6\left(\frac{(1+\nu)\varepsilon}{3}\right)^2$$

$$\boxed{e_e = \sqrt{\tfrac{2}{3}\mathbf{e} : \mathbf{e}} = \frac{2(1+\nu)}{3}\varepsilon}$$

Effective strain for 1D tension



Von Mises Criterion

- Material yields when $\sigma_e = \sigma_Y$

$$\sigma_e = \sqrt{\tfrac{3}{2}\mathbf{s}:\mathbf{s}} = \sqrt{3J_2}$$

$$J_2 = \tfrac{1}{2}\mathbf{s}:\mathbf{s}$$

2nd invariant of s

$$J_2 = \frac{1}{6}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2\right] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2$$

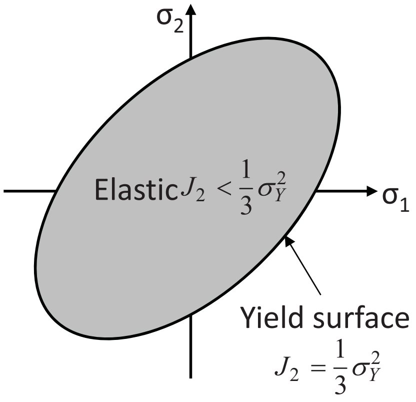
$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

In terms of principal stresses

- Yield criterion

$$\boxed{\sigma_e^2 - \sigma_Y^2 \equiv 3J_2 - \sigma_Y^2 = 0}$$

  - 1D test data σ_Y can be used for multi-dimensional stress state
  - Often called **J_2 plasticity model**



Von Mises Criterion cont.

- J_2: second invariant of s

$$J_2 = \tfrac{1}{2}[\mathbf{s}:\mathbf{s} - \mathrm{tr}(\mathbf{s})^2] = \tfrac{1}{2}\mathbf{s}:\mathbf{s}$$

- Von Mises yield function

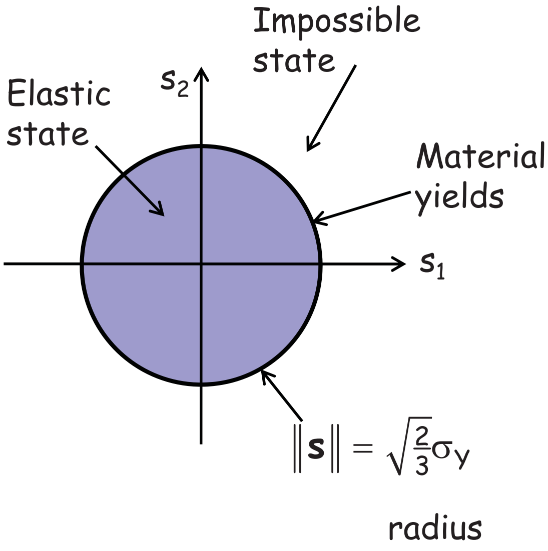
$$3J_2 - \sigma_Y^2 = 0$$

$$\Rightarrow \quad \tfrac{3}{2}\mathbf{s}:\mathbf{s} - \sigma_Y^2 = 0$$

$$\Rightarrow \quad \sqrt{\mathbf{s}:\mathbf{s}} - \sqrt{\tfrac{2}{3}}\sigma_Y = 0$$

$$\Rightarrow \quad \boxed{\|\mathbf{s}\| - \sqrt{\tfrac{2}{3}}\sigma_Y = 0}$$

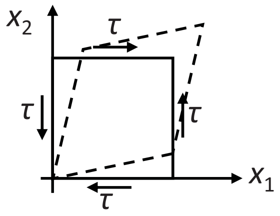
Yield function



Yield surface is circular in deviatoric stress space

Example

- Pure shear stress τ to yield



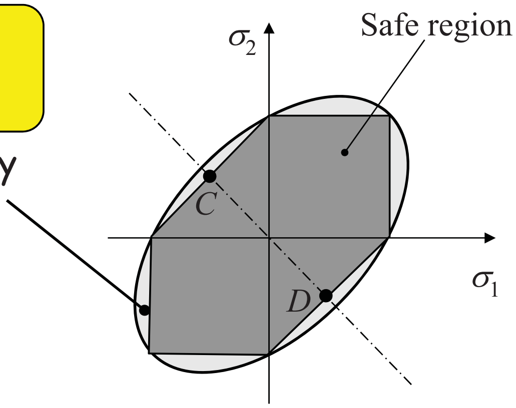
$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{s}$$

$$\|\mathbf{s}\| = \sqrt{\mathbf{s} : \mathbf{s}} = \sqrt{\tau^2 + \tau^2} = \tau\sqrt{2}$$

  - Yield surface:

$$\sqrt{2}\tau = \sqrt{\tfrac{2}{3}}\sigma_Y \quad \Rightarrow \quad \boxed{\tau = \frac{1}{\sqrt{3}}\sigma_Y}$$

  - Failure in max. shear stress theory
    Safe in distortion energy theory
  - Von Mises is more accurate, but
    Tresca is more conservative



Example

- Uniaxial tensile test

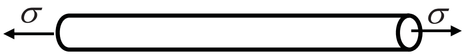
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} \tfrac{2}{3}\sigma & 0 & 0 \\ 0 & -\tfrac{1}{3}\sigma & 0 \\ 0 & 0 & -\tfrac{1}{3}\sigma \end{bmatrix}$$

$$\|\mathbf{s}\| = \sqrt{\tfrac{4}{9}\sigma^2 + \tfrac{1}{9}\sigma^2 + \tfrac{1}{9}\sigma^2} = \sqrt{\tfrac{2}{3}}\sigma$$

  - Yield surface

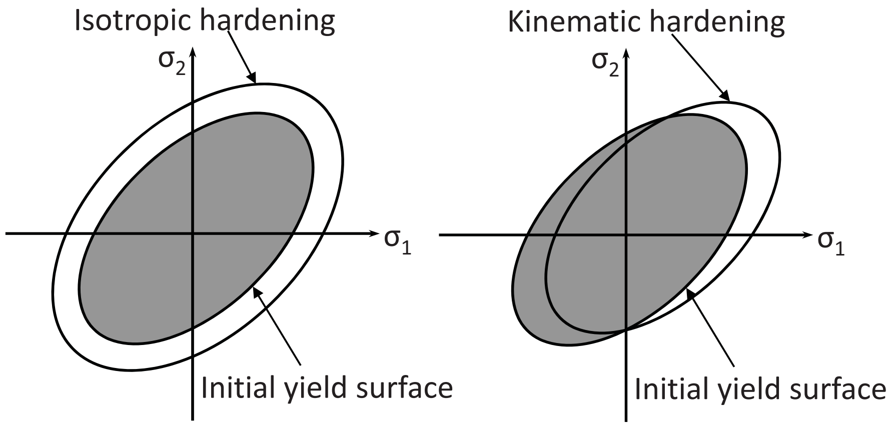
$$\sqrt{\tfrac{2}{3}}\sigma - \sqrt{\tfrac{2}{3}}\sigma_Y = 0 \quad \Rightarrow \quad \sigma = \sigma_Y$$

Consistent with uniaxial tension test



Hardening Model

- For many materials, the yield surface increases proportional to plastic deformation ⇒ **strain hardening**
- Isotropic hardening: Change in radius
- Kinematic hardening: Change in center



Hardening Model cont.

- Isotropic hardening model (linear)

$$\boxed{\sigma_Y = \sigma_Y^0 + He_p}$$

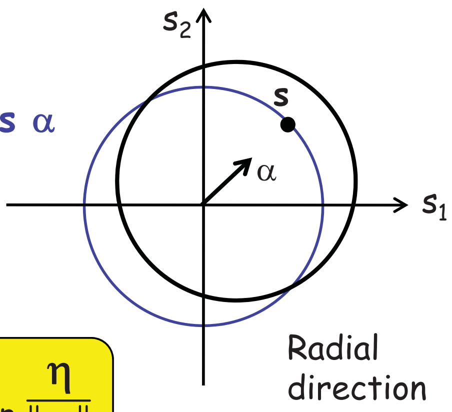
$H = \dfrac{\Delta\sigma}{\Delta e_p}$	Plastic modulus
e_p	Effective plastic strain
σ_Y^0	Initial yield stress

  - H = 0: elasto-perfectly-plastic material
- Kinematic hardening model (linear)
  - The center of yield surface : **back stress** α
  - **Shifted stress:** η **=** $\mathbf{s} - \alpha$

$$\boxed{\|\eta\| - \sqrt{\frac{2}{3}}\sigma_Y = 0}$$

  - α moves proportional to e_p

$$\boxed{\dot{\alpha} = \sqrt{\frac{2}{3}}H\dot{e}_p\frac{\eta}{\|\eta\|}}$$



Hardening Model cont.

- Combined Hardening
  - Many materials show both isotropic and kinematic hardenings
  - Introduce a parameter $\beta \in [0, 1]$ to consider this effect
  - **Baushinger effect**: The yield stress increases in one directional loading. But it decreases in the opposite directional load.
  - This is caused by dislocation pileups and tangles (back stress). When strain direction is changed, this makes the dislocations easy to move

$$\boxed{\|\eta\| - \sqrt{\tfrac{2}{3}}[\sigma_Y^0 + (1-\beta)He_p] = 0}$$

$$\boxed{\dot{\alpha} = \sqrt{\tfrac{2}{3}}\beta H\dot{e}_p \frac{\eta}{\|\eta\|}}$$

  - **Isotropic hardening: β = 0**
  - **Kinematic hardening: β = 1**

Ex) Uniaxial Bar with Hardening

- Calculate uniaxial stress s when e_p = 0.1, initial σ_Y = 400 MPa and H = 200 MPa (a) isotropic, (b) kinematic and (c) combined hardening with β = 0.5

$$\sigma = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix} \qquad \|\mathbf{s}\| = \sqrt{\tfrac{2}{3}}\sigma$$

a) Isotropic hardening

$$\|\mathbf{s}\| - \sqrt{\tfrac{2}{3}}(\sigma_Y^0 + He_p) = \sqrt{\tfrac{2}{3}}\sigma - \sqrt{\tfrac{2}{3}}(400 + 200 \times 0.1) = 0$$

$$\sigma = 420\,\text{MPa}$$

b) Kinematic hardening

$$\|\mathbf{s} - \alpha\| - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = 0$$

$$\Rightarrow \|\mathbf{s}\| - \|\alpha\| - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = \sqrt{\tfrac{2}{3}}\sigma - \sqrt{\tfrac{2}{3}}He_p - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = 0$$

$$\sigma = 420\,\text{MPa}$$

Ex) Uniaxial Bar with Hardening

c) Combined hardening

$$
\begin{aligned}
&\|\mathbf{s}-\alpha\| - \sqrt{\tfrac{2}{3}}\left[\sigma_y^0 + (1-\beta)He_p\right] \\
&= \|\mathbf{s}\| - \|\alpha\| - \sqrt{\tfrac{2}{3}}\left[\sigma_y^0 + (1-\beta)He_p\right] \\
&= \sqrt{\tfrac{2}{3}}\sigma - \sqrt{\tfrac{2}{3}}\beta He_p - \sqrt{\tfrac{2}{3}}\sigma_y^0 - \sqrt{\tfrac{2}{3}}(1-\beta)He_p \\
&= 0
\end{aligned}
$$

$$\sigma = \sigma_y^0 + He_p = (400 + 200 \times 0.1) = 420\text{MPa}$$

All three models yield the same stress (proportional loading)

Rate-Independent Elastoplasticity

- Additive decomposition

$\varepsilon = \varepsilon^e + \varepsilon^p \quad \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$ From small deformation assumption

- Strain energy (linear elastic)

$$W(\varepsilon^e) = \tfrac{1}{2}\varepsilon^e : \mathbf{D} : \varepsilon^e = \tfrac{1}{2}(\varepsilon - \varepsilon^p) : \mathbf{D} : (\varepsilon - \varepsilon^p)$$

- Stress (differentiating W w.r.t. strain)

$$\sigma = \frac{\partial W}{\partial \varepsilon^e} = \mathbf{D} : \varepsilon^e = \mathbf{D} : (\varepsilon - \varepsilon^p)$$

$$\dot{\sigma} = \mathbf{D} : (\dot{\varepsilon} - \dot{\varepsilon}^p)$$

$$\mathbf{D} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$$

Why we separate volumetric part from deviatoric part?

$$\mathbf{D} = (\lambda + \tfrac{2}{3}\mu)\mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}_{dev}$$

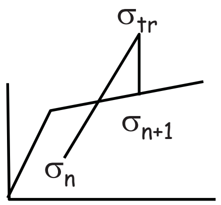
Rate-Independent Elastoplasticity cont.

- Stress cont.
  - Volumetric stress: $\dot{\sigma}_m = \frac{1}{3}\mathrm{tr}(\dot{\sigma}) = (\lambda + \frac{2}{3}\mu)\mathrm{tr}(\dot{\varepsilon}) = (3\lambda + 2\mu)\dot{\varepsilon}_m$
  - Deviatoric stress: $\dot{\mathbf{s}} = 2\mu(\dot{\mathbf{e}} - \dot{\mathbf{e}}^p)$

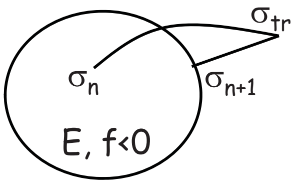
Why isn't this an elastic strain?

- Yield function
  - We will use von Mises, pressure insensitive yield function

$$\boxed{f(\eta, e_p) = \|\eta\| - \sqrt{\tfrac{2}{3}}\kappa(e_p) \le 0}$$



  - $\kappa(e_p)$: Radius of elastic domain
  - e_p: effective plastic strain
- Elastic domain (smooth, convex)



$$E = \left\{(\eta, e_p) \,\middle|\, f(\eta, e_p) \le 0\right\}$$

Rate-Independent Elastoplasticity cont.

- Flow rule (determine evolution of plastic strain)

$$\boxed{\dot{\varepsilon}^p = \dot{\gamma}\mathbf{r}(\sigma, \xi)}$$

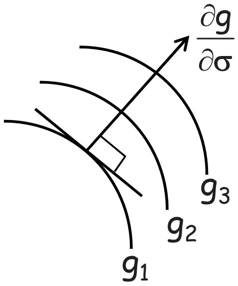
$\xi = (\alpha, e_p)$

Plastic variables

  - **Plastic consistency parameter** γ : γ > 0 (plastic), γ =0 (elastic)
- Flow potential $g(\sigma, \xi)$

$$\mathbf{r} = \frac{\partial g(\sigma, \xi)}{\partial \sigma} \quad \Rightarrow \quad \dot{\varepsilon}^p = \dot{\gamma}\frac{\partial g(\sigma, \xi)}{\partial \sigma}$$

  - Plastic strain increases in the normal direction to the flow potential



Rate-Independent Elastoplasticity cont.

- Associative flow rule
  - Flow potential = yield function

$$\boxed{\dot{\varepsilon}^p = \dot{\gamma}\frac{\partial f(\eta,\xi)}{\partial \eta}}$$

Unit deviatoric tensor normal to the yield surface

$$\frac{\partial f}{\partial \eta} = \frac{\partial \|\eta\|}{\partial \eta} = \frac{\partial \sqrt{\eta : \eta}}{\partial \eta} = \frac{\eta}{\sqrt{\eta : \eta}} = \frac{\eta}{\|\eta\|} = \mathbf{N}$$

$$\Rightarrow \boxed{\dot{\varepsilon}^p = \dot{\gamma}\frac{\partial f}{\partial \eta} = \dot{\gamma}\frac{\eta}{\|\eta\|} = \dot{\gamma}\mathbf{N}}$$

N determines the direction of plastic strain rate and $\dot{\gamma}$ determines the magnitude

---

Rate-Independent Elastoplasticity cont.

- Evolution of plastic variables (hardening model)
- Back stress α

$$\boxed{\dot{\alpha} = H_\alpha(e_p)\dot{\gamma}\frac{\partial f(\eta, e_p)}{\partial \eta} = H_\alpha(e_p)\dot{\gamma}\mathbf{N}}$$

Plastic modulus for kinematic hardening

- Effective plastic strain

$$\dot{e}^p = \sqrt{\frac{2}{3}\dot{\mathbf{e}}^p : \dot{\mathbf{e}}^p} = \sqrt{\frac{2}{3}}\left\|\dot{\mathbf{e}}^p\right\|$$

  - Note: plastic deformation only occurs in deviatoric components

$$\dot{\varepsilon}^p = \dot{\mathbf{e}}^p \qquad \left\|\dot{\varepsilon}^p\right\| = \left\|\dot{\gamma}\mathbf{N}\right\| = \dot{\gamma}$$

$$\boxed{\dot{e}^p = \sqrt{\frac{2}{3}}\dot{\gamma}}$$

$$\boxed{\begin{aligned} \xi &= (\alpha, e_p) \\ \Rightarrow \dot{\xi} &= \dot{\gamma}\mathbf{h}(\sigma, \xi) \end{aligned}}$$

Rate-Independent Elastoplasticity cont.

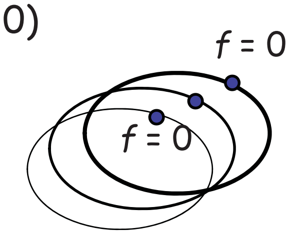
- Kuhn-Tucker conditions
  - The plastic consistency parameter must satisfy

$$\boxed{\dot{\gamma} \geq 0 \quad \dot{\gamma} f = 0 \quad f \leq 0}$$

1. Within elastic domain: $f < 0 \quad \dot{\gamma} = 0 \quad \Rightarrow \dot{\gamma} f = 0$
2. On the yield surface
   a. Elastic unloading $\dot{f} < 0 \quad \dot{\gamma} = 0 \quad \Rightarrow \dot{\gamma} f = 0$
   b. Neutral loading $\dot{f} = 0 \quad \dot{\gamma} = 0 \quad \Rightarrow \dot{\gamma} f = 0$
   c. Plastic loading (process attempt to violate $f \leq 0$)

$$\dot{f} = 0 \quad \dot{\gamma} > 0 \quad \Rightarrow \dot{\gamma} f = 0$$

$$\boxed{\Rightarrow \text{Equivalent to} \quad \dot{\gamma} \dot{f} = 0}$$



Classical Elastoplasticity

- Elastoplasticity boils down to how to calculate plasticity consistency parameter
- Classical plasticity uses the rate form of evolution relations to calculate it
- **Plastic consistency condition** $\boxed{\dot{\gamma} \dot{f} = 0}$
  - $\dot{\gamma}$ is only non-zero when continues plastic deformation

$$\dot{\gamma} > 0 \quad \dot{f}(\sigma, \xi) = 0$$

$$\dot{f}(\sigma, \xi) = \frac{\partial f}{\partial \sigma} : \dot{\sigma} + \frac{\partial f}{\partial \xi} \cdot \dot{\xi} = 0$$

$$\frac{\partial f}{\partial \sigma} : \mathbf{D} : (\dot{\varepsilon} - \dot{\varepsilon}^p) + \frac{\partial f}{\partial \xi} \cdot \dot{\gamma} \mathbf{h} = 0$$

$$\frac{\partial f}{\partial \sigma} : \mathbf{D} : \dot{\varepsilon} - \frac{\partial f}{\partial \sigma} : \mathbf{D} : \dot{\gamma} \mathbf{r} + \frac{\partial f}{\partial \xi} \cdot \dot{\gamma} \mathbf{h} = 0$$

Solve for plastic consistency parameter

Classical Elastoplasticity cont.

- Plastic consistency parameter

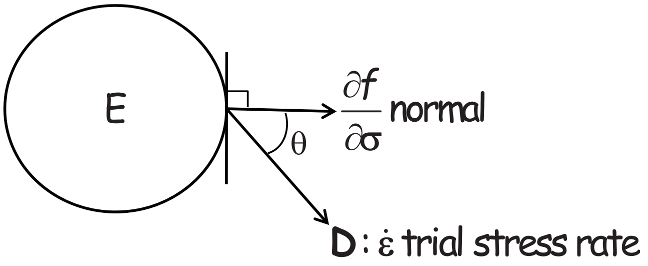
$$\boxed{\dot{\gamma} = \frac{\left\langle \frac{\partial f}{\partial \sigma} : \mathbf{D} : \dot{\varepsilon} \right\rangle}{\frac{\partial f}{\partial \sigma} : \mathbf{D} : \mathbf{r} - \frac{\partial f}{\partial \xi} \cdot \mathbf{h}}}$$

Assume the denominator is positive

$$\langle x \rangle = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

$$\dot{\gamma} > 0 \quad \Rightarrow \quad \frac{\partial f}{\partial \sigma} : \mathbf{D} : \dot{\varepsilon} > 0$$

$$\cos\theta = \frac{\frac{\partial f}{\partial \sigma} : \mathbf{D} : \dot{\varepsilon}}{\left\| \frac{\partial f}{\partial \sigma} \right\| \left\| \mathbf{D} : \dot{\varepsilon} \right\|}$$



θ < 90° : plastic loading
θ = 90° : neutral loading
θ > 90° : elastic unloading

Classical Elastoplasticity cont.

- Elastoplastic tangent stiffness (when $\dot{\gamma} > 0$)

$$\dot{\sigma} = \mathbf{D} : (\dot{\varepsilon} - \dot{\varepsilon}^p)$$

$$\dot{\sigma} = \mathbf{D} : \dot{\varepsilon} - \mathbf{D} : \dot{\gamma}\mathbf{r} = \mathbf{D} : \dot{\varepsilon} - \mathbf{D} : \mathbf{r} \frac{\left\langle \frac{\partial f}{\partial \sigma} : \mathbf{D} : \dot{\varepsilon} \right\rangle}{\frac{\partial f}{\partial \sigma} : \mathbf{D} : \mathbf{r} - \frac{\partial f}{\partial \xi} \cdot \mathbf{h}}$$

$$\dot{\sigma} = \left[\mathbf{D} - \frac{\left\langle \mathbf{D} : \mathbf{r} \otimes \frac{\partial f}{\partial \sigma} : \mathbf{D} \right\rangle}{\frac{\partial f}{\partial \sigma} : \mathbf{D} : \mathbf{r} - \frac{\partial f}{\partial \xi} \cdot \mathbf{h}} \right] : \dot{\varepsilon}$$

$$\boxed{\mathbf{D}^{ep} = \mathbf{D} - \frac{\left\langle \mathbf{D} : \mathbf{r} \otimes \frac{\partial f}{\partial \sigma} : \mathbf{D} \right\rangle}{\frac{\partial f}{\partial \sigma} : \mathbf{D} : \mathbf{r} - \frac{\partial f}{\partial \xi} \cdot \mathbf{h}}}$$

Elastoplastic tangent operator

In general, it is not symmetric, but for associative flow rule, it is

Nonlinear Hardening Models

- Nonlinear kinematic hardening model

$$\dot{\alpha} = H(e_p)\dot{\mathbf{e}}^p \quad H(e_p) = H_0 \exp\left(-\frac{e_p}{e_p^\infty}\right)$$

Saturated hardening

- Nonlinear isotropic hardening model

$$\kappa(e_p) = \sigma_Y^0 + (\sigma_Y^\infty - \sigma_Y^0)\left[1 - \exp(-e_p / e_p^\infty)\right]$$

Example: Linear hardening model

- Linear combined hardening model, associative flow rule
- 5 params: 2 elastic (λ, μ) and 3 plastic (β, H, $\sigma_Y{}^0$) variables

$$\kappa(e_p) = \sigma_Y^0 + (1-\beta)He_p \qquad \dot{\alpha} = \tfrac{2}{3}\beta H\dot{\mathbf{e}}^p$$

- Plastic consistency parameter

$$f(\mathbf{s},\alpha,e_p) = \left\|\mathbf{s}-\alpha\right\| - \sqrt{\tfrac{2}{3}}[\sigma_Y^0 + (1-\beta)He_p] = 0$$

$$\dot{f} = \frac{\partial f}{\partial \mathbf{s}} : \dot{\mathbf{s}} + \frac{\partial f}{\partial \alpha} : \dot{\alpha} + \frac{\partial f}{\partial e_p}\dot{e}_p = \mathbf{N} : \dot{\mathbf{s}} - \mathbf{N} : \dot{\alpha} - \sqrt{\tfrac{2}{3}}(1-\beta)H\dot{e}_p = 0$$

$$\dot{\mathbf{s}} = 2\mu\left(\dot{\mathbf{e}} - \dot{\mathbf{e}}^p\right) = 2\mu\dot{\mathbf{e}} - 2\mu\gamma\mathbf{N}$$

$$\dot{\alpha} = \tfrac{2}{3}\beta H\dot{\mathbf{e}}^p = \tfrac{2}{3}\beta H\gamma\mathbf{N}$$

$$\dot{e}_p = \sqrt{\tfrac{2}{3}}\gamma$$

Example: Linear hardening model cont.

- Plastic consistency parameter cont.

$$\dot{f} = 2\mu\mathbf{N}:\dot{\varepsilon} - 2\mu\gamma\mathbf{N}:\mathbf{N} - \tfrac{2}{3}\beta H\gamma\mathbf{N}:\mathbf{N} - \tfrac{2}{3}(1-\beta)H\gamma = 0$$

$$\mathbf{N}:\mathbf{N} = 1$$

$$\mathbf{N}:\dot{\boldsymbol{e}} = \mathbf{N}:\dot{\varepsilon}$$

$$\gamma = \frac{2\mu\mathbf{N}:\dot{\varepsilon}}{2\mu + \tfrac{2}{3}H}$$

  - No iteration is required

- Elastoplastic tangent stiffness

$$\dot{\sigma} = \mathbf{D}:\dot{\varepsilon} - \mathbf{D}:\dot{\varepsilon}^{p} = \mathbf{D}:\dot{\varepsilon} - \dot{\gamma}\mathbf{D}:\mathbf{N}$$

$$\mathbf{D} = (\lambda + \tfrac{2}{3}\mu)\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}_{dev}$$

$$\mathbf{D}:\mathbf{N} = 2\mu\mathbf{N}$$

$$\dot{\sigma} = \mathbf{D}:\dot{\varepsilon} - 2\mu\mathbf{N}\frac{2\mu\mathbf{N}:\dot{\varepsilon}}{2\mu + \tfrac{2}{3}H} = \underbrace{\left[\mathbf{D} - \frac{4\mu^{2}}{2\mu + \tfrac{2}{3}H}\mathbf{N}\otimes\mathbf{N}\right]}_{\mathbf{D}^{ep}}:\dot{\varepsilon}$$

Ex) Plastic Deformation of a Bar

E	μ	ν	σ_y	H	β
2.4GPa	1.0GPa	0.2	300MPa	100MPa	0.3

- At t_n: purely elastic, σ_{11} = 300 Mpa
- At t_{n+1}: $\Delta\varepsilon_{11}$ = 0.1, determine stress and plastic variables
- At t_n:

$$\sigma = \begin{bmatrix} 300 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\text{MPa}, \quad \mathbf{s} = \begin{bmatrix} 200 & 0 & 0 \\ 0 & -100 & 0 \\ 0 & 0 & -100 \end{bmatrix}\text{MPa}$$

- Strain increments

$$\Delta\varepsilon = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & -0.02 & 0 \\ 0 & 0 & -0.02 \end{bmatrix}, \quad \Delta\mathbf{e} = \begin{bmatrix} 0.08 & 0 & 0 \\ 0 & -0.04 & 0 \\ 0 & 0 & -0.04 \end{bmatrix}$$

Ex) Plastic Deformation of a Bar

- Purely elastic at t_n: ${}^n\alpha = 0$, ${}^n e_p = 0$. ${}^n\eta = {}^n\mathbf{s} - {}^n\alpha = {}^n\mathbf{s}$
- Trial states:

$$ {}^{tr}\eta = {}^{tr}\mathbf{s} = {}^{n}\mathbf{s} + 2\mu\Delta\mathbf{e} = \begin{bmatrix} 360 & 0 & 0 \\ 0 & -180 & 0 \\ 0 & 0 & -180 \end{bmatrix} \text{MPa} $$

$$ \left\| {}^{tr}\eta \right\| = \sqrt{360^2 + 180^2 + 180^2} = 180\sqrt{6}\text{MPa} $$

$$ \mathbf{N} = \frac{{}^{tr}\eta}{\left\| {}^{tr}\eta \right\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} $$

- Yield function

$$ f({}^{tr}\eta, {}^{tr}e_p) = \left\| {}^{tr}\eta \right\| - \sqrt{\tfrac{2}{3}}\kappa({}^{n}e_p) = 180\sqrt{6} - 300\sqrt{\tfrac{2}{3}} = 80\sqrt{6} > 0 $$

Plastic state!

Ex) Plastic Deformation of a Bar

- Plastic consistency parameter

$$ \gamma = \frac{2\mu\mathbf{N} : \Delta\varepsilon}{2\mu + \frac{2}{3}H} = 0.0948 $$

- Update stress and plastic variables

$$ {}^{n+1}\sigma = {}^{n}\sigma + \mathbf{D} : \Delta\varepsilon - 2\mu\gamma\mathbf{N} = \begin{bmatrix} 385.2 & 0 & 0 \\ 0 & 77.4 & 0 \\ 0 & 0 & 77.4 \end{bmatrix} \text{MPa} $$

No equilibrium!!

$$ {}^{n+1}\alpha = {}^{n}\alpha + \tfrac{2}{3}\beta H\gamma\mathbf{N} = \begin{bmatrix} 1.54 & 0 & 0 \\ 0 & -0.77 & 0 \\ 0 & 0 & -0.77 \end{bmatrix} \text{MPa} $$

$$ {}^{n+1}e_p = {}^{n}e_p + \sqrt{\tfrac{2}{3}}\gamma = 0.0774 $$

Numerical Integration

- Plastic evolution is given in the **rate form**
- We will use **backward Euler method** to integrate it

$$\dot{y} = f(t, y) \qquad \frac{y_{n+1} - y_n}{\Delta t} = f(t_{n+1} + y_{n+1})$$

$$\boxed{y_{n+1} = y_n + \Delta t \cdot f(t_{n+1} + y_{n+1})}$$

A-stable
Stable for all Δt

- Assumptions
  - We assume that all variables are known at load step n: σ^n, ξ^n
  - At the current time n+1, $\Delta\mathbf{u}$ or $\Delta\varepsilon$ is given
- We will use 2-step procedure
  1. Predictor: elastic trial
  2. Corrector: plastic return mapping (projection onto the yield surface)

Numerical Integration cont.

1. Elastic predictor

$$\mathbf{s}^{tr} = \mathbf{s}^n + 2\mu\underset{\substack{\uparrow \\ \text{dev. inc. strain}}}{\Delta\mathbf{e}} \qquad \underbrace{\alpha^{tr} = \alpha^n \qquad e_p^{tr} = e_p^n}_{\text{No plasticity}}$$

- Shifted stress: $\eta^{tr} = \mathbf{s}^{tr} - \alpha^{tr}$
- Yield function: $f(\eta^{tr}, e_p^n) = \left\| \eta^{tr} \right\| - \sqrt{\frac{2}{3}}\kappa(e_p^n)$

2. Plastic corrector
   1. If f < 0 (within the elastic domain)

$$\boxed{\mathbf{s}^{n+1} = \mathbf{s}^{tr} \qquad \alpha^{n+1} = \alpha^{tr} \qquad e_p^{n+1} = e_p^{tr}}$$

- Exit

Numerical Integration cont.

2. Plastic corrector cont.

   2. If f > 0 (return mapping to yield surface)

$$\mathbf{s}^{n+1} = \mathbf{s}^{tr} - 2\mu \boxed{\Delta\varepsilon_p} \xrightarrow{\text{unknown}} \Delta\varepsilon_p = \Delta\gamma\mathbf{N}$$

$$\mathbf{s}^{n+1} = \mathbf{s}^{tr} - 2\mu\Delta\gamma\mathbf{N}$$

$$\alpha^{n+1} = \alpha^{tr} - H_\alpha\Delta\gamma\mathbf{N}$$

$$\eta^{n+1} = \mathbf{s}^{n+1} - \alpha^{n+1} = \eta^{tr} - (2\mu + H_\alpha)\Delta\gamma\mathbf{N}$$

So far, unknowns are $\Delta\gamma$ and $\mathbf{N} = \left\|\eta^{n+1}\right\| / \eta^{n+1}$

- Trial direction is parallel to final direction

$$\eta^{n+1} = \eta^{tr} + \phi\eta^{n+1} \quad \Rightarrow \quad \eta^{tr} \parallel \eta^{n+1}$$

$$\boxed{\mathbf{N} = \frac{\eta^{n+1}}{\left\|\eta^{n+1}\right\|} = \frac{\eta^{tr}}{\left\|\eta^{tr}\right\|}}$$

Known from trial state

So, everything boils down to $\Delta\gamma$

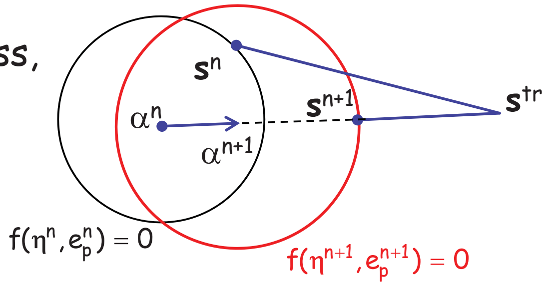
Numerical Integration cont.

2. Plastic corrector cont.

   - Now the plastic consistency parameter is only unknown!!
   - How to compute: **stress must stay on the yield surface**

$$\boxed{f(\eta^{n+1}, e_p^{n+1}) = 0}$$

   - While projecting the trial stress, the yield surface also varies
   - **But, both happen in the same direction N**



$$f(\eta^{n+1}, e_p^{n+1}) = \left\|\eta^{n+1}\right\| - \sqrt{\tfrac{2}{3}}\kappa(e_p^{n+1}) = 0$$

$$\left\|\eta^{n+1}\right\| = \left\|\eta^{tr} - (2\mu + H_\alpha)\Delta\gamma\mathbf{N}\right\| = \left\|\eta^{tr}\right\| - (2\mu + H_\alpha)\Delta\gamma$$

Numerical Integration cont.

2. Plastic corrector cont.

- Plastic consistency condition

$$\boxed{\left\|\eta^{tr}\right\| - \left(2\mu + H_\alpha(e_p^{n+1})\right)\Delta\gamma - \sqrt{\tfrac{2}{3}}\kappa(e_p^{n+1}) = 0}$$

- Nonlinear (scalar) equation w.r.t. $\Delta\gamma$ $\qquad e_p^{n+1} = e_p^n + \sqrt{\tfrac{2}{3}}\Delta\gamma$
- Use Newton-Raphson method (start with $\Delta\gamma = 0,\; e_p^{n+1} = e_p^n$)

$$f = \left\|\eta^{tr}\right\| - \left(2\mu + H_\alpha(e_p^{n+1})\right)\Delta\gamma - \sqrt{\tfrac{2}{3}}\kappa(e_p^{n+1})$$

$$\frac{df}{d\gamma} = -(2\mu + H_\alpha) - \sqrt{\frac{2}{3}}\frac{dH_\alpha}{de_p}\Delta\gamma - \frac{2}{3}\frac{d\kappa}{de_p}$$

$$\Delta\gamma = \Delta\gamma - \frac{f}{df\,/\,d\gamma}$$

$$e_p^{n+1} = e_p^{n+1} + \sqrt{\tfrac{2}{3}}\Delta\gamma$$

- Stop when f ~ 0

Numerical Integration cont.

- When N-R iteration is converged, update stress

$$\mathbf{s}^{n+1} = \mathbf{s}^n + 2\mu\Delta\mathbf{e} - 2\mu\Delta\gamma\mathbf{N}$$

$$\alpha^{n+1} = \alpha^n + H_\alpha\Delta\gamma\mathbf{N}$$

$$e_p^{n+1} = e_p^n + \sqrt{\tfrac{2}{3}}\Delta\gamma$$

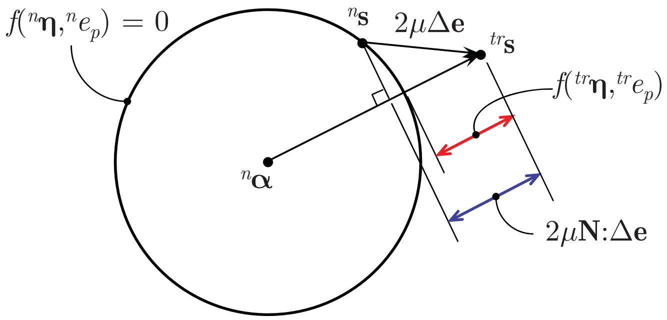
$$\sigma^{n+1} = \sigma^n + \underbrace{\mathbf{D} : \Delta\varepsilon - 2\mu\Delta\gamma\mathbf{N}}_{\mathbf{D}^{ep} : \Delta\varepsilon}$$

Tangent operator

Difference from the Rate Form

- Rate form (linear hardening)

$$\gamma = \frac{2\mu \mathbf{N} : \dot{\varepsilon}}{2\mu + \frac{2}{3}H}$$



- Incremental form

$$\Delta\gamma = \frac{\left\|\eta^{tr}\right\| - \sqrt{\frac{2}{3}}\kappa(e_p^n)}{2\mu + \frac{2}{3}H}$$

- Two formulations are equivalent when
  1. The material is in the plastic state at tn
  2. $\Delta\mathbf{e}$ is parallel to ${}^n\eta$
- When time increment is very small, these two requirements are satisfied

---

Consistent Tangent Operator

- Consistent tangent operator – tangent operator that is consistent with numerical integration algorithm

$$\underset{\substack{\uparrow \\ \text{Continuum tangent operator}}}{\mathbf{D}^{ep}} = \frac{\partial\dot{\sigma}}{\partial\dot{\varepsilon}} \qquad \underset{\substack{\uparrow \\ \text{Consistent tangent operator}}}{\mathbf{D}^{alg}} = \frac{\partial\Delta\sigma}{\partial\Delta\varepsilon}$$

- Differentiate stress update equation

$$\Delta\sigma = \mathbf{D} : \Delta\varepsilon - 2\mu\Delta\gamma\mathbf{N}$$

$$\frac{\partial\Delta\sigma}{\partial\Delta\varepsilon} = \mathbf{D} - 2\mu\mathbf{N} \otimes \underset{\substack{\uparrow \\ (1)}}{\frac{\partial\Delta\gamma}{\partial\Delta\varepsilon}} - 2\mu\Delta\gamma \underset{\substack{\uparrow \\ (2)}}{\frac{\partial\mathbf{N}}{\partial\Delta\varepsilon}}$$

Consistent Tangent Operator cont

- Term (1) $f(\eta^{n+1}, e_p^{n+1}) = f(\eta^n, e_p^n) = 0 \quad \Rightarrow \quad \dfrac{\partial f}{\partial \Delta\varepsilon} = 0$

$$\frac{\partial f}{\partial \Delta\varepsilon} = \frac{\partial}{\partial \Delta\varepsilon}\left[\left\|\eta^{tr}\right\| - (2\mu + H_\alpha(e_p^{n+1}))\Delta\gamma - \sqrt{\tfrac{2}{3}}\kappa(e_p^{n+1})\right] = 0$$

$$\frac{\partial\left\|\eta^{tr}\right\|}{\partial \Delta\varepsilon} = \frac{\partial(\eta^{tr} : \eta^{tr})^{1/2}}{\partial \Delta\varepsilon} = \frac{1}{2}\frac{1}{\left\|\eta^{tr}\right\|}2\eta^{tr} : \boxed{\frac{\partial \eta^{tr}}{\partial \Delta\varepsilon}}$$

$$\frac{\partial\left\|\eta^{tr}\right\|}{\partial \Delta\varepsilon} = \frac{\eta^{tr}}{\left\|\eta^{tr}\right\|} : \frac{\partial \eta^{tr}}{\partial \Delta\varepsilon} = 2\mu\mathbf{N}$$

$$\frac{\partial \eta^{tr}}{\partial \Delta\varepsilon} = \frac{\partial(\mathbf{s}^{tr} - \alpha^n)}{\partial \Delta\varepsilon} = 2\mu\mathbf{I}_{dev}$$

$$\frac{\partial f}{\partial \Delta\varepsilon} = 2\mu\mathbf{N} - (2\mu + H_\alpha(e_p^{n+1}))\frac{\partial \Delta\gamma}{\partial \Delta\varepsilon} - \frac{\partial H_\alpha}{\partial e_p}\frac{\partial e_p}{\partial \Delta\varepsilon}\Delta\gamma - \sqrt{\tfrac{2}{3}}\frac{\partial \kappa}{\partial e_p}\boxed{\frac{\partial e_p}{\partial \Delta\varepsilon}} = 0$$

$$\sqrt{\frac{2}{3}}\frac{\partial \Delta\gamma}{\partial \Delta\varepsilon}$$

$$2\mu\mathbf{N} - \left(2\mu + H_\alpha(e_p^{n+1}) + \sqrt{\tfrac{2}{3}}H_{\alpha,e_p}\Delta\gamma + \tfrac{2}{3}\kappa_{,e_p}\right)\frac{\partial \Delta\gamma}{\partial \Delta\varepsilon} = 0$$

$$\boxed{\frac{\partial \Delta\gamma}{\partial \Delta\varepsilon} = 2\mu A\mathbf{N}}$$

$$\frac{1}{A} = 2\mu + H_\alpha(e_p^{n+1}) + \sqrt{\tfrac{2}{3}}H_{\alpha,e_p}\Delta\gamma + \tfrac{2}{3}\kappa_{,e_p}$$

Consistent Tangent Operator cont

- Term (2):

$$\frac{\partial \mathbf{N}}{\partial \Delta\varepsilon} = \frac{\partial \mathbf{N}}{\partial \eta^{tr}} : \frac{\partial \eta^{tr}}{\partial \Delta\varepsilon}$$

$$\frac{\partial \eta^{tr}}{\partial \Delta\varepsilon} = \frac{\partial}{\partial \Delta\varepsilon}(\mathbf{s}^n + 2\mu\Delta\mathbf{e} - \alpha^n) = 2\mu\mathbf{I}_{dev}$$

$$\frac{\partial \mathbf{N}}{\partial \eta^{tr}} = \frac{\partial}{\partial \eta^{tr}}\left(\frac{\eta^{tr}}{\partial\left\|\eta^{tr}\right\|}\right) = \frac{\mathbf{I}}{\left\|\eta^{tr}\right\|} - \frac{\eta^{tr} \otimes \eta^{tr}}{\left\|\eta^{tr}\right\|^3} = \frac{1}{\left\|\eta^{tr}\right\|}\left[\mathbf{I} - \mathbf{N} \otimes \mathbf{N}\right]$$

$$\boxed{\frac{\partial \mathbf{N}}{\partial \Delta\varepsilon} = \frac{1}{\left\|\eta^{tr}\right\|}\left[\mathbf{I} - \mathbf{N} \otimes \mathbf{N}\right] : 2\mu\mathbf{I}_{dev} = \frac{2\mu}{\left\|\eta^{tr}\right\|}\left[\mathbf{I}_{dev} - \mathbf{N} \otimes \mathbf{N}\right]}$$

- Consistent tangent operator

$$\mathbf{D}^{alg} = \frac{\partial \Delta\sigma}{\partial \Delta\varepsilon} = \mathbf{D} - 2\mu\mathbf{N} \otimes (2\mu A\mathbf{N}) - 2\mu\Delta\gamma\frac{2\mu}{\left\|\eta^{tr}\right\|}\left[\mathbf{I}_{dev} - \mathbf{N} \otimes \mathbf{N}\right]$$

$$\boxed{\mathbf{D}^{alg} = \mathbf{D} - 4\mu^2 A\mathbf{N} \otimes \mathbf{N} - \underbrace{\frac{4\mu^2\Delta\gamma}{\left\|\eta^{tr}\right\|}\left[\mathbf{I}_{dev} - \mathbf{N} \otimes \mathbf{N}\right]}_{\text{Not existing in } \mathbf{D}^{ep}}}$$

Example

- Linear combined hardening

$$H_\alpha(e_p^{n+1}) = \tfrac{2}{3}\beta H \qquad H_{\alpha,e_p} = 0$$

$$\kappa(e_p^{n+1}) = \sigma_Y^0 + (1-\beta)He_p^{n+1} = \kappa(e_p^n) + \sqrt{\tfrac{2}{3}}(1-\beta)H\Delta\gamma$$

- Consistency condition

$$f = \left\|\eta^{tr}\right\| - \left(2\mu + \tfrac{2}{3}\beta H\right)\Delta\gamma - \sqrt{\tfrac{2}{3}}\left(\kappa(e_p^n) + \sqrt{\tfrac{2}{3}}(1-\beta)H\Delta\gamma\right) = 0$$

$$\Rightarrow \quad \Delta\gamma = \frac{\left\|\eta^{tr}\right\| - \sqrt{\tfrac{2}{3}}\kappa(e_p^n)}{2\mu + \tfrac{2}{3}H}$$

No iteration is required

$$\frac{1}{A} = 2\mu + \frac{2}{3}H$$

Variational Equation

- Variational equation

$$a({}^n\xi;{}^{n+1}\mathbf{u},\bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}), \quad \forall\bar{\mathbf{u}} \in \mathbb{Z}$$

$$a({}^n\xi;{}^{n+1}\mathbf{u},\bar{\mathbf{u}}) \equiv \iint_\Omega \varepsilon(\bar{\mathbf{u}}) : {}^{n+1}\sigma \, d\Omega$$

  - The only nonlinearity is from stress (**material nonlinearity**)
  - Small strain, small rotation

- Linearization

$$a^*({}^n\xi, {}^{n+1}\mathbf{u}^k; \delta\mathbf{u}^k, \bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}) - a({}^n\xi; {}^{n+1}\mathbf{u}^k, \bar{\mathbf{u}}), \quad \forall\bar{\mathbf{u}} \in \mathbb{Z},$$

$$a^*({}^n\xi, {}^{n+1}\mathbf{u}; \delta\mathbf{u}, \bar{\mathbf{u}}) \equiv \iint_\Omega \varepsilon(\bar{\mathbf{u}}) : \mathbf{D}^{alg} : \varepsilon(\delta\mathbf{u}) \, d\Omega.$$

- Update displacement

$${}^{n+1}\mathbf{u}^{k+1} = {}^{n+1}\mathbf{u}^k + \delta\mathbf{u}^k$$

Implementation of Elastoplasticity

- We will explain for a 3D solid element at a Gauss point
- Voigt notation

$$\{\sigma\} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{23} \quad \sigma_{13}]^T$$

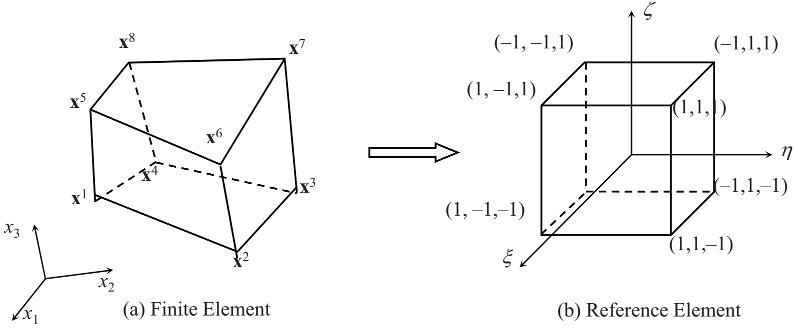
$$\{\Delta\varepsilon\} = [\Delta\varepsilon_{11} \quad \Delta\varepsilon_{22} \quad \Delta\varepsilon_{33} \quad 2\Delta\varepsilon_{12} \quad 2\Delta\varepsilon_{23} \quad 2\Delta\varepsilon_{13}]^T$$

- Inputs

$$\Delta\mathbf{d}_I = \left\{\Delta d_{I1} \quad \Delta d_{I2} \quad \Delta d_{I3}\right\}^T$$

$$\sigma^n = \left\{\sigma^n_{11} \quad \sigma^n_{22} \quad \sigma^n_{33} \quad \sigma^n_{12} \quad \sigma^n_{23} \quad \sigma^n_{13}\right\}^T$$

$$\xi^n = \left\{\alpha^n_{11} \quad \alpha^n_{22} \quad \alpha^n_{33} \quad \alpha^n_{12} \quad \alpha^n_{23} \quad \alpha^n_{13} \quad e^n_p\right\}^T$$



(a) Finite Element

(b) Reference Element

Implementation of Elastoplasticity cont.

- Displacement

$\xi = \{\xi, \eta, \zeta\}^T$ is the natural coordinates at an integration point

$$\Delta\mathbf{u} = \sum_{I=1}^{8} N_I(\xi)\Delta\mathbf{d}_I$$

- Strain

$$\boxed{\Delta\varepsilon = \sum_{I=1}^{8} \mathbf{B}_I \Delta\mathbf{u}_I}$$

$$\mathbf{B}_I = \begin{bmatrix} N_{I,1} & 0 & 0 \\ 0 & N_{I,2} & 0 \\ 0 & 0 & N_{I,3} \\ N_{I,2} & N_{I,1} & 0 \\ 0 & N_{I,3} & N_{I,2} \\ N_{I,3} & 0 & N_{I,1} \end{bmatrix}$$

- Update

$${}^{n+1}\mathbf{u} = {}^{n}\mathbf{u} + \Delta\mathbf{u}$$

$$\{{}^{n+1}\varepsilon\} = \{{}^{n}\varepsilon\} + \{\Delta\varepsilon\}$$

Return Mapping Algorithm

- Elastic predictor
  - Unit tensor $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}^T$
  - Trial stress $\sigma^{tr} = \sigma^n + \mathbf{C} \cdot \Delta\varepsilon$
  - Trace of stress $tr(\sigma) = \sigma_{11}^{tr} + \sigma_{22}^{tr} + \sigma_{33}^{tr}$
  - Shifted stress $\eta^{tr} = \sigma^{tr} - \frac{1}{3} tr(\sigma) - \alpha^n$
  - Norm $\left\| \eta^{tr} \right\| = \sqrt{(\eta_{11}^{tr})^2 + (\eta_{22}^{tr})^2 + (\eta_{33}^{tr})^2 + 2[(\eta_{12}^{tr})^2 + (\eta_{23}^{tr})^2 + (\eta_{13}^{tr})^2]}$
  - Yield function $f = \left\| \eta^{tr} \right\| - \sqrt{\frac{2}{3}} \left[\sigma_Y^0 + (1 - \beta) H e_p^n \right]$

Return Mapping Algorithm cont.

- Check yield status
  - If f < 0, then the material is elastic

    $\sigma^{n+1} = \sigma^{tr} \qquad \mathbf{D}^{alg} = \mathbf{D}$
  - Exit
- Consistency parameter $\Delta\gamma = f / (2\mu + \frac{2}{3} H)$
- Unit deviatoric tensor $\mathbf{N} = \eta^{tr} / \left\| \eta^{tr} \right\|$
- Update stress $\sigma^{n+1} = \sigma^{tr} - 2\mu\Delta\gamma\mathbf{N}$
- Update back stress $\alpha^{n+1} = \alpha^n + \frac{2}{3} \beta H \Delta\gamma \mathbf{N}$
- Update plastic strain $e_p^{n+1} = e_p^n + \sqrt{\frac{2}{3}} \Delta\gamma$
- Calculate consistent tangent matrix

Implementation of Elastoplasticity cont.

- Consistent tangent matrix

$$c_1 = \frac{4\mu^2}{2\mu + \frac{2}{3}H} \qquad c_2 = \frac{4\mu^2 \Delta\gamma}{\left\| \eta^{tr} \right\|}$$

$$\mathbf{I}^{dev} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{D}^{alg} = \mathbf{D} - (c_1 - c_2)\mathbf{N}\mathbf{N}^T - c_2 \mathbf{I}_{dev}$$

- Internal force and tangent stiffness matrix

$$\mathbf{f}^{int} = \sum_{I=1}^{4} \sum_{K=1}^{NG} (\mathbf{B}_I^T \sigma^{n+1} |\mathbf{J}|)_K \omega_K \qquad \mathbf{K}_T = \sum_{I=1}^{4} \sum_{J=1}^{4} \sum_{K=1}^{NG} (\mathbf{B}_I^T \mathbf{D}^{alg} \mathbf{B}_J |\mathbf{J}|)_K \omega_K$$

- Solve for incremental displacement

$$[\mathbf{K}_T]\{\Delta \mathbf{u}\} = \{\mathbf{f}^{ext}\} - \{\mathbf{f}^{int}\}$$

- The algorithm repeats until the residual reduces to zero
- Once the solution converges, save stress and plastic variables and move to next load step

Program combHard.m

```
%
% Linear combined isotropic/kinematic hardening model
%
function [stress, alpha, ep]=combHard(mp,D,deps,stressN,alphaN,epN)
% Inputs:
% mp = [lambda, mu, beta, H, Y0];
% D = elastic stiffness matrix
% stressN = [s11, s22, s33, t12, t23, t13];
% alphaN = [a11, a22, a33, a12, a23, a13];
%
Iden = [1 1 1 0 0 0]';
two3 = 2/3; stwo3=sqrt(two3);                   %constants
mu=mp(2); beta=mp(3); H=mp(4); Y0=mp(5);        %material properties
ftol = Y0*1E-6;                                 %tolerance for yield
%
stresstr = stressN + D*deps;                    %trial stress
I1 = sum(stresstr(1:3));                        %trace(stresstr)
str = stresstr - I1*Iden/3;                     %deviatoric stress
eta = str - alphaN;                             %shifted stress
etat = sqrt(eta(1)^2 + eta(2)^2 + eta(3)^2 ...
          + 2*(eta(4)^2 + eta(5)^2 + eta(6)^2));%norm of eta
fyld = etat - stwo3*(Y0+(1-beta)*H*epN);        %trial yield function
if fyld < ftol                                  %yield test
    stress = stresstr; alpha = alphaN; ep = epN;%trial states are final
    return;
else
    gamma = fyld/(2*mu + two3*H);               %plastic consistency param
    ep = epN + gamma*stwo3;                     %updated eff. plastic strain
end
N = eta/etat;                                   %unit vector normal to f
stress = stresstr - 2*mu*gamma*N;               %updated stress
alpha = alphaN + two3*beta*H*gamma*N;           %updated back stress
```

Program combHardTan.m

```
function [Dtan]=combHardTan(mp,D,deps,stressN,alphaN,epN)
% Inputs:
% mp = [lambda, mu, beta, H, Y0];
% D = elastic stiffness matrix
% stressN = [s11, s22, s33, t12, t23, t13];
% alphaN = [a11, a22, a33, a12, a23, a13];
%
Iden = [1 1 1 0 0 0]';
two3 = 2/3; stwo3=sqrt(two3);                   %constants
mu=mp(2); beta=mp(3); H=mp(4); Y0=mp(5);        %material properties
ftol = Y0*1E-6;                                 %tolerance for yield
stresstr = stressN + D*deps;                    %trial stress
I1 = sum(stresstr(1:3));                        %trace(stresstr)
str = stresstr - I1*Iden/3;                     %deviatoric stress
eta = str - alphaN;                             %shifted stress
etat = sqrt(eta(1)^2 + eta(2)^2 + eta(3)^2 ...
          + 2*(eta(4)^2 + eta(5)^2 + eta(6)^2));%norm of eta
fyld = etat - stwo3*(Y0+(1-beta)*H*epN);        %trial yield function
if fyld < ftol                                  %yield test
    Dtan = D; return;                           %elastic
end
gamma = fyld/(2*mu + two3*H);                   %plastic consistency param
N = eta/etat;                                   %unit vector normal to f
var1 = 4*mu^2/(2*mu+two3*H);
var2 = 4*mu^2*gamma/etat;                       %coefficients
Dtan = D - (var1-var2)*N*N' + var2*Iden*Iden'/3;%tangent stiffness
Dtan(1,1) = Dtan(1,1) - var2;                   %contr. from 4th-order I
Dtan(2,2) = Dtan(2,2) - var2;
Dtan(3,3) = Dtan(3,3) - var2;
Dtan(4,4) = Dtan(4,4) - .5*var2;
Dtan(5,5) = Dtan(5,5) - .5*var2;
Dtan(6,6) = Dtan(6,6) - .5*var2;
```

Program PLAST3D.m

```
function PLAST3D(MID, PROP, ETAN, UPDATE, LTAN, NE, NDOF, XYZ, LE)
%***********************************************************************
%  MAIN PROGRAM COMPUTING GLOBAL STIFFNESS MATRIX RESIDUAL FORCE FOR
%  PLASTIC MATERIAL MODELS
%***********************************************************************
%%
....
 %LOOP OVER ELEMENTS, THIS IS MAIN LOOP TO COMPUTE K AND F
  for IE=1:NE
    DSP=DISPTD(IDOF);
    DSPD=DISPDD(IDOF);
    %....
    % LOOP OVER INTEGRATION POINTS
    for LX=1:2, for LY=1:2, for LZ=1:2
      %
      % Previous converged history variables
      NALPHA=6;
      STRESSN=SIGMA(1:6,INTN);
      ALPHAN=XQ(1:NALPHA,INTN);
      EPN=XQ(NALPHA+1,INTN);
....
     % Computer stress, back stress & effective plastic strain
      if MID == 1
        % Infinitesimal plasticity
        [STRESS, ALPHA, EP]=combHard(PROP,ETAN,DDEPS,STRESSN,ALPHAN,EPN);
....
      %
      % Tangent stiffness
      if LTAN
        if MID == 1
          DTAN=combHardTan(PROP,ETAN,DDEPS,STRESSN,ALPHAN,EPN);
          EKF = BM'*DTAN*BM;
....
```

Summary

- 1D tension test data are used for 2D or 3D stress state using failure theories
  - All failure criteria are independent of coordinate system (must defined using invariants)
- Yielding of a ductile material is related to shear stress or deviatoric stress
- Kinematic hardening shift the center of elastic domain, while isotropic hardening increase the radius of it
- For rate-independent J_2 plasticity, elastic predictor and plastic correct algorithm is used
- Return mapping occurs in the radial direction of deviatoric stress
- During return mapping, the yield surface also changes

4.4

Elastoplasticity with Finite Rotation

Goals

- Understand the concept of objective rate and frame-indifference (why do we need objectivity?)
- Learn how to make a non-objective rate to objective one
- Learn different objective stress rates
- Learn how to maintain objectivity at finite rotation
- Understand midpoint configuration
- Understand how to linearize the energy form in the updated Lagrangian formulation
- Understand how to implement update Lagrangian frame

Elastoplasticity with Finite Rotation

- We studied elastoplasticity with infinitesimal deformation
  - Infinitesimal deformation means both strain and rotation are small

$$\nabla \mathbf{u} = \underbrace{\mathrm{sym}(\nabla \mathbf{u})}_{\text{strain}} + \underbrace{\mathrm{skew}(\nabla \mathbf{u})}_{\text{rotation}}$$

- We can relax this limitation by allowing finite rotation
- However, the engineering strain changes in rigid-body rotation (We showed in Chapter 3)
- How can we use engineering strain for a finite rotation problem?

$$\varepsilon = \begin{bmatrix} \cos\alpha - 1 & 0 & 0 \\ 0 & \cos\alpha - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

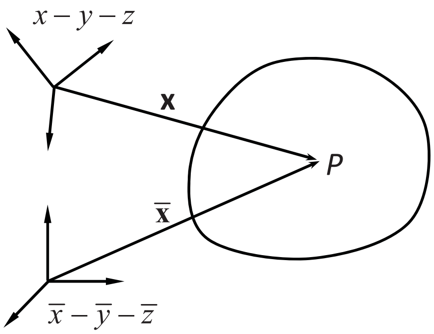
- **Instead of using X, we can use x^n as a reference (Body-fixed coordinate, not Eulerian but Lagrangian)**
- Can the frame of reference move?

Objective Tensor

- We want to take care of the issues related to the moving reference frame $\mathbf{x}^n$ (rotation and translation) using objectivity
- **Objective tensor**: any tensor that is not affected by superimposed rigid body translations and rotations of the spatial frame
- Rotation of a body is equivalent to rotation of coordinate frame in opposite direction
- Consider two frames in the figure (rotation + translation)

$$\boxed{\bar{\mathbf{x}} = \mathbf{Q}(t)\cdot\mathbf{x} + \mathbf{c}(t)}$$

$\mathbf{x}$ and $\bar{\mathbf{x}}$ are different by rigid-body motion, by relative motion between observers



Objective Tensor cont.

- Frame indifference (objectivity)
  - Quantities that depend only on $\mathbf{Q}$ and not on the other aspects of the motion of the reference frame (e.g., translation, velocity and acceleration, angular velocity and angular acceleration)
- Objective scalar $\boxed{\bar{f} = f}$
- Objective vector $\boxed{\bar{\mathbf{v}} = \mathbf{Q}\cdot\mathbf{v}}$
- Objective tensor $\boxed{\bar{\mathbf{T}} = \mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^T}$
- **In order to use a moving reference, we must use objective quantities**

Example

- Deformation gradient

$$\bar{\mathbf{F}} = \frac{\partial \bar{\mathbf{x}}}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}}(\mathbf{Q}(t)\cdot\mathbf{x} + \mathbf{c}(t)) = \mathbf{Q}(t)\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{Q}(t)\cdot\mathbf{F}$$

  - F transforms like a vector

- Right C-G deformation tensor

$$\bar{\mathbf{C}} = \bar{\mathbf{F}}^T\bar{\mathbf{F}} = (\mathbf{QF})^T(\mathbf{QF}) = \mathbf{F}^T\mathbf{Q}^T\mathbf{QF} = \mathbf{F}^T\mathbf{F} = \mathbf{C}$$

  - **Material tensors are not affected by rigid-body motion**

- Left C-G deformation tensor

$$\bar{\mathbf{b}} = \bar{\mathbf{F}}\bar{\mathbf{F}}^T = (\mathbf{QF})(\mathbf{F}^T\mathbf{Q}^T) = \mathbf{QFF}^T\mathbf{Q}^T = \mathbf{QbQ}^T$$

Objective tensor

- **Objectivity only applies to a spatial tensor, not material tensor**
- Deformation gradient transforms like a vector because it has one spatial component and one material component

---

Velocity Gradient

- In two different frames

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \qquad \bar{\mathbf{L}} = \frac{\partial \bar{\mathbf{v}}}{\partial \bar{\mathbf{x}}}$$

Velocity gradient is related to incremental displacement gradient in finite time step

$$\mathbf{L}\Delta t \approx \frac{\partial \Delta \mathbf{u}}{\partial \mathbf{x}}$$

- Time differentiate of $\bar{\mathbf{x}} = \mathbf{Q}\cdot\mathbf{x}$

$$\dot{\bar{\mathbf{x}}} = \mathbf{Q}\cdot\mathbf{v} + \dot{\mathbf{Q}}\cdot\mathbf{x}$$

$$\bar{\mathbf{v}} = \mathbf{Q}\cdot\mathbf{v} + \dot{\mathbf{Q}}\mathbf{Q}^T\bar{\mathbf{x}}$$

Velocity is not objective

- Spatial differentiation of $\bar{\mathbf{v}}$

$$\begin{aligned}\bar{\mathbf{L}} &= \frac{\partial \bar{\mathbf{v}}}{\partial \bar{\mathbf{x}}} \\ &= \mathbf{Q}\cdot\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}} + \dot{\mathbf{Q}}\cdot\mathbf{Q}^T\cdot\frac{\partial \bar{\mathbf{x}}}{\partial \bar{\mathbf{x}}} \\ &= \mathbf{Q}\cdot\mathbf{L}\cdot\mathbf{Q}^T + \dot{\mathbf{Q}}\cdot\mathbf{Q}^T\end{aligned}$$

Velocity gradient is not objective

Rate of Deformation and Spin Tensor

- Rate of Deformation

$$\mathbf{d} = \mathrm{sym}(\mathbf{L}) \qquad \bar{\mathbf{d}} = \mathrm{sym}(\bar{\mathbf{L}})$$

$$\bar{\mathbf{d}} = \mathrm{sym}(\bar{\mathbf{L}}) = \tfrac{1}{2}\left(\mathbf{Q}\cdot\mathbf{L}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\mathbf{L}^T\cdot\mathbf{Q}^T + \dot{\mathbf{Q}}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\dot{\mathbf{Q}}^T\right)$$

$$\mathbf{Q}\cdot\mathbf{Q}^T = \mathbf{1} \quad \Rightarrow \quad \dot{\mathbf{Q}}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\dot{\mathbf{Q}}^T = \mathbf{0}$$

$$\bar{\mathbf{d}} = \mathbf{Q}\cdot\tfrac{1}{2}(\mathbf{L}+\mathbf{L}^T)\cdot\mathbf{Q}^T = \mathbf{Q}\cdot\mathbf{d}\cdot\mathbf{Q}^T$$

Objective
This is incremental stain

- Spin tensor

$$\mathbf{W} = \tfrac{1}{2}(\mathbf{L}-\mathbf{L}^T) \qquad \bar{\mathbf{W}} = \tfrac{1}{2}(\bar{\mathbf{L}}-\bar{\mathbf{L}}^T)$$

$$\bar{\mathbf{W}} = \tfrac{1}{2}(\bar{\mathbf{L}}-\bar{\mathbf{L}}^T) = \tfrac{1}{2}(\mathbf{Q}\cdot\mathbf{L}\cdot\mathbf{Q}^T - \mathbf{Q}\cdot\mathbf{L}^T\cdot\mathbf{Q}^T + \dot{\mathbf{Q}}\cdot\mathbf{Q}^T - \mathbf{Q}\cdot\dot{\mathbf{Q}}^T)$$

$$\bar{\mathbf{W}} = \mathbf{Q}\cdot\mathbf{W}\cdot\mathbf{Q}^T + \tfrac{1}{2}(\dot{\mathbf{Q}}\cdot\mathbf{Q}^T - \mathbf{Q}\cdot\dot{\mathbf{Q}}^T)$$

Depends on the spin of rotating frame

Not Objective

Cauchy Stress Is an Objective Tensor

- Proof from the relation between stresses

$$\sigma = \frac{1}{J}\mathbf{FSF}^T$$

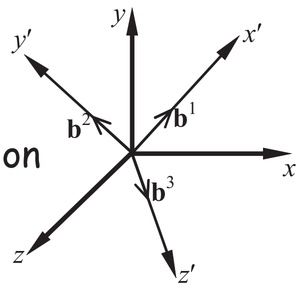
$$\bar{\sigma} = \frac{1}{J}\bar{\mathbf{F}}\bar{\mathbf{S}}\bar{\mathbf{F}}^T = \frac{1}{J}\mathbf{QFSF}^T\mathbf{Q}^T = \mathbf{Q}\left(\frac{1}{J}\mathbf{FSF}^T\right)\mathbf{Q}^T = \mathbf{Q}\sigma\mathbf{Q}^T$$

$$\bar{\mathbf{S}} = \mathbf{S}$$

- Proof from coordinate transformation of stress tensor

$$[\mathbf{T}^{(\mathbf{b}^1)} \quad \mathbf{T}^{(\mathbf{b}^2)} \quad \mathbf{T}^{(\mathbf{b}^3)}]_{xyz} = [\sigma]_{xyz}[\mathbf{b}^1 \quad \mathbf{b}^2 \quad \mathbf{b}^3] = [\sigma]_{xyz}[\mathbf{Q}]$$

$$[\sigma]_{x'y'z'} = [\mathbf{Q}]^T[\sigma]_{xyz}[\mathbf{Q}]$$



  - Coordinate transformation is opposite to rotation

$$[\bar{\sigma}]_{xyz} = [\mathbf{Q}][\sigma]_{xyz}[\mathbf{Q}]^T$$

Objective Rate

- If $\mathbf{T}$ is an objective tensor, will its rate be objective, too?
  - This is important because in plasticity the constitutive relation is given in terms of stress rate
- Differentiate an objective tensor $\bar{\mathbf{T}} = \mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^T$

$$\dot{\bar{\mathbf{T}}} = \dot{\mathbf{Q}}\cdot\mathbf{T}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\dot{\mathbf{T}}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\mathbf{T}\cdot\dot{\mathbf{Q}}^T$$

  - Not objective due to $\dot{\mathbf{Q}}$ and $\dot{\mathbf{Q}}^T$
- Remove non-objective terms using $\bar{\mathbf{L}} = \mathbf{Q}\cdot\mathbf{L}\cdot\mathbf{Q}^T + \dot{\mathbf{Q}}\cdot\mathbf{Q}^T$

$$\dot{\mathbf{Q}} = \bar{\mathbf{L}}\cdot\mathbf{Q} - \mathbf{Q}\cdot\mathbf{L} \qquad \dot{\mathbf{Q}}^T = \mathbf{Q}^T\cdot\bar{\mathbf{L}}^T - \mathbf{L}^T\cdot\mathbf{Q}^T$$

$$\begin{aligned}\dot{\bar{\mathbf{T}}} &= (\bar{\mathbf{L}}\mathbf{Q} - \mathbf{Q}\mathbf{L})\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}(\mathbf{Q}^T\bar{\mathbf{L}}^T - \mathbf{L}^T\mathbf{Q}^T) \\ &= \bar{\mathbf{L}}\mathbf{Q}\mathbf{T}\mathbf{Q}^T - \mathbf{Q}\mathbf{L}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\mathbf{Q}^T\bar{\mathbf{L}}^T - \mathbf{Q}\mathbf{T}\mathbf{L}^T\mathbf{Q}^T \\ &= \bar{\mathbf{L}}\bar{\mathbf{T}} - \mathbf{Q}\mathbf{L}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \bar{\mathbf{T}}\bar{\mathbf{L}}^T - \mathbf{Q}\mathbf{T}\mathbf{L}^T\mathbf{Q}^T\end{aligned}$$

Objective Rate cont.

- Objective rate

$$\dot{\bar{\mathbf{T}}} = \bar{\mathbf{L}}\bar{\mathbf{T}} - \mathbf{Q}\mathbf{L}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \bar{\mathbf{T}}\bar{\mathbf{L}}^T - \mathbf{Q}\mathbf{T}\mathbf{L}^T\mathbf{Q}^T$$

$$\boxed{\dot{\bar{\mathbf{T}}} - \bar{\mathbf{L}}\bar{\mathbf{T}} - \bar{\mathbf{T}}\bar{\mathbf{L}}^T = \mathbf{Q}(\dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T)\mathbf{Q}^T}$$

- Thus, $\boxed{\dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T}$ is an objective rate (Truesdell rate)
- Co-rotational rate (Jaumann rate)

$$\boxed{\dot{\mathbf{T}} - \mathbf{W}\cdot\mathbf{T} + \mathbf{T}\cdot\mathbf{W}}$$

- Convected rate

$$\boxed{\dot{\mathbf{T}} + \mathbf{L}^T\cdot\mathbf{T} + \mathbf{T}\cdot\mathbf{L}}$$

- These objective rates are different, but perform equally
- When $\mathbf{T}$ is stress, they are **objective stress rate**

Finite Rotation and Objective Rate

- Since constitutive relation should be independent of the reference frame, **it has to be given in terms of objective rate**
- Cauchy stress is an objective tensor, but **Cauchy stress rate is not objective rate**
- Instead of rate, we will use increment (**from previous converged load step to the current iteration**)
- Consider a unit vector $\mathbf{e}_j$ in spatial Cartesian coordinates under rigid body rotation from material vector $\mathbf{E}_j$

$$\mathbf{e}_j = \mathbf{Q} \cdot \mathbf{E}_j \qquad \mathbf{W} = \frac{1}{2}\left(\frac{\partial \Delta \mathbf{u}}{\partial \mathbf{x}} - \frac{\partial \Delta \mathbf{u}}{\partial \mathbf{x}}^{T}\right) = \Delta \mathbf{Q} \cdot \mathbf{Q}^{T}$$

$$\Delta \mathbf{e}_j = \Delta \mathbf{Q} \cdot \mathbf{E}_j = \mathbf{W} \cdot \mathbf{Q} \cdot \mathbf{E}_j = \mathbf{W} \cdot \mathbf{e}_j$$

W: spin tensor
Q: rotation tensor

Finite Rotation and Objective Rate cont.

- Cauchy stress in Cartesian coordinates

$$\sigma = \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

- Incremental Cauchy stress

$$\begin{aligned} \Delta\sigma &= \Delta\sigma_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j + \sigma_{ij}\, \Delta\mathbf{e}_i \otimes \mathbf{e}_j + \sigma_{ij}\, \mathbf{e}_i \otimes \Delta\mathbf{e}_j \\ &= \Delta\sigma_{ij}^{J}\, \mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{W} \cdot (\sigma_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j) + (\sigma_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j) \cdot \mathbf{W}^{T} \\ &= (\Delta\sigma_{ij}^{J} + W_{ik}\sigma_{kj} - \sigma_{ik}W_{kj})\, \mathbf{e}_i \otimes \mathbf{e}_j \end{aligned}$$

Effect of rigid body rotation

Jaumann or co-rotational Cauchy stress increment
Objective rate in the rotating frame

Only accurate for small, rigid body rotations

- Constitutive relation

$$\boxed{\Delta\sigma^{J} = \mathbf{D}^{alg} : \Delta\varepsilon} \qquad \boxed{\Delta\sigma^{J} = \Delta\sigma - \mathbf{W}\sigma + \sigma\mathbf{W}}$$

Finite Rotation and Objective Rate cont.

- For finite rotation, the spin tensor **W** is not constant throughout the increment
- Preserving objectivity for large rotational increments using midpoint configuration
- Instead of n+1, calculate strain increment and spin at n+$\frac{1}{2}$

$$\Delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial x_j^{n+\frac{1}{2}}} + \frac{\partial \Delta u_j}{\partial x_i^{n+\frac{1}{2}}}\right) \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial x_j^{n+\frac{1}{2}}} - \frac{\partial \Delta u_j}{\partial x_i^{n+\frac{1}{2}}}\right)$$

How to calculate these?

- Midpoint configuration

$$\mathbf{x}^{n+\frac{1}{2}} = \tfrac{1}{2}(\mathbf{x}^{n+1} + \mathbf{x}^{n}) = \mathbf{x}^{n} + \tfrac{1}{2}\Delta\mathbf{u} = \mathbf{x}^{n+1} - \tfrac{1}{2}\Delta\mathbf{u}$$

  - We want to rotation stress into the midpoint configuration

$$\frac{\partial\Delta\mathbf{u}}{\partial\mathbf{x}^{n+1/2}} = \frac{\partial\Delta\mathbf{u}}{\partial\mathbf{x}^{n}}\cdot\frac{\partial\mathbf{x}^{n}}{\partial\mathbf{x}^{n+1/2}}, \qquad \frac{\partial\mathbf{x}^{n+1/2}}{\partial\mathbf{x}^{n}} = \mathbf{1} + \frac{1}{2}\frac{\partial\Delta\mathbf{u}}{\partial\mathbf{x}^{n}}$$

Finite Rotation and Objective Rate cont.

- Rotational matrix to the midpoint configuration

$$\frac{d\mathbf{R}}{dt} = \mathbf{W}\cdot\mathbf{R}$$

$$\frac{\mathbf{R}_{n+1} - \mathbf{R}_{n}}{\Delta t} \approx \mathbf{W}(\dot{\mathbf{u}})\cdot\mathbf{R}_{n+\frac{1}{2}} \approx \mathbf{W}(\dot{\mathbf{u}})\cdot\frac{\mathbf{R}_{n+1} + \mathbf{R}_{n}}{2}$$

$$\mathbf{W}(\Delta\mathbf{u}) = \Delta t\mathbf{W}(\dot{\mathbf{u}})$$

$$\mathbf{R}_{n} = \mathbf{1}$$

$$\Rightarrow \quad \mathbf{R} = (\mathbf{1} - \tfrac{1}{2}\mathbf{W})^{-1}(\mathbf{1} + \tfrac{1}{2}\mathbf{W}) = \mathbf{1} + (\mathbf{1} - \tfrac{1}{2}\mathbf{W})^{-1}\mathbf{W}$$

- Rotation of stress and back stress

$$\bar{\sigma}^{n} = \mathbf{R}\cdot\sigma^{n}\cdot\mathbf{R}^{T}$$
$$\bar{\alpha}^{n} = \mathbf{R}\cdot\alpha^{n}\cdot\mathbf{R}^{T}$$

This takes care of rigid body rotation

- Now, return mapping with these stresses
  - Exactly same as small deformation plasticity

Program rotatedStress.m

```
%
% Rotate stress and back stress to the rotation-free configuration
%
function [stress, alpha] = rotatedStress(L, S, A)
%L = [dui/dxj] velocity gradient
%
str=[S(1) S(4) S(6);S(4) S(2) S(5);S(6) S(5) S(3)];
alp=[A(1) A(4) A(6);A(4) A(2) A(5);A(6) A(5) A(3)];
factor=0.5;
R = L*inv(eye(3) + factor*L);
W = .5*(R-R');
R = eye(3) + inv(eye(3) - factor*W)*W;
str = R*str*R';
alp = R*alp*R';
stress=[str(1,1) str(2,2) str(3,3) str(1,2) str(2,3) str(1,3)]';
alpha =[alp(1,1) alp(2,2) alp(3,3) alp(1,2) alp(2,3) alp(1,3)]';
```

Variational Principle for Finite Rotation

- Total Lagrangian is inconvenient
  - We don't know how 2nd P-K stress evolves in plasticity
  - plastic variables is directly related to the Cauchy stress
- Thus, we will use the updated Lagrangian formulation
- Assume the problem has been solved up to **n** load step, and we are looking for the solution at load step **n+1**
- Since load form is straightforward, we will ignore it
- Energy form

$$a(\xi^n; \mathbf{u}^{n+1}, \bar{\mathbf{u}}) \equiv \iint_{\Omega_{n+1}} \nabla_{n+1}\bar{\mathbf{u}} : \sigma^{n+1}\, d\Omega$$

  - Since the Cauchy stress is symmetric, it is OK to use $\nabla_{n+1}\bar{\mathbf{u}}$
  - **Both Ω_{n+1} and σ^{n+1} are unknown**
  - Nonlinear in terms of **u**

Variational Principle for Finite Rotation cont.

- Energy form cont.
  - Since the current configuration is unknown (depends on displacement) , let's transform it to the undeformed configuration Ω_0

$$a(\xi^n;\mathbf{u}^{n+1},\bar{\mathbf{u}}) = \iint_{\Omega_{n+1}} \nabla_{n+1}\bar{\mathbf{u}} : \sigma^{n+1}\, d\Omega = \boxed{\iint_{\Omega_0} (\nabla_0\bar{\mathbf{u}}\mathbf{F}^{-1}) : \sigma^{n+1} J\, d\Omega}$$

  - Integral domain can be changed by $\iint_{\Omega_{n+1}} d\Omega = \iint_{\Omega_0} J\, d\Omega \quad J = \det(\mathbf{F})$
  - This is only for convenience in linearization. Eventually, we will come back to the deformed configuration and integrate at there
  - The integrand is identical to $\mathbf{T} : \bar{\mathbf{F}}$ where $\mathbf{T} = J\mathbf{F}^{-1}\sigma$ is the first P-K stress
  - Nonlinearity comes from (a) constitutive relation (hypoelasticity), (b) spatial gradient (deformation gradient), and (c) Jacobian of deformation gradient (domain change)

Linearization

- Increment of deformation gradient

$$\Delta\mathbf{F} = \frac{\partial}{\partial\omega}\left[\frac{\partial(\mathbf{x}+\omega\Delta\mathbf{u})}{\partial\mathbf{X}}\right]_{\omega=0} = \frac{\partial\Delta\mathbf{u}}{\partial\mathbf{X}} = \nabla_0\Delta\mathbf{u}$$

$$\mathbf{F}\mathbf{F}^{-1} = \mathbf{1} \quad \Rightarrow \quad \Delta\mathbf{F}^{-1} = -\mathbf{F}^{-1}\nabla_0\Delta\mathbf{u}\mathbf{F}^{-1} = -\mathbf{F}^{-1}\nabla_{n+1}\Delta\mathbf{u}$$

- Increment of Jacobian

$$\Delta J = \Delta\left|\mathbf{F}\right| = J\,\mathrm{div}(\Delta\mathbf{u}) \qquad \left|F_{mn}\right| = \tfrac{1}{6} e_{ijk}e_{rst}F_{ir}F_{js}F_{kt}$$

$$e_{ijk}e_{ijr} = 2\delta_{kr}$$

Linearization cont.

- Linearization of energy form

$$
\begin{aligned}
&\Delta\iint_{\Omega_0}(\nabla_0\bar{\mathbf{u}}\mathbf{F}^{-1}):\sigma J\,d\Omega \\
&= \iint_{\Omega_0}\left[(\nabla_0\bar{\mathbf{u}}\Delta\mathbf{F}^{-1}):\sigma J+(\nabla_0\bar{\mathbf{u}}\mathbf{F}^{-1}):\Delta\sigma J+(\nabla_0\bar{\mathbf{u}}\mathbf{F}^{-1}):\sigma\Delta J\right]d\Omega \\
&= \iint_{\Omega_0}\left[-\nabla_0\bar{\mathbf{u}}\mathbf{F}^{-1}\nabla_{n+1}\Delta\mathbf{u}:\sigma+\nabla_0\bar{\mathbf{u}}\mathbf{F}^{-1}:\Delta\sigma+\nabla_0\bar{\mathbf{u}}\mathbf{F}^{-1}:\sigma\mathrm{div}(\Delta\mathbf{u})\right]J\,d\Omega \\
&= \iint_{\Omega_{n+1}}\left[-\nabla_{n+1}\bar{\mathbf{u}}\nabla_{n+1}\Delta\mathbf{u}:\sigma+\nabla_{n+1}\bar{\mathbf{u}}:\Delta\sigma+\nabla_{n+1}\bar{\mathbf{u}}:\sigma\mathrm{div}(\Delta\mathbf{u})\right]d\Omega \\
&= \iint_{\Omega_{n+1}}\nabla_{n+1}\bar{\mathbf{u}}:\left[\Delta\sigma+\sigma\mathrm{div}(\Delta\mathbf{u})-\sigma(\nabla_{n+1}\Delta\mathbf{u})^T\right]d\Omega
\end{aligned}
$$

$$\mathbf{AB}:\mathbf{C}=\mathbf{A}:\mathbf{CB}^T\qquad A_{ik}B_{kj}C_{ij}=A_{ik}C_{ij}B_{kj}$$

Use Jaumann objective rate

Linearization cont.

- Linearization of energy form cont.

$$
\begin{aligned}
L[a(\xi^n;\mathbf{u}^{n+1},\bar{\mathbf{u}})] &= \iint_{\Omega_{n+1}}\nabla_{n+1}\bar{\mathbf{u}}:\left[\Delta\sigma+\sigma\mathrm{div}(\Delta\mathbf{u})-\sigma(\nabla_{n+1}\Delta\mathbf{u})^T\right]d\Omega \\
&= \iint_{\Omega_{n+1}}\nabla_{n+1}\bar{\mathbf{u}}:\left[\Delta\sigma^J+\mathbf{W}\sigma-\sigma\mathbf{W}+\sigma\mathrm{div}(\Delta\mathbf{u})-\sigma(\nabla_{n+1}\Delta\mathbf{u})^T\right]d\Omega
\end{aligned}
$$

  - Express inside of [ ] in terms of $\nabla_{n+1}\Delta\mathbf{u}$
  - Constitutive relation: $\Delta\sigma^J=\mathbf{D}^{alg}:\Delta\varepsilon=\mathbf{D}^{alg}:(\nabla_{n+1}\Delta\mathbf{u})$
  - Spin term

$$
\begin{aligned}
W_{im}\sigma_{mj} &= \tfrac{1}{2}\left(\frac{\partial\Delta u_i}{\partial x_m}-\frac{\partial\Delta u_m}{\partial x_i}\right)\sigma_{mj}=\tfrac{1}{2}\sigma_{mj}(\delta_{ik}\delta_{ml}-\delta_{mk}\delta_{il})\frac{\partial\Delta u_k}{\partial x_l} \\
&= \tfrac{1}{2}(\sigma_{lj}\delta_{ik}-\sigma_{kj}\delta_{il})[\nabla_{n+1}\Delta\mathbf{u}]_{kl}
\end{aligned}
$$

$$-\sigma_{im}W_{mj}=\tfrac{1}{2}(\sigma_{il}\delta_{jk}-\sigma_{ik}\delta_{jl})[\nabla_{n+1}\Delta\mathbf{u}]_{kl}$$

$$\sigma_{ij}\frac{\partial\Delta u_k}{\partial x_k}=\sigma_{ij}\delta_{kl}[\nabla_{n+1}\Delta\mathbf{u}]_{kl}$$

$$-\sigma_{im}\frac{\partial\Delta u_j}{\partial x_m}=-\sigma_{il}\delta_{jk}[\nabla_{n+1}\Delta\mathbf{u}]_{kl}$$

Linearization cont.

- Linearization of energy form cont.

$$L[a(\mathbf{u},\bar{\mathbf{u}})] = \iint_{\Omega_{n+1}} \nabla_{n+1}\bar{\mathbf{u}} : \Big[\Delta\sigma^{J} + \underbrace{\mathbf{W}\sigma - \sigma\mathbf{W} + \sigma \operatorname{div}(\Delta\mathbf{u}) - \sigma(\nabla_{n+1}\Delta\mathbf{u})^{T}}_{\frac{1}{2}(\sigma_{jl}\delta_{ik} - \sigma_{jk}\delta_{il} - \sigma_{il}\delta_{jk} - \sigma_{ik}\delta_{jl}) + \sigma_{ij}\delta_{kl}} \Big] d\Omega$$

  - Initial stiffness term (we need to separate this term)

$$\sigma : \operatorname{sym}(\nabla_{n+1}\bar{\mathbf{u}}^{T}\nabla_{n+1}\Delta\mathbf{u}) = \sigma : \eta(\Delta\mathbf{u},\bar{\mathbf{u}})$$

$$\sigma_{rs}\frac{1}{2}\Big(\frac{\partial\bar{u}_m}{\partial x_r}\frac{\partial\Delta u_m}{\partial x_s} + \frac{\partial\bar{u}_m}{\partial x_r}\frac{\partial\Delta u_m}{\partial x_s}\Big) = \frac{\partial\bar{u}_i}{\partial x_j}\sigma_{jl}\delta_{ik}\frac{\partial\Delta u_k}{\partial x_l}$$

  - Define

$$\boxed{-\mathbf{D}^{*}_{ijkl} = \sigma_{ij}\delta_{kl} - \tfrac{1}{2}(\sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik} + \sigma_{ik}\delta_{jl} + \sigma_{jk}\delta_{il})}$$

Rotational effect of Cauchy stress tensor

Linearization cont.

- Linearization of energy form cont.

$$\boxed{\begin{aligned} L[a(\xi^{n};\mathbf{u}^{n+1},\bar{\mathbf{u}})] &= \iint_{\Omega_{n+1}} \Big[\nabla_{n+1}\bar{\mathbf{u}} : (\mathbf{D}^{alg} - \mathbf{D}^{*}) : \nabla_{n+1}\Delta\mathbf{u} + \sigma : \eta(\Delta\mathbf{u},\bar{\mathbf{u}}) \Big] d\Omega \\ &\equiv a^{*}(\xi^{n},\mathbf{u}^{n+1};\Delta\mathbf{u},\bar{\mathbf{u}}) \end{aligned}}$$

- N-R iteration

$$\boxed{a^{*}(\underbrace{{}^{n}\xi,\mathbf{u}_k^{n+1}}_{\text{History-dependent (implicit)}};\underbrace{\Delta\mathbf{u}_{k+1},\bar{\mathbf{u}}}_{\text{Bilinear}}) = \ell(\bar{\mathbf{u}}) - a({}^{n}\xi;\mathbf{u}_k^{n+1},\bar{\mathbf{u}}), \quad \forall\bar{\mathbf{u}} \in \mathbb{Z}}$$

Implementation

- We will explain using a 3D solid element at a Gauss point using **updated Lagrangian** form
- The return mapping and consistent tangent operator will be the same with infinitesimal plasticity
- Voigt Notation

$$\{\sigma\} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{23} \quad \sigma_{13}]^T$$

$$\{\Delta\varepsilon\} = [\Delta\varepsilon_{11} \quad \Delta\varepsilon_{22} \quad \Delta\varepsilon_{33} \quad 2\Delta\varepsilon_{12} \quad 2\Delta\varepsilon_{23} \quad 2\Delta\varepsilon_{13}]^T$$

- Inputs $\Delta \mathbf{d}_I = \{\Delta d_{I1} \quad \Delta d_{I2} \quad \Delta d_{I3}\}^T$

$$\sigma^n = \{\sigma_{11}^n \quad \sigma_{22}^n \quad \sigma_{33}^n \quad \sigma_{12}^n \quad \sigma_{23}^n \quad \sigma_{13}^n\}^T$$

$$\xi^n = \{\alpha_{11}^n \quad \alpha_{22}^n \quad \alpha_{33}^n \quad \alpha_{12}^n \quad \alpha_{23}^n \quad \alpha_{13}^n \quad e_p^n\}^T$$

Implementation cont.

- In the updated Lagrangian, the derivative is evaluated at the **current configuration** (unknown yet)
- Let the current load step is n+1 (unknown) and k+1 N-R iteration
- Then, we use the configuration at the **previous iteration (n+1, k)** as a reference
- This is not 'true' updated Lagrangian, but when the N-R iteration converges, k is almost identical to k+1
- **Caution: we only update stresses at the converged load step, not individual iteration**
- All derivatives and integration in updated Lagrangian must be evaluated at (n+1, k) configuration
- **Displacement increment Δu is from (n+1,0) to (n+1,k)**

Implementation cont.

- Stress-displacement matrix (Two approaches)
  1. Mapping between current (n+1, k) and reference configurations

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi} & \frac{\partial x_2}{\partial \xi} & \frac{\partial x_3}{\partial \xi} \\ \frac{\partial x_1}{\partial \eta} & \frac{\partial x_2}{\partial \eta} & \frac{\partial x_3}{\partial \eta} \\ \frac{\partial x_1}{\partial \zeta} & \frac{\partial x_2}{\partial \zeta} & \frac{\partial x_3}{\partial \zeta} \end{bmatrix} \qquad \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{bmatrix}$$

  2. Mapping between undeformed and reference configurations

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} = \mathbf{F}^{-1} \begin{bmatrix} \frac{\partial}{\partial X_1} \\ \frac{\partial}{\partial X_2} \\ \frac{\partial}{\partial X_3} \end{bmatrix} \qquad \nabla_{n+1}\mathbf{u} = \mathbf{F}^{-1}\nabla_0\mathbf{u}$$

Use this for B matrix

Implementation cont.

1. Obtain midpoint configuration (between k and k+1)

$$\frac{\partial \mathbf{x}^n}{\partial \mathbf{x}^{n+1/2}} = \left[\mathbf{1} + \frac{1}{2}\frac{\partial \Delta \mathbf{u}}{\partial \mathbf{x}^n}\right]^{-1} \qquad \frac{\partial \Delta \mathbf{u}}{\partial \mathbf{x}^{n+1/2}} = \frac{\partial \Delta \mathbf{u}}{\partial \mathbf{x}^n} \cdot \frac{\partial \mathbf{x}^n}{\partial \mathbf{x}^{n+1/2}}$$

$$\Delta\varepsilon = \mathrm{sym}(\nabla_{n+\frac{1}{2}}\Delta\mathbf{u}) \qquad \mathbf{W} = \mathrm{skew}(\nabla_{n+\frac{1}{2}}\Delta\mathbf{u})$$

2. Rotation matrix: $\mathbf{R} = \mathbf{1} + (\mathbf{1} - \frac{1}{2}\mathbf{W})^{-1}\mathbf{W}$
3. Rotate stresses: $\bar{\sigma}^n = \mathbf{R}\cdot\sigma^n\cdot\mathbf{R}^T \qquad \bar{\alpha}^n = \mathbf{R}\cdot\alpha^n\cdot\mathbf{R}^T$
4. Return mapping with $\bar{\sigma}^n, \bar{\alpha}^n$
   - This part is identical to the classical return mapping
   - Calculate stresses: $\sigma_{k+1}^{n+1}, \alpha_{k+1}^{n+1}$
   - Calculate consistent tangent operator $\mathbf{D}^{alg}$

Implementation cont.

- Internal force

$$\mathbf{f}^{int} = \sum_{I=1}^{4}\sum_{K=1}^{NG}(\mathbf{B}_I^T \sigma_{k+1}^{n+1} |\mathbf{J}|)_K \omega_K$$

This summation is similar to assembly (must be added to the corresponding DOFs)

- Tangent stiffness matrix

$$\mathbf{K}_T = \sum_{I=1}^{4}\sum_{J=1}^{4}\sum_{K=1}^{NG}[\mathbf{B}_I^T(\mathbf{D}^{alg} - \mathbf{D}^*)\mathbf{B}_J |\mathbf{J}|]_K \omega_K$$

$$\mathbf{D}^* = \begin{bmatrix} -\sigma_{11} & \sigma_{11} & \sigma_{11} & -\sigma_{12} & 0 & -\sigma_{13} \\ \sigma_{22} & -\sigma_{22} & \sigma_{22} & -\sigma_{12} & -\sigma_{23} & 0 \\ \sigma_{33} & \sigma_{33} & -\sigma_{33} & 0 & -\sigma_{23} & -\sigma_{13} \\ -\sigma_{12} & -\sigma_{12} & 0 & -\frac{1}{2}(\sigma_{11}+\sigma_{22}) & -\frac{1}{2}\sigma_{13} & -\frac{1}{2}\sigma_{23} \\ 0 & -\sigma_{23} & -\sigma_{23} & -\frac{1}{2}\sigma_{13} & -\frac{1}{2}(\sigma_{22}+\sigma_{33}) & -\frac{1}{2}\sigma_{12} \\ -\sigma_{13} & 0 & -\sigma_{13} & -\frac{1}{2}\sigma_{23} & -\frac{1}{2}\sigma_{12} & -\frac{1}{2}(\sigma_{11}+\sigma_{33}) \end{bmatrix}$$

- Initial stiffness matrix

$$\mathbf{K}_S = \sum_{I=1}^{4}\sum_{J=1}^{4}\sum_{K=1}^{NG}[\mathbf{B}_I^{G^T} \Sigma \mathbf{B}_J^G |\mathbf{J}|]_K \omega_K$$

$$[\Sigma] = \begin{bmatrix} \sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma \end{bmatrix}_{9\times 9}$$

$$[\mathbf{B}_I^G] = \begin{bmatrix} N_{I,1} & 0 & 0 \\ N_{I,2} & 0 & 0 \\ N_{I,3} & 0 & 0 \\ 0 & N_{I,1} & 0 \\ 0 & N_{I,2} & 0 \\ 0 & N_{I,3} & 0 \\ 0 & 0 & N_{I,1} \\ 0 & 0 & N_{I,2} \\ 0 & 0 & N_{I,3} \end{bmatrix}$$

Implementation cont.

- Solve for incremental displacement

$$[\mathbf{K}_T + \mathbf{K}_S]\{\delta \mathbf{d}_{k+1}\} = \{\mathbf{f}^{ext}\} - \{\mathbf{f}^{int}\}$$

- Update displacements

$$\mathbf{d}_{k+1}^{n+1} = \mathbf{d}_k^{n+1} + \delta \mathbf{d}_{k+1}$$
$$\Delta \mathbf{d}_{k+1} = \Delta \mathbf{d}_{k+1} + \delta \mathbf{d}_{k+1}$$

- When N-R iteration converges
  - Stress and history dependent variables are stored (updated) to the global array
  - Move on to the next load step

Program PLAST3D.m

```
function PLAST3D(MID, PROP, ETAN, UPDATE, LTAN, NE, NDOF, XYZ, LE)
%***********************************************************************
%  MAIN PROGRAM COMPUTING GLOBAL STIFFNESS MATRIX RESIDUAL FORCE FOR
%  PLASTIC MATERIAL MODELS
%***********************************************************************
....
     % Computer stress, back stress & effective plastic strain
      elseif MID == 2
        % Plasticity with finite rotation
        FAC=FAC*det(F);
        [STRESSN, ALPHAN] = rotatedStress(DEPS, STRESSN, ALPHAN);
        [STRESS, ALPHA, EP]=combHard(PROP,ETAN,DDEPS,STRESSN,ALPHAN,EPN);
....
      %
      % Tangent stiffness
    if LTAN
     elseif MID == 2
      DTAN=combHardTan(PROP,ETAN,DDEPS,STRESSN,ALPHAN,EPN);
      CTAN=[-STRESS(1) STRESS(1) STRESS(1) -STRESS(4) 0 -STRESS(6);
            STRESS(2) -STRESS(2) STRESS(2) -STRESS(4) -STRESS(5) 0;
            STRESS(3) STRESS(3) -STRESS(3) 0 -STRESS(5) -STRESS(6);
            -STRESS(4) -STRESS(4) 0 -0.5*(STRESS(1)+STRESS(2)) -0.5*STRESS(6) -0.5*STRESS(5);
            0 -STRESS(5) -STRESS(5) -0.5*STRESS(6) -0.5*(STRESS(2)+STRESS(3)) -0.5*STRESS(4);
            -STRESS(6) 0 -STRESS(6) -0.5*STRESS(5) -0.5*STRESS(4) -0.5*(STRESS(1)+STRESS(3))];
      SIG=[STRESS(1) STRESS(4) STRESS(6);
           STRESS(4) STRESS(2) STRESS(5);
           STRESS(6) STRESS(5) STRESS(3)];
      SHEAD=zeros(9);
      SHEAD(1:3,1:3)=SIG;
      SHEAD(4:6,4:6)=SIG;
      SHEAD(7:9,7:9)=SIG;
      EKF = BM'*(DTAN+CTAN)*BM + BG'*SHEAD*BG;
....
```

Ex) Simple Shear Deformation

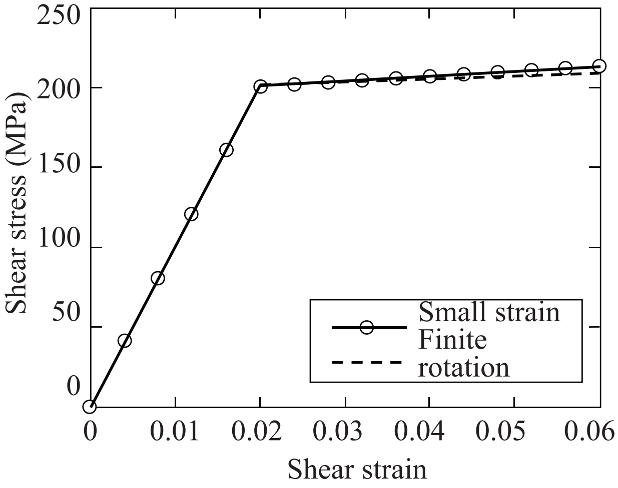
- Plane-strain square with the velocity gradient at each load step

$$\left[\frac{\partial \Delta \mathbf{u}}{\partial \mathbf{x}}\right] = \begin{bmatrix} 0 & 0.024 & 0 \\ -0.02 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E = 24\text{GPa},\ \nu = 0.2,$$
$$H = 1.0\text{GPa},\ \sigma_y^0 = 200\sqrt{3}\text{MPa}$$

```
Young = 24000; nu=0.2; mu=Young/2/(1+nu); lambda=nu*Young/((1+nu)*(1-2*nu));
beta = 0; H = 1000; sY = 200*sqrt(3);
mp = [lambda mu beta H sY];
Iden=[1 1 1 0 0 0]';
D=2*mu*eye(6) + lambda*Iden*Iden';
D(4,4) = mu; D(5,5) = mu; D(6,6) = mu;
L = zeros(3,3);
stressN=[0 0 0 0 0 0]';
deps=[0 0 0 0 0 0]';
alphaN = [0 0 0 0 0 0]';
epN=0;
stressRN=stressN; alphaRN=alphaN;epRN=epN;
for i=1:15
    deps(4) = 0.004; L(1,2) = 0.024; L(2,1) = -0.02;
    [stressRN, alphaRN] = rotatedStress(L, stressRN, alphaRN);
    [stressR, alphaR, epR]=combHard(mp,D,deps,stressRN,alphaRN,epRN);
    [stress, alpha, ep]=combHard(mp,D,deps,stressN,alphaN,epN);
    X(i) = i*deps(4); Y1(i) = stress(4); Y2(i) = stressR(4);
    stressN = stress; alphaN = alpha; epN = ep;
    stressRN = stressR; alphaRN = alphaR; epRN = epR;
end
X = [0 X]; Y1=[0 Y1]; Y2=[0 Y2]; plot(X,Y1,X,Y2);
```

Ex) Simple Shear Deformation



$$\text{stress} = [0 \quad 0 \quad 0 \quad 212.9 \quad 0 \quad 0]^T$$

$$\text{stressR} = [43.4 \quad -43.4 \quad 0 \quad 208.2 \quad 0 \quad 0]^T$$

Summary

- Finite rotation elastoplasticity is formulated using updated Lagrangian (reference frame moves with body)
- Finite rotation elastoplasticity is fundamentally identical to the classical plasticity. Only rigid-body rotation is taken into account using objective stress rate and integration
- We must use an objective stress rate to define the constitutive relation because the material response should be independent of coordinate system
- Objectivity only applies for spatial vectors and tensors
- In the finite rotation, the midpoint configuration is used to reduce errors involved in non-uniform rotation and spin
- Linearization is performed after transforming to the undeformed configuration

4.5

Finite Deformation Elastoplasticity with Hyperelasticity

Goals

- Understand the difference between hypoelasticity and hyperelasticity
- Learn the concept of multiplicative decomposition and intermediate configuration
- Understand the principle of maximum dissipation
- Understand the plastic evolution in strain space and stress space
- Learn J_2 plasticity in principal stress space

Finite Deformation Plasticity

- So far, we used small strain elastoplasticity theory
- Finite rotation has been taken care of using the deformed configuration with an objective rate
- However, still, the strain should be small enough so that the elastic and plastic strains are decomposed additively
- This is fundamental limitation of **hypoelasticity**
- How can we handle large strain problem?
- On the other hand, **hyperelasticity** can handle large strain
- However, it is not easy to describe plastic evolution in 2nd P-K stress. It is given in the current configuration (Cauchy stress)
- How can we handle it?**Transformation between references**

Intermediate Configuration

- Let's take one step back and discuss different references
- Lee (1967) proposed that the deformation gradient can be multiplicatively decomposed

$$\boxed{\mathbf{F}(\mathbf{X}) = \mathbf{F}_e(\mathbf{X})\mathbf{F}_p(\mathbf{X})}$$

  - Remember deformation gradient maps between deformed and undeformed configurations

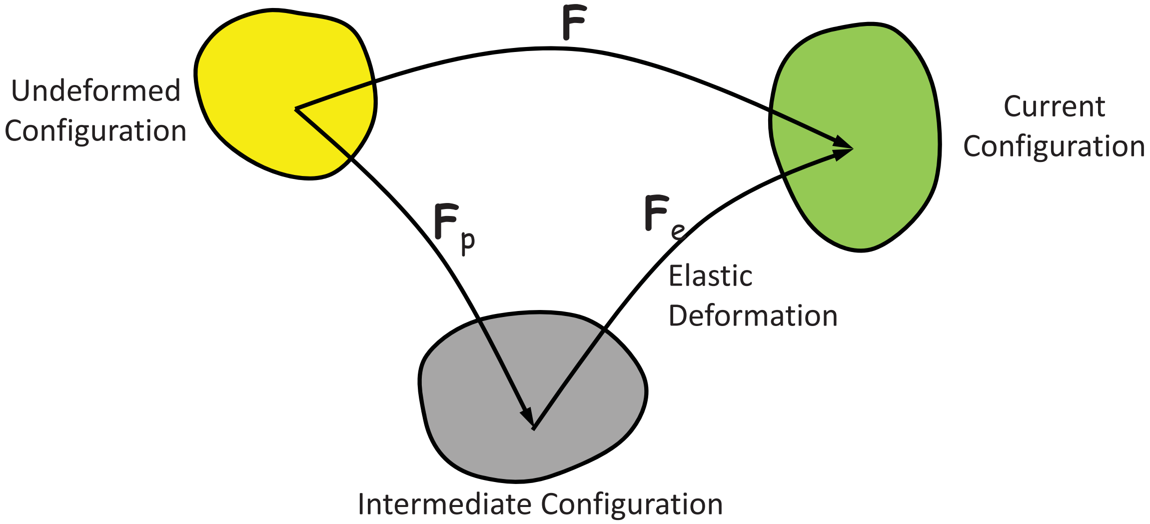
$$\mathbf{F}d\mathbf{X} = \mathbf{F}_e(\mathbf{F}_p d\mathbf{X}) = \mathbf{F}_e d\mathbf{x}_p$$

  - Instead of moving directly from Ω_0 to Ω_n, the deformation moves to an ***intermediate configuration*** **(Ω_p)** first and then goes to Ω_n
  - The intermediate configuration is an imaginary one and can be arbitrary

Additive decomposition: $\varepsilon = \varepsilon_e + \varepsilon_p$

Intermediate Configuration cont.

- $\mathbf{F}_p(\mathbf{X})$: deformation through the intermediate configuration (related to the internal plastic variables)
- $\mathbf{F}_e^{-1}(\mathbf{X})$: local, stress-free, unloaded process
- Decomposition of $\mathbf{F}(\mathbf{X})$ into the intermediate configuration followed by elastic deformation



Kirchhoff Stress – Matter of Convenience

- Kirchhoff stress $\boxed{\tau = J\sigma}$
  - This is different from 1st and 2nd P-K stress
  - It is defined using Cauchy stress with Jacobian effect (J = |**F**|)
  - When deformation is small $J \approx 1 \quad \Rightarrow \quad \tau \approx \sigma$
  - We assume the constitutive relation is given in terms of τ
- Why do we use different stress measure?
  - By including J into stress, we don't have to linearize it
  - We can integrate the energy form in Ω_0
  - But, still all integrands are defined in Ω_n

Elastic Domain and Free Energy

- Elastic domain

$$E \equiv \{(\tau, \mathbf{q}) \mid f(\tau, \mathbf{q}) \le 0\}$$

  - **q**: stress-like internal variables (hardening properties)
  - **Isotropy**: the yield function is independent of orientation of τ and **q** (**objectivity**)
- Free energy function (similar to strain energy density)

$$\psi = \psi(\mathbf{b}_e, \xi)$$

  - Elastic left C-G deformation tensor: $\mathbf{b}_e = \mathbf{F}_e\mathbf{F}_e^T$
  - strain-like internal variables vector: $\mathbf{q} = -\frac{\partial \psi}{\partial \xi}$
  - Free energy only depends on $\mathbf{F}_e$, and due to isotropy, $\mathbf{b}_e$

Dissipation Function

- Dissipation function (ignoring thermal part)

$$D \equiv \tau : \mathbf{d} - \frac{d}{dt}\psi(\mathbf{b}_e, \xi) \ge 0$$

  - Rate of stress work – rate of free energy change
  - Rate of deformation **d** = sym(**L**), where velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$
  - Dissipation is energy loss due to plastic deformation (irreversible)
- Rate of elastic left C-G tensor
  - We can't differentiate $\mathbf{b}_e$ because its reference is Ω_p
  - Transform to Ω_0 using $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ relation

$$\mathbf{b}_e = \mathbf{F}_e\mathbf{F}_e^T = (\mathbf{F}\mathbf{F}_p^{-1})(\mathbf{F}_p^{-T}\mathbf{F}^T) = \mathbf{F}(\mathbf{F}_p^{-1}\mathbf{F}_p^{-T})\mathbf{F}^T = \mathbf{F}\mathbf{C}_p^{-1}\mathbf{F}^T$$

$$\dot{\mathbf{b}}_e = \dot{\mathbf{F}}\mathbf{C}_p^{-1}\mathbf{F}^T + \mathbf{F}\mathbf{C}_p^{-1}\dot{\mathbf{F}}^T + \mathbf{F}\tfrac{d}{dt}\left(\mathbf{C}_p^{-1}\right)\mathbf{F}^T$$

Rate of Elastic Left C-G Tensor

- Rate of elastic left C-G tensor cont.

$$\dot{\mathbf{b}}_e = \dot{\mathbf{F}}\mathbf{C}_p^{-1}\mathbf{F}^T + \mathbf{F}\mathbf{C}_p^{-1}\dot{\mathbf{F}}^T + \mathbf{F}\tfrac{d}{dt}\left(\mathbf{C}_p^{-1}\right)\mathbf{F}^T$$

$$\longrightarrow \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\mathbf{F}_p^{-1}\mathbf{F}_p^{-T}\mathbf{F}^T = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{F}\mathbf{F}_p^{-1}\mathbf{F}_p^{-T}\mathbf{F}^T = \mathbf{L}\mathbf{F}_e\mathbf{F}_e^T = \mathbf{L}\mathbf{b}_e$$

$$\longrightarrow \mathbf{b}_e\mathbf{L}^T$$

$\mathbf{C}_p$: plastic right C-G deformation tensor

  - **Lie derivative**: $\mathbf{F}\tfrac{d}{dt}\left(\mathbf{C}_p^{-1}\right)\mathbf{F}^T = L_v\mathbf{b}_e$ pulling $\mathbf{b}_e$ back to the undeformed configuration, and after taking a time derivative, pushing forward to the current configuration (**plastic deformation**)

- Thus, we have

$$\boxed{\dot{\mathbf{b}}_e = \underbrace{\mathbf{L}\mathbf{b}_e + \mathbf{b}_e\mathbf{L}^T}_{\text{Elastic}} + \underbrace{L_v\mathbf{b}_e}_{\text{Plastic}}}$$

Dissipation Function cont.

- Dissipation function cont.

$$
\begin{aligned}
D &= \tau : \mathbf{d} - \frac{d}{dt}\psi(\mathbf{b}_e, \xi) \\
&= \tau : \mathbf{d} - \frac{\partial\psi}{\partial \mathbf{b}^e} : \dot{\mathbf{b}}_e - \frac{\partial\psi}{\partial\xi}\cdot\dot{\xi} \\
&= \tau : \mathbf{d} - \frac{\partial\psi}{\partial \mathbf{b}^e} : \left(\mathbf{L}\mathbf{b}_e + \mathbf{b}_e\mathbf{L}^T + L_v\mathbf{b}_e\right) + \mathbf{q}\cdot\dot{\xi} \\
&= \tau : \mathbf{d} - 2\frac{\partial\psi}{\partial \mathbf{b}_e}\mathbf{b}_e : \mathbf{L} + \left(2\frac{\partial\psi}{\partial \mathbf{b}_e}\mathbf{b}_e\right) : \left[-\tfrac{1}{2}(L_v\mathbf{b}_e)\mathbf{b}_e^{-1}\right] + \mathbf{q}\cdot\dot{\xi} \\
&= \left(\tau - 2\frac{\partial\psi}{\partial \mathbf{b}_e}\mathbf{b}_e\right) : \mathbf{d} + \left(2\frac{\partial\psi}{\partial \mathbf{b}_e}\mathbf{b}_e\right) : \left[-\tfrac{1}{2}(L_v\mathbf{b}_e)\mathbf{b}_e^{-1}\right] + \mathbf{q}\cdot\dot{\xi} \geq 0
\end{aligned}
$$

For a symmetric matrices, **A**:**BC** = **AC**:**B**
For a symmetric **S** and general **L**, **S**:**L** = **S**:sym(**L**)

Principle of Maximum Dissipation

- Principle of Maximum Dissipation
  - For all admissible stresses and internal variables, the inequality must satisfy

$$D = \left(\tau - 2\frac{\partial\psi}{\partial\mathbf{b}_e}\mathbf{b}_e\right):\mathbf{d} + \left(2\frac{\partial\psi}{\partial\mathbf{b}_e}\mathbf{b}_e\right):\left[-\tfrac{1}{2}(L_v\mathbf{b}_e)\mathbf{b}_e^{-1}\right] + \mathbf{q}\cdot\dot{\xi} \ge 0$$

  - If we consider the material is elastic, then no plastic variable will change $L_v\mathbf{b}_e = \dot{\xi} = 0$
  - In order to satisfy the inequality for any **d** (especially $\mathbf{d}_1 = -\mathbf{d}_2$)

$$\boxed{\tau = 2\frac{\partial\psi}{\partial\mathbf{b}_e}\mathbf{b}_e}$$

Constitutive relation

  - **Total form**: constitutive relation is given in terms of stress, not stress increment
  - In addition, we have $\dfrac{\partial\psi}{\partial\mathbf{b}_e} = \dfrac{1}{2}\tau\cdot\mathbf{b}_e^{-1}$

Principle of Maximum Dissipation cont.

- Reduced dissipation function

$$\left(2\frac{\partial\psi}{\partial\mathbf{b}_e}\mathbf{b}_e\right):\left[-\tfrac{1}{2}(L_v\mathbf{b}_e)\mathbf{b}_e^{-1}\right] + \mathbf{q}\cdot\dot{\xi} \ge 0$$

$$\Rightarrow \boxed{\tau:\left[-\tfrac{1}{2}(L_v\mathbf{b}_e)\mathbf{b}_e^{-1}\right] + \mathbf{q}\cdot\dot{\xi} \ge 0}$$

Plastic dissipation

- Principle of Maximum Dissipation
  - Plastic deformation occurs in the direction that maximizes D
  - In classical associative plasticity

$$D = \sigma : \dot{\varepsilon}_p + \mathbf{q}\cdot\dot{\xi} \ge 0$$

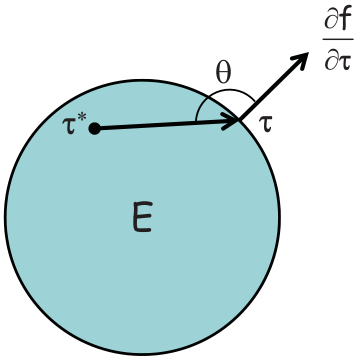
$$\varepsilon_p \parallel \frac{\partial f}{\partial \sigma} \quad \Rightarrow \quad \dot{\varepsilon}_p = \dot{\gamma}\frac{\partial f}{\partial \sigma}$$

Principle of Maximum Dissipation cont.

- Principle of Maximum Dissipation cont.
  - For given rates $\{L_v \mathbf{b}_e, \dot{\xi}\}$, state variables $\{\tau, \mathbf{q}\}$ maximize the dissipation function D

$$D = (\tau - \tau^*) : \left[-\tfrac{1}{2}(L_v \mathbf{b}_e)\mathbf{b}_e^{-1} \right] + (\mathbf{q} - \mathbf{q}^*) \cdot \dot{\xi} \geq 0, \quad \forall \left\{ \tau^*, \mathbf{q}^* \right\} \in E$$

  - **For classical variational inequality, the dissipation inequality satisfies if and only if the coefficients are in the normal direction of the elastic domain (defined by yield function)**
- Geometric interpretation
  - All τ^* should reside inside of E
  - Thus, the angle θ should be greater than or equal to 90°
  - In order to satisfy for all τ^*, $-\tfrac{1}{2}(L_v \mathbf{b}_e)\mathbf{b}_e^{-1}$ should be normal to yield surface



Principle of Maximum Dissipation cont.

- Evolution equations for multiplicative decomposition

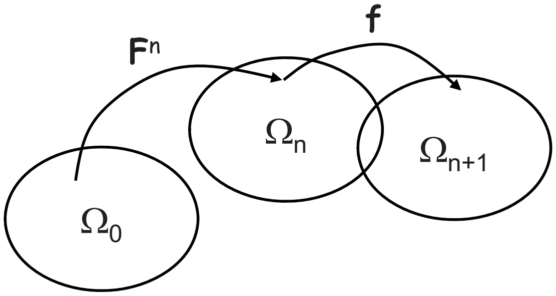
$$\boxed{-\tfrac{1}{2} L_v \mathbf{b}_e = \dot{\gamma} \frac{\partial f(\tau, \mathbf{q})}{\partial \tau} \mathbf{b}_e} \qquad \boxed{\dot{\xi} = \dot{\gamma} \frac{\partial f(\tau, \mathbf{q})}{\partial \mathbf{q}}}$$

$$\dot{\gamma} \geq 0, \quad f(\tau, \mathbf{q}) \leq 0, \quad \dot{\gamma} f(\tau, \mathbf{q}) = 0.$$

  - Plastic evolution is still in a rate form
  - Stress is hyperelastic (total form)
  - Plastic evolution is given in terms of strain ($\mathbf{b}_e$ and ξ)
  - We need to integrate these equations

Time Integration

- Given: $\{\mathbf{F}^n \quad \mathbf{b}_e^n \quad \xi^n\}$ and $\Delta\mathbf{u}$
- **Relative deformation gradient**



$$\mathbf{f}(\mathbf{x}) = \frac{\partial \mathbf{x}^{n+1}}{\partial \mathbf{x}^n} = \mathbf{1} + \nabla_n \Delta\mathbf{u}$$

$$\mathbf{F}^{n+1} = \mathbf{f} \cdot \mathbf{F}^n$$

$$\dot{\mathbf{f}} = \frac{\partial \dot{\mathbf{u}}}{\partial \mathbf{x}^n} = \frac{\partial \dot{\mathbf{u}}}{\partial \mathbf{x}^{n+1}} \frac{\partial \mathbf{x}^{n+1}}{\partial \mathbf{x}^n} = \mathbf{L}\mathbf{f}$$

- First-order evolution equations

$$\dot{\mathbf{b}}_e = \left[\mathbf{L}\mathbf{b}_e + \mathbf{b}_e\mathbf{L}^T \right] - 2\dot{\gamma} \frac{\partial f(\tau, \mathbf{q})}{\partial \tau} \mathbf{b}_e$$

$$\dot{\xi} = \dot{\gamma} \frac{\partial f(\tau, \mathbf{q})}{\partial \mathbf{q}}$$

$$\dot{\gamma} \ge 0, \quad f(\tau, \mathbf{q}) \le 0, \quad \dot{\gamma} f(\tau, \mathbf{q}) = 0$$

Initial conditions

$$\{\mathbf{f}, \mathbf{b}_e, \xi\}\big|_{t=t_n} = \{\mathbf{1}, \mathbf{b}_e^n, \xi^n\}$$

Strain-based evolution

Time Integration cont.

- Constitutive law

$$\tau = 2 \frac{\partial \psi}{\partial \mathbf{b}_e} \mathbf{b}_e \qquad \mathbf{q} = -\frac{\partial \psi(\mathbf{b}_e, \xi)}{\partial \xi}$$

  - **The constitutive relation is hyperelastic**
  - Once $\mathbf{b}_e$ is found, stress can be calculated by differentiating the free energy function. Same for the internal variables
- Elastic predictor (no plastic flow)
  - Similar to classical plasticity, we will use elastic predictor and plastic corrector algorithm
  - For given incremental displacement, eliminate plastic flow and push the elastic, left C-G tensor forward to the current configuration

$$\mathbf{f}^{tr} = \mathbf{f} \;\Rightarrow\; \mathbf{F}_e^{tr} = \mathbf{f} \cdot \mathbf{F}_e^n \quad \mathbf{F}_p^{tr} = \mathbf{F}_p^n$$

Time Integration cont.

- Elastic predictor cont.

$$\mathbf{b}_e^{tr} = \mathbf{F}_e^{tr} \cdot \mathbf{F}_e^{trT} = \mathbf{f} \cdot \mathbf{F}_e^{n} \cdot \mathbf{F}_e^{nT} \cdot \mathbf{f}^T = \mathbf{f}\mathbf{b}_e^n\mathbf{f}^T$$

$$\boxed{\mathbf{f}^{tr} = \mathbf{f} \quad \mathbf{b}_e^{tr} = \mathbf{f}\mathbf{b}_e^n\mathbf{f}^T \quad \xi^{tr} = \xi^n}$$

- Check for yield status

$$\tau^{tr} = 2\frac{\partial\psi}{\partial \mathbf{b}_e^{tr}}\mathbf{b}_e^{tr} \qquad \mathbf{q}^{tr} = -\frac{\partial\psi(\mathbf{b}_e^{tr}, \xi^{tr})}{\partial \xi^{tr}}$$

  - If $\tau^{tr} < f$, trial state is final state and stop

$$\boxed{\begin{aligned} \mathbf{b}_e^{n+1} &= \mathbf{b}_e^{tr} & \xi^{n+1} &= \xi^{tr} \\ \tau^{n+1} &= \tau^{tr} & \mathbf{q}^{n+1} &= \mathbf{q}^{tr} \end{aligned}}$$

Time Integration cont.

- Plastic corrector (in the fixed current configuration)
  - The solution of $\dot{y} = Ay$ is $y = y_0\exp(At)$

$$\dot{\mathbf{b}}_e = \underbrace{\mathbf{L}\mathbf{b}_e + \mathbf{b}_e\mathbf{L}^T}_{\text{Elastic}} \underbrace{-2\dot{\gamma}\frac{\partial f}{\partial \tau}\mathbf{b}_e}_{\text{Plastic}}$$

$$\boxed{\begin{aligned} \mathbf{b}_e^{n+1} &= \mathbf{b}_e^{tr}\exp\left[-2\Delta\gamma\frac{\partial f(\tau^e, \mathbf{q})}{\partial \tau}\right] \\ \xi^{n+1} &= \xi^{tr} + \Delta\gamma\frac{\partial f(\tau^e, \mathbf{q})}{\partial \mathbf{q}} \\ \Delta\gamma \geq 0, &\quad f(\tau, \mathbf{q}) \leq 0, \quad \Delta\gamma f(\tau, \mathbf{q}) = 0 \end{aligned}}$$

$$\Delta\gamma = \dot{\gamma}\Delta t$$

  - First-order accuracy and unconditional stability
  - **return-mapping algorithms for the left Cauchy-Green tensor**

Spectral Decomposition

- Objective: want to get a similar return mapping algorithm with classical plasticity
- Return-mapping algorithm for principal Kirchhoff stress
- For isotropic material, the principal direction of τ is parallel to that of $\mathbf{b}_e$
- Spectral decomposition

$$\mathbf{b}_e = \sum_{i=1}^{3} \lambda_i^2 \, \hat{\mathbf{n}}^i \otimes \hat{\mathbf{n}}^i \qquad \tau = \sum_{i=1}^{3} \tau_{pi} \, \hat{\mathbf{n}}^i \otimes \hat{\mathbf{n}}^i$$

$$\mathbf{b}_e^{n+1} = \mathbf{b}_e^{tr} \exp\left[\cdots\right]$$

$\mathbf{b}_e$ and $\mathbf{b}_e{}^{tr}$ have the same eigenvectors!!

λ_i : principal stretch

τ_{pi} : principal Kirchhoff stress

$\hat{\mathbf{n}}^i$: spatial eigenvector

$\hat{\mathbf{N}}^i$: material eigenvector

Do you remember that η // η^{tr} in classical plasticity?

Return Mapping in Principal Stress Space

- Principal stress vector $\tau_p = [\tau_{p1}, \tau_{p2}, \tau_{p3}]^T$
- Logarithmic elastic principal strain vector

$$\mathbf{e} = [e_1 \quad e_2 \quad e_3]^T = [\log\lambda_1 \quad \log\lambda_2 \quad \log\lambda_3]^T$$

Good for large elastic strain

- Free energy for J_2 plasticity

$$\psi(\mathbf{e}, \xi) = \tfrac{1}{2}\lambda[e_1 + e_2 + e_3]^2 + \mu[e_1{}^2 + e_2{}^2 + e_3{}^2] + \hat{K}(\xi)$$

- Constitutive relation in principal space

$$\tau_p = \frac{\partial \psi}{\partial \mathbf{e}} = \mathbf{c}^e \cdot \mathbf{e} \qquad \mathbf{c}^e = (\lambda + \tfrac{2}{3}\mu)\hat{\mathbf{1}} \otimes \hat{\mathbf{1}} + 2\mu\mathbf{1}_{dev}$$

$$\hat{\mathbf{1}} = [1, 1, 1]^T \qquad \mathbf{1}_{dev} = \mathbf{1} - \tfrac{1}{3}(\hat{\mathbf{1}} \otimes \hat{\mathbf{1}})$$

  - Linear relation between principal Kirchhoff stress and logarithmic elastic principal strain

Return Mapping in Principal Stress Space cont.

- Take log on return mapping for $\mathbf{b}_e$ and pre-multiply with $\mathbf{c}^e$

$$\mathbf{b}_e = \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{n}}^i \otimes \hat{\mathbf{n}}^i \qquad \Rightarrow \quad \log(\mathbf{b}_e) = \sum_{i=1}^{3} 2\log(\lambda_i) \hat{\mathbf{n}}^i \otimes \hat{\mathbf{n}}^i = \sum_{i=1}^{3} 2 e_i \hat{\mathbf{n}}^i \otimes \hat{\mathbf{n}}^i$$

$$\mathbf{b}_e^{n+1} = \mathbf{b}_e^{tr} \exp\left[-2\Delta\gamma \frac{\partial f(\tau^e, \mathbf{q})}{\partial \tau} \right]$$

$$\Rightarrow \quad \log(\mathbf{b}_e^{n+1}) = \log(\mathbf{b}_e^{tr}) + \log\exp\left[-2\Delta\gamma \frac{\partial f(\tau, \mathbf{q})}{\partial \tau} \right]$$

$$f(\tau, \mathbf{q}) = \hat{f}(\tau_p, \mathbf{q}) \quad \Rightarrow \quad \frac{\partial f}{\partial \tau} = \sum_{i=1}^{3} \frac{\partial \hat{f}}{\partial \tau_{pi}} \hat{\mathbf{n}}^i \otimes \hat{\mathbf{n}}^i$$

$$\Rightarrow \quad 2\mathbf{e}^{n+1} = 2\mathbf{e}^{tr} - 2\Delta\gamma \frac{\partial \hat{f}(\tau_p, \mathbf{q})}{\partial \tau_p}$$

$$\Rightarrow \quad \mathbf{c}^e \cdot \mathbf{e}^{n+1} = \mathbf{c}^e \cdot \mathbf{e}^{tr} - \Delta\gamma \mathbf{c}^e \cdot \frac{\partial \hat{f}(\tau_p, \mathbf{q})}{\partial \tau_p}$$

Return Mapping in Principal Stress Space cont.

- Plastic evolution in principal stress space

$$\tau_p = \tau_p^{tr} - \Delta\gamma\, \mathbf{c}^e \cdot \frac{\partial \hat{f}(\tau_p, \mathbf{q})}{\partial \tau_p}$$

$$\xi^{n+1} = \xi^n + \Delta\gamma \frac{\partial \hat{f}(\tau_p, \mathbf{q})}{\partial \mathbf{q}}$$

$$\Delta\gamma \geq 0, \quad \hat{f}(\tau_p, \mathbf{q}) \leq 0, \quad \Delta\gamma\, \hat{f}(\tau_p, \mathbf{q}) = 0$$

$$\tau_p^{tr} = \mathbf{c}^e \cdot \mathbf{e}^{tr}$$

  - Fundamentally the same with classical plasticity: Classical plasticity [σ(6×1) and **C**(6×6)], but here [τ_p(3×1) and $\mathbf{c}^e$(3×3)]
  - **During the plastic evolution, the principal direction remains constant (fixed current configuration)**
  - **Only principal stresses change**

Return Mapping Algorithm

- Deviatoric principal stress

$$\mathbf{s} = \tau_p - \tfrac{1}{3}(\tau_p \cdot \hat{\mathbf{1}})\hat{\mathbf{1}} = \mathbf{1}_{dev} \cdot \tau_p$$

- Yield function

$$f(\eta, e_p) = \|\eta\| - \sqrt{\tfrac{2}{3}}\kappa(e_p) \le 0$$

$$\eta = \mathbf{s} - \alpha$$

$$e_p = \int_0^t \sqrt{\tfrac{2}{3}}\|\dot{\mathbf{e}}_p(t)\| dt$$

- Return mapping

$$\tau_p^{tr} = \mathbf{c}^e \cdot \mathbf{e}^{tr}$$

$$\tau_p^{n+1} = \tau_p^{tr} - 2\mu\Delta\gamma\mathbf{N}$$

$$\alpha^{n+1} = \alpha^{tr} + \Delta\gamma H_\alpha \mathbf{N}$$

$$e_p^{n+1} = e_p^{tr} + \sqrt{\tfrac{2}{3}}\Delta\gamma$$

$$\alpha^{tr} = \alpha^n$$

$$e_p^{tr} = e_p^n$$

$$\mathbf{N} = \frac{\eta^{n+1}}{\|\eta^{n+1}\|} = \frac{\eta^{tr}}{\|\eta^{tr}\|}$$

Identical to the classical plasticity

Return Mapping Algorithm cont.

- Plastic consistency parameter

$$f(\eta^{n+1}, e_p^{n+1}) = \|\eta^{n+1}\| - \sqrt{\tfrac{2}{3}}\kappa(e_p^{n+1})$$
$$= \|\eta^{tr}\| - (2\mu + H_\alpha)\Delta\gamma - \sqrt{\tfrac{2}{3}}\kappa(e_p^{n+1}) = 0$$

  - Solve for $\Delta\gamma$ using N-R iteration, or directly for linear hardening
  - Derivative $\frac{\partial f}{\partial \Delta\gamma} = -(2\mu + H_\alpha + \sqrt{\tfrac{2}{3}} H_{\alpha,e_p}\Delta\gamma + \tfrac{2}{3}\kappa_{,e_p}) = -\frac{1}{A}$

- Recovery
  - Once return mapping converged, recover stress and strain

$$\boxed{\tau^{n+1} = \sum_{i=1}^{3} \tau_{pi}^{n+1}\hat{\mathbf{n}}^i \otimes \hat{\mathbf{n}}^i = \sum_{i=1}^{3} \tau_{pi}^{n+1}\mathbf{m}^i}$$

$$\mathbf{m}^i = \hat{\mathbf{n}}^i \otimes \hat{\mathbf{n}}^i$$

$$\mathbf{b}_e^{n+1} = \sum_{i=1}^{3} \exp(2e_i^{n+1})\mathbf{m}^i \qquad \mathbf{e}^{n+1} = \mathbf{e}^{tr} - \Delta\gamma\mathbf{N}$$

Ex) Incompressible Elastic Cube

- Elastic deformation: $x_1 = \alpha X_1, \quad x_2 = \beta X_2, \quad x_3 = \beta X_3$
- Deformation gradient

$$\mathbf{F} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^T = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^2 \end{bmatrix}$$

- Incompressibility: det(**F**)=1 $\quad \beta = 1/\sqrt{\alpha}$
- Eigenvalues and eigenvectors:

$$\begin{aligned} \lambda_1 &= \alpha^2, & \mathbf{n}^1 &= [1 \quad 0 \quad 0]^T \\ \lambda_2 &= \alpha^{-1}, & \mathbf{n}^2 &= [0 \quad 1 \quad 0]^T \\ \lambda_3 &= \alpha^{-1}, & \mathbf{n}^3 &= [0 \quad 0 \quad 1]^T \end{aligned}$$

- Logarithmic stretches:

$$\mathbf{e} = \{2\log\alpha \quad -\log\alpha \quad -\log\alpha\}^T$$

Ex) Incompressible Elastic Cube

- Stress-strain relation (principal space)

$$\tau^p = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{bmatrix} \begin{Bmatrix} 2\log\alpha \\ -\log\alpha \\ -\log\alpha \end{Bmatrix} = \begin{Bmatrix} 4\mu\log\alpha \\ -2\mu\log\alpha \\ -2\mu\log\alpha \end{Bmatrix}$$

- Kirchhoff stress

$$\tau = \sum_{i=1}^{3} \tau_i^p \mathbf{n}^i \otimes \mathbf{n}^i = 2\mu\log\alpha \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Consistent Tangent Operator

- Relation b/w material and spatial tangent operators

$$\mathbf{D} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} \qquad \Rightarrow \qquad \bar{\mathbf{E}} : \mathbf{D} : \Delta\mathbf{E} = [\mathbf{F}^T \varepsilon(\bar{\mathbf{u}})\mathbf{F}] : \mathbf{D} : [\mathbf{F}^T \varepsilon(\Delta\mathbf{u})\mathbf{F}] = \varepsilon(\bar{\mathbf{u}}) : \mathbf{c} : \varepsilon(\Delta\mathbf{u})$$

$$c_{ijkl} = F_{ir}F_{js}F_{km}F_{ln}D_{rsmn} = F_{ir}F_{js}F_{km}F_{ln}\frac{\partial S_{rs}}{\partial E_{mn}}$$

  - $F_{ir}F_{js}$: transform stress to material frame $\tau = \mathbf{FSF}^T$
  - $F_{km}F_{ln}$: differentiate w.r.t. E and then transform to spatial frame

- But, $c_{ijkl} \neq \dfrac{\partial \tau_{ij}}{\partial \varepsilon_{kl}}$

$$\frac{\partial w}{\partial \varepsilon} = \mathbf{F}\frac{\partial w}{\partial \mathbf{E}}\mathbf{F}^T$$

- Let $\tau^{n+1} = \sum_{i=1}^{3} \tau_{pi}^{n+1}\mathbf{m}^i$

- We want $\mathbf{c} = \partial\tau/\partial\varepsilon$, but we have

$$\frac{\partial \tau_p}{\partial \mathbf{e}} = \mathbf{c}^{alg} = \mathbf{c}^e - 4\mu^2 A\mathbf{N} \otimes \mathbf{N} - \frac{4\mu^2\Delta\gamma}{\left\|\eta^{tr}\right\|}[\mathbf{1}_{dev} - \mathbf{N} \otimes \mathbf{N}]$$

Consistent Tangent Operator cont.

- How to obtain $\mathbf{c} = \partial\tau/\partial\varepsilon$ using $\mathbf{c}^{alg} = \partial\tau_p/\partial\mathbf{e}$?
- Remember $\partial\tau_p/\partial\mathbf{e}$ contains all plasticity
- Since intermediate frame is reference, we have to use $\mathbf{F}_e$
- Start from stress expression

$$\tau^{n+1} = \sum_{i=1}^{3} \tau_{pi}^{n+1}\mathbf{m}^i$$

$$\mathbf{c} = \frac{\partial \tau}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon}\left(\sum_{i=1}^{3}\tau_{pi}^{n+1}\mathbf{m}^i\right) = \sum_{i=1}^{3}\left[\frac{\partial \tau_{pi}^{n+1}}{\partial \varepsilon} \otimes \mathbf{m}^i + \tau_{pi}^{n+1}\frac{\partial \mathbf{m}^i}{\partial \varepsilon}\right]$$

$$\mathbf{c} = \sum_{i=1}^{3}\left[\sum_{j=1}^{3}\underset{\underset{(1)}{\uparrow}}{\frac{\partial \tau_{pi}^{n+1}}{\partial e_j^{tr}}}\,\underset{\underset{(2)}{\uparrow}}{\frac{\partial e_j^{tr}}{\partial \varepsilon}} \otimes \mathbf{m}^i + \tau_{pi}^{n+1}\underset{\underset{(3)}{\uparrow}}{\frac{\partial \mathbf{m}^i}{\partial \varepsilon}}\right]$$

Consistent Tangent Operator cont.

(1) $\frac{\partial \tau_p}{\partial \mathbf{e}^{tr}} = \mathbf{c}^{alg}$

consistent tangent operator in principal stress
Same as classical return mapping (3×3)

(2) $\frac{\partial e_j^{tr}}{\partial \varepsilon} = 2\mathbf{F}_e \frac{\partial e_j^{tr}}{\partial \mathbf{C}_e} \mathbf{F}_e^T = \mathbf{m}^j$

(3) $\frac{\partial \mathbf{m}^i}{\partial \varepsilon} = 2\mathbf{F}_e \frac{\partial \mathbf{m}^i}{\partial \mathbf{C}_e} \mathbf{F}_e^T = 2\hat{\mathbf{c}}^i$

These are elastic

- Using (1), (2), and (3),

$$\boxed{\mathbf{c} = \frac{\partial \tau}{\partial \varepsilon} = \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}^{alg} \mathbf{m}^i \otimes \mathbf{m}^j + \sum_{i=1}^{3} 2\tau_{pi} \hat{\mathbf{c}}^i}$$

Incremental Variational Principle

- Energy form (nonlinear)

$$a({}^n\xi; \mathbf{u}, \bar{\mathbf{u}}) = \iint_{\Omega_n} \sigma : \varepsilon(\bar{\mathbf{u}})\, d\Omega = \iint_{\Omega_0} \sigma : \varepsilon(\bar{\mathbf{u}}) J\, d\Omega$$
$$= \iint_{\Omega_0} \tau : \varepsilon(\bar{\mathbf{u}})\, d\Omega$$

- Linearization

$$\boxed{a^*({}^n\xi, {}^n\mathbf{b}^e, \mathbf{u}; \Delta\mathbf{u}, \bar{\mathbf{u}}) \equiv \iint_{{}^0\Omega} \left[\varepsilon(\bar{\mathbf{u}}) : \mathbf{c} : \varepsilon(\Delta\mathbf{u}) + \tau : \eta(\Delta\mathbf{u}, \bar{\mathbf{u}}) \right] d\Omega}$$

- N-R iteration

$$\boxed{a^*({}^n\xi, {}^{n+1}\mathbf{u}^{k+1}; \Delta\mathbf{u}^{k+1}, \bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}) - a({}^n\xi; {}^{n+1}\mathbf{u}^k, \bar{\mathbf{u}}), \quad \forall \bar{\mathbf{u}} \in \mathbb{Z}}$$

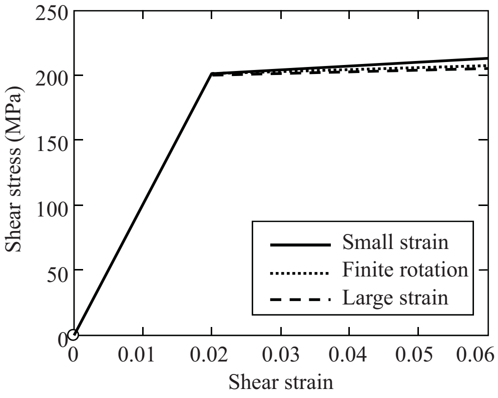
MATLAB Code MULPLAST

```
function [stress, b, alpha, ep]=mulPlast(mp,D,L,b,alpha,ep)
%mp = [lambda, mu, beta, H, Y0];
%D = elasticity matrix b/w prin stress & log prin stretch (3x3)
%L = [dui/dxj] velocity gradient
%b = elastic left C-G deformation vector (6x1)
%alpha = principal back stress (3x1)
%ep = effective plastic strain
%
EPS=1E-12;
Iden = [1 1 1]'; two3 = 2/3; stwo3=sqrt(two3); %constants
mu=mp(2); beta=mp(3); H=mp(4); Y0=mp(5);       %material properties
ftol = Y0*1E-6;                                %tolerance for yield
R = inv(eye(3)-L);                             %inc. deformation gradient
bm=[b(1) b(4) b(6);b(4) b(2) b(5);b(6) b(5) b(3)];
bm = R*bm*R';                                  %trial elastic left C-G
b=[bm(1,1) bm(2,2) bm(3,3) bm(1,2) bm(2,3) bm(1,3)]';
[~,P]=eig(bm);                                 %eigenvalues
eigen=sort(real([P(1,1) P(2,2) P(3,3)]))';           %principal stretch
%
% Duplicated eigenvalues
TMP=-1;
for I=1:2
  if abs(eigen(1)-eigen(3)) < EPS
    eigen(I)=eigen(I)+TMP*EPS;
    TMP=-TMP;
  end
end
if abs(eigen(1)-eigen(2)) < EPS; eigen(2) = eigen(2) + EPS; end;
if abs(eigen(2)-eigen(3)) < EPS; eigen(2) = eigen(2) + EPS; end;
%
% EIGENVECTOR MATRIX N*N' = M(6,*)
M=zeros(6,3);                                  %eigenvector matrices
```

```
for K=1:3
  KB=1+mod(K,3);
  KC=1+mod(KB,3);
  EA=eigen(K);
  EB=eigen(KB);
  EC=eigen(KC);
  D1=EB-EA;
  D2=EC-EA;
  DA = 1 / (D1 * D2);
  M(1,K)=((b(1)-EB)*(b(1)-EC)+b(4)*b(4)+b(6)*b(6))*DA;
  M(2,K)=((b(2)-EB)*(b(2)-EC)+b(4)*b(4)+b(5)*b(5))*DA;
  M(3,K)=((b(3)-EB)*(b(3)-EC)+b(5)*b(5)+b(6)*b(6))*DA;
  M(4,K)=(b(4)*(b(1)-EB+b(2)-EC)+b(5)*b(6))*DA;
  M(5,K)=(b(5)*(b(2)-EB+b(3)-EC)+b(4)*b(6))*DA;
  M(6,K)=(b(6)*(b(3)-EB+b(1)-EC)+b(4)*b(5))*DA;
end
%
eigen=sort(real([P(1,1) P(2,2) P(3,3)]))';     %principal stretch
deps = 0.5*log(eigen);                         %logarithmic
sigtr = D*deps;                                %trial principal stress
eta = sigtr - alpha - sum(sigtr)*Iden/3;       %shifted stress
etat = norm(eta);                              %norm of eta
fyld = etat - stwo3*(Y0+(1-beta)*H*ep);        %trial yield function
if fyld < ftol                                 %yield test
   sig = sigtr;                                %trial states are final
   stress = M*sig;                             %stress (6x1)
else
   gamma = fyld/(2*mu + two3*H);               %plastic consistency param
   ep = ep + gamma*stwo3;                      %updated eff. plastic strain
   N = eta/etat;                               %unit vector normal to f
   deps = deps - gamma*N;                      %updated elastic strain
   sig = sigtr - 2*mu*gamma*N;                 %updated stress
   alpha = alpha + two3*beta*H*gamma*N;        %updated back stress
   stress = M*sig;                             %stress (6x1)
   b = M*exp(2*deps);                          %updated elastic left C-G
end
```

Ex) Shear Deformation of a Square

```
Young = 24000; nu=0.2; mu=Young/2/(1+nu); lambda=nu*Young/((1+nu)*(1-2*nu));
beta = 0; H = 1000; sY = 200*sqrt(3);
mp = [lambda mu beta H sY];
Iden=[1 1 1 0 0 0]';
D=2*mu*eye(6) + lambda*Iden*Iden';
D(4,4) = mu; D(5,5) = mu; D(6,6) = mu;
Iden=[1 1 1]';
DM=2*mu*eye(3) + lambda*Iden*Iden';
L = zeros(3,3);
stressN=[0 0 0 0 0 0]';
deps=[0 0 0 0 0 0]';
alphaN = [0 0 0 0 0 0]';
epN=0;
stressRN=stressN; alphaRN=alphaN;epRN=epN;
bMN=[1 1 1 0 0 0]';
alphaMN = [0 0 0]';
epMN=0;
for i=1:15
    deps(4) = 0.004; L(1,2) = 0.024; L(2,1) = -0.02;
    [stressRN, alphaRN] = rotatedStress(L, stressRN, alphaRN);
    [stressR, alphaR, epR]=combHard(mp,D,deps,stressRN,alphaRN,epRN);
    [stress, alpha, ep]=combHard(mp,D,deps,stressN,alphaN,epN);
    [stressM, bM, alphaM, epM]=mulPlast(mp,DM,L,bMN,alphaMN,epMN);
    X(i)=i*deps(4);Y1(i)=stress(4);Y2(i)=stressR(4);Y3(i)=stressM(4);
    stressN = stress; alphaN = alpha; epN = ep;
    stressRN = stressR; alphaRN = alphaR; epRN = epR;
    bMN=bM; alphaMN = alphaM; epMN = epM;
end
X = [0 X]; Y1=[0 Y1]; Y2=[0 Y2]; Y3 = [0 Y3]; plot(X,Y1,X,Y2,X,Y3);
```



Summary

- In multiplicative decomposition, the effect of plasticity is modeled by intermediate configuration
- The total form stress-strain relation is given by hyperelasticity between intermediate and current config.
- We studied principle of max dissipation to derive constitutive relation and plastic evolution
- Similar to classical plasticity, the return mapping algorithm is used in principal Kirchhoff stress and principal logarithmic elastic strain
- It is assumed that the principal direction is fixed during plastic return mapping