What is Neologicism?*

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1. Introduction

Logicism is a thesis about the foundations of mathematics, roughly, that mathematics is derivable from logic alone. It is now widely accepted that the thesis is false and that the logicist program of the early 20th century was unsuccessful. Frege's (1893/1903) system was inconsistent and the Whitehead and Russell (1910–13) system was not thought to be logic, given its axioms of infinity, reducibility, and choice. Moreover, both forms of logicism are in some sense non-starters, since each asserts the existence of objects (courses of values, propositional functions, etc.), something which many philosophers think logic is not supposed to do. Indeed, the tension in the idea underlying logicism, that the axioms and theorems of mathematics can be derived as theorems of logic, is obvious: on the one hand, there are numerous existence claims among the theorems of mathematics, while on the other, it is thought to be impossible to prove the existence of anything from logic alone. According to one well-received view, logicism was replaced by a very different account of the foundations of mathematics, in which mathematics was seen as the study of axioms and their consequences in models consisting of the sets described by Zermelo-Fraenkel set theory (ZF). Mathematics, on this view, is just applied set theory.

Recently, 'neologicism' has emerged, claiming to be a successor to the original project. It was shown to be (relatively) consistent this time and is claimed to be based on logic, or at least logic with analytic truths added. However, we argue that there are a variety of positions that might properly be called 'neologicism', all of which are in the vicinity of logicism. Our project in this paper is to chart this terrain and judge which forms of neologicism succeed and which come closest to the original logicist goals. As we look back at logicism, we shall see that its failure is no longer such a clear-cut matter, nor is it clear-cut that the view which replaced it (that mathematics is applied set theory) is the proper way to conceive of mathematics. We shall be arguing for a new version of neologicism, which is embodied by what we call third-order non-modal object theory. We hope to show that this theory offers a version of neologicism that most closely approximates the main goals of the original logicist program.

In the positive view we put forward in what follows, we adopt the distinctions drawn in Shapiro 2004, between metaphysical foundations for mathematics (2004, 17ff), epistemic foundations for mathematics (2004, 21ff), and mathematical foundationalism (2004, 27ff). We shall be concerned primarily with the first two and plan to remain neutral on the last. The version of neologicism we defend will be a metaphysical foundation for mathematics, in the sense that it (a) provides an ontology of mathematical objects and relations, and (b) identifies not only the denotations of mathematical terms and predicates, but also the truth conditions of mathematical statements, in terms of that ontology, and (c) does so without appealing to any mathematical notions. Moreover, our version of neologicism will constitute an epistemic foundation, in the sense that it shows how we can have knowledge of mathematical claims. These positions will be consistent with whatever position a mathematician might take with respect to mathematical foundationalism (i.e., with respect to any attempt to distinguish some mathematical theory, such as set theory or category theory, as one in which all other mathematical theories should be constructed).

2. From Logicism to Neologicism

Before we start our investigation, we should examine in more detail why it appears to be so widely accepted today that logicism was not success-

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ful. On the one hand, logicism was rejected by Hilbert in the 1920s, while on the other, Hempel maintained that logicism was vindicated as late as 1945.[1] Hilbert and others abandoned the search for an external foundation, and looked instead to mathematics itself to provide its own foundation, using metamathematics, formalism, and finitism. Others, however, looked to set theory as the only foundation mathematics needed. While it is clear that the Hilbert program is not a form of neologicism, one might wonder, why does Zermelo-Fraenkel (ZF) set theory not count as a successful form of logicism? It offers a notion of reducibility by which all of mathematics is reducible. Though there is no proof of its consistency, it is still the standard by which the consistency of other theories is measured.

Indeed, a long time after logicism's heyday, Hempel declared that membership and class talk are logical notions and that the truth of logicism had been established (1945, 378):

> . . . , the following conclusion has been obtained, which is also known as *the thesis of logicism concerning the nature of mathematics*:
>
> Mathematics is a branch of logic. It can be derived from logic in the following sense:
>
> a. All the concepts of mathematics, i.e., of arithmetic, algebra, and analysis, can be defined in terms of four concepts of pure logic.
>
> b. All the theorems of mathematics can be deduced from those definitions by means of the principles of logic (including the axioms of infinity and choice).

[1]For Hilbert, see Sieg 1999. For Hempel, see below. Clearly Gödel's Incompleteness Theorem was not a factor in Hilbert's decision to reject logicism nor did it affect Hempel's conclusion that it was successful. Hilbert's decision to abandon logicism was based on his view that *Principia Mathematica* was not logic, and Hempel's conclusion that logicism was successful was based on the view that logicism is a thesis about the derivability of all the *axioms and theorems* of mathematics as theorems of logic. See our discussion below.

By contrast, some early logicists took the thesis to be the broader claim that all the *truths* of mathematics are derivable as theorems of logic. This explains Kneale and Kneale's 1962 assessment of logicism's failure, for they see Gödel's Theorem as showing that mathematical theories involve "a notion or notions which cannot be characterized exhaustively by the laying down of rules of inference; and this seems to be a very good reason for excluding them from the scope of logic" (724).

We note that even if one takes the broader view of logicism as a thesis about the derivability of the truths of mathematics as theorems of logic, there is still a question as to whether Gödel's Theorem counts against that thesis, since one might just as well conclude from this theorem that the system of logic needed for the reduction is not recursively axiomatizable. See Hale and Wright 2001 (4, fn. 5), for a similar point. However, in what follows, we bypass this issue entirely, for we take logicism in its narrower sense as a thesis about the derivability of the axioms and theorems of mathematics as theorems of logic. This is a significant and worthy thesis in its own right.

So as recently as 1945, some philosophers took set theory to be a part of logic and thought logicism had been vindicated.

Why does this sound so alien to our ears? In part under Quine's influence, a picture emerged which made it natural to think that set theory is not a part of logic. On that view, the existential commitments of logic are kept to a minimum and the membership relation is conceived as a distinguished two-place non-logical relation within the first-order predicate calculus. So how did this change come about?

To begin with, our understanding of which concepts are logical changed. The four concepts of pure logic Hempel refers to are: *neither-nor*, *every*, x is an *element of* class C, and *the class of all things x such that . . . .* Why would Hempel think that 'element' and 'the class of . . . ' are logical notions? Based on the suppressed footnote from the above quotation from Hempel (which concerns the principles of logic used in Quine's work), we conjecture that Hempel was impressed by Tarski (1935) and Quine (1937a, 1937b), in which '$x \in y$' is the basic statement form in a fundamental system of logic. Hempel may have seen Quine's NF (1937b) as a kind of logicism because set comprehension is subject only to a logical restriction (stratified formulas). Indeed, Hempel agrees with Quine, who said (1937b, 81):

> Subsequent investigations have shown that the array of logical notions required is far more meager than was supposed even in *Principia*. We need only these three: *membership*, expressed by interposing the sign '$\in$' and enclosing the whole in parentheses; *alternative denial*, expressed by interposing the sign '$|$' and enclosing the whole in parentheses; and *universal quantification*, expressed by prefixing a variable enclosed in parentheses. All logic in the sense of *Principia*, and hence all mathematics as well, can be translated into a language which consists only of an infinity of variables . . . and these three modes of notational composition.

A more recent version of this reason for thinking that membership is a logical notion is that in an extensional second-order logic, '$\in$' could be conceived as a 'logical particle' expressing predication (e.g., Quine 1970,

65) so that Fx and $x \in F$ are synonymous. All of this stands in contrast to the view prevalent today, namely, that membership is a distinguished two-place (non-logical) relation symbol added to first-order logic.

A second change in the conception of logicism from that expressed by Hempel concerns which principles are logical. It was supposed that the basic principles governing these three modes of notational composition were logical principles, including the principle (Quine 1937b, 92):

> If ϕ is stratified and does not contain 'x', $\exists x \forall y(y \in x \equiv \phi)$ is a theorem.

The view was that although unrestricted set comprehension in naïve set theory had the character of a logical principle, it unfortunately led to contradiction. This led Quine to suppose that his (1937b) restricted set comprehension axiom (restricting the existence of sets to stratified formulas) was part of logic.

Nowadays, however, the axioms of ZF, NF, NBG, and other set theories are considered non-logical principles. This change was ushered in by Tarski, whose model-theoretic semantics for the predicate calculus required logical truths to be true in all models. Models are set-theoretic structures in which the domain of objects need contain nothing more than a single object. Consequently, there are interpretations of the language of set theory under which existence assertions for sets are false. Thus, none of the existence axioms of ZF and other set theories were considered true in all models, and as such, were re-conceptualized as non-logical principles.

In light of these changes as to which concepts and which principles are logical, it is no wonder that ZF is not today considered a form of logicism. What replaced the earlier picture of logic was Quine's later view that the concepts and principles of the first-order predicate calculus (possibly with identity) are all there is to logic. Thus, the prospects for logicism vaporized. On this picture, the foundations of mathematics becomes centralized in the proper theory of sets, and the reduction of mathematics consists of relative interpretation of other mathematical theories in set theory. On this view some parts of mathematics, such as group theory, for example, consist of the study of the logical consequences of axioms, within set theory, i.e., applied set theory. Clearly, this picture gives up the idea that mathematics is reducible to logic.

Our analysis suggests that to understand the idea of neologicism, we

have to return to the original conception of logicism. Historically, logicism has been spelled out as follows:

1. mathematical concepts can be defined in terms of logical ones (no mathematical primitives are used), and
2. mathematical principles can be derived from logical axioms, given the definitions of mathematical concepts.

However, let us focus on the intuitive understanding of logicism, namely, the idea that:

> L: Mathematics is reducible to logic alone

Given this idea, the goal of bringing mathematics within the fold of logic might still be achieved by holding mathematics fixed and reconceiving the three other concepts in L. A view which might be called 'neologicism' can be produced by (any combination of) the following three strategies.

1. Expand the conception of what counts as 'logic'
2. Allow more resources than 'logic alone'
3. Reconceive the notion of 'reducible'

The idea would be to relax L in at least one of these dimensions so as to yield a system that is in the spirit of the original logicist program. Thus, an expanded conception of logic must be close to current conceptions of logic; the added resources should be in the form of analytic truths; and any new concept of reduction must provide some kind of analysis. We will chart the terrain along these three dimensions and then assess the resulting forms of neologicism, in part, by examining how much of mathematics they capture.

Our view is that some revised, modest form of logicism is worth pursuing, if only to help make the ontological presuppositions of logic more precise. Logic seems committed to the existence of something — it must have some ontological commitments — if only to truth-values, the consequence relation, sentence-types, propositions, or possible worlds. The current conception of logic has yet to come to grips with its own existential presuppositions. The move to study these commitments with more model theory involves endless ontological relativity and avoids accounting for the foundations of mathematics, for the set theory used in model theory is part of mathematics. Therefore, the study of neologicism may help to fill in the gaps in the standard picture of logic.

Here, then, is how we categorize and discuss the forms of neologicism in the remainder of this paper. In Section 3, we examine those forms which arise primarily as a result of pursuing strategy (1) above (i.e., expanding the conception of logic). In Section 4, we examine those forms which arise primarily as a result pursuing strategy (2) above (i.e., adding analytic truths to logic). This section includes a discussion of second-order modal object theory. Its interest lies in the fact that, unlike the other forms of neologicism, it uses no mathematical primitives in the analytic truths it adds to logic, yet has some limited mathematical power. However, in all the cases discussed in this section, the resulting theories only recapture some portion of mathematics and run into "limits of abstraction". Finally, in Section 5, we examine our preferred form of neologicism, namely, third-order non-modal object theory, which arises primarily as a result of pursuing strategy (3) above (i.e., reconceiving the notion of 'reduction'). Its distinguishing feature is that it *does not* restrict the amount of mathematics that can be reduced. We shall then conclude with some observations about how our preferred version of neologicism best captures the goals of the original logicist program.

3. Expand the Conception of What Counts as Logic

A natural way of expanding the conception of logic is to accept second-order logic as part of logic. Many of the systems discussed in this section and the next assume this extension of logic is legitimate. Indeed, some of the systems to be discussed assume *full* second-order logic; they depend on the fact that the domain of the properties contains at least as many properties as there are members of the power set of the set of individuals. Though we are in sympathy with Shapiro's (1991) reasons for thinking that full second-order logic is a part of logic, this view about second-order logic is not required by the forms of neologicism we defend later in the paper.[2] As we shall see, one can get by just fine without full second-order logic or the view that it is a part of logic.[3] But some of the systems studied in what follows do require full second-order logic.

We think quantification over properties (or concepts) is justified on the grounds that such entities are required for any realistic understanding of predication. Logic should be committed to any entities needed for a robust understanding of predication, and so should be committed to properties. But this does not commit us to much, since minimal models of the second-order comprehension principle (without any further axioms about properties) require only two properties in the domain.

Can any second-order or higher-order logic alone serve as a neologicist foundation for mathematics? Most philosophers agree that none can. The first consistent attempt, Whitehead and Russell's 'no-classes' theory from $*20$ of *Principia Mathematica*, only goes so far. To see why, let us briefly review the 'no-classes' theory, in which talk about classes starts with the contextual definition:

$$\phi(\{y|Fy\}) \ =_{df} \ \exists G[\forall x(Fx \equiv Gx) \ \& \ \phi(G)]$$

This shows how to eliminate $\{y|Fy\}$ from contexts in which it appears. Note that in the type theory of *Principia*, the class $\{y|Fy\}$ has the same logical type as the function G, and so it makes sense to substitute the latter for the former in ϕ.[4] The no-classes theory then continues with a definition which substitutes talk about membership in a property with having a property (*Principia Mathematica*, $*20 \cdot 02$):

$$x \in F \equiv Fx$$

[2] Indeed, we do not adopt all of Shapiro's conclusions in 1991. For example, we believe that a *philosophical* foundation for mathematics can be developed which leaves it open whether the mathematicians should or should not accept any particular *mathematical* foundations for mathematics (in the sense of preferring one mathematical theory as the one in which all of mathematics can and should be cast). So, if Shapiro is anti-foundational in the latter sense, we would agree with him, but not if he is anti-foundational in the former sense. We believe that no matter how the mathematicians decide the question of whether there is a distinguished mathematical theory at the foundations of mathematics, philosophers can and should offer an interpretation of the language of mathematics which shows how that language is meaningful, and which answers metaphysical and epistemological questions about the truth conditions of that language.

[3] Indeed, the version of neologicism we discuss in Section 4.2 is incompatible with full second-order logic. Our 'second-order modal object theory' will require only second-order syntax (with names and variables for properties and relations), the logic of the second-order quantifiers, and the comprehension principle for properties and relations. Similarly, in Section 5, our third-order, non-modal object theory has only general models, and is incompatible with full third-order logic. We discuss this further in Sections 4.2 and 5.

[4] G must be a 'predicative' property, and is guaranteed to exist by the Axiom of Reducibility. This existence claim is required by the Whitehead and Russell form of logicism.

Note that membership in a class now becomes defined in context.[5] The no-classes theory concludes with definitions allowing for the elimination of the remaining class expressions, including quantification over classes.

Now to see why this theory only gets us so far towards a neologicist foundation, note that Whitehead and Russell famously defined the natural numbers, for example, as classes of equinumerous classes. But to define expressions for classes of classes, third-order logic is needed, and so on through the full theory of types, for each increase in the 'rank' of classes. To get more complex number theories (such as rational and real number theory), one needs classes of natural numbers, and that's why the hierarchy of the simple theory of types must be invoked in the formulation of a 'Russellian' neologicism. Moreover, the neologicism from the no-classes theory and theory of types is not complete until one combines classes of classes with an axiom of infinity, to ensure that every number has a successor.

Many have thought that this does not succeed as version of neologicism because the axiom of infinity exceeds the bounds of logic, even when we consider the expanded conception of logic which includes simple type theory and the no-classes theory. Whitehead and Russell directly postulate the individuals needed for the construction of the natural numbers with the notorious axiom of infinity (1910–13, ∗120·03).[6] Most philosophers have concluded that the axiom of infinity is *obviously* not a principle of logic, given its strong existential commitments. However, is it so obvious?

For one thing, it is weaker than the set-theoretic axiom of infinity in ZF, which directly asserts the existence of an infinite set, since the Whitehead and Russell axiom only asserts the existence of finitely many objects, for each finite number. As many have pointed out (e.g. Boolos 1994), there are two notions of infinite: the weaker notion of a set not being equinumerous with any initial segment of the natural numbers and the stronger notion (Dedekind infinite) of a set being equinumerous with a proper subset of itself. ZF's axiom of infinity directly postulates a Dedekind infinite set, whereas the Whitehead and Russell axiom is not that strong; one needs notions from the no-classes theory to construct even the smallest Dedekind infinite class of all the 'inductive' classes postulated by the Whitehead and Russell axiom of infinity.

Moreover, two things should be noted about this axiom which brings it closer to logic. First, it can be stated solely in terms of logical notions. Assuming that the notions of type theory are logical notions, and assuming that the no-classes theory successfully reduces talk of membership in a class in terms of the logical notion of predication, then the notion of an inductive class, which is crucial to the statement of the Whitehead and Russell axiom of infinity, can be defined in purely logical terms. Thus, the axiom of infinity itself is expressible solely in terms of logical notions.

Second, a modal version of the axiom of infinity *is* philosophically justifiable, even if a particular non-modal version is not, on the grounds that the modal version is a *presupposition* of logic. Russell himself told us that logic should not tell us how many objects there are, and his grounds for saying this were that there might have been any number of objects. This intuition, that there might have been any number of objects, has been accepted by countless logicians as a presupposition of (our conception of) logic. It may therefore be argued that a modal version of the axiom of infinity should be accepted as part of a properly expanded conception of logic. Thus, the Whitehead and Russell version of neologicism does not stray too far from the bounds of logic.[7] And if one remains firm in the belief that the axiom of infinity is not a logical principle, then Whitehead and Russell's neologicism should simply be recategorized as of the kind to be discussed in the next section, where we investigate the version of neologicism in which one eliminates the appeal to logic *alone*.

Before we turn to the next section, however, it is worthwhile mentioning three other examples of neologicism based on an expanded conception of logic: Hodes 1984 and 1991, Tennant 2004 and Martin-Löf 1984. We discuss only the first two here, since they have been explicitly positioned

[5] In other words, $x \in \{y|Fy\}) =_{df} \exists G[\forall x(Fx \equiv Gx) \,\&\, x \in G]$.

[6] By contrast, Frege stays within the confines of second-order logic to define the natural numbers, but at the cost of adding his fatal theory of extensions. To prove that every number has a successor Frege relies on the fact that a number n will have n predecessors (starting with 0) and so to find a class with $n+1$ members (to represent a class of equinumerous classes each with $n+1$ members), one only needs to gather n with its predecessors into one class. For Russell the class containing the predecessors of a given number will be one type higher than any of those predecessors, and so each successor would move up one type. Consequently there would be no class of all numbers. Frege, however, treats extensions as of the same type as individuals, and so is able to treat all classes of numbers as further individuals. Of course this is what leads directly to the contradiction for his system. Somehow a consistent 'neo' Fregean system must indicate which extensions there are and so it must be involved in a seemingly extra-logical consideration of objects.

[7] For a discussion of whether the axiom of reducibility is part of logic, see Linsky 1990.

as a kind of neologicism.

Hodes (1984) holds that the mathematics of numbers can be derived from third-order logic. He claims (1984):

> In making what appears to be a statement about numbers, one is really making a statement primarily about cardinality object-quantifiers; what appears to be a first-order theory about objects of a distinctive sort really is an encoding of a fragment of third-order logic. (143)
>
> ...
>
> From the start, the ur-mathematician is beholden to a body of truths, e.g., truths of third-order logic. (145)

Though this form of neologicism extends only to arithmetic, Hodes later extends it to some, but not all, principles of set theory. He says (1991):

> ... fundamental set-theoretic principles are encodings of validities in an appropriate second-order logic. (151)
>
> ...
>
> The Alternative theory construes mathematical principles as validities in appropriate logics. (161)
>
> ...
>
> I regard the Alternative theory as partly in the spirit of logicism. (163)

However, Hodes acknowledges that his Alternative theory not only presupposes the existence of infinitely many objects, but also two principles which are neither logical nor analytic. The two principles are that there are 'acceptably' many objects and that the universe has 'strong limit size'.[8] Though Hodes says that his theory can construe set-theoretic principles as logical truths 'modulo' these assumptions, what does this mean?

[8] Hodes uses branching quantifiers to formulate the first as (1991, 163):

$$\begin{matrix}(\forall X)(\exists x)\\(\forall Y)(\exists y)\end{matrix}\,[x = y \equiv \forall z(X(z) \equiv Y(z))]$$

(The branching quantifiers here are not to be evaluated in the same way as the linear string of quantifiers $(\forall X)(\exists x)(\forall Y)(\exists y)$. With linear quantifiers, the value of the variable y in $\exists y$ is dependent on the value of the variable X in $\forall X$, but with branching quantifiers, the value of the variable y in $\exists y$ is dependent only on the value of the variable Y in $\forall Y$. As an example of branching first-order quantifiers, consider "Most cousins of each villager and most cousins of each townsman hate each other." Here the values of 'townsman's cousin' are relative only to 'each townsman' and are independent of 'each villager'. See Henkin (1961) and Barwise (1979).)

If a principle A can be regarded as logical only if one assumes a principle B which is neither logical nor analytic, then is not A both non-logical and synthetic? Hodes' system does show how some mathematical truths can be reduced to truths containing only logical vocabulary (at least after they have been revealed to be encodings of higher-order logical forms), but the reducing theory involves non-logical, synthetic truths. Moreover, the non-logical assumptions Hodes adds to higher-order logic force the domain to be infinite in various ways. By contrast, the version of neologicism we defend in this paper will be based on assumptions that place minimal conditions on the domain and have a certain claim to analyticity (and, indeed, maybe even some claim to being logical). In addition, our theory preserves Hodes' idea that mathematical statements are in some way 'encodings' of higher-order statements, and indeed, on our view, these are encodings in a technical sense (Zalta 1999, 643).

Tennant's system is neologicist in the sense discussed in this section since it 'expands the conception of logic' and attempts to derive a philosophically interesting portion of mathematics. Tennant expands the conception of what counts as logic by allowing that introduction and elimination rules governing abstracts for mathematical objects are part of logic. Tennant's general method is to find rules for statements of the form '$t = \alpha x \Phi(x)$', where t is any singular term that is presumed to be understood, α is an abstraction operator and Φ is a concept. His method assumes that there is some appropriate binary relation R that holds between t and the objects categorized by Φ (2004, 115). The introduction and elimination rules appeal to such an R. For example, with respect to number abstracts of the form $\#x\Phi(x)$, the relevant R is the relation that obtains between the number of Φs and the individuals falling under Φ. In the case of definition descriptions of the form '$\imath x\Phi(x)$', the R in question is that of identity. In the case of set abstracts of the form $\{x|\Phi(x)\}$, the R in question is the set membership relation (2004, 115–9).

How much mathematics does this method of abstraction buy us? Very little, it seems, without a lot of extra work. For example, formulating the introduction and elimination rules for successor ('s') and number of

Hodes then formulates the second as (1991, 163):

$$(\forall X)(\exists Y)\begin{matrix}(\forall U)(\exists u)\\(\forall V)(\exists v)\end{matrix}\,((\forall x)(Ux \supset Xx) \supset [Yu \,\&\, ((\forall x)(Ux \equiv Vx) \supset u = v)])$$

Given that these principles have factual content about the actual cardinality of the universe, we can not see how either can be construed as logical or analytic truths.

('#') so that the same numbers apply to equinumerous classes requires all of the work of constructing the sequence of integers with the successor relation (2004, 113). The successor of the number of Fs has to be specially introduced so that it is the number of things which are F or equal to some given object r, and so on. Rather than simply abstracting objects to represent equivalence classes of objects falling under concepts, we are allowed to abstract with respect to any concept Φ (to produce $\alpha_R x\Phi(x)$), but then we must add new rules governing the objects which all bear the relation R to that abstract (see the α-introduction and elimination rules in 2004, 116).

Moreover, Tennant freely admits that his theory of numbers presupposes the existence of an infinite progression (124-5). The presupposition is implicit in the introduction and eliminations rules for a numerical abstraction operator α which has been introduced with respect to a relation $<$ that meets a complex conditions Γ for being a progression.[9] The non-analytic conditions in Γ guarantee that that every number abstract will have a unique successor. Tennant says:

> This account of numerical abstraction, to be sure, involves quite a heavy presuppositional burden: a presupposition to the effect that there are indeed progressions — that is, domains orderable by relations $<$ satisfying the condition Γ. . . .
>
> I am happy to premise my logicist thinking about number on the logically possible existence of at least one progression (a domain with an ordering $<$ satisfying condition Γ).

To us, these passages sound like an apology for an axiom of infinity, for despite the language in the previous quotation, Tennant premises his logicism on the assumption that there actually exists at least one progression of the kind in question, not the mere logical possibility that there is one such progression.

Tennant's method does yield some weak set theory (2004, 118–120), and some real number theory, though it is not clear to us what existence claims are required for his theory of the real numbers. These facts stand in contrast to some of the systems discussed below.

We conclude with one worry and one observation about Tennant's approach. The worry is whether all mathematical theories can be captured

[9]The complex condition Γ is a conjunction of the conditions: existence of an initial element, irreflexivity, connectedness, transitivity, unique left- and right-immediacy, and finite connectivity. See Tennant 2004, 121.

in the way Tennant suggests, namely, in terms of a special relation R by which abstracts are to be introduced. It is not clear that Tennant's model of abstraction operators is general enough for the representation of all mathematical theories in terms of introduction and elimination rules, since it is not clear whether the appropriate R will be available for each mathematical theory.

Finally, there is an important observation to make about this approach to neologicism, namely, that for each new mathematical object to be reduced, a new logic is needed. That is, new abstraction operators, and the introduction and elimination rules governing them, must be added. Indeed, these new rules do not all have the same form; the forms vary according to the kind of object that is to be reduced. This strikes us as a piecemeal approach to the method of 'expanding the logic' so as to produce a form of neologicism. In contrast to the system we defend in Section 5 of the paper, Tennant's system does not yet offer a uniform analysis of all of mathematics.

4. Allow more resources than logic alone

4.1 Neologicism Based on New Abstraction Principles

In this section we consider those forms of neologicism which result from adding analytic-sounding abstraction principles to some fixed logical basis such as second-order logic. We allow that second-order logic is a natural background for such an enterprise, since it allows one to quantify over objects and concepts (properties) and thereby treats both of the elements involved in predication.

One might think that a place to look for this kind of neologicism is to consider those systems which preserve Frege's Basic Law V but which place constraints on second-order comprehension for concepts (where 'ϵF' denotes the extension of F):

Basic Law V: $\epsilon F = \epsilon G \equiv \forall x(Fx \equiv Gx)$

Comprehension for Concepts: $\exists F \forall x(Fx \equiv \phi)$

The constraints on comprehension for concepts typically restrict the kinds of formulas that can be substituted for ϕ. These are clearly systems which use more resources than logic alone, if we acknowledge that Frege's Basic

Law V is not, strictly speaking, a logical truth but rather is an analytic-sounding abstraction principle.[10] However, the systems of this kind, investigated by Heck (1996), Wehmeier (1999), and Ferreira and Wehmeier (2002), will not be considered here because they are too weak for the reconstruction of much mathematics.[11] The main problem is that if one limits second-order comprehension by elminating impredicative formulas from comprehension, then one cannot define the notion of successor or the notion of membership. Thus, we think of these systems as 'neo-Fregean' without being 'neologicist'.

In this section, however, we will examine the kind of neologicism defended by Wright and Hale and also look at similar forms of neologicism, such as those suggested by Boolos, Cook, and Fine (none of whom considered themselves a 'neologicist' but who have all tried in some way to extend the ideas of Wright and Hale to their limit). We will contrast these forms of neologicism with that of second-order modal object theory, which can also be conceived as a logic supplemented by an analytic-sounding abstraction principle.

The form of neologicism studied by Wright and Hale starts with the idea of replacing Basic Law V with 'Hume's Principle' for the purpose of defining mathematical concepts and deriving principles of mathematics:

Hume's Principle: $\#F = \#G \equiv F \approx G$,

where '$\#F$' denotes the number of Fs and where $F \approx G$ asserts in purely logical (second-order definable) terms that F and G are equinumerous (i.e., that there is a relation R which witnesses the one-to-one correspondence of the Fs and the Gs). Wright's 1983 was a study of the consequences of adding this principle to second-order logic — he sketched how the basic axioms of arithmetic could be derived from Hume's Principle in second-order logic, and Heck 1993 confirmed that even in Frege's own system, the derivations of the basic principles of arithmetic all went by way of Hume's Principle without making any essential appeal to Basic Law V (other than to 'establish' Hume's Principle). Hale's 1987 was a

[10] Thus, it is our position that even if Frege's Basic Law V had been consistent, his system would *not* have constituted a pure form of logicism! Basic Law V goes beyond the resources of logic alone, and indeed, Frege seems to have suspected as much.

[11] *A fortiori*, those systems that investigate Basic Law V and its extensions in first-order logic will not be considered here, for the same reason. See the work of Parsons (1987), Burgess (1998), Bell (1994), and Goldfarb (2001a).

study of adding other 'abstraction' principles, and in 2000, he proposed a reconstruction of the real numbers using such abstraction principles.[12]

Most of the discussions of neologicism in the literature have focused on these theories. As a result, their virtues and limitations are rather well-known. In addition to the worry over whether Hume's Principle is analytic (Boolos 1997), this form of neologicism is subject to the infamous Julius Caesar problem, the 'bad company' objection[13] and the 'embarrassment of riches' objection.[14] We shall not rehearse these here, other than to make three observations about this form of neologicism.

The first is that without a solution to the Julius Caesar problem, Hume's Principle may be no better off than the Whitehead and Russell axiom of infinity. The Julius Caesar problem presents itself for any system consisting of second-order logic supplemented by Fregean biconditionals having the form of Hume's Principle. The problem is, Hume's Principle does not tell us how to prove whether or not $\#F = x$ for arbitrary x. If Hume's Principle is taken as a contextual definition, then the system as a whole leaves it indeterminate how to prove that $\#F$ (for some given F) is *not* identical to, say, Julius Caesar ('j'). Now one might claim that given Hume's Principle, we know that to prove $\#F = j$, we have to prove '$\exists G(j = \#G \& F \approx G)$'. But, then, the system is still indeterminate, since we still do not know what we have to prove in order to prove that Julius Caesar is or is not identical to the number of some concept.[15] Thus, given the Caesar problem, it is not clear how to prove that Hume's Principle does *not* introduce an infinite number of concrete objects, since one can not rule out that numbers are concrete objects.

[12] See Cook 2001, Batitsky 2001 for a discussion of the problems with this approach, and Hale 2002 for a reply.

[13] Noted by Boolos (1990, 214), Field (1989, 158), and Dummett (1991, 188-189).

[14] This was put in its strongest form by Weir (2003, 16), who notes: "there are indefinitely many consistent but pairwise inconsistent abstraction principles. If all consistent abstraction principles are analytic, then both of two such principles are analytic and presumably true which is absurd." Weir was relying on the results in Heck 1992.

[15] In Hale and Wright 2001 (Chapter 14), there is a machinery involving sortals, categories, and criteria of identity for objects of different sorts. Wright and Hale use this machinery to conclude that Caesar is not a number, since numbers are the kind of thing whose identity is governed by equivalence relations and abstraction principles whereas Caesar is not. In response to this, we make only two points: (1) it remains unclear how a *proof* that Julius Caesar is not a number would go in the system of second-order logic and Hume's Principle, and (2) our system of neologicism, which we describe in Section 4.2, makes it clear how such a proof would go.

Now to complete our first observation, if one can not prove that Hume's Principle isn't a principle that postulates an infinity of ordinary individuals, then this would seem to put Hume's Principle on a par with the Whitehead and Russell axiom of infinity, which implies an infinite number of 'individuals'. In the system of *Principia Mathematica*, one can not prove that such individuals are not concrete things like Julius Caesar. Since logic should not dictate the size of the domain of concrete objects, the axiom of infinity was criticized as being a non-logical addition to the system in *Principia Mathematica*, as we've mentioned. Indeed, one might say that any successful version of neologicism should be able to prove that it does *not* dictate the size of the domain of concrete objects. The axiom of infinity in *Principia Mathematica* offers no such guarantee, and given the Julius Caesar problem, neither does neologicism based on Hume's Principle.

Second, the 'bad company' and the 'embarrassment of riches' objections both point to the underlying piecemeal nature of this form of neologicism. This form of neologicism requires one to add abstraction principles to the background logic whenever suitable ones are discovered. There is no formal criterion for rejecting the inconsistent principles and selecting among the various, and competing, consistent alternatives. As we shall see, Fine's work attempts to address the problem underlying this observation.

A third observation is that this form of neologicism is based not only on the addition of an analytic-sounding principle (Hume's Principle) to second-order logic, but also on the fact that the principle in question is formulated in terms of a primitive mathematical notion, namely, 'the number of'. Although one might accept that there is a kind of analyticity to Hume's Principle on the grounds that it appropriately captures the concept of 'the number of Fs', the fact remains that the concept in question is a distinctively mathematical concept. Nor is it eliminable, since Hume's Principle allows the elimination of that term only from certain contexts. So, there is an additional step here, in going from the formulation of a neologicism which adds principles expressible solely in terms of logical notions and identity (such as the claim 'there are exactly n things', when this is defined solely in logical terms) to the formulation of a neologicism which adds principles that are expressed in terms of mathematically primitive notions.[16]

[16] In this respect, the Whitehead and Russell view counts as closer to logicism than the Wright and Hale view, since the former does not use any primitive mathematical notions. As noted above, Whitehead and Russell could state the axiom of infinity using only logical concepts (cf. Boolos 1994, 271). By contrast, Hume's Principle employs the primitive concept 'the number of Fs', as we just noted.

We turn next to a second form of neologicism, which reconstructs the set-theoretic universe by developing restricted versions of Frege's Law V while asserting unrestricted comprehension over concepts in second-order logic. Though Boolos did not consider himself to be defending neologicism, he did investigate systems which should be considered neologicist given the definition explored in this paper. Boolos (1986) considered replacing Basic Law V with 'New V', so as to yield a consistent theory of extensions. Call a concept F *small* if it is not in 1–1 correspondence with a universal concept, and call F and G *similar* concepts ('$F \sim G$') iff either F and G both fail to be small or F and G are coextensive. Boolos then examines the consequences of adding the following principle to second-order logic to systematize the concept of the extension of F ('$\textsc{ext}(F)$'):

New V: $\textsc{ext}(F) = \textsc{ext}(G) \equiv F \sim G$,

(Boolos also proposed a variant of this in [1993], based on a idea of Terence Parsons.) Boolos himself shows how a version of arithmetic can be captured using New V (179), in terms of the hereditarily finite sets. Shapiro and Weir point out (1999, 301), however, that New V can not be used to reconstruct Frege Arithmetic, since the finite cardinals would be identified as the extensions of concepts that are not small (so by New V and the definition of similarity, such cardinals would be identified). They also note that the axioms of infinity and power set are not derivable (301). Cook (2003), however, develops a system which extends New V and which addresses the problems raised by Shapiro and Weir. He adds two other abstraction principles and an axiom asserting an infinity of urelements (InfNonSets). One abstraction principle, SOAP, introduces the restricted-size ordinal abstract for relation R ($\textsc{ord}(R)$), and the other, Newer V, extends the characterization of $\textsc{ext}(F)$.[17] Cook then claims that "SOAP + Newer V + New V + InfNonSets provides the neologicist

[17] Cook's principles are as follows. First, SOAP is the Size-Restricted Ordinal Abstraction Principle, and it introduces 'the size-restricted order-type of relation R' ('$\textsc{ord}(R)$') as follows (2003, 67):

Size-Restricted Ordinal Abstraction Principle (= SOAP):

$\textsc{ord}(R) = \textsc{ord}(S) \leftrightarrow$

$[(\neg WO(R) \lor Big(R)) \,\&\, (\neg WO(S) \lor Big(S))] \lor$

$WO(R) \,\&\, WO(S) \,\&\, Isomorphic(R,S) \,\&\, \neg Big(R) \,\&\, \neg Big(S)$.

with a set theory that is (roughly) as strong as full second-order ZFC" (2003, 86).

We believe that many of the problems that arose for the Wright and Hale versions of neologicism apply to the Boolos and Cook approaches: (a) New V, SOAP, and Newer V have far less claim to analyticity than Hume's Principle; (b) the Julius Caesar problem still applies to New V, SOAP, and Newer V; (c) the procedure for adding abstraction principles to neologicism still seems to be somewhat piecemeal; and (d) these systems use primitive mathematical concepts in the basic abstraction principles.

But instead of considering these problems in more detail, let us focus on some issues that will serve as points of contrast with the version of neologicism we defend in the next subsection and in Section 5 of the paper. Notice first that Cook 2003 uses a simple and direct assertion of the infinite size of the domain of urelements. This axiom of infinity opens the system to the same objection that plagued Whitehead and Russell. Second, in addition to the worry about whether New V, SOAP, and Newer V are analytic, there is a worry about whether Cook can

Here, WO(R) abbreviates the claim that R is a well-ordering (2003, 89), i.e.,

$$\begin{aligned}&\mathrm{WO}(R) \leftrightarrow\\ &\quad \forall x(\neg R(x,x))\ \&\ \forall x\forall y\forall z((R(x,y)\ \&\ R(y,z)) \rightarrow R(x,z))\ \&\\ &\quad \forall P[(\exists x(Px)\ \&\ \forall x(Px \rightarrow A_R(x))) \rightarrow \exists y(Py\ \&\ \forall z(Pz \rightarrow (z=y \vee R(y,z))))],\end{aligned}$$

and $A_R(x)$ ('x is in the domain or range of R') in this latter definition abbreviates $\exists y(R(x,y) \vee R(y,x))$ (2003, 88). In addition, the notion $\mathit{Big}(R)$ used in SOAP is an abbreviation for $\mathit{Big}(A_R)$, where $\mathit{Big}(F)$, for arbitrary F, is defined in terms of the second-order formula asserting that the Fs are equinumerous with the entire domain, i.e.,

$$\exists f\forall x\exists y(Fy\ \&\ f(y)=x).$$

The above understanding fills a minor gap in Cook 2003 (66-7, and 88, note 5), where $\mathit{Big}(R)$ (i.e., the notion of bigness for relations R, as opposed to properties F) used in SOAP is not strictly speaking defined.

Second, the principle Newer V (2003, 69) governs $\textsc{ext}(F)$:

Newer V:
$$\textsc{ext}(F) = \textsc{ext}(G) \leftrightarrow [\forall x(Fx \leftrightarrow Gx) \vee (\mathit{Bad}(F)\ \&\ \mathit{Bad}(G))],$$

where $\mathit{Bad}(F)$ asserts that there is no ordinal α such that all the members of F are elements of α's stage (2003, 69), i.e.,

$$\mathit{Bad}(F) \leftrightarrow \neg\exists\alpha(\mathit{ON}(\alpha)\ \&\ \forall x(Fx \leftrightarrow x \in_S \mathit{Stg}(\alpha))).$$

Cook's recent unpublished work suggests that New V + Newer V + Hume's Principle + Fine's principle of abstract identity (i.e., that 'abstracts' which correspond to the same equivalence class of concepts are identical) constitute the most promising technical strategy for a neo-Fregean set theory.

offer a single abstraction principle which states the identity conditions for extensions and which thereby provides the epistemological benefits that are supposed to attach to this form of neologicism. For the conclusion, at the end of Cook 2003, is with respect to a system in which there is no single abstraction principle that governs $\textsc{ext}(F)$. Cook is aware of the problem and makes a few remarks intended to address the worry (2003, 79):

> The reader should keep in mind that this conjunction of 'definitions' of the extension operator 'EXT' can be replaced by a richer (but formally less tractable) account of identity conditions across distinct abstraction principles.

But it is unclear to us whether one can claim that logic has been extended with analytic abstraction principles if no one *single* principle governs $\textsc{ext}(F)$.[18]

Finally, we have two global concerns regarding Cook's system. The first arises from the combination of facts that the system (1) takes the mathematical notions of 'extension' and 'ordinal' as primitive,[19] (2) includes several non-analytic axioms governing these mathematical notions, (3) has an axiom of infinity, and (4) includes as many principles as are needed to secure the strength of second-order ZFC. This combination has the following effect, namely, that the system comes across more as an alternative method of axiomatizing the mathematical theory of sets (and ordinals) with something like abstraction axioms, rather than a way of demonstrating the truth behind the slogan "mathematics is reducible to logic supplemented by analytic truths". Cook's system, therefore, no longer seems to count as a form of neologicism. The second concern is that the system as a whole is subject to the form of the Julius Caesar problem that Cook and Ebert (2005) call the 'C-R' problem. The C-R problem is that when more than one abstraction principle is added to second-order logic, it is not clear how to prove that the abstractions introduced by one principle are identical with those of another. Thus, in Cook 2003, it is

[18] How can New V (something of the form: $\textsc{ext}(F) = \textsc{ext}(G) \equiv \phi$) be analytic and Newer V (something of the form: $\textsc{ext}(F) = \textsc{ext}(G) \equiv \psi$) be analytic if '$\phi \equiv \psi$' is not analytic? Given the particular ϕ and ψ used in New V and Newer V, it does not seem reasonable to claim that '$\phi \equiv \psi$' is analytic — New V and Newer V have different content and are logically independent.

[19] The axioms New V and Newer V both take 'EXT' ('extension') as primitive and the axiom SOAP takes the mathematical notion '$\textsc{ord}(R)$' as primitive.

not clear how to prove that $\textsc{ord}(R) = \textsc{ext}(F)$, for any R and F.

Recently, Fine (2002) studied the question of just how far one can go with the method of abstraction, i.e., the method of introducing 'abstracts' corresponding to the cells of an equivalence relation on concepts by means of abstraction principles. He addressed the question, how do we determine which of the many possible abstraction principles can be added to our background logic to produce a neologicism? To answer this question, he developed a more general theory of abstraction principles, using complex criteria to identify all and only the acceptable ones. To a first approximation, Fine counts an abstraction principle based on an equivalence relation R as acceptable only when R is a second-level equivalence relation on first-level concepts that is both (1) invariant with respect to permutations of the domain of individuals, and (2) 'non-inflationary' in the sense that there are no more equivalence classes of concepts under R than there are individuals in the domain (Fine 2002). One constraint on the theory is that the totality of acceptable abstraction principles must not jointly be inflationary (this avoids the problem of *hyper-inflation*).

In a recent review of Fine's work (2003), and in a forthcoming monograph (2005), Burgess notes that the limits of abstraction on Fine's approach "turn out to be those of third-order Peano arithmetic ... so that Fine's approach can be said to get us all of mathematics except higher set theory" (2005, 129). This fact serves as the most remarkable point of contrast with the neologicism offered by third-order object theory, which we investigate in Section 5 of the paper. For now, we offer a few other observations that will prove to be interesting points of comparison in the remainder of the paper.

First, there is an explicit non-analytic assumption that there are at least two individual objects in the domain of individuals (Fine 2002, 189). Fine justifies this on the grounds that it is akin to Frege's assumption that there are two truth values, and so something that might even be considered a logical truth (though he admits that a more orthodox understanding would take this to be a non-logical principle). This will be contrasted with object theory, in which (a) no (concrete) individuals are assumed to exist, (b) truth values are explicitly defined as abstract objects of a certain kind, and (c) the existence of exactly two truth values is a theorem (Anderson and Zalta 2004, Section 3.4).

Second, Fine admits (2002, 192) that his theory of abstraction is incomplete in the sense that it offers no account of abstraction with respect to equivalence relations on individuals as opposed to equivalence relations on concepts; i.e., there is no mechanism for identifying acceptable abstraction principles of the form: the direction of line a = the direction of line b iff $a \| b$. Moreover, his theory applies only to second-order equivalence conditions on first-order, monadic concepts, since the Burali-Forti paradox prevents one from supposing that there are abstracts corresponding to isomorphic first-order relations. These two limitations can be contrasted with object theory, in which the existence of directions (and other abstract individuals corresponding to the cells of arbitrary equivalence relations on individuals) can be derived, and one can define abstracts corresponding to relations that are isomorphic with respect to *ordinary* objects.[20] Finally, it has been noted by Burgess and others that Fine's system still does not offer straightforward solutions to the Julius Caesar problem, or to the problem of how we have knowledge of numbers. In object theory, by contrast, there are solutions to both problems, as we explain in the next section.[21]

4.2 Neologicism Based on Object Theory

In this subsection we investigate how second-order modal object theory, as a form of neologicism, fits into the above picture, and we identify both its virtues and its limitations of abstraction. In Section 5, we then show how the limitations can be overcome in third-order non-modal object theory. Much of the discussion that follows in this subsection is grounded in Anderson & Zalta 2004, and Zalta 1999, and we shall assume some familiarity with these. In the former, it was shown that second-order *non-modal* object theory yields a modest theory of extensions (with a version of Basic Law V falling out as a theorem), as well as a theory of various other logical objects such as directions, shapes, and truth-values (with their governing abstraction principles also derivable). In the latter, a derivation of the Dedekind/Peano axioms for number theory was accomplished in second-order modal object theory, together with some natural additional assumptions.

Second-order modal object theory is couched in a second-order modal logic having a second kind of atomic formula, namely, 'encoding' formulas of the form xF (read: x encodes F). Two kinds of complex terms are

[20] For the former, see Anderson and Zalta 2004 (Section 3.3), and for the latter, see footnote 27.

[21] See Cook and Ebert 2004, for other criticisms of Fine's view.

used: λ-expressions of the form $[\lambda x_1 \ldots x_n\, \phi]$ and definite descriptions of the form $\imath x\phi$, the latter being interpreted rigidly. The system assumes the axioms and rules of classical S5 second-order quantified modal logic, modified only to accomodate the facts that the descriptions are rigid and that (complex terms containing) descriptions may fail to denote. The underlying second-order logic includes the following abstraction principle for relations:

$[\lambda x_1 \ldots x_n\, \phi]y_1 \ldots y_n \equiv \phi^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}$, where ϕ has no encoding subformulas and no descriptions

From this λ-conversion principle, comprehension for relations is derivable as a simple consequence.[22] Object theory then becomes a form of neologicism as soon as one replaces its fundamental comprehension principle for abstract objects ('$A!x$'):

(A) $\exists x(A!x \,\&\, \forall F(xF \equiv \phi))$,

with the following abstraction principles for abstract objects and relations:

(B) $\imath x(A!x \,\&\, \forall F(xF \equiv \phi))G \equiv \phi^G_F$.

In standard second-order object theory, (A) is an axiom and (B) is a theorem. Note here that encoding formulas in object theory have some similarities with the η formulas that Boolos uses in the Numbers axiom of Frege Arithmetic (Boolos 1987): $\forall G \exists x \forall F(F\eta x \equiv F \approx G)$. Whereas Frege Arithmetic employs an unrestricted comprehension principle for properties and a comprehension principle for numbers restricted to equinumerosity conditions, object theory employs an *unrestricted* comprehension principle for abstract objects and combines it with a comprehension principle for relations that bars only encoding subformulas.

A version of neologicism arises when we reformulate object theory by taking (B) as an axiom instead of (A). Metatheoretically, these are two equivalent ways of formulating object theory, since anything provable in one formulation of the system is provable in the other and vice versa. The reformulation replaces the 'synthetic' existence principle which comprehends the domain of abstract objects with a principle having more

[22]That is, the following claim,

$\exists F \forall x_1 \ldots \forall x_n(Fx_1 \ldots x_n \equiv \phi))$, where ϕ has no encoding subformulas and no descriptions,

can be derived from λ-abstraction.

of the character of an analytic abstraction principle. The reformulated system also speaks to the interests of the neologicists who look back to Frege's Context Principle for insight, since it introduces singular terms for abstract objects in the context of a larger sentence.

Unlike Hume's Principle and other Fregean biconditionals, the abstraction principle for abstract objects (B) does not introduce objects with a biconditional having an identity between two abstracts (singular terms) on one side and a 'recognition statement' on the other. Instead, each instance of abstraction introduces a single abstract object by a canonical description, and it leaves the question of the identity of abstracts to a separate definition of identity.[23] Where 'x is ordinary' ('$O!x$') is defined as 'x is possibly concrete' ('$\Diamond E!x$'), and 'x is abstract' ('$A!x$') is defined as 'x could not be concrete' ('$\neg\Diamond E!x$'), the following biconditionals may be construed as definitions:

$$x =_E y \equiv O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)$$

$$x = y \equiv x =_E y \;\vee\; (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)$$

The first principle defines a notion of identity on 'ordinary' objects, while the second defines a general notion of identity which relies on the fact that the domain is partitioned into two mutually exclusive subdomains: ordinary and abstract objects.

The first observation to make about this reformulation of the theory is that no primitive mathematical notions are used. Second-order modal object theory just uses second-order modal logic and the notion of encoding, along with descriptions and λ-expressions. Other than encoding, the primitive notions of this system are: *exemplifies* ($F^n x_1 \ldots x_n$), *not* ($\neg$), *if-then* ($\rightarrow$), *every* ($\forall$), *necessarily* ($\Box$), *being such that* (λ) and *the* ($\imath$). We shall assume that these are not mathematical notions. But what about encoding (xF)? Encoding is no more mathematical than exemplification, and exemplification can be viewed as mathematical only if the second-order variables are interpreted as ranging over the full power set of the domain of the individual variables. In such an interpretation, one might argue that exemplification is really the mathematical notion of set membership. However, note that one can not interpret the predicate variables of (modal) object theory as ranging over the full power

[23]This is in fact a more standard use of the term 'abstraction principle'. (B) is more similar in form to the set-theoretic abstraction principle ($y \in \{x|Fx\} \equiv Fy$) and λ-abstraction than to Hume's Principle.

set of the domain of the individual variables, as can be seen from the restrictions on λ-formation and the derived comprehension principle for relations.[24] Since the intended interpretation of object theory is that the predicate variables range over properties (intensionally conceived), we think a strong case can be made for thinking that exemplification and encoding are both logical notions, rather than mathematical. Thus, second-order modal object theory contains no mathematical primitives and this stands in contrast with all the previous forms of neologicism (with the possible exception of Hodes), which either use mathematical primitives in the formation rules of the logic (e.g., Tennant), or in the formulation of the relevant abstraction principles.[25]

The second observation to make about this formulation of second-order object theory concerns the analyticity of the abstraction principle (B). We would argue that object abstraction has at least as much claim to being analytic as Hume's Principle, if not more. Consider how one would read (B): the-abstract-object-that-encodes-exactly-the-properties-satisfying-ϕ encodes G iff G satisfies ϕ. This certainly sounds like an analytic truth about the nature of abstract objects. Of course, one might argue that since it is equivalent to the comprehension principle (A) and that (A) is clearly synthetic, (B) must be synthetic as well. But this just shows that the question of analyticity is now revealed simply as a decision about whether one can accept that an existence claim can be analytic. Note that (B)'s analyticity is not subject to Boolos's objection to the analyticity of Hume's Principle (1997, 304-5). Boolos objected that no principle implying an infinity of objects can be analytic. By contrast, the simplest models of second-order object theory do not require an infinity of

[24] Restrictions on λ-formation prohibit $[\lambda y_1 \ldots y_n\, \phi]$ from including encoding subformulas in ϕ. Thus, the version of comprehension over relations derived in footnote 22 explicitly includes the restriction that ϕ not contain encoding subformulas. Hence, in models of second-order modal object theory, the second-order quantifiers do not range over the full power set of the domain of ordinary and abstract objects, but only of the domain of 'individuals'. This can be seen by inspecting the Aczel-models of the theory described in Zalta 1999 (626–628), where abstract objects are represented at the third level (i.e., as sets of properties) but are combined with first-level individuals to form the domain D of quantification for the individual variables. The domain of properties is only as large as the power set of the set of first-level individuals, and can not be as large as the power set of D.

[25] Hume's Principle uses 'the number of' as primitive, and the various implementations of Boolos's and Cook's theory use 'the extension of' and other mathematical primitives. Those who add biconditionals such as those for directions and shapes import primitive mathematical notions like 'the direction of' and 'the shape of', etc.

objects; indeed, in these minimal models, there are no ordinary objects, two properties, and four abstract objects. At the very least, this fact gives (B) a stronger claim to being analytic than Hume's Principle. But, still, we hope to leave it an open question as to whether (B) is indeed analytic. The important question is whether (B) in combination with second-order modal logic constitutes a form of neologicism. And given what have been accepted as forms of neologicism in the literature, we think there is little room to object on this score.

A third observation about second-order object theory based on abstraction is that there is no Julius Caesar problem. The Julius Caesar problem for the system of second-order logic and Hume's Principle does not arise in object theory, for several reasons. (1) Once '$\#_F$' is explicitly defined as the abstract object that encodes all and only the properties G equinumerous with F on the ordinary objects (Zalta 1999), the formula '$\#_F = x$' has well-defined truth conditions: you simply plug in '$\#_F$' into the object-theoretic definition of identity to get:

$$\#_F \!=\! x \;\equiv\; [\#_F \!=_E\! x \,\vee\, (A!\#_F \;\&\; A!x \;\&\; \Box\forall G(\#_F G \equiv xG))]$$

In other words, $\#_F = x$ iff either $\#_F$ and x are both ordinary objects that necessarily exemplify the same properties or they are both abstract objects that necessarily encode the same properties. So the system is explicit about what has to be proved if we are to prove whether $\#_F$ is identical to Julius Caesar. (2) Since $\#_F$ is defined as an abstract object, the first disjunct never obtains when x is an ordinary object. For by definition, ordinary objects ($O!x$) are possibly concrete ($\Diamond E!x$) and abstract objects ($A!x$) necessarily fail to be concrete ($\neg\Diamond E!x$). Thus, it follows that $\#_F$ is identical to x iff both $\#_F$ and x are abstract objects that encode the same properties. So the theory decides the question "Is $\#_F$ identical to Julius Caesar?" (for any F) in the negative as soon as (a) $\#_F$ is defined as above and (b) Julius Caesar is asserted to be an ordinary object ($O!j$).[26]

[26] As we saw earlier, the distinction between abstract and ordinary objects partitions the domain of objects. The abstraction principle (B) discussed above only comprehends the domain of abstract objects and does not attempt to partially define identity conditions for abstract objects. Identity conditions are provided by the separate identity principles outlined above.

Note also that in object theory, the claim that Julius Caesar is an ordinary object, is a necessary truth if true (since $\Box\Diamond E!x$ follows from $\Diamond E!x$ in S5), and moreover, one that is discovered *a posteriori*. It is thus similar in kind to identity statements, which are necessary if true, though discovered *a posteriori*.

Note the contrast between object theory and the system of second-order logic with Hume's Principle. In the latter, the definition of 'x is a number' as '$\exists F(x = \#_F)$' is not perfectly general, since '$\#_F$' has been defined via Hume's Principle only for certain contexts. In object theory, however, the definition of 'x is a number' as '$\exists F(x = \#_F)$' is perfectly general; the notion '$\#_F$' is explicitly defined independently of Hume's Principle, as described in the previous paragraph. This directly engages Frege's point in §66 of the *Grundlagen*, where he says, in connection with the principle "the direction of line a is identical with the direction of line b iff a is parallel to b":

> It will not, for instance, decide for us whether England is the same as the direction of the Earth's axis. ... our definition ... says nothing as to whether the proposition,
>
> "the direction of a is identical with q"
>
> should be affirmed or denied, except for the one case where q is given in the form of "the direction of b". What we lack is the concept of direction; for if we had that, then we could lay it down that, if q is not a direction, our proposition is to be denied, while if it is a direction, our original definition will decide whether it is to be denied or affirmed. So the temptation is to give as our definition,
>
> q is a direction, if there is a line b whose direction is q.
>
> But then we have obviously come around in a circle. For in order to make use of this definition, we should have to know already in every case whether the proposition,
>
> "q is identical with the direction of b"
>
> was to be affirmed or denied.

Object theory breaks out of Frege's circle because it provides a completely general definition of the concept of 'number' which makes no use of any notions contextually defined by Hume's Principle. That stops the Julius Caesar problem. Indeed, the theory answers Cook and Ebert's (2005) C-R problem as well, since there are determinate conditions under which $\#F$ is identical to x even when x is some other kind of abstract object that can be defined in object theory (such as those definable on the basis of an abstraction over an equivalence relation).

It is instructive here to compare second-order object theory's simple statement of comprehension with Fine's theory of abstraction. Whereas Fine limits abstraction principles to those which are non-inflationary (and jointly non-hyperinflationary), object theory does not. Indeed, models of object comprehension show that there can be *as many or more* individuals as there are properties or sets of properties, as long as the objects are built in terms of a second mode of predication. This shields their defining properties from the conditions that lead to paradox. The Aczel-models of second-order object theory (Zalta 1999, 626–8) show that the sets representing abstract objects are clearly more numerous than the domain of ordinary individuals in the model. But in these models, first-order quantifier ranges over a domain which includes both the ordinary individuals and the sets representing abstracta! These models demonstrate that one can have a domain of individuals (consisting of both ordinary objects and abstracts) that is inflationary as long as there is (1) a new mode of predication and (2) no means of constructing arbitrary new properties in terms of that new mode of predication.

Another observation relevant to the comparison concerns the generality of the theory. Recall Fine admits that his theory of abstraction is incomplete in two ways: (a) there are no abstracts corresponding to the cells of equivalence relations on individuals and (b) there are no abstracts corresponding to the cells of equivalence relations on relations. However, there are both kinds of abstracta in object theory. For (a), see Anderson and Zalta 2004, which describes the method for deriving the Fregean biconditionals for directions and shapes once directions and shapes are identified as abstract objects. For (b), consider that abstraction can take place over relations, either by projecting the relations into properties or by using equivalence relations on relations themselves, without paradox.[27]

[27] For example, here is how to define abstracts for isomorphism types, without paradox. We use the following notions: $x =_E y$ (defined above), $F \approx_E G$ (defined in Zalta 1999), and $F \equiv G$ (i.e., F and G are materially equivalent). For arbitrary relation R, we want to define an abstract object that encodes the (properties whose extensions are the domains of) relations isomorphic to R with respect to the ordinary objects. First we require 3 preliminary definitions:

$\mathbf{D}_R$ ('being in the domain of R') $=_{df}$ $[\lambda x\, \exists y Rxy]$

$\mathbf{R}_S$ ('being in the range of S') $=_{df}$ $[\lambda x\, \exists y Syx]$

$\mathit{Isomorphic}_E(R, S)$ $=_{df}$ (a) $\mathbf{D}_R \approx_E \mathbf{D}_S$, and (b) there exists a relation T such that:

(a) $\mathbf{D}_T \equiv \mathbf{D}_R$ and $\mathbf{R}_T \equiv \mathbf{D}_S$,

(b) T is one-one and onto (relative to $=_E$), and

(c) $\forall x, y[R(x, y) \equiv S(\imath z Txz, \imath z Tyz)]$

Though we have focused on formulating object theory in terms of the abstraction principle (B) instead of the comprehension principle (A), there are reasons for preferring (A) in the formulation. The latter shows how object theory responds to the paradox that defeated Frege's original logicist program. One open question left by the logicists is, exactly when can one introduce objects corresponding to sets of concepts? A proper new form of logicism ought to be explicit about how it avoids the paradoxes. Similarly, Fine's investigations of the limits of abstraction return to the issue of inflation raised by the original, inconsistent Fregean set theory, so as to reconcile its jointly inconsistent constraints (a) that the domain of individuals be at least as large as the domain of properties, and (b) that the latter must be strictly larger than the former. It is only by using an explicit comprehension principle that a theory directly reveals its solution to the paradoxes.

A second reason for formulating object theory with comprehension rather than abstraction is that it provides answers to questions Boolos raises in connection with the analyticity of Hume's Principle. In 1997, Boolos asks (306), "Why believe there is a function that maps concepts to objects like octothorpe?", and asks (307), "How do we know that for every concept there is such a thing as a number of that concept?".[28] The answer to these questions, in object theory, traces back to comprehension; we know there is such a function as octothorpe and that every concept has a number because the relevant existence claims are *theorems* directly provable from the comprehension principle. We have attempted to justify the comprehension principle elsewhere.[29] Indeed, how could one know the answer to such questions unless they were implied by a more general theory? Of course, a neologicist could point out that Boolos's questions nevertheless still apply to the reformulation of object theory that uses abstraction (B) instead of comprehension (A). They might argue that one could answer these questions as applied to Hume's Principle by pointing to Boolos's reformulation (using the η relation) of Hume's Principle as the comprehension principle Numbers (in his development of Frege Arithmetic); since Numbers is similarly a comprehension principle, the answers we just gave to Boolos's questions are available to a defender of Hume's Principle. But here, a difference emerges because comprehension over objects as in (A) is much more general than the Numbers principle underlying Frege Arithmetic, which simply asserts the existence of numbers. It is easier to answer the question, "How do we know that comprehension principle for abstract objects is true?" than it is to answer the question "How do we know that the principle underlying Frege Arithmetic is true?" The former question concerns a general principle of abstract objects which reflects their very nature and which may be part of the metaphysical foundations of logic itself, whereas the latter is simply a piecemeal, non-logical theory of one kind of abstract object, namely, the Frege numbers.

Despite these reasons for employing (A), the formulation of object theory using the abstraction principle (B) has the following virtue: since minimal models of second-order object theory show that it has no greater ontological commitment than that of pure third-order logic (which commits us to at least one individual, two first-level properties, and four second-level properties), we might conceive of the abstraction principle (B) not just as an analytic truth, but as a *logical* truth as well. This would cast second-order object theory as a kind of *logic*.[30] Indeed, what (non-question-begging) reason is there to think that such an unrestricted, general abstraction principle is not part of logic?[31,32]

Finally, we define *the isomorphism type of* R (with respect to ordinary objects):

$$^{\equiv}R = \imath x(A!x \,\&\, \forall F(xF \equiv \exists S(F = \mathbf{D}_S \,\&\, \mathit{Isomorphic}_E(S, R))))$$

It now follows that $^{\equiv}R = {}^{\equiv}S$ iff $\mathit{Isomorphic}_E(S,R)$.

[28] He asks the question again (1997, 308), "What guarantee have we that every concept has a number?"

[29] See Linsky and Zalta 1995, where we attempt to show that the comprehension principle is justiable on naturalist grounds because it is (a) simple, (b) non-arbitrary, (c) parsimonious, and (d) required for our understanding of any possible scientific theory.

[30] One can even justify the modal axiom used in the proof of the infinity of numbers (Zalta 1999) as part of logic, since it simply tells us that the domain *might* be of any size, not what its size is. But we shall discuss this idea in more detail in the next paragraph.

[31] Boolos has formulated an objection to thinking that logic commits us to objects. He says (1997, 502): "No conception of logic commits us to the existence of two distinct objects (on any understanding of logic now available to us)". Our reply is that there is an understanding of logic now available which does commit us to the existence of two distinct objects (where 'object' is construed broadly here). Second-order logic under general models commits one to at least one individual and two properties. We think this is justified as part of logic because you need at least this much for an understanding of predication. Boolos here seems to be thinking here that the only conception of logic that is available to us is first-order logic.

[32] It is also worth replying here to another argument one might have for thinking that object abstraction is not part of logic, namely, that there is an interpretation of the language of object theory under which it is false. We actually mentioned this earlier,

The view that second-order object theory with abstraction is a logic applies even when that theory is extended with the modal axiom needed to derive the infinity of natural numbers. The modal axiom used in Zalta 1999 to prove that every number has a successor was presented after the definition of octothorpe, predecessor, and natural number. This modal axiom asserts (where '$\mathcal{A}$' in the formal version represents the actuality operator used in the informal version):

> If natural number n numbers the Fs, then there might have been a concrete object distinct from all the actual Fs.

$$\exists x(\mathit{NaturalNumber}(x)\ \&\ x = \#_F) \rightarrow \\ \quad \Diamond \exists y(E!y\ \&\ \forall u(\mathcal{A}Fu \rightarrow u \neq_E y))$$

This principle can be used to prove that every number n has a successor, by identifying the number which is the successor of n as the number of the concept $[\lambda z\, Fz \vee z =_E a]$, where a is some (possibly concrete) object which might have been distinct from all the actual Fs.

It is important here to make a series of observations about this modal claim and its relationship to logic. First, it should be noted that this principle only commits one to the possible existence of certain concrete objects, not to the existence of an infinite number of concrete objects. So, it is not on a par with 'axioms of infinity', which assert that there is an infinite number of objects by asserting either the existence of an infinite set or the unboundedness of some domain of objects. The above modal axiom does not do either of these things. Rather, the fact that every number has a successor can be proved only once the above axiom is located within a fixed domain modal logic, for then the Barcan formula becomes the essential additional element for requiring that the possible existence of certain concrete objects implies the existence of possibly concrete (i.e., ordinary) objects. Thus, the above modal axiom is rather weak, and postulates not an actual infinity of objects but a potential infinity, in the sense that no matter how many concrete objects there are, there might have been more.

as a reason many logicians now use for thinking that the axioms of set theory are not logical axioms. However, we did not offer that reason as a conclusive argument for thinking that the axioms of set theory are non-logical. For indeed, one could just as easily claim that the '$p \vee \neg p$' is not a logical truth because there are non-standard (e.g., many-valued, intuitionistic, etc.) interpretations of the language of the propositional calculus on which this formula turns out to be false and so fails to be a logical theorem. Thus, the fact that you can investigate principles from a model-theoretic point of view and see what consequences they have (e.g., some are true in empty domains, some in non-empty domains, some in domains with more than one object, etc.), does not show whether something is or is not a logical truth, since the relevance logicians or intuitionists can study the properties of classical logical principles in a similar way.

Second, we think this modal axiom is not just consistent with the elements of the traditional conception of logic, but also grounds one of those elements. One traditional (semantic) conception of logic is that logic is a topic-neutral account of the consequence relation, and so it is (1) consistent with any *possible* domain, and (2) concerned with concepts that remain invariant under any permutation of the domain. Given that all the terms in our modal axiom were defined solely in terms of the modal logic of encoding (i.e., second-order modal logic and the encoding mode of predication), we would argue that it is not only consistent with this understanding of logic, but also that it captures the idea that the domain could be of any size, and thereby expresses one of the intuitions that grounds the above conception of logic.

If we are right, then logic is committed to an infinite number of *contingently nonconcrete* objects; they are ordinary but they are not concrete.[33] We therefore take it that there is a modal component to logic which has not been made explicit, though logicians often implicitly express it when they say things like "The domain might be of any size." At present, we are committed only to the modal component described above, namely, that logic should be consistent with any possible domain of concrete objects.[34]

We agree that there is a difficult line to draw between the bounds of logic and metaphysics, but our broader conception of logic is one which admits that logic must be committed to the existence of *something*, whether it is propositions, truth-values, sentence-types, a consequence relation, or possible worlds, etc. Thus logic has some metaphysical presuppositions

[33] As we shall see, there is a number which numbers the contingently concrete (i.e., ordinary) objects but it is not a finite number. See the theorem displayed below. Moreover, we should note here that modal considerations have played a role in recent discussions of what logic is. See Etchemendy 1990.

[34] Boolos does restrict the argument mentioned in footnote 31 to conceptions of logic 'currently available', but presumably the force of the argument comes from the likelihood that there is not a better account on offer. We suspect that the intuition that logic should be consistent with a domain of any size is also part of the formal or 'schematic' view of logic as identifying those uninterpreted schemata which come out true on all interpretations (see Goldfarb 2001b), as well as part of the normative conception of logic as defining what is constitutive of correct inference (see MacFarlane 2002).

implicit in the tools used by logicians, and these should be explicitly identified in the form of existence principles or abstraction principles which are part of logic as well as being metaphysical in character. We've seen that the line is not always sharp. For example, we've noted the fact that object abstraction ((B) above) looks more like a logical principle than the equivalent principle of object comprehension ((A) above), which seems clearly non-logical. Still, both object abstraction and the above modal axiom might be part of the metaphysical presuppositions underlying logic. The resulting metaphysical theory is applicable no matter which concrete objects there are. It is topic neutral with respect to the things there actually are — abstract objects are correlated with arbitrary conditions on properties. And it has some modal force — it acknowledges that there might be concrete domains of any size.

There is one technical result which shows that the statement of object theory which comes closest to being an axiom of infinity turns out to be a theorem and not an axiom! Note that Frege defines a 'finite number' (i.e., natural number) to be any individual to which 0 bears the weak ancestral of the Predecessor relation. Thus, the statement of object theory which corresponds to the Russellian axiom of infinity is: the number of ordinary individuals is not a finite number.

In object theory (where the definition of $NaturalNumber(x)$ is just Frege's definition of $FiniteNumber(x)$), we can prove this corresponding statement as a theorem:[35]

[35] Here is a proof, which uses the numbering scheme in Zalta 1999. (Although not strictly necessary, note that the fact that $\#O!$ is well-defined is established by Theorem 9 and by Definition 10 in Zalta 1999.)

Now suppose, for reductio, that $NaturalNumber(\#O!)$. Call this number n. Then since $n = \#O!$, it will follow by the Modal Axiom (39), that $\Diamond\exists y(E!y \,\&\, \forall u(\mathcal{A}O!u \rightarrow u \neq_E y))$. (In other words, where $\mathcal{A}$ is the actuality operator, the Modal Axiom will imply (under our reductio hypothesis): there might be a concrete object distinct from all the actual ordinary objects.) By the Barcan Formula, $\exists y\Diamond(E!y \& \forall u(\mathcal{A}O!u \rightarrow u \neq_E y))$. Call an arbitrary such ordinary object b. Then we know $\Diamond(E!b \,\&\, \forall u(\mathcal{A}O!u \rightarrow u \neq_E b))$. By the laws of possibility, it follows that:

$$\Diamond E!b \,\&\, \Diamond\forall u(\mathcal{A}O!u \rightarrow u \neq_E b). \qquad (\theta)$$

From the first conjunct of (θ), it follows by definition that $O!b$, and by the laws of actuality, that $\mathcal{A}O!b$.

From the second conjunct of (θ), it follows by Modal Lemma (40) (Zalta 1999) that $\forall u(\mathcal{A}O!u \rightarrow u \neq_E b)$. (If it is possible that b is distinct$_E$ from all the actual ordinary objects, it *is* distinct$_E$ from all the actual ordinary objects. This Lemma was proved by the necessity of identity$_E$.) So, instantiating to b, we have $\mathcal{A}O!b \rightarrow b \neq_E b)$. But, since $O!b$, it is a theorem (3) that $b =_E b$. So, $\neg\mathcal{A}O!b$. Contradiction.

Theorem: $\neg NaturalNumber(\#O!)$

From the theorem that $\#O!$ is not a finite (natural) number, it follows that there is a natural cardinal which is not a finite number:

$$\exists x(NaturalCardinal(x) \,\&\, \neg Natural\ Number(x))$$

Thus, the closest statement in object theory to Russell's axiom of infinity is a theorem. This theorem guarantees that the domain of possibly concrete objects is infinite, not that the domain of concrete objects is infinite.

Having now described the virtues of second-order modal object theory, we conclude this subsection by comparing its limits of abstraction with those of the other forms of neologicism discussed previously. We shall argue in the next section, however, that our preferred form of neologicism, third-order non-modal object theory, is not subject to these limitations.

Though second-order modal object theory has numerous philosophical applications (not discussed here), and offers a very general theory of abstraction that reifies *any* condition on properties into abstract individuals, its limitation as a form of neologicism concerns the fact that the reconstruction of interesting mathematical notions in object theory requires that encoding formulas be used in definitions. For example, the *predecessor* relation is defined as:

$$Precedes(x,y) \ =_{df} \ \exists F\exists u(Fu \,\&\, y\!=\!\#_F \,\&\, x\!=\!\#_{[\lambda z\ Fz \,\&\, z\neq_E u]})$$

Note that when '=' (in the second conjunct of the quantified claim in the definiens) is replaced by primitive notation, the resulting formula contains encoding subformulas. As such, there is no guarantee that a *predecessor* relation exists, since the defining formula is not allowed in relation comprehension. The existence of such a relation has to be asserted as an axiom and the resulting theory proved to be consistent (as in Zalta 1999). So whereas the forms of neologicism in Section 4.1 use unrestricted property comprehension and must add various new abstraction principles to get to new kinds of mathematical objects, second-order modal object theory has an unrestricted abstraction principle for objects but has to add existence assertions for properties to get new kinds of mathematical objects.

Thus, all of the forms of neologicism described thus far give up generality in some important way — none of them can capture the full range of mathematics. Each version of neologicism runs into some limit of abstraction. Only a new approach can get past these limits, as we now show.

5. Reconceive what counts as a reduction

Having compared second-order modal object theory with other forms of neologicism, we now argue that third-order (nonmodal) object theory is the best way to reconstruct logicism. Second-order modal object theory was of interest precisely because it involved *no* mathematical primitives and yet had some limited mathematical power. Some of the other systems we discussed had more mathematical power, though the more mathematical power they had the less they seemed like a form of neologicism. There is, however, a rather different way to approach mathematics and address the problems that motivated logicism and neologicism, namely, by adding the (expressive) power of third-order (non-modal) object theory. The goal here is to find a neologicism that overcomes the limitations of abstraction by being so general as to be applicable to all of mathematics rather than just to some part of it. Instead of concluding that some parts of mathematics can not be epistemically justified, we plan to show that all of mathematics (no matter how it turns out) can be justified in a uniform way.

If it is legitimate to find a form of neologicism by weakening the claim "mathematics is reducible to logic alone" in the ways explored in previous sections, then it is certainly also legimitate to find a form that weakens the claim by reconceiving the notion of 'reduction'. Our view is that philosophy itself should not be concerned with 'mathematical' foundations for mathematics. We should let the mathematicians decide which mathematical theories and tools are best suited for the pursuit of mathematics and the investigation of its power and resources. So we have nothing to offer by way of mathematical reductions and foundations. But philosophers should be concerned with metaphysical and epistemological foundations for mathematics, and we therefore plan to offer a notion of reduction that provides answers to the metaphysical question, "What is mathematics about?", and to the epistemological question, "How do we know its claims are true?" Indeed, a unique feature of our program is that it yields no proper mathematics on its own, and so makes no judgments about which parts of mathematics are philosophically justified! Instead, it takes as data any arbitrary mathematical theory that mathematicians may formulate, and provides a more general explanation and analysis of the subject as a whole. This analysis encounters no limits of abstraction.

The theory of mathematical theories, objects, and relations formulated in Linsky and Zalta 1995, and developed in Zalta 2000, identifies a mathematical theory T as an abstract object that encodes the propositions p that are true in T, where 'p is true in T' is defined as 'T encodes the property *being such that* p' (i.e., $\text{T}[\lambda y\, p]$). We hereafter write $\text{T} \models p$ to more vividly indicate that p is true in T. A special Rule of Closure guarantees that $\text{T} \models p$ is closed under proof-theoretic consequence (i.e., we may infer $\text{T} \models q$ whenever q is syntactically derivable from propositions $p_1, \ldots, p_n$ all of which are true in T). Then, each axiom ϕ of T is imported into object theory by adding to object theory analytic truths of the form: $\text{T} \models \phi^*$, where ϕ^* is the result of indexing all the well-defined terms and predicates of T with the symbol that names the theory T. For example, the axiom of Peano Arithmetic which asserts that 0 is a number gets imported into object theory as: $\text{PA} \models N_{\text{PA}}0_{\text{PA}}$. And the ZF axiom that asserts that there is a null set would be imported into object theory as: $\text{ZF} \models \exists x \neg\exists y(y \in_{\text{ZF}} x)$. Then, for any well-defined object term κ in T, object theory uses the following principle:

$$\kappa_{\text{T}} \;=\; \imath x(A!x \,\&\, \forall F(xF \equiv \text{T} \models F\kappa_{\text{T}}))$$

This principle guarantees that within object theory, the object κ of theory T is the abstract individual x which encodes all and only the properties that κ has in theory T. This is not a definition which introduces an object using an instance of abstraction, but is rather a theoretical claim which tells us, in principle, how to identify κ_{T} in terms of the role it plays in T.

Whereas this principle for identifying the mathematical objects of theory T employs only second-order object theory, we need third-order object theory to similarly identify the properties and relations of T. So, to make the adjustments necessary, consider an n-place predicate 'Π' in theory T. Let '$\mathbf{A!}$' be the second-level property of being abstract that applies to first-level n-place relations such as Π. And let $\mathbf{F}$ be a variable ranging over (second-level) properties of n-place relations such as Π. Then, where 'R' is a variable ranging over first-order n-place relations, third-order object theory specifies, for any T:

$$\Pi_{\text{T}} \;=\; \imath R(\mathbf{A!}R \,\&\, \forall \mathbf{F}(R\mathbf{F} \equiv \text{T} \models \mathbf{F}\Pi_{\text{T}}))$$

In other words, the property Π of theory T is the abstract relation R which encodes all and only those second-level properties $\mathbf{F}$ such that in theory T, Π exemplifies $\mathbf{F}$. Again, this does not *introduce* the relation Π but rather is a principle that identifies Π in terms of its role in T.

So, although third-order object theory imports primitive mathematical notions and indexes them to their respective theories, it characterizes these notions by *analytic* claims of the form 'In theory T, *p*', and then proceeds to identify the primitive objects and relations as abstract objects and relations, respectively. So to each primitive mathematical notion there corresponds a principle that identifies the object or relation it denotes. Moreover, as sketched in Linsky and Zalta 1995 and developed in detail in Zalta 2000, object theory offers an analysis of the truth conditions of ordinary mathematical claims. We will not go into detail here how this is to be done, but only note that each ordinary mathematical claim (i.e., unprefaced by the theory operator) will get a reading on which it is true. The truth of ordinary mathematical statements is captured using the encoding mode of predication.[36]

It is important to observe here that this offers a new notion of 'reduction'. This notion, that of 'metaphysical' or 'ontological' reduction, was discussed in detail in Zalta 2000, but the main idea is that an ontological reduction of mathematics within third-order object theory gives us a general treatment of mathematics which offers an analysis of both the denotation of the terms of mathematics and the truth conditions of mathematical statements. Whereas the traditional notions of 'reducibility' (such as proof-theoretic, interpretability, model-theoretic, etc.) are needed when one is trying to find a *mathematical* foundation for mathematics, they need not be in play for giving metaphysical or epistemological reductions of mathematics. Indeed the set theory used in model theory will be just one more mathematical theory and so have no particular foundational significance. Thus, in the classification that Burgess uses in 2005, set theory serves as mathematical foundations and a linear hierarchy of systems can be arranged in terms of interpretability power. We are here suggesting that a different notion of reducibility is needed for metaphysical and epistemological foundations, as opposed to mathematical foundations.

To see how third-order object theory constitutes a form of neologicism, we need to show how it is related to the principle of logicism, namely, that 'mathematics is reducible to logic alone'. Previously we have discussed ways in which our conceptions of 'logic' and 'logic alone' could be relaxed so as to form a neologicism. Our claim is:

> Third-order object theory is a neologicism because it reduces (in the sense just described) all of mathematics to 'third-order' logic and some analytic truths.

[36]It will also get a reading on which it is false. Thus '2 is algebraic' is true if analyzed as '2A' and false if analyzed as 'A2'. However, 'In real number theory, 2 is algebraic' is true simpliciter when 'is' is read as 'exemplifies', and is represented as 'RNT $\models A2$'

By quoting the phrase 'third-order', we are calling attention to the fact that the theory is weaker than full third-order logic. Though our theory is most naturally formulated using third-order syntax, its logical strength is no greater than multi-sorted first-order logic. [It is important to mention here that in previous work (Linsky and Zalta 1995, Zalta 2000), the object theory used to analyze mathematics was formulated in the framework of simple type theory. We typed the language and axioms of object theory, as well as its comprehension principles, using a standard simple type theory. Then we asserted the existence of abstract objects at each type t. However, in the interests of finding the weakest theory needed to do the job, we are now using only the third-order formulation of object theory.]

Now, to establish the above claim, we note that third-order object theory consists of third-order logic with (a) (arguably analytic) abstraction principles for abstract individuals and for first-level abstract properties and relations, and (b) analytic truths about what's true in mathematical theories. First, we formulate the abstraction principles for abstract individuals and for abstract properties and relations as follows. Let 'R' be a variable ranging over the domain of first-level properties and relations, '$\mathbf{F}$' and '$\mathbf{G}$' be variables ranging over properties of first-level properties and relations, and '$\mathbf{A!}$' be the second-level property of being abstract. Then we have:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \phi))G \equiv \phi^G_F, \text{ where } \phi \text{ has no free } x\text{s.}$$

$$\imath R(\mathbf{A}!R \,\&\, \forall \mathbf{F}(R\mathbf{F} \equiv \phi))\mathbf{G} \equiv \phi^{\mathbf{G}}_{\mathbf{F}}, \text{ where } \phi \text{ has no free } R\text{s}$$

The first implies the existence of abstract individuals in terms of the first-level properties they encode, and the second implies the existence of abstract first-level properties and relations in terms of the second-level properties they encode. So, far then, our third-order logic plus abstraction principles is free of primitive mathematical notions and axioms.

Second, we add analytic truths of the form 'In theory T, *p*', for arbitrary mathematical theories T. As noted above, these axioms are added by importing the constants and predicates of each mathematical theory T into the language of third-order object theory, indexing them to their respective theories. Then, each of these constants and predicates will be subject to their respective identification principle, as described above.

Note that by adding the primitive analytic truths of this form, we are not adding mathematical axioms such as New V or Hume's Principle. We distinguish primitive mathematical axioms, such as '0 is a number', 'there is a set having no members', Hume's Principle, New V, etc., from analytic truths of the form 'In mathematical theory T, p'. A case can be made for thinking that the latter are *not* mathematical principles, at least not in the same sense as the former, though we shall not pursue the point here. (Intuitively, our view is that the former are about mathematical objects and their properties whereas the latter are about mathematical theories.)

So we have a form of neologicism; the original logicist claim "mathematics is reducible to logic alone" is reconstructed as: all of mathematics is ontologically reducible to (syntactically) third-order object theory (which has no greater logical power than multi-sorted, first-order object theory with two atomic forms of predication), analytic abstraction principles, and analytic truths about the content of mathematical theories. This reconstruction does not restrict the amount of mathematics to which it is applicable. It applies to all mathematical theories, with no limits of abstraction.

Note that our neologicism does not suffer from a bad company objection or an embarrassment of riches objection. There is a single abstraction principle for each domain of quantification, rather than many. There are no other abstraction principles with the same form but which lead to contradiction, nor are there too many individually consistent, but jointly incompatible, alternative abstraction principles. The theory is therefore not piecemeal in its approach to abstract objects. Moreover, we are not just finding alternative axiomatizations of mathematical theories, but rather exhibiting a way in which mathematics is reducible to logic plus analytic truths.

It is important to recognize that we now have an answer to the first of the questions posed earlier, namely, "What is mathematics about?" Our answer is that mathematics is about abstract objects (indeed, objects that bear some resemblance to the 'indeterminate elements' (Benacerraf 1965) required by structuralist analyses of mathematics) and the properties that they encode. If the mathematicians come along and decide that ZF is the proper mathematical foundations for the rest of mathematics, then we are prepared to agree with them that mathematical objects are ZF sets. But we would go one step further and give a philosophical answer to the question, "What is ZF about?", namely, that it is about abstract objects

that 'are' ZF sets in the sense that they encode the abstract property *being a set*$_{ZF}$ (where the abstract property of *being a set*$_{ZF}$ is itself a property that encodes all and only the higher-order properties **F** such that, in the theory ZF, the property *being a set*$_{ZF}$ exemplifies **F**).

Moreover, third-order object theory provides the epistemological foundations for pure mathematics that the logicists sought. Recall Benacerraf's description of this epistemological goal in [1981] (42-43):

> But in reply to Kant, logicists claimed that these propositions are a priori because they are analytic—because they are true (false) merely "in virtue of" the meanings of the terms in which they are cast. Thus to know their meanings is to know all that is required for a knowledge of their truth. No empirical investigation is needed. The philosophical point of establishing the view was nakedly epistemological: logicism, if it could be established, would show that our knowledge of mathematics could be accounted for by whatever would account for our knowledge of language. And, of course, it was assumed that knowledge of language could *itself* be accounted for in ways consistent with empiricist principles, that language was itself entirely learned. Thus, following Hume, all our knowledge could once more be seen as concerning either "relations of ideas" (analytic and a priori) or "matters of fact".[37]

If we are right, then our answer to the question "How do we know that the claims of mathematics are true?" depends on whether we are considering an axiom or a theorem. If the claim in question is an axiom of some theory, say T, then we know that it is true in virtue of our knowledge of language (and we therefore account for that knowledge in terms of whatever faculty accounts for our knowledge of language). So, no special faculty of intuition is needed. If the claim in question is a theorem of T, then we know that it is true in virtue of our ability to derive claims about analytic truths from more basic analytic truths. Again, no special faculty of intuition is needed for this.

Whereas Benacerraf thought that mathematical knowledge had to be analytic, we think the operative point in the above passage is this: to know that a mathematical claim is true is know that it is an axiom or

[37] We recognize that this passage comes in the context where Benacerraf is presenting the "myth he learned as a youth", but this bit is not the mythical part!

theorem of some theory of mathematics. Thus, our knowledge of what mathematical claims mean is *grounded* or systematized by our abstraction principle. Given such a principle, then, we only need to understand analytic claims of the form 'In the theory T, p' in order to formulate theoretical descriptions of mathematical objects and relations. This allows us to have knowledge of all the axioms and theorems of mathematical theories, including for example the truths of ZF and those of alternatives to ZF such as Aczel's nonwellfounded set theory. (The fact that these latter two theories are inconsistent with one another does not mean that we can not have knowledge of their claims. In each case, the inconsistency is undermined by the fact that both get imported into object theory under the scope of the theory operator: ZF sets will encode the property of being well-founded, while some Aczel-sets will encode the property of being nonwellfounded.) Thus, unlike other neologicist schools, we are not in the business of saying which mathematical theories can be epistemically justified and which are not; we think they are all epistemically justified in the appropriate way.[38]

We conclude this section with a brief discussion of how this view squares with the Platonized naturalism defended in Linsky and Zalta 1995. We recognized there that knowledge of abstract objects (and thus, mathematics) was knowledge involving (canonical) descriptions of the form $\imath x(A!x \;\&\; \forall F(xF \equiv \phi))$, where these canonical descriptions were grounded in the comprehension principle for abstract objects. We argued that no special faculty of intuition was needed for our knowledge of mathematics, and that since each well-defined mathematical term could be identified with a description, knowledge by acquaintance with mathematical objects and relations collapses to knowledge by description. But, in the present piece, we are recognizing that the main principle of the theory can be reformulated as an abstraction principle rather than a comprehension principle. This lends itself to the suggestion that the abstraction principle is analytic, given the present climate in which principles like Hume's Principle have been labeled as such. If object abstraction were analytic, it would need a different justification than that of a synthetic *a priori* comprehension principle.

[38]We even think we can give an appropriate justification for our knowledge that certain mathematical theories and objects are inconsistent. We know, for example, that in naïve set theory, one can prove that there exists a Russell set (i.e., a set which has as members all and only those sets which are not members of themselves) and prove that it is a member of itself iff not. So, on our view, the expression for the Russell set will denote an object that encodes the property of being a set which is a member of itself iff it is not. (Of course, it will be a somewhat uninteresting object, since all properties definable in naïve set theory will also be encoded by this object.) So, we can account for our knowledge of the facts concerning inconsistent mathematical theories.

But we have tried to avoid drawing a final conclusion about the status of object abstraction as an analytic, or even logical, truth. We are not sure it really matters whether one takes it to be analytic, or logical or simply a reformulation of a synthetic *a priori* comprehension principle. For the important questions are whether the addition of object abstraction to logic constitutes a neologicism, and whether this version of neologicism is consistent with the kind of naturalism we have defended earlier. We think that the answer to both questions is 'Yes'. Adding object abstraction to logic produces a form of neologicism if adding Hume's Principle to logic does. Moreover, no one has yet explained how a version of logicism that implies and acknowledges existential claims might be consistent with naturalism. But, if Linsky and Zalta 1995 is right, the current version of neologicism is consistent with naturalism, on the grounds outlined in that paper. We see the result as a naturalist account of the existence claims that form an essential part of mathematics and our neologicism.

Conclusion

Our answer to the question, 'What is neologicism?', has been to map out the ways that one can carry out the program of 'reducing mathematics to logic alone' by reconceptualizing the notions involved in this seminal logicist claim. By comparing the various ways of reconstructing the logicist program, we have been led to defend one particular version of neologicism, on the grounds that it best addresses the underlying motives of the early neologicists. If epistemological concerns about how we can have knowledge of mathematics were the principal driving force of the early logicists, then the version of neologicism defended here is the only one that addresses those concerns for the entire range of (possible) mathematical theories. For we have shown how one can (a) start with a classical notion of logic, (b) add some fundamental truths (i.e., the instances of object abstraction) which sound analytic and which may be analytic in some important sense even if they are not analytic in the classical sense, (c) add other genuinely analytic truths of the form 'In theory T, p', (d) give a 'reductive' analysis of arbitrary mathematical theories that accounts for the denotations and truth conditions of their terms and sentences, respectively, and finally, (e) account for our knowledge of mathematics in terms of our knowledge of object abstraction. Our knowledge of mathematics is to be explained in terms of the faculty we use to understand language, since that is the only faculty we need to understand object abstraction. None of the other forms of neologicism offer this level of generality in their explanation of mathematical knowledge.

From the point of view of object theory there are two ways of approaching mathematics. One is to view mathematical objects as arising from equivalence relations on concepts and individuals, by introducing an individual corresponding to each cell of the equivalence relation. This is captured in 'second-order' object theory as a special case of comprehension or abstraction, for these latter principles introduce a new individual corresponding to each expressible property of properties, and not just a new individual for each equivalence relation. The other approach is to view mathematical objects as creatures of mathematical theories, and to view both those theories and their objects, as abstract objects. This second approach is captured in third-order object theory, by abstracting over the role each well-defined mathematical term and predicate plays in its respective theory. We have compared object theory with those other forms of neologicism which take the first approach to mathematics, hoping to cast new light on such theories. The second approach, however, is what is needed to have a completely general account, one that goes beyond the inevitable 'limits of abstraction' of the current neologicist views.

To confirm this, note how our view falls outside the categorization offered by Burgess in 2005. Though Burgess was attempting to classify all the theories in the business of 'fixing Frege', the above work shows that his classification is incomplete. Here is how. At the beginning of Chapter 3, Burgess reviews his classification of systems as follows:

> Predicative theories: extensions for every concept, but formulas with quantifiers over all concepts do not determine concepts.
>
> Impredicative theories: accept full second-order logic with quantification over all concepts freely admitted in formulas, and with all formulas assumed to determine concepts, but that restrict the assumption of the existence of extensions, or replace it by the assumption of the existence of abstracts for some equivalence other than coextensiveness.

But consider the ways in which our neologicism, which in some sense fixes Frege's attempt to reduce mathematics to logic, fails to be covered by this classification scheme. (1) Models of 'second-order' object theory show that the theory does not allow *full* second-order logic, nor do models of third-order object theory allow full third-order logic. Thus, the theory falls outside the second classification for impredicative theories, since it does not accept full second-order logic. But (2), the theory does not fall under the first classification above, since formulas with quantifiers *do* determine concepts. In addition, there are other ways that our theory fails this classification scheme: (a) The abstract objects postulated in object theory are not extensions — they correspond to groups of properties and not just to single properties. If anything, they can be represented as extensions of higher-order properties. But in either case, the Burgess classification seems to imply that the abstracts are extensions of concepts or, at least, must correspond to equivalence relations on concepts. And (b) though object theory is impredicative (properties can be formed which quantify over relations), not all open conditions on individuals in object theory are assumed to determine concepts — those with encoding subformulas do not. So object theory does not fall within Burgess's second categorization of impredicative theories.

We believe that the scheme for classifying the forms of neologicism introduced here offers a wider perspective on the problem of reconstructing logicism or 'fixing Frege'. We have tried to focus on the philosophical (i.e., metaphysical and epistemological) power of theories, and not just on their mathematical power. Though logicians are often interested in interpretability relations among theories, the more mathematical power a theory has, the more likely it is to be a piece of mathematics rather than a new form of logicism. By contrast, we believe that logicism is more closely approximated by theories that have *little* mathematical power while having serious philosophical power, and we believe that the approach followed here should be of interest primarily because of this point.

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