Comments, corrections, suggestions and
additional references welcomed, as always!

Why do we have three sorts of adjunctions?

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The three sorts of pairs of adjoint functors (covariant, contravariant right, and contravariant left) arise as a double coset space $H \backslash D_4 / K$, where D_4 is the dihedral group of order 8, and H and K are certain 2-element subgroups.

1. It is well-known that adjunctions come in three flavors: covariant adjunctions (involving one left and one right adjoint functor), contravariant right adjunctions (where both contravariant functors play the same role), and contravariant left adjunctions (ditto). Ultimately, of course, these are the same phenomenon, differently labeled. If we start with the description of a covariant adjunction,

$$\mathbf{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{D} \tag{1}$$

where U is the right and F the left adjoint, then the sorts of relabeling that we can apply are to replace $\mathbf{C}$ and/or $\mathbf{D}$ by its opposite, and/or to interchange the roles of $\mathbf{C}$ and $\mathbf{D}$. Thus, the four category-names $\mathbf{C}$, $\mathbf{D}$, $\mathbf{C}^{\mathrm{op}}$, $\mathbf{D}^{\mathrm{op}}$ may be permuted in all ways that respect the correspondence between opposite categories. If we arrange these symbols in a square with opposite categories at antipodal vertices,

$$\begin{array}{ccc} \mathbf{C} & \text{———} & \mathbf{D} \\ | & & | \\ \mathbf{D}^{\mathrm{op}} & \text{———} & \mathbf{C}^{\mathrm{op}} \end{array} \tag{2}$$

then the above relabelings constitute the symmetry group, D_4, of that square.

Note that if in (1) we replace each of $\mathbf{C}$ and $\mathbf{D}$ by its opposite (and regard our two functors, which now connect $\mathbf{C}^{\mathrm{op}}$ and $\mathbf{D}^{\mathrm{op}}$, as functors between $\mathbf{C}$ and $\mathbf{D}$), then the resulting structure is again a covariant adjunction between $\mathbf{C}$ and $\mathbf{D}$, but their roles have been reversed: $\mathbf{C}$ is now the domain of the left adjunction and $\mathbf{D}$ of the right adjunction. Hence, if we combine this relabeling with the relabeling that interchanges $\mathbf{C}$ and $\mathbf{D}$ (adjusting the names of U and F accordingly), we get exactly the original description of a covariant adjunction. (It does not matter whether we interchange $\mathbf{C}$ and $\mathbf{D}$ before or after replacing them by their opposites, since the operation of replacing both categories by their opposites, the $180°$ rotation of (2), is in the center of D_4.)

Within D_4, the above composite operation is the reflection of (2) through its horizontal axis of symmetry; and the subgroup K that it generates is in fact the isotropy subgroup of the description (1) of an adjoint pair. Thus, the set of descriptions of adjunctions obtained by applying D_4 to (1) can be identified with the 4-element set D_4 / K.

However, once we have applied some element of D_4 to (1), and gotten a description of a type of adjunction, we really do not care which category is called $\mathbf{C}$ and which is called $\mathbf{D}$ – those symbols are

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This note is an elaboration of a remark in [**2**, near the end of §7.12].
URLs of this preprint: http://math.berkeley.edu/~gbergman/papers/3adjs.ps and ... .pdf .

just stand-ins for whatever category-names we will use in our applications of the concept. Hence if we write H for the 2-element group whose nonidentity element interchanges $\mathbf{C}$ and $\mathbf{D}$, i.e., reflects (2) about its vertical axis, then *distinct* concepts we are interested in form the orbit space $H\backslash D_4/K$.

Now H interchanges one pair of elements of D_4/K (namely, the element corresponding to (1), and the element corresponding to same diagram with $\mathbf{C}$ and $\mathbf{D}$ transposed), but fixes each of the other two elements (corresponding to the two sorts of contravariant adjunctions). So the orbit space has cardinality 3, explaining the three sorts of adjunction. In particular, though a coset-space G/K of a finite group must have cardinality dividing the order of G, we see that this is not true of a double coset space $H\backslash G/K$, since the orbits of H, which are collapsed to single points, can have different cardinalities.

To see group-theoretically how this happens in our particular case, note that if we take an element $gK\in D_4/K$ such that g does not normalize K, then g must conjugate K to H. (Reflection about the horizontal and vertical axes can only be conjugated to one another.) So $gK = Hg$, which is fixed under the left action of H. On the other hand, for g normalizing K we have $gK = Kg$, which is not fixed under H, hence belongs to a 2-element orbit.

(As noted in [**1**], nontrivial contravariant left adjunctions among varieties of algebras are rare. But this rarity is irrelevant to the abstract taxonomy of adjunctions.)

2. There is a certain amount of arbitrariness in the choice to consider phenomena ‘‘the same’’ if and only if they can be obtained from one another by permuting our unsuperscripted category-names. As observed in the second sentence of this note, from a wider perspective all of these sorts of adjunctions are essentially the same. This viewpoint corresponds to allowing the replacement of $\mathbf{C}$ and $\mathbf{D}$ by their opposites as well as each other in deciding what phenomena we will put under one hat, and leads to the 1-element orbit space $D_4\backslash D_4/K$. This corresponds to the approach of Mac Lane [**3**, §IV.2].

In the opposite direction, we might, in some situation, want to think of adjunctions in terms of the ‘‘first’’ and ‘‘second’’ categories involved, i.e., not divide out by H. This, as noted, gives four kinds of adjunctions, corresponding to the set D_4/K.

In still other situations we might, for some reason, choose to identify types of adjunctions that could be obtained from one another by replacing both categories simultaneously by their opposites, but not by so replacing just one. This corresponds to the action of the center Z of D_4, giving us the 2-element orbit space $Z\backslash D_4/K \cong ZH\backslash D_4/K$, which simply classifies adjunctions as covariant or contravariant.

3. Can our description of the orbit of (1) under D_4 as D_4/K be visualized in terms of the picture (2)? In that picture, there are indeed entities having K as their isotropy group: either of the two vertical lines. Thus, if we associate the description (1) with, say, the left-hand vertical line of (2), we get an isomorphism of D_4-sets between the orbit of (1) and the edge-set of (2). Can we interpret this isomorphism?

Recall that an adjunction (1) can be characterized as an isomorphism of bifunctors

(3) $$\mathbf{C}(F(-),-) \cong \mathbf{D}(-,U(-)) \text{ as functors } \mathbf{D}^{\mathrm{op}}\times\mathbf{C} \to \mathbf{Set}.$$

Similarly, the adjunction we get by reversing the roles of $\mathbf{C}$ and $\mathbf{D}$ in (1) corresponds to a bifunctor $\mathbf{C}^{\mathrm{op}}\times\mathbf{D}\to\mathbf{Set}$, while a contravariant right adjunction corresponds to a bifunctor $\mathbf{C}^{\mathrm{op}}\times\mathbf{D}^{\mathrm{op}}\to\mathbf{Set}$, and a contravariant left adjunction to a bifunctor $\mathbf{C}\times\mathbf{D}\to\mathbf{Set}$. Thus, each sort of adjunction corresponds to a bifunctor on a pair of categories – and the pairs in question are precisely those labeled by the two ends of one or another edge of (2). This yields the indicated isomorphism of D_4-sets.

4. What sort of natural symmetries does the double coset space $H\backslash D_4/K$ have?

It is well-known that for any group G and subgroup K, the set G/K has a structure of G-set; i.e., an action of G on the left. A fact not as well known to non-specialists, but easy to verify, is that it has an action of $N_G(K)$, the normalizer of K in G, on the right; or better, an action of $N_G(K)/K$, since right multiplication by elements of K fixes all points. This right action commutes with the left action of G; indeed, $N_G(K)/K$ is the automorphism group of the G-set G/K.

Similarly, given a group G and two subgroups H and K, the double coset space $H\backslash G/K$ has both a right action of $N_G(K)/K$ and a left action of $N_G(H)/H$, which commute with one another.

In the case we are interested in, where $G = D_4$ and H and K are the 2-element subgroups noted above, the normalizers of these subgroups are HZ and KZ respectively, where Z is the center of D_4. (For HZ and KZ clearly normalize H and K, and they each have order 4, hence are maximal proper subgroups of D_4; so, since neither H nor K is normal, these must be their normalizers.) The induced actions on $H\backslash D_4/K$ are both, therefore, the action of Z, which interchanges the two sorts of contravariant adjunction, and fixes covariant adjunction.

5. Appendix. Since introductory discussions of coset-spaces G/K do not generally mention the right action of $N_G(K)/K$, I will give a couple of examples.

First, let $G = S_5$, and let $K = D_5 < A_5$, the symmetry group of the regular pentagon with vertices labeled consecutively 1, 2, 3, 4, 5, or equivalently, of the undirected 5-vertex cyclic graph with vertices again so labeled. The set G/K can then be identified with the set of *all* structures of undirected cyclic graph on $\{1,2,3,4,5\}$, with S_5 acting by permuting those vertices. One finds that K has index 2 in its normalizer, and that the nonidentity element of $N_G(K)/K$ acts on the above set of graphs by sending the cyclic graph with successive vertices p_1, p_2, p_3, p_4, p_5 to the cyclic graph with successive vertices p_1, p_3, p_5, p_2, p_4, i.e., with the original edges erased, and new edges drawn between what were previously alternate vertices.

Next, suppose that G is the group of proper motions (orientation-preserving symmetries) of the 3-cube, isomorphic to S_4, and that K is the isotropy group of one vertex v. Since $N_G(K)/K$ is the automorphism group of the 8-element G-set G/K, it will carry v to all vertices having the same isotropy group. These are just v and its antipodal vertex, and we find that $N_G(K)/K$ is Z_2, acting by central reflection. Together with G, this generates the group $S_4\times Z_2$ of all motions, proper and improper, of the cube. The same thing happens if we take the same G, but let K be the isotropy group of any of the 6 faces of the cube, or of any of its 12 edges, since each face or edge has an antipodal face or edge, which is the only one with the same isotropy group.

On the other hand, the group of proper motions of the cube also acts faithfully on the set of 4 solid diagonals (as their full group of permutations); and if we take for K the isotropy group of one of these, which will be $S_3 < S_4$, we find that it is its own normalizer; so here the automorphism group $N_G(K)/K$, and hence its action, are trivial.

At the opposite extreme, if we let K be the isotropy group of a half-edge (the segment from the center of an edge to one of its vertices), then K is trivial, so $N_G(K)/K = G$, and the induced action on the set of half-edges is by $S_4\times S_4$. Here the second factor does not act by isometries of the cube; e.g., a typical element is the map sending every half-edge to the other half of the same edge, which in particular sends the three half-edges adjacent to any vertex to non-adjacent half-edges.

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