Subharmonic functions

U always denotes a connected open set in $\mathbb{C}$.
We recall the definition:
A subharmonic function is an uppersemicontinuous function $f : U \to [-\infty, +\infty)$ such that $f \not\equiv -\infty$ and for all disks $\overline{D(P,r)} \subset \mathrm{U}$ and for all harmonic h on a neighborhood of $\overline{D(P,r)}$,

$$f \leq h \quad \text{on} \quad \partial D(P,r) \implies f \leq h \quad \text{on} \quad D(P,r)\,.$$

small circle sub-mean value property

Theorem

Let $f : U \to [-\infty, +\infty)$ be uppersemicontinuous. Then f is subharmonic $\iff \forall\, P \in U,\ \exists \varepsilon_P > 0$ such that $\overline{D(P, \varepsilon_P)} \subset U$ and

$$f(P) \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + \varepsilon\, e^{i\theta})\, d\theta\,, \quad \forall\, \varepsilon \in (0, \varepsilon_P)\,.$$

We could call this the *small circle sub-mean value property (SCsMV).*

We begin the proof with a technical lemma (to deal with the semicontinuity):

Lemma

Let $\varphi : \partial D \to [-\infty, +\infty)$ be uppersemicontinuous. Then there exists a sequence of continuous functions $\varphi_n : \partial D \to \mathbb{R}$ such that $\varphi_n \searrow \varphi$ pointwise.

Proof: Assume that $\varphi \not\equiv -\infty$. Let

$$\varphi_n(e^{i\theta}) = \sup_{t \in \mathbb{R}} \{\varphi(e^{i(\theta+t)}) - n|t|\} \in \mathbb{R} .$$

Clearly $\varphi_n \geq \varphi_{n+1} \geq \varphi$.

Claim 1: φ_n is continuous. (Exercise)

Claim 2: $\varphi_n \to \varphi$.
Proof of Claim 2: Let θ_0 be fixed. Let

$$g_n(t) = \varphi(e^{i(\theta_0+t)}) - n|t|$$

so that $\varphi_n(e^{i\theta_0}) = \sup g_n$. Let $c > \varphi(e^{i\theta_0})$ be arbitrary. Since φ is uppersemicontinuous, we can choose $\delta > 0$ so that
$\varphi(e^{i(\theta_0+t)}) < c \,, \ \forall t \in (-\delta, \delta)$.

$$g_n(t) = \varphi(e^{i(\theta_0+t)}) - n|t|$$

Let $A = \sup\varphi < +\infty$. Therefore

$$g_n(t) \leq \begin{cases} A - n\,\delta & \text{for} \quad |t| \geq \delta \\ c & \text{for} \quad |t| < \delta \end{cases}$$

Therefore there exists $n_0 \in \mathbb{Z}^+$ such that

$$\varphi_n(e^{i\theta_0}) = \sup g_n \leq c \quad \text{for} \quad n \geq n_0.$$

Since $c > \varphi(e^{i\theta_0})$ is arbitrary, it follows that $\varphi_n(e^{i\theta_0}) \to \varphi(e^{i\theta_0})$, verifying Claim 2 and completing the proof of the lemma. □

Theorem

Let $f : U \to [-\infty, +\infty)$ be uppersemicontinuous. Then f is subharmonic $\iff$ SCsMV.

Proof: ($\Longrightarrow$): Let $\overline{D} := \overline{D(P, \varepsilon)} \subset U$ be arbitrary.
Suppose that $f(P) > \frac{1}{2\pi} \int_{\partial D} f \, d\theta$.
Let $\varphi_n \in \mathcal{C}(\partial D)$ such that $\varphi_n(P + \varepsilon e^{i\theta}) \searrow f(P + \varepsilon e^{i\theta})$.
By Lebesgue monotone convergence, $\int_{\partial D} \varphi_n \to \int_{\partial D} f$.
Therefore we can chose n_0 such that $f(P) > \frac{1}{2\pi} \int_{\partial D} \varphi_{n_0} \, d\theta$.
Let h be the solution to the Dirichlet problem on D with $h = \varphi_{n_0}$ on ∂D. Let $\delta = f(P) - \frac{1}{2\pi} \int_{\partial D} \varphi_{n_0} \, d\theta > 0$. Since ∂D is compact and $f - h \leq 0$ is uppersemicontinuous on $\overline{D}$, $\exists s < \varepsilon$ such that $f - h \leq \delta/2$ on $\partial D(P, s)$.
Therefore, (1) $f - \delta/2 \leq h$ on $\partial D(P, s)$; (2) h is harmonic on $D(P, \varepsilon)$.
f subharmonic $\Longrightarrow$ $f(P) - \delta/2 \leq h(P) = \frac{1}{2\pi} \int_{\partial D} h \, d\theta = \frac{1}{2\pi} \int_{\partial D} \varphi_{n_0} \, d\theta = f(P) - \delta$. Contradiction □

Lemma

Let $f : U \to [-\infty, +\infty)$ be uppersemicontinuous. If f satisfies the SCsMV property, then f satisfies the "maximum principle":

$$P_0 \in U, f(P_0) = \sup f \implies f \equiv \textit{constant}.$$

The proof of the lemma is similar to the proof that SCMV $\Longrightarrow$ maximum principle.

Outline of proof that SCsMV $\Longrightarrow$ subharmonic: Let f satisfy SCsMV. Suppose on the contrary that h harmonic on $D = D(P, r)$, $f \leq h$ on ∂D, $f(Q) > h(Q)$, $Q \in D$. Let $g = f - h$ on $\overline{D}$. Then g attains its maximum at a point of D and hence is constant, by the above lemma. Contradiction □