Eisenstein Series, Crystals and Ice

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1 Introduction

Automorphic forms are generalizations of periodic functions; they are functions on a group that are invariant under a discrete subgroup. A natural way to arrange this invariance is by averaging. Eisenstein series are an important class of functions obtained in this way. It is possible to give explicit formulas for their Fourier coefficients. Such formulas can provide clues to deep connections with other fields. As an example, Langlands' computation of the constant Fourier coefficients of Eisenstein series inspired his far-reaching conjectures that dictate the role of automorphic forms in modern number theory.

In this article, we present two combinatorial models for the Fourier coefficients of (certain) Eisenstein series: crystal graphs and square ice models. Crystal graphs combinatorially encode important data associated to Lie group representations while ice models arise in the study of statistical mechanics. Both will be described from scratch in subsequent sections.

We were led to these surprising combinatorial connections by studying Eisenstein series not just on a group, but more generally on a family of arithmetic covers of the group. We will present formulas for their Fourier coefficients which hold even in this generality. In the simplest case, the Fourier coefficients of Eisenstein series are described in terms of symmetric functions known as Schur polynomials, so that is where our story begins.

2 Schur polynomials

The symmetric group S_n acts on the ring of polynomials in n variables $\mathbb{Z}[x_1, \ldots, x_n]$ by permuting the variables. A polynomial is *symmetric* if it is invariant under this action. Classical examples are the familiar elementary symmetric functions

$$e_j = \sum_{1 \leq i_1 < \ldots < i_j \leq n} x_{i_1} \ldots x_{i_j}.$$

Since the property of being symmetric is preserved by sums and products, the symmetric polynomials make up a subring Λ_n of $\mathbb{Z}[x_1, \ldots, x_n]$. The e_j, $1 \leq j \leq n$, generate the ring Λ_n.

Since Λ_n is also an abelian group under polynomial addition, it is natural to seek a set that generates Λ_n as an abelian group. One such set is given by the Schur polynomials (first considered

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by Jacobi), which are attached to partitions. A *partition* of a positive integer k is a non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $k = \sum \lambda_i$; necessarily only a finite number of terms in the sequence are nonzero. Partitions are added componentwise. If $\lambda = (\lambda_i)$ is a partition with $\lambda_i = 0$ for $i > n$, let $\rho = (n-1, n-2, \ldots, 0, \ldots)$, and let

$$a_{\lambda+\rho} = \det(x_i^{\lambda_j+n-j})_{1 \le i,j \le n}.$$

Then a_ρ divides $a_{\lambda+\rho}$ in Λ_n and the quotient is the Schur polynomial s_λ: $s_\lambda = a_{\lambda+\rho}/a_\rho$. This polynomial is in Λ_n and is homogenous of degree k. For example, we have

$$s_{(k,0)}(x_1, x_2) = x_1^k + x_1^{k-1}x_2 + \cdots + x_1 x_2^{k-1} + x_2^k \quad (1)$$

$$s_{(2,1,1)}(x_1, x_2, x_3) = x_1 x_2 x_3 (x_1 + x_2 + x_3). \quad (2)$$

Then the s_λ, with λ as above, form a basis for Λ_n. Schur showed that these polynomials describe the characters of representations of the symmetric and general linear groups. See Macdonald [16] for more details.

3 Eisenstein series on $\mathrm{SL}(2)$

Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ denote the complex upper half plane. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$ by linear fractional transformation:

$$\gamma(z) = \frac{az+b}{cz+d}, \qquad \text{where} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

It is of interest to find functions that are *automorphic*—invariant under the action of a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$. The modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is of particular importance. One may create a family of automorphic functions on Γ by averaging. To this end for each $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, define the *unnormalized Eisenstein series*

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma(z))^s, \quad \text{where} \quad \Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}.$$

Note that we must quotient out by the subgroup Γ_∞ since this is an infinite group that stabilizes the imaginary part of z. The definition makes clear that the Eisenstein series is automorphic – $E(\gamma(z), s) = E(z, s)$ for all $\gamma \in \Gamma$. Using the identity $\mathrm{Im}(\gamma z) = y/|cz + d|^2$, we may reparametrize the sum in terms of integer pairs (c, d). Indeed each pair of relatively prime integers (c, d) is the bottom row of a matrix in Γ and two matrices γ_1 and $\gamma_2 \in \Gamma$ have the same bottom row if and only if $\gamma_1 \gamma_2^{-1} \in \Gamma_\infty$. Thus the Eisenstein series may be expressed in the form

$$E(z, s) = \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{y^s}{|cz + d|^{2s}}, \quad (3)$$

from which one may deduce that the series converges absolutely for $\mathrm{Re}(s) > 1$.

The series $E(z, s)$ has many spectacular analytic properties. To describe them, define the *normalized Eisenstein series*,

$$E^*(z, s) = \tfrac{1}{2} \pi^{-s} \Gamma(s) \zeta(2s) E(z, s),$$

where $\zeta(s)$ is the Riemann zeta function and $\Gamma(s)$ is the Gamma function. One can show that $E^*(z,s)$ has analytic continuation to a meromorphic function for $s \in \mathbb{C}$ and satisfies the functional equation $E^*(z,s) = E^*(z,1-s)$. This may be proved by spectral methods, as $E(z,s)$ is an eigenfunction of the Laplace-Beltrami operator on $\mathcal{H}$.

This fact has far reaching consequences for the theory of automorphic forms. As an illustration in our present case, observe that since $E^*(z+1,s) = E^*(z,s)$, the Eisenstein series admits a Fourier series with respect to the real variable x as follows:

$$E^*(z,s) = \sum_{r=-\infty}^{\infty} a(r,y,s)e^{2\pi inx}, \quad \text{where} \quad a(r,y,s) = \int_0^1 E^*(x+iy,s)e^{-2\pi irx}\,dx.$$

In the special case $r=0$, one can show that

$$a(0,y,s) = y^s\xi(2s) + y^{1-s}\xi(2-2s),$$

where $\xi(s) = \pi^{-s}\Gamma(s)\zeta(2s)$. Because $a(0,y,s)$ inherits the analytic properties of the Fourier series, the analytic continuation and functional equation of the Riemann zeta function follow.

What about the remaining Fourier coefficients? A calculation (see for example [6], Section 1.6) shows that if $r \neq 0$ then

$$a(r,y,s) = 2|r|^{s-1/2}\sigma_{1-2s}(|r|)y^{1/2}K_{s-1/2}(2\pi|r|y)$$

where K denotes a K-Bessel function and σ is the divisor function

$$\sigma_{1-2s}(r) = \sum_{m|r} m^{1-2s}.$$

Let us shift s to $s+\frac{1}{2}$ and examine the arithmetic parts $a(r) \stackrel{\text{def}}{=} |r|^s\sigma_{-2s}(|r|)$ of the non-constant Fourier coefficients of $E^*(z,s+\frac{1}{2})$. They are *multiplicative*. That is, if $\gcd(r_1,r_2)=1$, then $a(r_1r_2) = a(r_1)a(r_2)$. Thus they are completely determined by their values at prime powers $r=p^k$. These values are easy to describe:

$$a(p^k) = p^{ks} + p^{(k-2)s} + \ldots + p^{-ks}.$$

A fundamental theme of automorphic forms identifies these coefficients with *values of characters* of a representation. Let V denote the standard representation of $\mathrm{SL}(2,\mathbb{C})$ and let $\vee^{k-1}V$ denote the $(k-1)$-st symmetric power. Thus if $A \in \mathrm{SL}(2,\mathbb{C})$ has eigenvalues α,β then $\vee^{k-1}A$ has eigenvalues $\alpha^k, \alpha^{k-1}\beta, \ldots, \alpha\beta^{k-1}, \beta^k$. The character χ_k of the representation $\vee^{k-1}V$ is given by

$$\chi_k(A) = \mathrm{tr}(\vee^{k-1}(A)) = \sum_{k_1+k_2=k} \alpha^{k_1}\beta^{k_2}.$$

Comparing with our earlier expression for the arithmetic piece $a(p^k)$, we find

$$a(p^k) = \chi_k\left(\begin{pmatrix} p^s & \\ & p^{-s} \end{pmatrix}\right). \tag{4}$$

Notice that $a(p^k)$ is thus the Schur polynomial in (1) evaluated at $(x_1,x_2) = (p^s,p^{-s})$:

$$a(p^k) = s_{(k,0)}(p^s,p^{-s}).$$

This identity has substantial generalizations. Indeed, one can define Eisenstein series analogous to $E(z,s)$ for any reductive group G. In this generality, the notion of Fourier coefficient is replaced by that of Whittaker coefficient. The Casselman-Shalika formula [7], first proved for $\mathrm{GL}(n)$ by Shintani [17], states that the values on prime powers of these coefficients may be captured by characters of a representation. For $\mathrm{GL}(n)$, these characters are expressed in terms of Schur polynomials. For more general groups, the representation is not of the complex points of G, but rather a representation of the Langlands dual group of G.[1]

These generalizations are usually stated in a different language. The coefficients $a(p^k)$ above are expressible as integrals on groups over p-adic fields known as p-adic Whittaker functions. The local version of the Eisenstein series is an induced representation, and the Whittaker function is a p-adic integral evaluated on a canonical vector in the representation space. Similarly, one may study the Whittaker functions attached to more general Eisenstein series, corresponding to more general induced representations. These may be shown to be products of Langlands L-functions, and this observation is important in the study of those L-functions.

These constructions have been known for many years. The goal of this article is to put them in a new context, by considering a group *together with its covers.* When we do this, we find that the formula (4) and its generalizations may be reinterpreted in terms of crystal graphs, which are combinatorial structures introduced by Kashiwara in the context of representations of quantum groups. We begin by illustrating this for covers of $\mathrm{SL}(2)$ before discussing higher rank.

4 Eisenstein series on covers of $\mathrm{SL}(2)$

The classical metaplectic group is a two-sheeted cover of a symplectic group over either the reals or a p-adic field. This group was introduced by Weil and explains the transformation formulas for theta functions. More generally, Kubota and Matsumoto defined a family of n-sheeted covers of $\mathrm{SL}(2)$ (or any simply connected group) for each $n \geqslant 1$. This is most conveniently described in terms of the adèles $\mathbb{A}_L$ of a number field L, defined as an appropriately restricted product over all completions of L. For L containing a full set μ_n of n-th roots of unity, the *n-fold metaplectic group* is a central extension of $\mathrm{SL}_2(\mathbb{A}_L)$ by μ_n:

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G} \longrightarrow \mathrm{SL}_2(\mathbb{A}_L) \longrightarrow 1.$$

This extension is described by means of a two-cocycle which is constructed using the arithmetic of L. (It is not the adelic points of an algebraic group). See [3] for details.

Informally, we may think of $\tilde{G}$ as follows: it is an n-sheeted cover, where the sheets are indexed by the n-th roots of unity. The group law requires moving between the sheets, and the n-th root of unity that arises in taking the product of two group elements is computed using number theory.

For these groups, one may define an Eisenstein series $E_n(z,s)$ as an average, similar to (3). The definition is modified by adding an extra factor in the average that keeps track of the sheets of the cover. The Fourier coefficients of $E_n(z,s)$ turn out to be of great interest: they are Dirichlet series made with Gauss sums.

A *Gauss sum* is a discrete analogue of the Gamma integral $\Gamma(s) = \int_0^\infty y^s e^{-y}\,\frac{dy}{y}$ – a product of multiplicative and additive characters summed over the invertible elements of a finite ring. For example if the cover degree is $n=3$, we may take $L = \mathbb{Q}(e^{2\pi i/3})$ with ring of integers $\mathfrak{o}_L = \mathbb{Z}[e^{2\pi i/3}]$.

[1] In fact, the dual group enters subtly into the computation above. The Eisenstein series $E(z,s)$ may be regarded as a function on PGL_2 and the Langlands dual of this group is $\mathrm{SL}_2(\mathbb{C})$.

Let $e(\cdot)$ be an additive character of L which is trivial on $\mathfrak{o}_L$ but no larger fractional ideal. Then for integers $m, c \in \mathfrak{o}_L$ with $c \neq 0$, let

$$g_3(m,c) = \sum_{\substack{a (\mathrm{mod}\ c) \\ \gcd(a,c)=1}} \left(\frac{a}{c}\right)_3 e(ma/c), \tag{5}$$

where the sum is over $a \in \mathfrak{o}_L$ that are invertible mod c and $(-)_3$ is the cubic residue symbol. For general n and L, we may define a Gauss sum $g_n(m,c)$ made with n-th power residue symbols. To do so, we must pass from the ring of integers $\mathfrak{o}_L$ to a localization $\mathfrak{o}_{L,S}$ where denominators are allowed at a finite set of places S, and some additional technicalities result.

Kubota computed the Fourier expansion of $E_n(z,s)$, whose m-th coefficient is a K-Bessel function times an arithmetic part $a(m)$. In the special case $n = 3$,

$$a(m) = \sum_{\substack{c \in \mathfrak{o}_L \\ c \equiv 1 \ (\mathrm{mod}\ 3)}} \frac{g_3(m,c)}{\mathrm{N}c^{2s}}, \tag{6}$$

where $\mathrm{N}c$ denotes the absolute norm of c. The form for general n is much the same with an arithmetic part involving $g_n(m,c)$ in place of g_3. The series is easily seen to converge absolute for $\Re(s) > 3/4$, and since $E(z,s)$ has analytic continuation and functional equation, $a(m)$ inherits these properties as well. This series (and its generalizations) are basic objects of interest.

Let us recall two properties of Gauss sums valid for any $n \geq 1$. By the Chinese Remainder Theorem, if $\gcd(c_1, c_2) = 1$, then

$$g_n(m, c_1c_2) = \left(\frac{c_1}{c_2}\right)_n \left(\frac{c_2}{c_1}\right)_n g_n(m,c_1) g_n(m,c_2) \tag{7}$$

and if $\gcd(m_1, c) = 1$ then an easy change of variables shows that for any integer m_2

$$g_n(m_1m_2, c) = \left(\frac{m_1}{c}\right)_n^{-1} g_n(m_2, c).$$

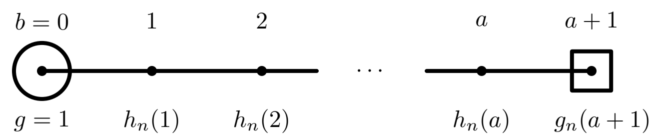
In particular, (7) shows that the Dirichlet series in (6) is not expressible as an Euler product – a product over primes – when $n > 2$. This is quite different from the situation for $n = 1, 2$ and, more generally, for Langlands L-functions. Instead, we see that to combine contributions from relatively prime c_1 and c_2, we must introduce n-th roots of unity depending on arithmetic. For these reasons, we call series with such a property *twisted Euler products*. See [11] for more information and further examples.

Though not strictly multiplicative, these two properties allow one to reconstruct $g_n(m,c)$ from its values at prime powers $g_n(p^a, p^b)$ for non-negative integers a, b. Thus we may restrict to these simpler cases in describing the Fourier coefficients.

Let us consider the coefficients $g_n(p^a, p^b)$ at a given prime p. Here a is fixed (it is the order of m at p) and b is varying. These coefficients come in three flavors. First, there is the case $b = 0$, where the coefficient is simply $1 = p^b$. Second, there are the coefficients for $1 \leq b \leq a$. The inequality $b \leq a$ makes the additive character in (5) trivial, and so this coefficient is the function

$$h_n(b) = \begin{cases} \phi(p^b) & \text{if } n | b, \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi(p^b) = |(\mathfrak{o}_{L,S}/p^b\mathfrak{o}_{L,S})^\times|$ is the Euler phi function for $\mathfrak{o}_{L,S}$. Finally, there is the case $b = a+1$. In this case, the Gauss sum is always nonzero and it is not possible to evaluate it in closed form (unless $b \equiv 0 \bmod n$ or $n = 2$). We write this sum simply $g_n(a+1)$ for short. For $b \geq a+2$, the sum can be shown to be zero (one is summing a nontrivial character over a group). Hence the entire contribution can be summarized in the following diagram.



We have circled the location $b = 0$ and boxed the location $b = a+1$ since the contributions are special at these locations, while at b such that $1 \leq b \leq a$, the contribution is $h_n(b)$. This is the most common situation. Notice that the diagram is the same for any n, it is only the functions g_n and h_n that depend on n.

For the non-metaplectic Eisenstein series (the special case $n = 1$), the coefficients at p could be described as values of a character. In fact, the picture above also is related to representation theory. This is the picture of a crystal graph associated to a representation of the quantum group $U_q(\mathfrak{sl}_2)$; each vertex represents a canonical basis element, and the lines between successive vertices are given by Kashiwara operators. The vertices at $b = 0$ and $b = a+1$ have special properties. And it is this description that applies to SL_{r+1} for any r and any cover degree n!

The attentive reader may be concerned: for $n > 1$ we have been working with the unnormalized coefficients, while in the classical case we used normalized coefficients. We will account for this disparity below. Even when $n = 1$ the formula above is not the classical formula in terms of Schur polynomials, but rather obtainable from it as a deformation, using a theorem of Tokuyama.

5 Eisenstein series on covers of SL_{r+1} and crystal graphs

One can define the n-fold cover of $\mathrm{SL}_{r+1}(\mathbb{A}_L)$ for any r, and a corresponding Eisenstein series E_n for this group. It is an average of a suitable function, this time a function of r complex variables $s_1, \dots, s_r$, over a discrete subgroup.[2]

Fourier coefficients generalize to Whittaker coefficients. These are defined by integrating E_n against a character of U, the subgroup of upper triangular unipotent matrices of $\mathrm{SL}_{r+1}(\mathbb{A}_L)$. The characters of U are indexed by r-tuples $\mathbf{m} = (m_1, \dots, m_r)$ of elements of $\mathfrak{o}_L$. Indeed, a character of U depends only on the r locations just above the main diagonal since everything else is in $[U, U]$. Then the Whittaker coefficients are defined by integration against this character.

The main theorem of [3] expresses the arithmetic part $a(\mathbf{m})$ of these Whittaker coefficients as multiple Dirichlet series

$$a(\mathbf{m}) = \sum \frac{H_n(\mathbf{m}; c_1, \dots, c_r)}{Nc_1^{2s_1} \dots Nc_r^{2s_r}}. \tag{8}$$

This is a generalization of (6). The coefficients H_n are once again twisted multiplicative, and this allows one to reduce to their study to that of the coefficients $H_n(p^{\ell_1}, \dots, p^{\ell_r}; p^{k_1}, \dots, p^{k_r})$ attached to a given prime p of $\mathfrak{o}_L$. Here the ℓ_i and k_i are non-negative integers. The coefficients H turn out

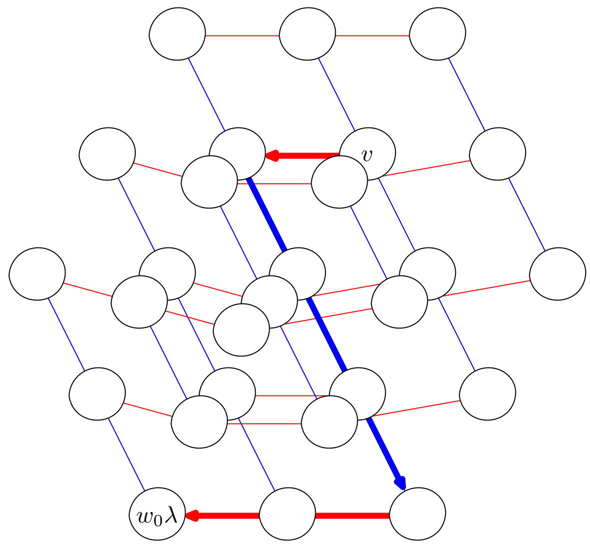
[2]There are more general Eisenstein series built from automorphic forms on lower rank groups, but we do not consider them here.

to be built out of the functions g_n, h_n, and powers of Np, that already appeared in the previous section for the n-fold cover of $\mathrm{SL}(2)$. However, the exact description is considerably more subtle. It involves the theory of crystal graphs.

To explain the description, we first give a brief account of some properties of crystal graphs. A *weight* of GL_{r+1} is a rational character of the diagonal torus T of GL_{r+1}. We may identify the weights with the lattice $\Lambda = \mathbb{Z}^{r+1}$: if $\mu = (\mu_1, \cdots, \mu_{r+1}) \in \Lambda$ then $\boldsymbol{t}^{\mu} := \prod t_i^{\mu_i}$ is such a character. The weight is called *dominant* if $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{r+1}$, and strictly dominant if the above inequalities are all strict. Weights are ordered: $\lambda > \mu$ if $\lambda_i = \mu_i + h_i - h_{i+1}$ for each i, where the h_i are non-negative (and $h_0 = h_{r+2} = 0$ by definition). A weight for a representation V of the associated Lie algebra $\mathfrak{gl}_{r+1}(\mathbb{C})$ is a weight μ such that there exists a non-zero vector in V that transforms under the torus by μ; it is highest if no larger weight satisfies this property. Cartan's Theorem of the Highest Weight states that every finite-dimensional irreducible complex representation of $\mathfrak{gl}_{r+1}(\mathbb{C})$ (or more generally, of any complex semisimple finite-dimensional Lie group) has a unique highest weight vector (up to scalars) and that the highest weight classifies the representation. There is similarly a theorem of the lowest weight.

The quantum group $U_q(\mathfrak{gl}_{r+1}(\mathbb{C}))$ is a deformation of the universal enveloping algebra of $\mathfrak{g}l_{r+1}(\mathbb{C})$ that is obtained when a parameter q is introduced into the relations that describe the universal enveloping algebra. See for example Hong and Kang [13]. Such representations are once again classified by highest weight. Let λ be a dominant weight. Then Kashiwara [14] associated with λ a *crystal graph* $\mathcal{B}_\lambda$ which is a directed graph whose vertices correspond to certain basis vectors for the representation of $U_q(\mathfrak{gl}_{r+1}(\mathbb{C}))$ with highest weight λ. The edges of this graph are colored with one color for each simple root, and describe the action of the unipotents in the Lie algebra on this basis as $q \to 0$. The crystal graph $\mathcal{B}_\lambda$ comes endowed with a weight map wt to the weight lattice Λ such that $\sum_{v \in \mathcal{B}_\lambda} \boldsymbol{t}^{\mathrm{wt}(v)}$ is the character of the irreducible representation of $\mathrm{GL}_{r+1}(\mathbb{C})$ with highest weight λ. The map wt is compatible with the graph structure: walking one step along an edge of $\mathcal{B}_\lambda$ in the direction of the highest weight vector (resp. lowest weight vector) corresponds to increasing (resp. decreasing) the weight of the vertex by the simple root with which it is labeled.

Here is an example, the $\mathfrak{gl}_3$ crystal with highest weight $\lambda = (4, 2, 0)$ and lowest weight $w_0\lambda = (0, 2, 4)$. It is drawn so that elements of the same weight are clustered together in overlapping vertices.



Berenstein and Zelevinsky and Littelmann associate a path to each vertex in $\mathcal{B}_\lambda$. To associate a path to a vertex, one must choose an order to walk the edges. To do this, choose a reduced factorization of the long element w_0 of the Weyl group into simple reflections (i.e., one of minimal length). Walk the graph in the order that the simple reflections appear in the factorization, going as far in a given direction as the graph will allow before using the next operator. It turns out that such a factorization always leads to a path to the lowest weight vector if one moves in the direction of the negative roots. The sequence $\mathrm{BZL}(v)$ of path lengths so obtained parametrizes the vertex v of $\mathcal{B}_\lambda$.

For example, in the figure above we have indicated a walk from a vertex v to the lowest weight vector $w_0\lambda$ corresponding to the factorization of the long element $w_0 = s_1s_2s_1$ of the symmetric group S_3, the Weyl group of GL_3. Thus we walk along the graph in order s_1, s_2, s_1 (=red,blue,red). The lengths of the corresponding paths are $1, 3, 2$, so $\mathrm{BZL}(v) = (1, 3, 2)$.

The main theorem of [3] computes the coefficients $H_n(p^{\ell_1}, \cdots, p^{\ell_r}; p^{k_1}, \cdots, p^{k_r})$ by attaching products of Gauss sums to BZL sequences. Let $\lambda_{r+1} = 0$, $\lambda_r = \ell_r$, and $\lambda_i = \ell_i + \lambda_{i+1}$ when $i < r$, and let λ be the dominant weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{r+1})$.[3] Let ρ denote the Weyl vector, that is, half the sum of the positive roots, or in coordinates $(r, r-1, \ldots, 1, 0)$. Since our conditions guarantee λ is dominant, $\lambda + \rho$ is strictly dominant.

Theorem 1 *The coefficient H_n is given by*

$$H_n(p^{\ell_1}, \cdots, p^{\ell_r}; p^{k_1}, \cdots, p^{k_r}) = \sum_{\substack{v \in \mathcal{B}_{\lambda+\rho} \\ \mathrm{wt}(v) = \mu}} G_n(v), \tag{9}$$

where μ is the weight related to $(k_1, \cdots, k_r)$ by the condition that $\sum_{i=1}^{r} k_i\alpha_i = \mu - w_0(\lambda + \rho)$ where α_i are the simple roots and the function $G_n(v)$ is described below.

[3] By fixing $\lambda_{r+1} = 0$, we parametrize representations of $\mathrm{SL}_{r+1}(\mathbb{C})$, the Langlands dual group of PGL_{r+1}.

The definition of $G_n(v)$ depends on a recipe for walking the graph, so it depends on the choice of a reduced expression for w_0 in the symmetric group S_{r+1}. In terms of the standard reflections s_i (recorded simply by their index i) corresponding to simple roots, let us choose

$$\Sigma = \Sigma_1 := (r, r-1, r, r-2, r-1, r, \cdots, 1, 2, 3, \cdots, r) \tag{10}$$

or

$$\Sigma = \Sigma_2 := (1, 2, 1, 3, 2, 1, \cdots, r, r-1, \cdots, 3, 2, 1) \tag{11}$$

and take the associated path lengths $\mathrm{BZL}(v) = (b_1, \ldots, b_N)$ to the lowest weight vector. We decorate the entries b_i, boxing (resp. circling) b_i if the i-th leg of the path to the lowest weight vector (resp. to the highest weight vector) is maximal for the associated Kashiwara operator. Then we prove that

$$G_n(v) = G_{n,\Sigma}(v) = \prod_{b_i \in \mathrm{BZL}(v)} \begin{cases} Np^{b_i} & \text{if } b_i \text{ is circled (but not boxed),} \\ g_n(b_i) & \text{if } b_i \text{ is boxed (but not circled),} \\ h_n(b_i) & \text{if neither,} \\ 0 & \text{if both.} \end{cases} \tag{12}$$

The equality of the expression in (9) for Σ_1 and Σ_2 is not formal, and is established directly in [2] by an elaborate blend of number-theoretic and combinatorial arguments.

It is noteworthy that the dependence on the degree of the cover n is reflected in the functions g_n and h_n, but that the description in terms of crystal graphs is otherwise independent of n.

In closing this section, we mention that there are not one but two distinct generalizations of the Casselman-Shalka formula to the metaplectic case. Chinta and Gunnells [8] and Chinta and Offen [9] show that the p-parts of the Whittaker coefficients of metaplectic Eisenstein series on covers of SL_{r+1} can also be expressed as quotients of sums over the Weyl group, directly analogous to the Weyl character formula.

6 The case $n = 1$: Tokuyama's deformation formula

When $n = 1$, we are concerned with the points of an algebraic group. In that case, the Whittaker coefficients of Eisenstein series may be computed in two different ways. First, Theorem 1 provides an answer in terms of crystal graphs. This result holds for any $n \geq 1$. Second, the formula of Shintani [17] and Casselman and Shalika [7] (which holds only for $n = 1$) expresses the Whittaker coefficients of normalized Eisenstein series as the values of the characters of irreducible representations of the L-group $\mathrm{SL}_{r+1}(\mathbb{C})$. These characters are given by Schur polynomials, as described above.

These two expressions for the Whittaker coefficients are related by the following result (cf. [2]).

Theorem 2 *Let Φ^+ denote the positive roots of SL_{r+1}. For any dominant weight λ,*

$$\left[\prod_{\alpha \in \Phi^+} (1 - q^{-1}\boldsymbol{z}^\alpha)\right] \chi_\lambda(g) = \sum_{v \in \mathcal{B}_{\rho+\lambda}} G_1(v) q^{-\langle \mathrm{wt}(v) - w_0(\lambda+\rho), \rho\rangle} \boldsymbol{z}^{\mathrm{wt}(v) - w_0\rho},$$

where the $G_1(v)$ are computed as in (12) using the reduced word Σ_1.

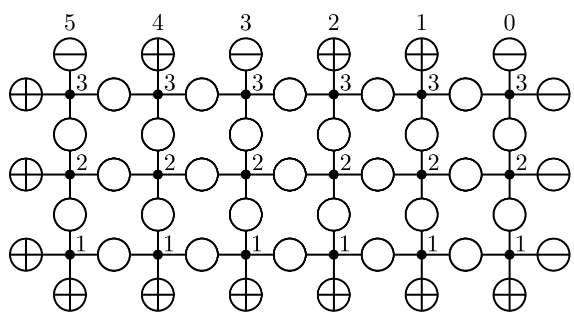
After taking into account the normalizing factors that appear in the Casselman-Shalika formula, Theorem 2 shows that the Casselman-Shalika formula and Theorem 1 in the case $n = 1$ are equivalent. In the application to the Casselman-Shalika formula, $q = Np$.

Theorem 2 is equivalent to an earlier result of Tokuyama [18], and may be viewed as a deformation of the Weyl character formula. Tokuyama's formulation is expressed in terms of combinatorial arrays called Gelfand-Tsetlin patterns. We highlight that the character with highest weight λ is expressed as a combinatorial sum with respect to highest weight $\lambda + \rho$.

7 Ice models for Whittaker coefficients

In this final section, we describe another combinatorial representation of the p-parts of Whittaker coefficients. These can be described using *square ice*, a particular example of a two-dimensional *lattice model*. We describe these in detail when $n = 1$; that is, when the Whittaker coefficients at the prime p are given by the values of a Schur polynomial. An ice model description for arbitrary covers is presented in [5].

Lattice models arise in statistical mechanics, where they can be used to represent thin sheets of matter such as ice. Consider a rectangular array of lattice points in the plane. Add vertical and horizontal edges from each lattice point, so the points are embedded in a rectangular array of line segments. Label the boundary edges of this configuration by a fixed set of signs $\pm$.

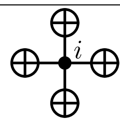
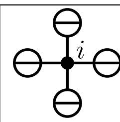
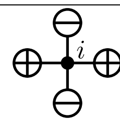
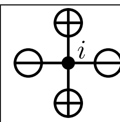
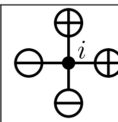
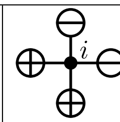


(13)

Signs on interior segments are assigned in all possible ways ('states') such that the number of plusses at each vertex is even. Each possible configuration of signs at a given vertex is assigned a weight, called a *Boltzmann weight*, and the weight attached to a given state is the product of the Boltzmann weights at all vertices. The *partition function* of the configuration is the sum of the weights over all possible states. Lattice models for which the partition function may be explicitly evaluated are called *exactly solved* and are of particular interest. See Baxter [1].

Hamel and King [12] found ice models whose partition functions are Schur polynomials. In [4], the authors gave two such choices, including the one of Hamel and King, and gave a new approach to these results based on the Yang-Baxter equation. The fact that there are two different choices is related to the two factorizations (10) and (11) of the long Weyl group element.

We describe one of these, called *Gamma Ice*. It is a six-vertex model: only six configurations have nonzero Boltzmann weights. In the i-th row of the lattice, the Boltzmann weights are:

Gamma Ice						
Boltzmann weights	1	z_i	t_i	z_i	$z_i(t_i+1)$	1

The variables t_i, z_i are complex parameters. To use these Boltzmann weights to compute a Schur polynomial, we specify boundary conditions on a finite lattice depending on the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$. Consider a rectangular lattice having at least $\lambda_1 + r + 1$ columns and $r+1$ rows. Label the columns by non-negative integers in ascending order from right to left, beginning with zero. To every edge of the lattice, we assign a sign $\pm$ as follows. On the left and bottom edges, we put $+$; on the right edge we put $-$. On the top, we put $-$ at every column labeled $\lambda_i + r - i + 1$, $1 \leq i \leq r+1$, that is, for the columns labeled by the components of $\lambda + \rho$, and $+$ at the remaining columns. For example, suppose that $r = 2$ and $\lambda = (3, 2, 0)$, so that $\lambda + \rho = (5, 3, 0)$. Then we have the configuration (13) above.

Denote the resulting partition functions Z_λ^Γ and Z_λ^Δ for the two types of ice, where the superscript corresponds to the choice of Boltzmann weights. Then, for λ a partition with at most n non-zero parts, we prove in [4] that the partition functions are

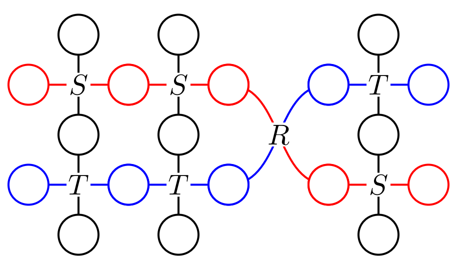
$$Z(\mathfrak{S}_\lambda^\Gamma) = \prod_{i<j}(t_i z_j + z_i) s_\lambda(z_1, \cdots, z_n), \qquad Z(\mathfrak{S}_\lambda^\Delta) = \prod_{i<j}(t_j z_j + z_i) s_\lambda(z_1, \cdots, z_n).$$

To establish these evaluations of the partition function define

$$s_\lambda^\Gamma(z_1, \cdots, z_n; t_1, \cdots, t_n) = \frac{Z(\mathfrak{S}_\lambda^\Gamma)}{\prod_{i<j}(t_i z_j + z_i)}.$$

Then one seeks to show that s_λ^Γ is symmetric in the sense that it is unchanged if the same permutation is applied to both z_i and t_i. Once this is known, it is possible to show that it is a polynomial in the z_i and t_i, then that it is independent of the t_i; finally, taking $t_i = -1$ one may invoke the Weyl character formula and conclude that it is equal to the Schur polynomial.

In order to prove the symmetry property of s_λ^Γ we use an instance of the Yang-Baxter equation. This relation allows one to put a twist between two rows of ice and use it to interchange them. Schematically, the process looks like this:



Here R is an additional type of vertex with its own carefully chosen Boltzmann weights. Algebraically, the Yang-Baxter equation is the identity

$$R_{12} S_{13} T_{23} = T_{23} S_{13} R_{12}, \tag{14}$$

where R, S, T are endomorphisms of $V \otimes V$ for an abstract two-dimensional vector space V. The nonzero coefficients in the matrices of S, T and R are the Boltzmann weights corresponding to the two rows (red and blue) to be interchanged, and to the the new type of Boltzmann weight, respectively. In (14), R_{ij} is the endomorphism of $V \otimes V \otimes V$ in which R is applied to the i-th and j-th copies of V and the identity map to the k-th component, where i, j, k are $1, 2, 3$ in some order. Concretely, it means that the special vertex labeled R may be slid left or right in the lattice without changing the partition function. Thus moving it across the entire lattice from left to right interchanges the red and blue rows. This gives a new proof of Tokuyama's result.

A second instance of the Yang-Baxter equation solves the same problem for the analogously defined s_λ^Δ, and a third instance shows directly, without using the above evaluations, that $s_\lambda^\Gamma = s_\lambda^\Delta$. See [4] for details. Two-dimensional lattice models may also be used to give a description of the general metaplectic Whittaker coefficient at a prime p, though its description is more complicated (see [5]).

The study of ice models and the Yang-Baxter equation was advanced by ideas of representation theory and ultimately led to the discovery of quantum groups. See Faddeev [10] for a history. It is intriguing that our earlier description in terms of crystal graphs involved quantum groups as well. However, the two combinatorial models we have presented here lead to rather different phenomena. In particular, their defining data are not in bijection. Exploration of the surprising relationship between automorphic forms and quantum groups is ongoing.

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