Math 554 — Autumn 2005

Homework Set 8 — Due Friday December 2

Reading Horn & Johnson, Appendix D

(1) In this problem, let $\langle A, B\rangle_F = \operatorname{tr}(B^H A)$ denote the Frobenius (or Hilbert-Schmidt) inner product on $\mathbb{C}^{n\times n}$, and $\|A\|_F = \left(\sum_{i,j} |a_{ij}|^2\right)^{\frac{1}{2}}$ the associate norm. Consider the problem of minimizing $\|A-B\|_F$ for a given $A \in \mathbb{C}^{n\times n}$, where B ranges over all scalar multiples α of unitary matrices $W \in \mathbb{C}^{n\times n}$.

   (a) Show that if $\alpha \in \mathbb{C}$ and $W \in \mathbb{C}^{n\times n}$ is unitary, then $\|A - \alpha W\|_F^2 \geq \|A\|_F^2 - \frac{1}{n}|\langle A, W\rangle_F|^2$.

   (b) Show that $\sup_{W \text{ unitary}} |\langle A, W\rangle| = \sum_{i=1}^n \sigma_i$, where the sup is taken over all unitary matrices $W \in \mathbb{C}^{n\times n}$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are the singular values of A.

   (c) Identify α, W so that $B = \alpha W$ solves the minimization problem, and identify the minimum value of $\|A - B\|_F$.

(2) Let $L = \mathcal{L}(V)$ where $\dim V < \infty$, and let λ be an eigenvalue of L. Show that the geometric multiplicity of λ equals its algebraic multiplicity if and only if λ is a simple pole of the resolvent of L.

(3) Let V be a finite-dimensional inner product space, and let $L \in \mathcal{L}(V)$ be normal. Show that $\|R(\zeta)\| = (d(\zeta, \sigma(L)))^{-1}$, where $\|\cdot\|$ is the operator norm on $\mathcal{L}(V)$ induced by the inner product on V, and $d(\zeta, \sigma(L)) = \min_{\lambda\in\sigma(L)} |\zeta - \lambda|$.

(4) Let $\dim V < \infty$ and let $L \in \mathcal{L}(V)$ have spectral decomposition $L = \sum_{i=1}^k (\lambda_i P_i + N_i)$.

   (a) Show that e^{tL} has spectral decomposition $e^{tL} = \sum_{i=1}^k e^{t\lambda_i}[P_i + (e^{tN_i} - I)]$.

   (b) Let $\alpha = \max_{1\leq i\leq k} \mathcal{R}e\lambda_i$, and let m be the largest algebraic multiplicity of those λ_i's for which $\mathcal{R}e\lambda_i = \alpha$. Given a norm $\|\cdot\|$ on $\mathcal{L}(V)$, show $\exists$ a constant M for which $\|e^{tL}\| \leq M(1 + |t|)^{m-1}e^{\alpha t}$ for $t \geq 0$. α is called the *spectral abscissa of* L.

   (c) Show by example that it is not necessarily true that $\|e^{tL}\| \leq Me^{\alpha t}$ for some constant M.

   (d) Compute e^{tJ} explicitly, where

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ \ddots & \ddots & 1 \\ 0 & \dots & \lambda \end{bmatrix} \in \mathbb{C}^{\ell\times\ell}$$

   is an $\ell \times \ell$ Jordan block.

(5) Let $\dim V < \infty$, let $L \in \mathcal{L}(V)$, and suppose all eigenvalues of L satisfy $\mathcal{R}e\lambda < 0$.

   (a) Use the estimate from 4(b) to show that $\lim_{N\to\infty} \int_0^N e^{tL}dt$ exists (where the limit is in some norm $\|\cdot\|$ on $\mathcal{L}(V)$). Define the limit to be $\int_0^\infty e^{tL}dt$.

   (b) Show that $\int_0^\infty e^{tL}dt = -L^{-1}$. Be sure to justify all steps.

(6) Using deflation techniques and Householder reflections, show, by construction, that every element of $\mathbb{C}^{n\times n}$ is unitarily equivalent to a matrix in upper Hessenberg form.

(7) Let $A \in \mathbb{C}^{n\times n}$.

   (a) Show that A is nonderogatory if and only if the characteristic and minimal polynomials of A coincide.

   (b) Show that the nonderogatory matrices in $\mathbb{C}^{n\times n}$ form an open dense subset of $\mathbb{C}^{n\times n}$.

Hint: You may use the continuity of the spectrum to prove this fact. One way to interpret the phrase *continuity of the spectrum* is as follows. Consider the mapping $\lambda : \mathbb{C}^{n\times n} \mapsto \mathbb{C}^n$ where components of $\lambda(A)$ are the eigenvalues of A (including multiplicities) ordered lexicographically from largest to smallest. Let Π_n be the set of $n \times n$ permutation matrices. Then, by the continuity of the spectrum, we mean that

$$\lim_{A\to A_0} \min_{P\in\Pi_n} \|\lambda(A_0) - P\lambda(A)\| \;=\; 0 \;.$$

(Why do we need to introduce the permutaion matrices here?)

(8) Prove the following formula from the Spectral Representation Theorem using the resolvent theory:

$$L = S + N$$

where

$$S = \sum_{j=1}^{s} \lambda_j P_j \quad \text{and} \quad N = \sum_{j=1}^{s} N_j.$$

Hint: Consider the integral

$$-\frac{1}{2\pi i}\oint_\Gamma \zeta R(\zeta) d\zeta$$

and make use of the expansion of $R(\zeta)$ at infinity,

$$R(\zeta) = -\sum_{n=0}^{\infty} \zeta^{-(n+1)} L^n$$

and its partial fractions decomposition,

$$R(\zeta) = -\sum_{j=1}^{s}\left[(\zeta-\lambda_j)^{-1}P_j + \sum_{n=1}^{m_j-1}(\zeta-\lambda_j)^{-(n+1)}N_j^n\right] ,$$

where $\sigma(L) = \{\lambda_1, \ldots, \lambda_s\}$ is the spectrum of L and Γ be a simple closed curve about the origin containing $\sigma(L)$ in its interior.

(9) Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \sigma(A)$ with associated eigenvector v. Let $b \in \mathbb{C}^n$ be any vector for which $\langle v, b\rangle \neq 0$. Consider solving the equation $(A - \lambda I)x = b$ numerically (i.e, subject to numerical error), and let $\bar{x}$ be the numerically computed solution. Give an arguement that explains why the vector $v = \bar{x}/\|\bar{x}\|$ is an excellent approximation to an eigenvector associated with the eigenvalue λ.

To illustrate this problem try the following Matlab commands:

$$\begin{aligned}
A &= 10 * (0.5 * ones(50,50) - rand(50,50));\\
ev &= eig(A);\\
b &= 10 * (0.5 * ones(50,1) - rand(50,1));\\
x &= (A - ev(1) * eye(50,50))\backslash b;\\
xn &= x/norm(x,2);\\
r &= (A - ev(1) * eye(50,50)) * xv;\\
& \ norm(r)
\end{aligned}$$

The first command generates a random 50×50 matrix with entries between -10 and 10. The semicolon at the end of each line prevents Matlab from printing the results of the command line. The final line will print the 2-norm of the residual vector r. The norm of r should be on the order of 10^{-13} indicating that v is an eigenvector for the eigenvalue $ev(1)$ within the numerical precision of the floating point computations used in Matlab (14 decimal places). When you execute the command

$$x = (A - ev(1) * eye(50, 50)) \backslash b;$$

Matlab will send a warning message telling you about the near singularity of the matrix $(A - ev(1) * eye(50, 50))$. The warning is to be expected in this case (why?) so just ignore it. What happens when you repeat the last four of the Matlab commands given above but with the vector b replaced by the eigenvector approximation v? In particular, what should happen to the new residual norm $\|r\|$?