SECANT DIMENSIONS OF MINIMAL ORBITS: COMPUTATIONS AND CONJECTURES

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ABSTRACT

We present an algorithm for computing the dimensions of higher secant varieties of minimal orbits. Experiments with this algorithm lead to many conjectures on secant dimensions, especially of Grassmannians and Segre products. For these two classes of minimal orbits we give a short proof of the relation, known from the work of Ehrenborg, Catalisano-Geramita-Gimigliano, and Sturmfels-Sullivant, between the existence of certain codes and non-defectiveness of certain higher secant varieties.

1. INTRODUCTION

A generic polynomial of degree d in $\mathbb{C}[x]$ can be written as a sum of $\lfloor (d+1)/2 \rfloor$ powers $(ax+b)^d$, $a,b \in \mathbb{C}$. A generic $n \times n$-matrix of rank k and trace 0 is the sum of k matrices of rank 1 and trace 0; in fact, this is true for *any* trace zero matrix of rank k, though that doesn't matter here. But what is the generic rank of a tensor in $(\mathbb{C}^2)^{\otimes 10}$, i.e., if we want to write a generic element of this tensor power as a sum of decomposable tensors, then how many do we need?

These are instances of a general type of problem, which has been solved only in very few cases. In this paper we do not solve many instances, either, but we do present a program for investigating small concrete instances. Also, we will boldly state some conjectures that our experiments with this program suggest. We hope that this paper will be an incentive for people working in this field, either to prove or disprove our conjectures, or to use our program and experiment for themselves.

To be more concrete on the type of problem that our program can handle, let G be a connected reductive complex algebraic group and let V be a non-trivial irreducible module for G; for the theory of algebraic groups we refer to [5]. The projective space $\mathbb{P}V$ contains a unique (Zariski-)closed orbit X, consisting of the highest weight lines and called the *minimal orbit*; see Section 2 for the short argument. Denote by $C \subseteq V$ the affine cone over X. For any natural number k, we write kX for the Zariski closure of the union of all projective $(k-1)$-spaces spanned by k points on X; kX is called the kth *secant variety* of X (by some other authors the $(k-1)$st secant variety). Often the term *secant variety* itself is used for the second secant variety, while those for $k > 2$ are referred to as *higher secant varieties*. More concretely, the affine cone over kX is the Zariski closure of

$$kC := \{v_1 + \ldots + v_k \mid v_i \in C\}.$$

In the examples above G is equal to SL_2 acting on the space V of binary forms of degree d, or to SL_n acting on its Lie algebra, or to SL_2^{10} acting on the space $(\mathbb{C}^2)^{\otimes 10}$ of 10-tensors, respectively. Accordingly, C is the set of pure d-th powers of linear forms, or the set of trace zero matrices of rank at most 1, or the set of all pure 10-tensors.

One can ask many questions about the sets kX or kC. For instance: how do we find polynomial equations defining $\overline{kC}$? For the matrix example we know them: $kC = \overline{kC}$ is the set of trace zero matrices of rank at most k, and these are characterised (even scheme-theoretically, see [7]) by the vanishing of all $(k+1)$-minors. For the example of binary forms, too, equations defining $\overline{kC}$ set-theoretically are known: certain minors of Hankel (or catalecticant) matrices [15, Proposition 9.7] (indeed, these equations even define $\overline{kC}$ scheme-theoretically; see, e.g., [19]). But for the example of 10-tensors we do not know equations. Of course, equations can in theory be found by a Gröbner basis computation: one computes the closure of the image of the addition morphism $C^k \to kC$. However, these computations quickly become totally infeasible, so sophisticated algebraic geometry is already needed to find equations for small secant varieties. There is recent progress in this direction: in [24] scheme-theoretic equations for some secant varieties of 3-fold and 4-fold Segre powers are given, and [27] shows how combinatorial methods help in finding such equations.

Another question: are the sets kC closed? In the matrix case they are, for the 10-tensors we do not know, and for the binary forms they are not. One can show, in fact, that a polynomial with a zero of multiplicity m, $0 < m < d$, cannot be written as a sum of less than $m + 1$ pure d-th powers, so that the sets kC with $\lfloor (d+1)/2 \rfloor \le k < d$ cannot possibly be closed.

But by far the most modest property of kC to want to determine is its dimension $\dim kC := \dim \overline{kC} = \dim kX + 1$, and this is precisely what our algorithm does. That such an algorithm is useful is clear from the vast literature dealing exactly with these secant dimensions; see, e.g., [1, 2, 3, 8, 9, 10, 11, 16, 27]. As the addition map $C^k \to \overline{kC}$ is dominant, $\dim kC$ is at most $k \dim C$; we call the minimum of the latter number and $\dim V$ the *expected dimension* of kC. If kC has the expected dimension, then kC (and $\overline{kC}$ and kX) are called *non-defective*. Otherwise, kC, $\overline{kC}$, and kX are called *defective*. The difference $\min\{\dim V, k \dim C\} - \dim kC$ is called the k-*defect* (note that these defects are the *sums* of the defects as defined in [29]). If kC is not defective for any $k \ge 1$, then we call C and X themselves non-defective; otherwise, we call them defective.

As we will see below, calculating $\dim kC$ in concrete cases boils down to straightforward linear algebra computations—at least if one allows for a small error probability—and only in rare concrete cases does kC *not* have the expected dimension. And yet it is very difficult to prove anything substantial in this direction.

First, however, we list some important things that *are* known about these higher secant varieties. The standard reference for secant varieties, containing a wealth of classification results on varieties with constraints on their secant dimensions, is [29].

(1) Take $G = \mathrm{SL}_n$ and let V be the space of homogeneous polynomials in $x_1, \ldots, x_n$ of degree d. Then C is the set of d-th powers of linear forms. A simple duality shows that $\dim kC$ is the *codimension* of the space of homogeneous polynomials in $x_1, \ldots, x_n$ of degree d that vanish together

    with all their first partial derivatives on k fixed, generic points. This relates higher secant varieties to the problem of multivariate interpolation, which was solved in the series of papers [16, 1, 2, 3].

(2) For G a simple algebraic group acting on its Lie algebra $\mathfrak{g}$, the set C consists of all "extremal elements", that is: elements $X \in \mathfrak{g}$ for which $\mathrm{ad}(X)^2 \mathfrak{g} \subseteq \mathbb{C}X$. The first secant variety is known in this case ([21, 22]), and for classical G the higher secant varieties were completely determined in [4].

(3) For $G = \mathrm{SL}_n$ and V the d-fold exterior power of $\mathbb{C}^n$ the set C is the affine cone over the Grassmannian, in its Plücker embedding, of d-dimensional vector spaces in $\mathbb{C}^n$. The paper [9] lists some defective Grassmannians, and proves that for $d > 2$ and $kd \leq n$ the variety kC is not defective. In Subsection 4.1 we generalise this latter result, and conjecture that the list of defective Grassmannians in [9] is complete.

(4) For $G = \mathrm{SL}_{n_1} \times \ldots \times \mathrm{SL}_{n_d}$ and $V = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$ the cone C consists of the decomposable powers, and is the cone over the Segre product of the $\mathbb{C}^{n_i}$. For small d and some concrete values of the n_i secant dimensions of C, or bounds on these, are known; see [8] and the references there. The case where all n_i are equal is treated in more detail in Section 4.

From here we proceed as follows: in Section 2 we present our algorithm for computing $\dim kC$, Section 3 deals with some implementation issues, and Section 4 lists our conjectures based on experiments with that implementation. These conjectures concern Grassmannians and Segre products, as well as a general finiteness statement. Our algorithm, called `sedimo`, can be downloaded from the following address: `http://www.win.tue.nl/~jdraisma/`

2. The algorithm

We retain the notation G, V, X, C from the Introduction, i.e., G is a connected reductive algebraic group, V is a non-trivial irreducible G-module, X is the closed orbit in $\mathbb{P}V$—whose uniqueness will be dealt with soon—and C is the cone over X in V.

2.1. **Required representation theory.** For those whose background is not in representation theory we briefly sketch how one can put the objects under investigation into a computer algebra system to compute secant dimensions: Let B be a Borel subgroup of G, i.e., maximal under the closed, connected subgroups of G, and let T be a maximal torus in B. Then B stabilises a unique 1-dimensional subspace in V, spanned by a *highest weight vector* v_0, and the character of T on this line determines V up to isomorphism. This character is the *highest weight* and can be given by a tuple of natural numbers relative to a certain basis of the (free abelian) character group of T. Conversely, algorithms are already available in [12] to construct, given a highest weight as a tuple of natural numbers, the corresponding module V—strictly speaking not as a G-module, but as a module over the Lie-algebra $\mathfrak{g}$ of G. The tangent space to the orbit Gv_0 in V is $\mathfrak{g}v_0$, and transporting this space around by elements of G constructed with the exponential map $\mathfrak{g} \to G$ gives tangent spaces that are used in the computation of $\dim kC$—eventually, the rank of some large matrix is our lower bound on $\dim kC$. All these matters can be found in [5, 18, 17].

Unfortunately, the computations are rather slow in characteristic 0, and therefore we indicate in Section 3 how to reduce them modulo a prime. Very roughly speaking, one first does the construction above over $\mathbb{Z}$, i.e., one constructs elements of G that have integral matrices relative to some appropriate basis of V. Then the large matrix whose rank needs to be computed is also integral, and hence its rank modulo p is a lower bound to its rank over $\mathbb{Q}$. The insight that these constructions can *in theory* be done over $\mathbb{Z}$ (and hence over any field) goes back to Chevalley [26]. The results on quantum groups that the *actual implementation* in Section 3 uses, are beyond extensive description in the present paper.

2.2. **The highest weight orbit.** We reduce the computation of $\dim kC$ to straightforward linear algebra as follows. First, we recall that the closed G-orbit in $\mathbb{P}V$ is unique and equal to the orbit of highest weight lines: Indeed, suppose that $x \in \mathbb{P}V$ has a closed orbit Gx, and let P be the stabiliser of x in G. Then $G/P \cong Gx$ is a projective variety, and therefore P contains a Borel subgroup B of G [5, §11.2, Corollary]. By the representation theory of algebraic groups and their Lie algebras [5, 17, 18] B stabilises a *unique* line in the irreducible representation V, namely, the highest weight line $\mathbb{C}v_0$, which therefore corresponds to x. Hence $X = Gx$ is the orbit of highest weight lines. Then the cone C over the minimal orbit X equals $C = Gv_0 \cup \{0\}$—recall that V was assumed non-trivial, so that this really is a cone. In what follows we need the Lie algebras $\mathfrak{g}, \mathfrak{p}$ of G and P, and also a maximal torus T of B.

2.3. **A dense orbit under a unipotent subgroup.** Now let $\mathfrak{u}_-$ be the direct sum of all T-root spaces in $\mathfrak{g}$ that are not in $\mathfrak{p}$. Then $\mathfrak{u}_-$ is the Lie algebra of a unique connected (unipotent) subgroup U_- of G. Let $X_1, \ldots, X_r$ be a basis of $\mathfrak{u}_-$ consisting of T-root vectors. Then the following statements are well known:

(1) The map $\Psi : \mathbb{C}^r \to U_-$ sending $(t_1, \ldots, t_r)$ to $\exp(t_1 X_1) \cdots \exp(t_r X_r)$ is an isomorphism of varieties (combine §14.4, Proposition with §7.3 from [5]).
(2) The U_--orbit $U_- v_0$ is the intersection of C with the affine hyperplane where the v_0-coordinate is 1 (relative to a T-weight basis of V containing v_0). This follows easily from the Bruhat decomposition [5, §14.12, Theorem].
(3) Hence the image of $U_- v_0$ in $\mathbb{P}V$ is dense in X.

Our program works, in fact, with elements in $U_- v_0$ rather than all of C.

2.4. **Terracini's lemma.** We work here with $\tilde{C} := Gv_0 = C \setminus \{0\}$, which is a smooth variety since it is homogeneous. Consider the addition map $\pi : \tilde{C}^k \to \overline{kC}$, $(v_1, \ldots, v_k) \mapsto \sum_i v_i$ and denote its differential at the k-tuple $\mathbf{v} = (v_1, \ldots, v_k) \in \tilde{C}^k$ by $d_v \pi$. By elementary algebraic geometry the map sending $\mathbf{v} \in \tilde{C}^k$ to the rank of $d_{\mathbf{v}} \pi$ is lower semi-continuous (as $\tilde{C}^k$ is smooth), and its generic value is $\dim(kC)$ by the dominance of π. On the other hand, the image of $d_{\mathbf{v}} \pi$ equals

$$\sum_{i=1}^{k} T_{v_i} \tilde{C},$$

where $T_{v_i} \tilde{C}$ denotes the tangent space to C at v_i, regarded as a linear subspace of V. We conclude that the dimension of this latter space is always a lower bound for $\dim kC$, while it is equal to $\dim kC$ for generic tuples $\mathbf{v}$. This observation is, in fact, one of the first results in the theory of join and secant varieties, and due to Terracini [28].

2.5. **The algorithm.** Now our algorithm, which we have implemented in `GAP` [12], is as follows:

> **Input:** $(\mathfrak{g}, \lambda, k)$, where $\mathfrak{g}$ is a split rational semisimple Lie algebra with a distinguished split Cartan subalgebra $\mathfrak{h}$ and a distinguished Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$; $\lambda \in \mathfrak{h}^*$ is a $\mathfrak{b}$-dominant weight; and k is a natural number.
>
> **Output:** a lower bound for $\dim kC$ which with high probability equals $\dim kC$. (Here $C = Gv_0 \cup \{0\}$ where G is the simply connected complex algebraic group associated to $\mathfrak{g}$ and v_0 is a $\mathfrak{b}$-highest weight vector in the irreducible $\mathfrak{g}$-(and G-) module of highest weight λ.)
>
> **Method:**
> (1) Construct the irreducible representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of highest weight λ.
> (2) Denote by v_0 the highest weight vector of V, and compute representatives $(X_1, \ldots, X_r)$ of the negative $\mathfrak{h}$-root spaces in $\mathfrak{g}$ that do not vanish on v_0 (these span $\mathfrak{u}_-$).
> (3) For $i \in \{1, \ldots, r\}$ compute those divided powers $\rho(X_i)^d/(d!)$ that are non-zero.
> (4) Set $T := \{0\}$, the zero subspace of V.
> (5) Compute $T_{v_0}C := Kv_0 + \mathfrak{u}_- v_0$.
> (6) Repeat k times the following steps:
>   (a) Choose rational numbers $t_1, \ldots, t_r$ at random.
>   (b) Compute $u := \exp(t_1\rho(X_1)) \cdots \exp(t_r\rho(X_r))$ using the divided powers of the $\rho(X_i)$ for faster computation of the exponentials.
>   (c) Set $T := T + uT_{v_0}C$.
> (7) return $\dim T$.

Proof of the algorithm. The isomorphism Ψ of 2.3 identifies U_- as an affine space, and moreover gives U_- a $\mathbb{Q}$-structure, corresponding to $\mathbb{Q}^r$ via Ψ. In the ith step of the algorithm a random element u_i of $U_-(\mathbb{Q})$ is computed, as well as the corresponding tangent space $T_{u_i v_0}C = u_i T_{v_0}C$. The algorithm returns the dimension of the sum of these $T_{u_i v_0}C$. We claim that this dimension is only smaller than $\dim kC$ if $(u_1, \ldots, u_k)$ lies in some proper Zariski-closed subset of $U_-(\mathbb{Q})^k$—hence only with "small probability". Indeed, as $T_{sv}C = T_vC$ for all $s \in \mathbb{C}^*$ and $v \in U_-v_0$, and as $\{sv \mid s \in \mathbb{C}^*, v \in U_-v_0\}$ is dense in C, the generic rank of $d_\mathbf{v}\pi$ for $\mathbf{v} \in C^k$ is equal to the generic rank of $d_\mathbf{v}\pi$ for $\mathbf{v} \in (U_-v_0)^k$. Hence the $(u_1, \ldots, u_k) \in U_-^k$ where $d_{(u_1 v_0, \ldots, u_k v_0)}\pi$ does not have rank $\dim kC$ form a proper closed subset S of U_-^k. Finally, $U_-(\mathbb{Q})^k$ is Zariski-dense in U_-^k, so that it cannot be contained in S. This proves the correctness of the algorithm. $\qquad\square$

It is hard to estimate, in an implementation of this algorithm, the precise probability with which the selected tuple $(u_1, \ldots, u_k)$ lies outside S, as it depends on the random generator supplying the rational numbers t_i. However, if the algorithm returns the expected dimension, then we are sure that the output is correct. Otherwise, by repeating the procedure, we can make the probability that the result is correct arbitrarily close to 1.

3. Implementation

We have implemented the algorithm in the computer algebra system GAP4 ([12]), using the built-in functionality for semisimple Lie algebras and their representations. All steps are rather straightforward to implement. It turns out that when working over the field $\mathbb{Q}$, the main bottleneck of the algorithm is the computation of a basis of the space T in Step 6(c). For example the computation of the secant dimensions of the Grassmannian of 5-dimensional subspaces in an 11-dimensional vector space took 362 seconds, of which 309 were spent in the basis computation of Step 6(c).[1] This is due to the fact that the coefficients of the vectors grow very fast (probably because of their random nature). In the example mentioned before, the vectors in a triangularised basis were dense, and contained rational numbers of up to 70 digits in both denominator and numerator.

For this reason we have included in our implementation the possibility to compute modulo a prime p. This however presents a new problem: the coefficients of the matrices of the divided power in Step (3) may not be integral. We can get around this by computing an "admissible lattice" in the highest weight module V (cf. [17]). An algorithm for this purpose is not present in GAP4. We have implemented an algorithm for this based on the theory of crystal bases (cf. [20]). Roughly this works as follows. First we note that V is also a module for the quantum group $U_q(\mathfrak{g})$. Now from the crystal graph of V we get a set of elements F_i in the negative part of $U_q(\mathfrak{g})$, with the property that $\{F_i \cdot v_0\}$ spans an admissible lattice (for details we refer to [13], [23]). Each F_i can be mapped to an element F_i' of the negative part of the universal enveloping algebra $U(\mathfrak{g})$. Then $\{F_i' \cdot v_0\}$ spans an admissible lattice of V. This approach has the advantage that we do not need to check linear independence of the basis elements. The necessary algorithms for quantum groups are implemented in the GAP4 package QuaGroup ([14]). With this the computation in the example above took 71 seconds, with only 3 seconds spent in Step 6(c).

When computing modulo a prime the computed dimensions may be smaller than the ones over $\mathbb{Q}$. However, we have an upper bound for the dimension of kC (namely $k \dim C$) which "usually" gives the correct dimension. Furthermore, if this bound is attained by our algorithm, then we are sure to have the correct value. It rarely happens that this upper bound is not reached. However, if this happens to be the case, then we perform the computation modulo a bigger prime, and eventually over $\mathbb{Q}$. If we still do not attain the upper bound in that case, we conclude that we are in a defective situation with high probability.

Another problem occurs when the dimension of V gets large (e.g., close to 1000). Then storing the matrices in Step (3) may lead to memory problems. To get around this we used an ad-hoc implementation of sparse matrices (only storing the nonzero entries). This greatly reduces the memory requirements, and for dimensions greater than roughly 500 leads to a speed-up for the matrix multiplications in Step 6 (b) as well.

Table 1 contains some run-times of the algorithm, when computing the secant dimensions of the Grassmannian of d-dimensional subspaces of n-space. We see that the running times increase rather sharply, mainly because the same holds for the dimensions of the $\mathfrak{g}$-modules. Most of the time is spent on the construction of

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[1]The computations in this section were done on a 2GHz processor, with 500M RAM memory for GAP

n	d	total	module	basis	$\dim V(\lambda)$	k_{max}
11	5	71	60	3	462	14
12	6	222	165	23	924	24
13	6	973	686	160	1716	39
14	7	5456	3249	1380	3432	68

TABLE 1. Time (in seconds) for the computation of the secant dimensions of the Grassmannian of d-spaces in n-space. The third column lists the total time spent; the fourth and fifth, respectively, the time used for the construction of the module along with the matrices in Step (3), and for the computation of the basis in Step 6(c). The sixth column displays the dimension of the $\mathfrak{g}$-module, and the last column contains k_{max}, the maximal k for which $\dim kC$ has been computed.

the module $V(\lambda)$ and the matrices of Step (3). For small n the time used in Step 6(c) is negligible, but as n increases the percentage of the time spent in that step also increases.

4. CONJECTURES

4.1. **Grassmannians.** For $G = \mathrm{SL}_n$ and $V = \bigwedge^d(\mathbb{C}^n)$ the minimal orbit X is the Grassmannian, in its Plücker embedding, of d-dimensional vector subspaces of $\mathbb{C}^n$, and the cone C over X is the set of (completely) decomposable wedge-products in V. For this setting our algorithm more or less reduces to the algorithm presented in [25], in which article the following conjecture is also tested up to $n = 14$. For $d = 2$ the set kC equals is the set of all skew-symmetric matrices of (usual matrix-) rank at most $2k$ (see, e.g., [29] or [4]); we therefore exclude $d = 2$ in the following conjecture.

Conjecture 4.1. *Suppose that $d > 2$ and also that $2d \leq n$. Then C is defective in exactly the following cases:*

   (1) *$n = 7$ and $d = 3$, in which case $(\dim kC)_k$ equals $(13, 26, *34, 35)$;*
   (2) *$n = 8$ and $d = 4$, in which case $(\dim kC)_k$ equals $(17, 34, *50, *64, 70)$; or*
   (3) *$n = 9$ and $d = 3$, in which case $(\dim kC)_k$ equals $(19, 38, 57, *74, 84)$,*

where $*$ indicates the defective dimensions.

We have verified this conjecture with our program for all n up to 15. The defective Grassmannians in the list above were already found in [9]. For each of them, the first defect occurs at the smallest k with $kd > n$. The following proposition shows that there are no other defective Grassmannians with this property.

Proposition 4.2. *In the setting of Conjecture 4.1, write $n = qd+r$ with $0 \leq r < d$. Then kC is not defective for $k = 1, \ldots, q$, and $(q + 1)C$ is defective if and only if $(n, d, q + 1) \in \{(7, 3, 3), (8, 4, 3), (9, 3, 4)\}$.*

The statement for $k \leq q$ is already present in [9].

Proof. For $k \leq q+1$ the group SL_n has a dense orbit on the k-fold Cartesian power X^k. Indeed, choose a basis

$$
\begin{array}{ccc}
e_{11} & \cdots\cdots\cdots & e_{1d} \\
e_{21} & \cdots\cdots\cdots & e_{2d} \\
\vdots & & \vdots \\
e_{q,1} & \cdots\cdots\cdots & e_{q,d} \\
e_{q+1,1} & \cdots & e_{q+1,r}
\end{array}
$$

of $\mathbb{C}^n$. For every d-subset S of

$$
A := \{(i,j) \mid 1 \leq i \leq q+1, 1 \leq j \leq d, \text{ and } i = q+1 \Rightarrow j \leq r\}
$$

we let e_S denote the wedge product (say in normal reading order) of the e_{ij} with $(i,j) \in S$; these form a basis of $V = \bigwedge^d \mathbb{C}^n$. For $i = 1, \ldots, q$ let v_i be wedge product of the i-th row in the array above. For $j = r+1, \ldots, d$ let s_j be the sum of the d-th column in the array above, and set

$$
v_k := e_{q+1,1} \cdots e_{q+1,r} s_{r+1} \cdots s_d.
$$

One readily verifies that the GL_n-orbit of the tuple $(\mathbb{C}v_i)_{i=1}^k$ is dense in X^k for all $k = 1, \ldots, q+1$. Therefore, the generic rank of the summation map $C^k \to kC$ equals its rank at the tuple $(v_1, \ldots, v_k)$. In other words, the dimension of kC is equal to the dimension of the sum

$$
T_{v_1}C + \ldots + T_{v_k}C.
$$

Now for $i \leq q$, the tangent space $T_{v_i}C$ is the span of all e_S where S is a d-subset of A containing at least $d-1$ elements from row i. Using this description, the assumption $d > 2$ readily implies that the sum

$$
T := T_{v_1}C + \ldots + T_{v_q}C
$$

is direct. This proves that $\dim(kC) = k\dim(C)$ for $k \leq q$.

Now suppose that $k = q+1$. Then the above shows that kC is defective if and only if T_{v_k} contains a non-zero element of T. An element of T_{v_k} is of the form

$$
w := \sum_{j=1}^r e_{k,1} \cdots b_j \cdots e_{k,r} s_{r+1} \cdots s_d + \sum_{j=r+1}^d e_{k,1} \cdots e_{k,r} s_{r+1} \cdots b_j \cdots s_d,
$$

where the $b_j \in \mathbb{C}^n$ replaces $e_{k,j}$ in the first sum and s_j in the second sum. Let S be a d-set in A; we will compute the coefficient of e_S in w. First, e_S can only occur in w if S is of one of the following four types:

Type 1: S contains the entire last (k-th) row of A, and furthermore S hits all $d-r$ last columns of A.

Type 2: S is obtained from a type 1 set by replacing a single position in the k-th row, say (k,j) with $j \leq r$, by position (i,j) with $i < k$ in the same column.

Type 3: S is obtained from a type 1 set by replacing a single position in the k-th row, say (k,j), by position (i,j') with $i < k$ and $j' \neq j$, i.e., in a different column.

Type 4: S is obtained from a type 1 set by replacing a single position in the last $d-r$ columns, say (i,j) with $i < k$ and $j > r$, with a position (i',j') with $i' < k$ and $j' \neq j$, i.e., in a different column.

For computing the coefficient of e_S in w in these four cases, we let $B : A \to \mathbb{C}$ be the function mapping (i, j) to the (i, j)-th component of b_j; note that B only records the entries of b_j corresponding to basis elements of $\mathbb{C}^n$ in column j. Then the coefficients are as follows:

(1) If S is of type 1, then e_S can occur in all terms of w (and, in fact, also in v_k), and its coefficient in w is $\pm \sum_{(i,j) \in S} B(i, j)$ (the sign is irrelevant).

(2) If S is of type 2, then e_S can only occur in the term of w with index j, and its coefficient is $\pm B(i, j)$.

(3) If S is of type 3 or 4, then e_S can only occur in the term of w with index j, and there is a unique column of A containing two elements of S, say $(i_1, j'), (i_2, j')$ with $i_1 < i_2$ and $j' \neq j$. Now there are two cases:

(a) If $j' \leq r$, then i_2 is equal to $k = q+1$, and e_S has coefficient $\pm (b_j)_{(i_1, j')}$ (which coefficient is not recorded by B!).

(b) If $j' > r$, then the coefficient of e_S is $\pm ((b_j)_{(i_1, j')} - (b_j)_{(i_2, j')})$: either one chooses $e_{i_1, j'}$ as the j-th factor and $e_{i_2, j'}$ as the factor at position j', or vice versa. Again, these coefficients are not in B.

Now suppose that $w \in T$. Then the only e_S that can have non-zero coefficient in w are those for which S is obtained from one of the first q rows of A by replacing a single entry by another entry of A. Such d-sets S we will call *good*, others *bad*. For every bad S of types 1–4, the above yields a linear condition on the coefficients of the b_j reflecting that e_S has coefficient 0 in w. We will first argue that we may assume the coefficients $(b_j)_{(i, j')}$ with $j' \neq j$, i.e., those not recorded by B, to be zero.

By assumption, $d \geq 3$ and $n \geq 2d \geq 6$, so $q \geq 2$. The case $n = 6 = 2d$ needs special attention: note that then $T_{v_1}C \oplus T_{v_2}C = V$, so that C is not defective. So we assume that either $d \geq 4$ or $n > 2d$. Then it is somewhat tedious, but straightforward, to verify that the linear conditions for S of types 3 and 4 show that for all $j = 1, \ldots, d$ one has:

(1) For all $j' \leq r$ with $j' \neq j$, and for all $i = 1, \ldots, q$, one has $(b_j)(i, j') = 0$. Indeed, for this one need only exhibit a bad S of type 3 or 4 containing (i, j') and (k, j') and no element from column j. If, for instance, $n = 8, d = 3, r = 2, k = q + 1 = 3, j = 1, j' = 2, i = 2$, then S indicated by the stars in the following picture is such a bad d-set, of type 3:

$$
\begin{matrix}
. & . & * \\
. & * & . \\
. & *
\end{matrix}
$$

(2) for all $j' > r$ with $j' \neq j$, and for all $i_1, i_2 = 1, \ldots, q$ with $i_1 \neq i_2$ one has $(b_j)_{(i_1, j')} = (b_j)_{(i_2, j')}$. Indeed, for this one needs only exhibit a bad S of type 3 or 4 containing $(i_1, j'), (i_2, j')$ and no element from column j. If, for instance, $n = 8, d = 4, r = 0, k = q + 1 = 3, j = 2, j' = 3, i_1 = 1, i_2 = 2$, then

$$
\begin{matrix}
. & . & * & * \\
* & . & * & .
\end{matrix}
$$

(with an empty last row) is an example of such a bad S, of type 4.

This shows that the components of the b_j that are not recorded by B together only contribute a scalar multiple of v_k to w. We may therefore just as well assume that

those components are all 0, i.e., that b_j only has non-zero coefficients on positions in the j-th column of A. Hence now all coefficients of e_S in w are zero for S of type 3 or 4.

Thus we end up looking for $B : A \to \mathbb{C}$ with the following properties:

(1) For every bad S of type 1 the sum $\sum_{(i,j) \in S} B(i,j)$ is 0. Pictorially: the sum of B along subsets of the form

$$
\begin{array}{ccccc}
. & . & . & . & * \\
. & . & * & * & . \\
* & * & & &
\end{array}
$$

is zero.

(2) For every bad S of type 2 the entry of B at position (i, j) (as in the definition of type 2) is 0. Pictorially: if S contains the positions of the stars in

$$
\begin{array}{ccccc}
. & . & . & . & * \\
. & * & * & * & . \\
* & . & & &
\end{array}
$$

then $B(2, 2) = 0$.

(3) For some good S of type 1 the sum above is non-zero, or for some good S of type 2 the entry $B(i, j)$ is non-zero—this to assure that w is a non-zero element of T. This condition will be expressed by calling B non-trivial.

For $(n, d, q + 1) \in \{(8, 4, 3), (9, 3, 4)\}$ (where the $(q + 1)$-st row of A is empty) there certainly are such B:

$$
(1) \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}.
$$

Note that indeed, the sum along any subset S of the indices which hits all columns is zero for bad S, while it can be non-zero if S is good. For $(n, d, q + 1) = (7, 3, 3)$ we can take the B for $(n, d, q + 1) = (9, 3, 4)$ and delete the (empty last row and) positions $(3, 2)$ and $(3, 4)$. This proves defectiveness of the Grassmannians in the proposition. (And with a little extra work, one shows that all B with the required properties lead to scalar multiples of the *same* vector in $T_{v_k} C \cap T$, so that the defect is 1.)

Finally, we have to show that such B only exist for the parameters above. First we argue that r is at most 1. Indeed, suppose that $r \geq 2$. As $k - 1 = q \geq 2$, every choice of a position (i, j) in $\{1, \ldots, k-1\} \times \{1, \ldots, r\}$, together with all entries (k, j') with $j' \leq r$ and $j' \neq j$, can be extended to a bad S—having, in fact, positions in three different rows. This implies that $B(i, j) = 0$ for all $i \leq k - 1$ and $j \leq r$. Furthermore, all S of type 1 are bad (since $r \geq 2$), so that B cannot be non-trivial.

So $r \leq 1$. First we handle $r = 0$. We distinguish 3 cases:

(1) If $d \geq 5$, then one readily sees that B must be constant on each column of A: for instance, for $(n, d) = (10, 5)$ this follows for the third column of A from the fact that the sum of B is zero along both of the following bad d-sets:

$$
\begin{array}{ccccc}
* & * & * & . & . \\
. & . & . & * & *
\end{array} \text{ and }
\begin{array}{ccccc}
* & * & . & . & . \\
. & . & * & * & *
\end{array}
$$

Now if B is constant on each column, then its sums along all S of type 1 are identical, hence if that sum is zero for bad S, then it is also zero for all good S—but then B cannot be non-trivial. This settles $d \geq 5$.

(2) If $d = 4$ and $q \geq 3$, then again B is necessarily constant on every column of A; for instance, the fact that the sum of B is zero along the following two bad sets:

$$
\begin{array}{cccc}
* & * & . & . \\
. & . & * & . \\
. & . & . & *
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
* & * & . & . \\
. & . & * & * \\
. & . & . & .
\end{array}
$$

shows that $B(2,4) = B(3,4)$. As in the case where $d \geq 5$, this implies that B cannot be non-trivial.

(3) Finally, if $d = 3$ and $n \geq 12$, then a similar reasoning using bad sets such as

$$
\begin{array}{ccc}
* & . & . \\
. & * & . \\
. & . & * \\
. & . & .
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
* & . & . \\
. & * & . \\
. & . & . \\
. & . & *
\end{array}
$$

shows that again B is constant on each column, whence trivial.

Next take $r = 1$. Now we distinguish two cases:

(1) $d \geq 4$. Then using bad d-sets of type 2 such as

$$
\begin{array}{cccc}
. & . & * & * \\
* & * & . & . \\
. & & &
\end{array}
$$

one finds that $B(i,1) = 0$ for all $i < k$. Also, the fact that B sums up to 0 along d-sets of type 1 such as the following:

$$
\begin{array}{cccc}
. & . & * & * \\
. & * & . & . \\
* & & &
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
. & . & * & . \\
. & * & . & * \\
* & & &
\end{array}
$$

shows that B is constant on each of the $d-1$ last columns of A. As before, this implies that B does not meet the non-triviality condition.

(2) $d = 3$ and $n \geq 10$. Then bad d-sets of type 2 such as

$$
\begin{array}{ccc}
. & . & * \\
. & * & . \\
* & . & . \\
. & &
\end{array}
$$

show that $B(i,1) = 0$ for all $i < k$. And finally, pairs of d-sets such as

$$
\begin{array}{ccc}
. & . & * \\
. & . & . \\
. & * & . \\
* & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
. & . & . \\
. & . & * \\
. & * & . \\
* & &
\end{array}
$$

show that B is constant on each of the $d-1$ last columns of A, hence not non-trivial.

This proves the proposition. $\square$

We cannot resist pointing out an interesting link between secant dimensions of Grassmannians and coding theory: a *binary code of length n and constant weight d* is a subset of $\{0,1\}^n$ where every element has exactly d entries equal to 1. The *(Hamming) distance* between two elements of $\{0,1\}^n$ is the number of coordinates where they differ.

Theorem 4.3. *Retain the setting of Conjecture 4.1, and let B be a binary code of length n and constant weight d with $|B| = k$. Then the following holds.*

> (1) *If the distance between any two distinct elements of B is at least 6, then kC is not defective.*
> (2) *If every word in $\{0,1\}^n$ of weight d has an element of B at distance at most 2, then $kC = V$.*

The first observation slightly generalises [9, Theorem 2.1.ii]; the second observation is also stated, in a slightly different form, in [11, Proposition 7.5]. More relations between secant varieties and combinatorics are given in [27].

Proof. Let $e_1, \ldots, e_n$ be a basis of $\mathbb{C}^n$. To every word w in $\{0,1\}^n$ of weight d we associate an element of V as follows: if $i_1 < \ldots < i_d$ are the coordinates i where $b_i = 1$, then we set

$$e_w := e_{i_1} \cdots e_{i_d} \in V.$$

Now

$$T_{e_w} C = \sum_{j=1}^{d} e_{i_1} \cdots e_{i_{j-1}} \mathbb{C}^n e_{i_{j+1}} \cdots e_{i_d}$$

is precisely the span of all e_u where u is a word of weight d at distance at most 2 from w. Hence the dimension of $\sum_{b \in B} T_{e_b} C$ (and therefore that of kC) is at least the cardinality of the set of all weight-d words in $\{0,1\}^n$ at distance at most 2 from B. This implies both statements. □

4.2. Segre powers. For $G = \mathrm{SL}_n^d$ and $V = (\mathbb{C}^n)^{\otimes d}$ the minimal orbit X is $(\mathbb{P}^{n-1})^d$ in its Segre embedding, and the affine cone C over X is the set of decomposable tensors in V. For $d = 2$ the set kC corresponds to the set of $n \times n$-matrices of rank $\leq k$, so we leave out this well-understood case from our study. Already for $d = 3$, or for d large and $n = 2$ the secant dimensions are not known—though there are some results, for a good overview of which we refer to [8]. Our conjecture is as follows.

Conjecture 4.4. *Suppose that $d \neq 2$. The variety C is defective if and only if*

> (1) $n = 2$ *and* $d = 4$, *in which case* $(\dim kC)_k$ *equals* $(5, 10, *14, 16)$; *or*
> (2) $n = 3$ *and* $d = 3$, *in which case* $(\dim kC)_k$ *equals* $(7, 14, 21, *26, 27)$,

where $*$ indicates the defective secant dimensions.

We have verified this conjecture with our program for $d + n \leq 8$ as well as for $n = 2$ and $d = 9, 10$ and for $d = 3$ and $n \leq 9$. Again, it is not hard but tedious to prove the following proposition.

Proposition 4.5. *In the setting of Conjecture 4.4, kC is not defective for $k \leq n$, and $(n + 1)C$ is defective if and only if $(n, d) \in \{(2, 4), (3, 3)\}$.*

The proof is completely analogous to that of Proposition 4.2: here $(\mathrm{SL}_n)^d$ has a dense orbit on X^k, hence the rank of the differential of the summation map $C^k \to kC$ need only be computed in a point of C^k over this orbit. We omit the details, but do report a funny numeric coincidence: the same array B that shows that $3C$ is defective for the Grassmannian of 4-dimensional subspaces of an 8-dimensional space, also shows that $3C$ is defective for the 4th Segre power of $\mathbb{PC}^2$. The same (numeric) connection exists between the defect in $4C$ for the Grassmannian of 3-dimensional subspaces of a 9-dimensional space and the defect in $4C$ for the 3rd Segre power of $\mathbb{PC}^3$.

We recall another interesting link to coding theory, which can be proved in the same manner as Theorem 4.3, but is also well known from [8, 11, 27].

Theorem 4.6. *In the setting of Conjecture 4.4 let B be a subset of $\{1, \ldots, n\}^d$ of size k. Then the following holds.*

(1) *If the Hamming distance between any two elements of B is at least 3, then kC is not defective.*
(2) *If every element of $\{1, \ldots, n\}^d$ is at distance at most 1 from an element of B, then $kC = V$.*

4.3. **A finiteness question.** The experiments with our program suggest the following question: Fix the complex semisimple group G. Is it true that the set of all irreducible representations of G whose minimal orbit is defective is *finite*? Though we dare not formulate this as a conjecture at this stage, the question is a very natural one. For instance, from the fundamental work of Alexander and Hirschowitz [1, 2, 3], we know that for each n only finitely many symmetric powers of the natural representation of SL_n have defective minimal orbits.

We give a conjecturally complete list of "defective highest weights" for some small groups; note that for $G = \mathrm{SL}_2$ the minimal orbit in *no* irreducible representation is defective.

Conjecture 4.7. *The only irreducible representations of G for which the minimal orbit is defective are those with the following highest weights (in the labelling of [6]):*

(1) *for G of type A_2:*
  (a) *$2\omega_1$ and $2\omega_2$ with secant dimensions $(3, *5, 6)$ (X is the quadratic Veronese embedding of $\mathbb{P}^2$),*
  (b) *$4\omega_1$ and $4\omega_2$ with secant dimensions $(3, 6, 9, 12, *14, 15)$ (the quartic Veronese embedding of $\mathbb{P}^2$),*
  (c) *$\omega_1 + \omega_2$ with secant dimensions $(4, *7, 8)$ (a hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$), and*
  (d) *$2\omega_1 + 2\omega_2$ with secant dimensions $(4, 8, 12, 16, 20, 24, *26, 27)$ (the image of the previous variety under the quadratic Veronese re-embedding of $\mathbb{P}(V_{\omega_1 + \omega_2})$—which image spans only a subspace of projective dimension 26 in the space $\mathbb{P}S^2(V_{\omega_1 + \omega_2})$ of dimension 35);*
(2) *for G of type A_3:*
  (a) *$\omega_1 + \omega_2$ and $\omega_2 + \omega_3$ with secant dimensions $(6, 12, *17, 20)$ (the flag variety of flags $V_1 \subseteq V_2 \subseteq \mathbb{C}^4$ with $\dim V_i = i$, in its minimal-dimensional embedding),*
  (b) *$2\omega_1$ and $2\omega_3$ with secant dimensions $(4, *7, *9, 10)$ (the quadratic Veronese embedding of $\mathbb{P}^3$),*

(c) $\omega_1 + \omega_3$ with secant dimensions $(6, *11, *14, 15)$ (a hyperplane section of the Segre embedding of $\mathbb{P}^3 \times \mathbb{P}^3$),

(d) $2\omega_2$ with secant dimensions $(5, 10, *14, *17, *19, 20)$ (the image of the Grassmannian of 2-spaces in 4-space under the quadratic Veronese re-embedding of $\mathbb{P}V_{\omega_2}$; this image spans a codimension 1 subspace), and

(e) $4\omega_1$ and $4\omega_3$ with secant dimensions $(4, 8, 12, 16, 20, 24, 28, 32, *34, 35)$ (the quartic Veronese embedding of $\mathbb{P}^3$);

(3) for G of type B_2:

(a) $2\omega_1$ with secant dimensions $(4, 8, *11, *13, 14)$ (the image of a quadric in $\mathbb{P}^4$ under the quadratic Veronese re-embedding—which image spans a codimension 1 subspace),

(b) $2\omega_2$ with secant dimensions $(4, *7, *9, 10)$ (the image of the Grassmannian of isotropic 2-dimensional vector spaces under the quadratic Veronese re-embedding),

(c) $\omega_1 + \omega_2$ with secant dimensions $(5, 10, *14, 16)$ (the flag variety of isotropic flags $V_1 \subseteq V_2 \subseteq \mathbb{C}^5$ with $\dim V_i = i$, in its minimal embedding), and

(d) $4\omega_2$ with secant dimensions $(4, 8, \dots, 28, 32, *34, 35)$ (the quartic Veronese re-embedding of the Grassmannian of isotropic 2-dimensional vector spaces in $\mathbb{C}^4$); and, finally,

(4) for G of type G_2:

(a) $2\omega_1$ with secant dimensions $(6, 12, *17, *21, 24)$ (the quadratic Veronese re-embedding of a quadric in $\mathbb{P}^6$, spanning a codimension 1 subspace),

(b) ω_2 with secant dimensions $(6, *11, 14)$ (the orbit of long-root vectors in G_2), and

(c) $2\omega_2$ with secant dimensions $(6, \dots, 72, *76, 77)$ (the quadratic Veronese re-embedding of the previous variety—which image spans a proper subspace).

The conjecture for A_2 and B_2 has been verified for weights $i\omega_1 + j\omega_2$ with $i+j \leq 6$. For A_3 the conjecture has been checked for highest weights $i\omega_1 + j\omega_2 + k\omega_3$ with $i + j + k \leq 4$, and for G_2 the conjecture has been verified for all highest weights $i\omega_1 + j\omega_2$ with $i + j \leq 4$. To attack the question posed in this section, one would need completely new techniques, far beyond our easy algorithm. But we hope that the challenges boldly posed in this paper as conjectures will be taken up by some of our readers!

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