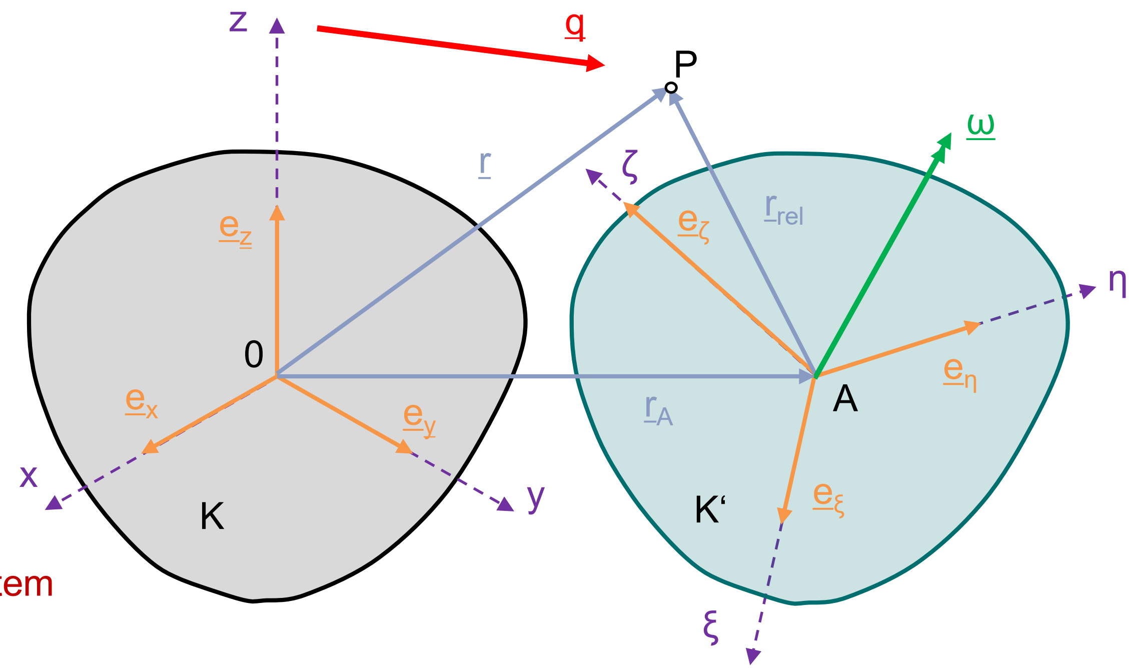
23. Relativkinematik

23.1 Grundbegriffe



K… festes System = Absolutsystem
→ $\underline{e}_x, \underline{e}_y, \underline{e}_z$ in K fest

K'… bewegtes System = Führungssystem
→ $\underline{e}_\xi, \underline{e}_\eta, \underline{e}_\zeta$ in K' fest

Bewegung von P in K = Absolutbewegung: $\underline{r}(t)$, $\underline{v}(t)$, $\underline{a}(t)$

Bewegung von P in K' = Relativbewegung: $\underline{r}_{rel}(t)$, $\underline{v}_{rel}(t)$, $\underline{a}_{rel}(t)$

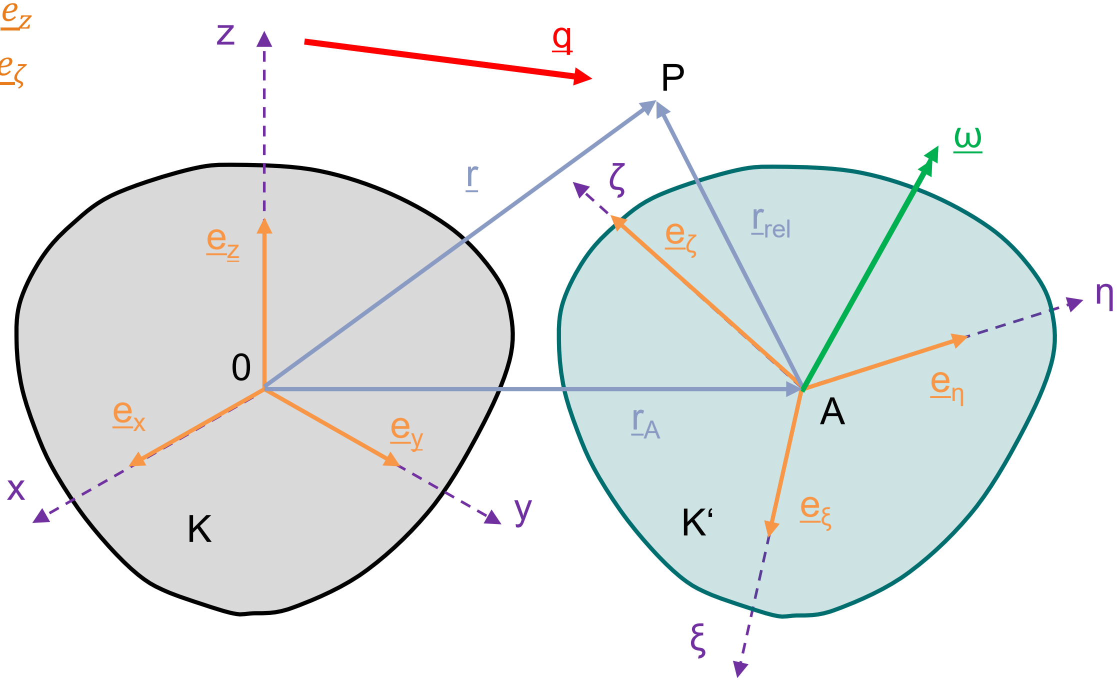
K … in der Kinetik ein Inertialsystem, z.B.: Erde

Zerlegung in K: $\underline{q} = q_x \underline{e}_x + q_y \underline{e}_y + q_z \underline{e}_z$

Zerlegung in K‘: $\underline{q} = q_\xi \underline{e}_\xi + q_\eta \underline{e}_\eta + q_\zeta \underline{e}_\zeta$

Zeitliche Änderung in K: $\frac{d}{dt}$

zeitliche Änderung in K‘: $\frac{\partial}{\partial t}$ oder $\frac{d'}{dt}$



$$\frac{d\underline{q}}{dt} = \frac{d}{dt}\left(q_x \underline{e}_x + q_y \underline{e}_y + q_z \underline{e}_z\right) = \frac{dq_x}{dt}\underline{e}_x + \frac{dq_y}{dt}\underline{e}_y + \frac{dq_z}{dt}\underline{e}_z$$

$$\frac{d\underline{q}}{dt} = \frac{d}{dt}\left(q_\xi \underline{e}_\xi + q_\eta \underline{e}_\eta + q_\zeta \underline{e}_\zeta\right) = \underbrace{\left(\frac{dq_\xi}{dt}\underline{e}_\xi + \frac{dq_\eta}{dt}\underline{e}_\eta + \frac{dq_\zeta}{dt}\underline{e}_\zeta\right)}_{} + \left(q_\xi \frac{d\underline{e}_\xi}{dt} + q_\eta \frac{d\underline{e}_\eta}{dt} + q_\zeta \frac{d\underline{e}_\zeta}{dt}\right)$$

$\frac{\partial \underline{q}}{\partial t}$ … zeitliche Änderung von $\underline{q}$ in K‘

$$\frac{d\underline{q}}{dt} = \frac{d}{dt}(q_\xi \underline{e}_\xi + q_\eta \underline{e}_\eta + q_\zeta \underline{e}_\zeta) = \left(\frac{dq_\xi}{dt}\underline{e}_\xi + \frac{dq_\eta}{dt}\underline{e}_\eta + \frac{dq_\zeta}{dt}\underline{e}_\zeta\right) + \left(q_\xi \frac{d\underline{e}_\xi}{dt} + q_\eta \frac{d\underline{e}_\eta}{dt} + q_\zeta \frac{d\underline{e}_\zeta}{dt}\right)$$

$\frac{d\underline{e}_\xi}{dt} = \underline{\omega} \times \underline{e}_\xi$ detto für η, ζ

$$\frac{d\underline{q}}{dt} = \frac{\partial \underline{q}}{\partial t} + \left[\underline{\omega} \times \overbrace{(q_\xi \underline{e}_\xi + q_\eta \underline{e}_\eta + q_\zeta \underline{e}_\zeta)}^{\underline{q}}\right]$$

$$\boxed{\frac{d\underline{q}}{dt} = \frac{\partial \underline{q}}{\partial t} + \underline{\omega} \times \underline{q}}$$ vgl. Kollergang (Mechanik IB)

für $\underline{q} = \underline{\omega}$: $\frac{d\underline{\omega}}{dt} = \frac{\partial \underline{\omega}}{\partial t} + \overbrace{\underline{\omega} \times \underline{\omega}}^{0} \rightarrow \frac{d\underline{\omega}}{dt} = \frac{\partial \underline{\omega}}{\partial t} = \underline{\dot{\omega}}$

$$\underline{r} = \underline{r}_A + \underline{r}_{rel}$$

$$\frac{d\underline{r}}{dt} = \frac{d\underline{r}_A}{dt} + \frac{d\underline{r}_{rel}}{dt} \qquad \frac{d\underline{r}_{rel}}{dt} = \frac{\partial \underline{r}_{rel}}{\partial t} + \underline{\omega} \times \underline{r}_{rel}$$

$$\underbrace{\frac{d\underline{r}}{dt}}_{\underline{v}} = \underbrace{\frac{d\underline{r}_A}{dt}}_{\underline{v}_A} + \underline{\omega} \times \underline{r}_{rel} + \underbrace{\frac{\partial \underline{r}_{rel}}{\partial t}}$$

Geschwindigkeit von P in K'…. Relativgeschwindigkeit $\underline{v}_{rel}$

$\underline{v}$ …Absolutgeschwindigkeit (Geschwindigkeit von P im System K)

$$\underline{v} = \underbrace{\underline{v}_A + \underline{\omega} \times \underline{r}_{rel}} + \underline{v}_{rel} \rightarrow \underline{v} = \underline{v}_F + \underline{v}_{rel}$$

$\underline{v}_F$ …. Führungsgeschwindigkeit (= Geschwindigkeit des Systemdeckpunkts)
=Stelle P zur Zeit t

$$\frac{d\underline{v}}{dt} = \frac{d\underline{v}_A}{dt} + \left(\underbrace{\frac{d\underline{\omega}}{dt}}_{\dot{\underline{\omega}}} \times \underline{r}_{rel}\right) + \left(\underline{\omega} \times \underbrace{\frac{d\underline{r}_{rel}}{dt}}\right) + \underbrace{\frac{d\underline{v}_{rel}}{dt}}$$

$$\underline{\omega} \times \underline{v}_{rel} + \frac{\partial \underline{v}_{rel}}{\partial t}$$

$$\frac{d\underline{v}}{dt} = \frac{d\underline{v}_A}{dt} + \dot{\underline{\omega}} \times \underline{r}_{rel} + \underline{\omega} \times \left(\underbrace{\frac{\partial \underline{r}_{rel}}{\partial t}}_{\underline{v}_{rel}} + \underline{\omega} \times \underline{r}_{rel}\right) + \underbrace{\frac{\partial \underline{v}_{rel}}{\partial t}} + \underline{\omega} \times \underline{v}_{rel}$$

$\frac{\partial \underline{v}_{rel}}{\partial t}$ … Relativbeschleunigung = $\underline{a}_{rel}$(Beschleunigung von P im K‘)

$\frac{d\underline{v}}{dt}$ … Beschleunigung von P in K: …. Absolutbeschleunigung

$\frac{d\underline{v}_A}{dt}$ … Beschleunigung von A in K

$\frac{\partial \underline{r}_{rel}}{\partial t} = \underline{v}_{rel}$ Geschwindigkeit von P in K‘: Relativgeschwindigkeit

$$\boxed{\underline{v} = \underline{v}_A + \underline{\omega} \times \underline{r}_{rel} + \underline{v}_{rel}}$$

$$\boxed{\underline{a} = \underbrace{\underline{a}_A + \dot{\underline{\omega}} \times \underline{r}_{rel} + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}_{rel}\right)}_{\underline{a}_F} + \underbrace{\underline{a}_{rel}}_{\underline{a}_{rel}} + \underbrace{2\underline{\omega} \times \underline{v}_{rel}}_{\underline{a}_{cor}}}$$

$$\boxed{\underline{a} = \underline{a}_F + \underline{a}_{rel} + \underline{a}_{cor}}$$

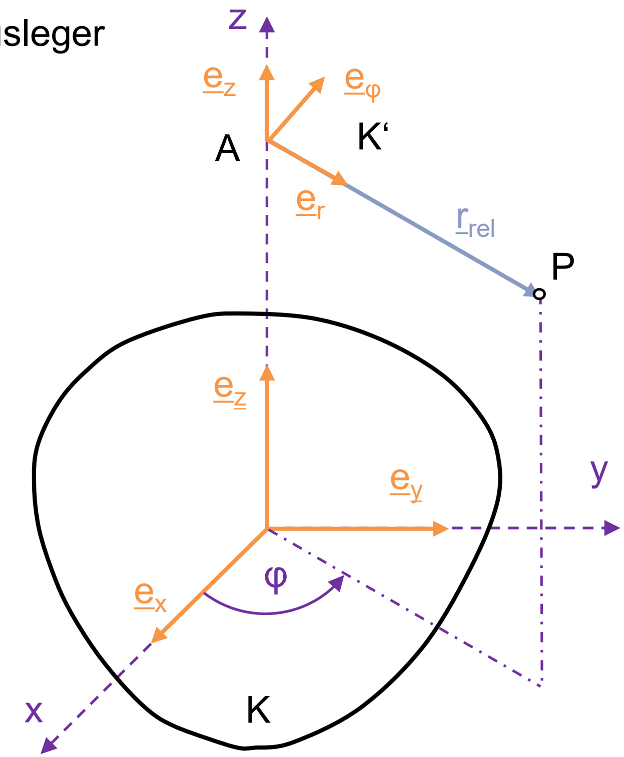
$\underline{a}_F$ ... Führungsbeschleunigung (Beschleunigung des Systemdeckpunkts in K)

$\underline{a}_{rel}$ ... Relativbeschleunigung (Beschleunigung von P im System K‘)

$\underline{a}_{cor}$ ... Coriolisbeschleunigung $= 2\underline{\omega} \times \underline{v}_{rel}$

23.2 Beispiele

Geg.: Kranausleger
Ges.: $\underline{v}$, $\underline{a}$



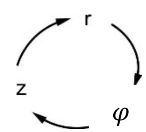
$$\frac{d\underline{r}_A}{dt} = \underline{v}_A = \dot{z}\underline{e}_z \qquad \underline{a}_A = \ddot{z}\underline{e}_z$$

$$\underline{\omega} = \omega\underline{e}_z \qquad \underline{\dot{\omega}} = \dot{\omega}\underline{e}_z$$

$$\underline{r}_{rel} = r\underline{e}_r \qquad \underline{v}_{rel} = \frac{\partial \underline{r}_{rel}}{\partial t} = \dot{r}\underline{e}_r \qquad \underline{a}_{rel} = \ddot{r}\underline{e}_r$$

$$\underline{v} = \underline{v}_A + \underline{\omega} \times \underline{r}_{rel} + \underline{v}_{rel} = \dot{z}\underline{e}_z + \omega r\overbrace{\left(\underline{e}_z \times \underline{e}_r\right)}^{\underline{e}_\varphi} + \dot{r}\underline{e}_r$$

$$\rightarrow \underline{v} = \dot{r}\underline{e}_r + r\omega\underline{e}_\varphi + \dot{z}\underline{e}_z$$



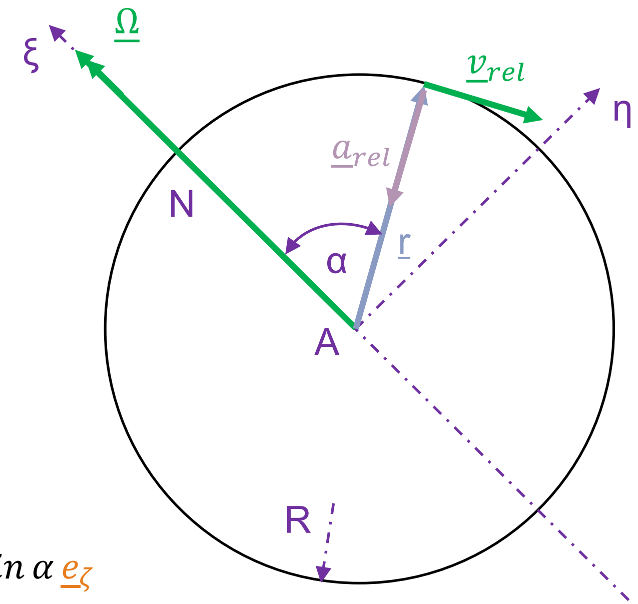
$$\underline{a} = \underline{a}_A + \underline{\dot{\omega}} \times \underline{r}_{rel} + \left[\underline{\omega} \times \left(\underline{\omega} \times \underline{r}_{rel}\right)\right] + \underline{a}_{rel} + \left(2\underline{\omega} \times \underline{v}_{rel}\right)$$

$$= \ddot{z}\underline{e}_z + \dot{\omega}r\underbrace{\left(\underline{e}_z \times \underline{e}_r\right)}_{\underline{e}_\varphi} + \mathrm{r}\omega^2\underbrace{\left(\underline{e}_z \times \left(\underline{e}_z \times \underline{e}_r\right)\right)}_{-\underline{e}_r} + \ddot{r}\underline{e}_r + 2\omega\dot{r}\underbrace{\left(\underline{e}_z \times \underline{e}_r\right)}_{\underline{e}_\varphi}$$

$$\rightarrow \underline{a} = \underbrace{\left(\ddot{r} - r\omega^2\right)}_{a_r}\underline{e}_r + \underbrace{\left(r\dot{\omega} + 2\dot{r}\omega\right)}_{a_\varphi}\underline{e}_\varphi + \underbrace{\ddot{z}}_{a_z}\underline{e}_z$$

Beispiel: Bewegung der Erde in N-S-Richtung

Geg.: α, Ω, v = const. (Fahrgeschwindigkeit des Autos)



$$\underline{r} = R\cos\alpha\,\underline{e}_\xi + R\sin\alpha\,\underline{e}_\eta$$

$$\underline{\Omega} = \Omega\underline{e}_\xi \qquad \underline{\dot{\Omega}} = \underline{0}$$

$$\underline{v}_{rel} = -v\sin\alpha\,\underline{e}_\xi + v\cos\alpha\,\underline{e}_\eta = \frac{\partial\underline{r}}{\partial t}$$

$$\underline{a}_{rel} = -\frac{v^2}{R}\cos\alpha\,\underline{e}_\xi - \frac{v^2}{R}\sin\alpha\,\underline{e}_\eta$$

$$\underline{v} = \underline{v}_F + \underline{v}_{rel}, \quad \underline{a} = \underline{a}_F + \underline{a}_{rel} + \underline{a}_{cor}$$

$$\underline{v}_F = \underline{v}_A + \underline{\omega}\times\underline{r} = \Omega\underline{e}_\xi \times \left(R\cos\alpha\,\underline{e}_\xi + R\sin\alpha\,\underline{e}_\eta\right) = R\,\Omega\sin\alpha\,\underline{e}_\zeta$$

$$\underline{v} = -v\sin\alpha\,\underline{e}_\xi + v\cos\alpha\,\underline{e}_\eta + R\Omega\sin\alpha\,\underline{e}_\zeta$$

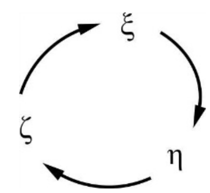
$$\underline{v} = \begin{pmatrix} -v\sin\alpha \\ v\cos\alpha \\ R\Omega\sin\alpha \end{pmatrix}$$

Beispiel: Bewegung der Erde in N-S-Richtung

$$\underline{a}_F = \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = \Omega \underline{e}_\xi \times (R\,\Omega \sin\alpha\, \underline{e}_\zeta) = -R\,\Omega^2 \sin\alpha\, \underline{e}_\eta$$

$$\underline{a}_{cor} = 2\underline{\Omega} \times \underline{v}_{rel} = 2\Omega \underline{e}_\xi \times (-v\sin\alpha\, \underline{e}_\xi + v\cos\alpha\, \underline{e}_\eta) = 2\Omega v \cos\alpha\, \underline{e}_\zeta$$

$$\underline{a} = -\frac{v^2}{R}\cos\alpha\, \underline{e}_\xi - \left(\frac{v^2}{R} + R\Omega^2\right)\sin\alpha\, \underline{e}_\eta + 2\Omega v\cos\alpha\, \underline{e}_\zeta$$



$$a = \begin{pmatrix} -\dfrac{v^2}{R}\cos\alpha \\ -\left(\dfrac{v^2}{R} + R\Omega^2\right)\sin\alpha \\ 2\Omega v\cos\alpha \end{pmatrix}$$

z.B:

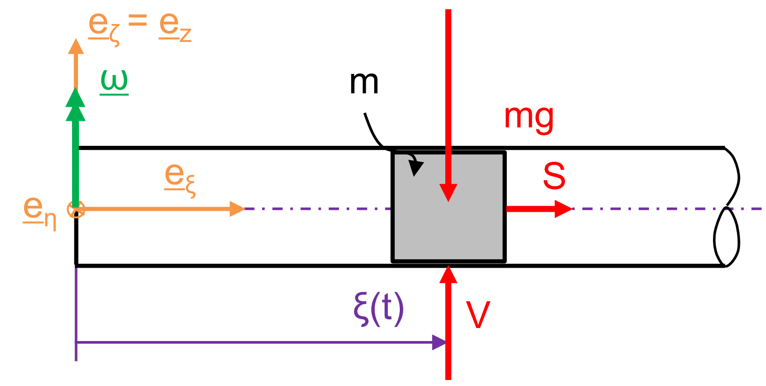
$$\Omega = 2\pi \frac{1}{24h} = 7.27 \times 10^{-5} s^{-1} \qquad R = 6400\,km \qquad v = 100\,\frac{km}{h} \mathrel{\widehat{=}} 27.78\,m/s$$

Äquator: $\alpha = \frac{\pi}{2}$: $\quad |\underline{a}_{cor}| = 0 \quad |\underline{a}_F| = 0.003\,m/s^2$

Nordpol: $\alpha = 0$: $\quad |\underline{a}_{cor}| = 0.004\,m/s^2 \quad |\underline{a}_F| = 0$

Beispiel: Masse in horizontal rotierendem Rohr

Geg.: S, Masse m, ω =const.
Ges.: Bewegung $\xi(t)$



$$\underline{v}_{rel} = \dot{\xi}\underline{e}_\xi, \qquad \underline{v}_F = \xi\omega\underline{e}_\eta$$

$$\underline{a}_{rel} = \ddot{\xi}\underline{e}_\xi, \qquad \underline{a}_F = -\xi\omega^2\underline{e}_\xi$$

$$\underline{a}_{cor} = 2\omega\underline{e}_\zeta \times \dot{\xi}\underline{e}_\xi = 2\omega\dot{\xi}\underline{e}_\eta$$

Schwerpunktsatz:

$$m(\ddot{\xi} - \xi\omega^2)\underline{e}_\xi + 2m\omega\dot{\xi}\underline{e}_\eta = S\underline{e}_\xi + H\underline{e}_\eta + (V - mg)\underline{e}_\zeta$$

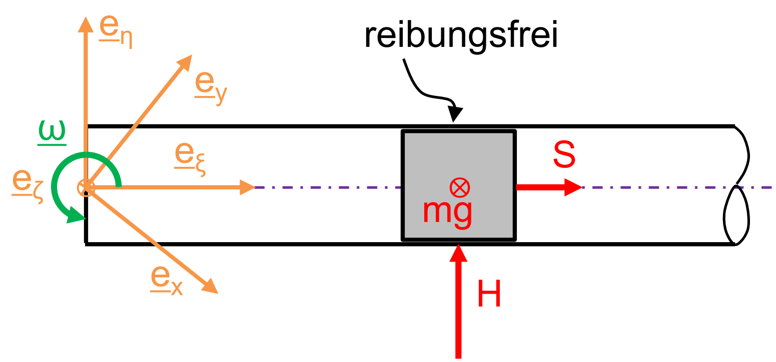
(1) $m(\ddot{\xi} - \xi\omega^2) = S$

(2) $2m\omega\dot{\xi} = H$

(3) $0 = V - mg$

3 Gleichungen für ξ, H und V

$V = mg$



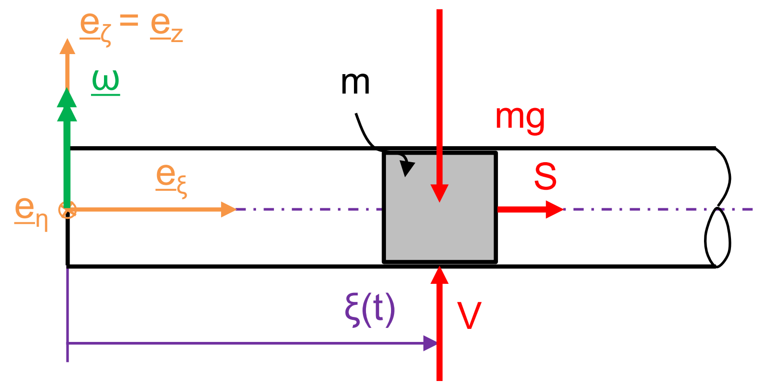
aus (1) $\xi(t) = C_1 e^{\omega t} + C_2 e^{-\omega t} - \dfrac{S}{\omega^2 m}$

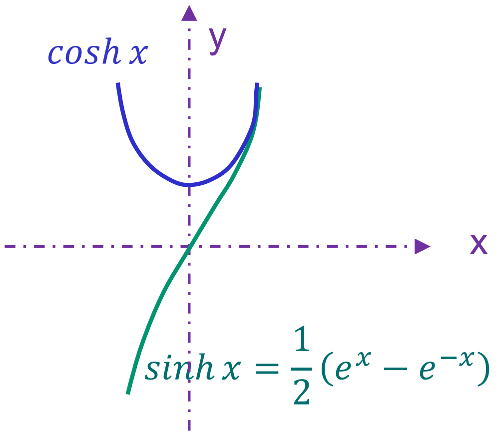
Beispiel: Masse in horizontal rotierendem Rohr, Fortsetzung

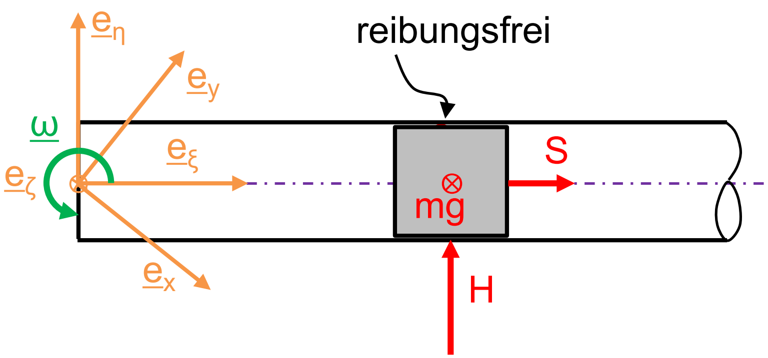
$$AB: t = 0: \ \xi = \xi_o, \ \dot{\xi} = v_o$$

$$C_1 = \frac{1}{2}\left(\xi_o + \frac{S}{\omega^2 m} + \frac{v_o}{\omega}\right)$$
$$C_2 = \frac{1}{2}\left(\xi_o + \frac{S}{\omega^2 m} - \frac{v_o}{\omega}\right)$$

$$\xi(t) = \left(\xi_o + \frac{S}{\omega^2 m}\right) \cosh \omega t + \frac{v_o}{\omega} \sinh \omega t$$







Für t → groß: $\xi(t) = \frac{1}{2}\left(\xi_o + \frac{S}{\omega^2 m} + \frac{v_o}{\omega}\right) e^{\omega t}$ → $H = 2m\omega^2 \, \xi(t)$