Geometry & Topology Monographs 9 (2006) 17–32

The Bryant–Ferry–Mio–Weinberger construction of generalized manifolds

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Following Bryant, Ferry, Mio and Weinberger we construct generalized manifolds as limits of controlled sequences $\{X_i \xrightarrow{p_i} X_{i-1} : i = 1, 2, \ldots\}$ of controlled Poincaré spaces. The basic ingredient is the ε–δ–surgery sequence recently proved by Pedersen, Quinn and Ranicki. Since one has to apply it not only in cases when the target is a manifold, but a controlled Poincaré complex, we explain this issue very roughly. Specifically, it is applied in the inductive step to construct the desired controlled homotopy equivalence $p_{i+1}\colon X_{i+1} \to X_i$. Our main theorem requires a sufficiently controlled Poincaré structure on X_i (over X_{i-1}). Our construction shows that this can be achieved. In fact, the Poincaré structure of X_i depends upon a homotopy equivalence used to glue two manifold pieces together (the rest is surgery theory leaving unaltered the Poincaré structure). It follows from the ε–δ–surgery sequence (more precisely from the Wall realization part) that this homotopy equivalence is sufficiently well controlled. In the final section we give additional explanation why the limit space of the X_i 's has no resolution.

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1 Preliminaries

A *generalized n–dimensional manifold* X is characterized by the following two properties:

(i) X is a Euclidean neighborhood retract (ENR); and

(ii) X has the local homology (with integer coefficients) of the Euclidean n–space $\mathbb{R}^n$, ie
$$H_*(X, X \setminus \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}).$$

Since we deal here with locally compact separable metric spaces of finite (covering) dimension, ENRs are the same as ANRs.

Generalized manifolds are Poincaré spaces, in particular they have the Spivak normal fibrations ν_X. The total space of ν_X is the boundary of a regular neighborhood

Published: 22 April 2006 DOI: 10.2140/gtm.2006.9.17

$N(X) \subset \mathbb{R}^L$ of an embedding $X \subset \mathbb{R}^L$, for some large L. One can assume that $N(X)$ is a mapping cylinder neighborhood (see Lacher [5, Corollary 11.2]).

The global Poincaré duality of Poincaré spaces does not imply the local homology condition (ii) above. The local homology condition can be understood as the "controlled" global Poincaré duality (see Quinn [9, p270], and Bryant–Ferry–Mio–Weinberger [1, Proposition 4.5]). More precisely, one has the following:

Theorem 1.1 *Let X be a compact ANR Poincaré duality space of finite (covering) dimension. Then X is a generalized manifold if and only if for every $\delta > 0$, X is a δ–Poincaré space (over X).*

The definition of the δ–Poincaré property is given below. The following basic fact about homology manifolds was proved by Ferry and Pedersen [4, Theorem 16.6].

Theorem 1.2 *Let X be an ANR homology manifold. Then ν_X has a canonical* TOP *reduction.*

This statement is equivalent to existence of degree-one normal maps $f\colon M^n \to X$, where M^n is a (closed) topological n–manifold, hence the structure set $\mathcal{S}^{\mathrm{TOP}}(X)$ can be identified with $[X, G/\mathrm{TOP}]$.

Let us denote the 4–periodic simply connected surgery spectrum by $\mathbb{L}$ and let $\widehat{\mathbb{L}}$ be the connected covering of $\mathbb{L}$. There is a (canonical) map of spectra $\widehat{\mathbb{L}} \to \mathbb{L}$ given by the action of $\widehat{\mathbb{L}}$ on $\mathbb{L}$. Note that $\widehat{\mathbb{L}}_0$ is G/TOP.

If M^n is a topological manifold there exists a fundamental class $[M]_{\mathbb{L}} \in H_n(M; \mathbb{L}^\bullet)$, where $\mathbb{L}^\bullet$ is the symmetric surgery spectrum (see Ranicki [11, Chapters 13 and 16]).

Theorem 1.3 *If M^n is a closed oriented topological n–manifold, then the cap product with $[M]_{\mathbb{L}}$ defines a Poincaré duality of $\mathbb{L}$–(co)homology*

$$H^p(M; \mathbb{L}) \xrightarrow{\cong} H_{n-p}(M; \mathbb{L})$$

and $\widehat{\mathbb{L}}$–(co)homology

$$H^p(M; \widehat{\mathbb{L}}) \xrightarrow{\cong} H_{n-p}(M; \widehat{\mathbb{L}}).$$

Since $H^0(M; \mathbb{L}) = [M, \mathbb{Z} \times G/\mathrm{TOP}]$ and $H^0(M; \widehat{\mathbb{L}}) = [M, G/\mathrm{TOP}]$, we have

$$H_n(M; \mathbb{L}) = \mathbb{Z} \times H_n(M; \widehat{\mathbb{L}})$$

and the map $\widehat{\mathbb{L}} \to \mathbb{L}$ has the property that the image of

$$H_n(M; \widehat{\mathbb{L}}) \to H_n(M; \mathbb{L}) = \mathbb{Z} \times H_n(M; \widehat{\mathbb{L}})$$

is $\{1\}\times H_n(M;\widehat{\mathbb{L}})$ (see Ranicki [11, Appendix C]). Moreover, the action of $H^0(M;\widehat{\mathbb{L}})$ on $H^0(M;\mathbb{L})=\mathbb{Z}\times H^0(M;\widehat{\mathbb{L}})$, induced by the action of $\widehat{\mathbb{L}}$ on $\mathbb{L}$, preserves the $\mathbb{Z}$–sectors.

If X is a generalized n–manifold we get similar results by using the fundamental class $f_*([M]_{\mathbb{L}})=[X]_{\mathbb{L}}\in H_n(X;\mathbb{L}^\bullet)$, where $f\colon M\to X$ is the canonical degree-one normal map. So the composition map

$$\Theta\colon [X,G/\mathrm{TOP}]\to H_n(X;\widehat{\mathbb{L}})\to H_n(X;\mathbb{L})=\mathbb{Z}\times H_n(X;\widehat{\mathbb{L}})$$

has the property that $\operatorname{Im}\Theta$ belongs to a single $\mathbb{Z}$–sector, denoted by $I(X)\in\mathbb{Z}$.

The following is the fundamental result of Quinn on resolutions of generalized manifolds [10].

Theorem 1.4 *Let X be a generalized n–manifold, $n\geq 5$. Then X has a resolution if and only if $I(X)=1$.*

Remark The integer $I(X)$ is called the *Quinn index* of the generalized manifold X. Since the action of $\widehat{\mathbb{L}}$ on $\mathbb{L}$ preserves the $\mathbb{Z}$–sectors, arbitrary degree-one normal maps $g\colon N\to X$ can be used to calculate $I(X)$. Alternatively, we can define $I(X)$ using the fibration $\widehat{\mathbb{L}}\to\mathbb{L}\to\mathbb{K}(\mathbb{Z},0)$, where $\mathbb{K}(\mathbb{Z},\cdot\,)$ is the Eilenberg–MacLane spectrum, and define $I(X)$ as the image of (see Ranicki [11, Chapter 25]):

$$\{f\colon M\to X\}\in H_n(X;\mathbb{L})\to H_n(X;\mathbb{K}(\mathbb{Z},0))=H_n(X;\mathbb{Z})=\mathbb{Z}.$$

We assume that X is oriented. Therefore $I(X)$ is also defined for Poincaré complexes, as long as we have a degree-one normal map $f\colon M\to X$, determining an element in $H_n(X;\mathbb{L})$. In this case $I(X)$ is not a local index. In fact, for generalized manifolds one has local $\mathbb{L}$–Poincaré duality using locally finite chains, hence we can define $I(\mathcal{U})$ for any open set $\mathcal{U}\subset X$. It is also easy to see that $I(\mathcal{U})=I(X)$. On the algebraic side $I(X)$ is an invariant of the controlled Poincaré duality type (see Ranicki [11, p283]).

2 Constructing generalized manifolds from controlled sequences of Poincaré complexes

Beginning with a closed topological n–manifold M^n, for $n\geq 5$, and $\sigma\in H_n(M;\mathbb{L})$, we shall construct a sequence of closed Poincaré duality spaces $X_0,X_1,X_2,\ldots$, and maps $p_i\colon X_i\to X_{i-1}$ and $p_0\colon X_0\to M$.

We assume that M is a PL manifold, or that M has a cell structure. The X_i are built by gluing manifolds along boundaries with homotopy equivalences, and by doing some surgeries outside the singular sets. Hence all the X_i have cell decompositions.

We can assume that the X_i lie in a (large enough) Euclidean space $\mathbb{R}^L$ which induces the metric on X_i. So the cell chain complex $C_\#(X_i)$ can be considered as a geometric chain complex over X_{i-1} with respect to $p_i\colon X_i \to X_{i-1}$, ie the distance between two cells of X_i over X_{i-1} is the distance between the images of the centers of these two cells in X_{i-1}. Let us denote the distance function by d.

We now list five properties of the sequence $\{(X_i, p_i)\}_i$, including some definitions and comments. For each $i \geq 0$ we choose positive real numbers ξ_i and η_i.

(i) $p_i\colon X_i \to X_{i-1}$ and $p_0\colon X_0 \to M$ are $\mathcal{U}V^1$–maps. This means that for every $\varepsilon > 0$ and for all diagrams

$$\begin{tikzcd}
K_0 \arrow[r, "\alpha_0"] \arrow[d, hook] & X_i \arrow[d, "p_i"] \\
K \arrow[r, "\alpha"'] \arrow[ur, dashed, "\overline{\alpha}"] & X_{i-1}
\end{tikzcd}$$

with K a 2–complex, $K_0 \subset K$ a subcomplex and maps α_0, α, there is a map $\overline{\alpha}$ such that $\overline{\alpha}\big|_{K_0} = \alpha_0$ and $d(p_i \circ \overline{\alpha}, \alpha) < \varepsilon$. (This is also called $\mathcal{U}V^1(\varepsilon)$ property.)

(ii) X_i is an η_i–Poincaré complex over X_{i-1}, ie

  (a) all cells of X_{i-1} have diameter $< \eta_i$ over X_{i-1}; and

  (b) there is an n–cycle $c \in C_n(X_i)$ which induces an η_i–chain equivalence $\cap_c\colon C^\#(X_i) \to C_{n-\#}(X_i)$.

Equivalently, the diagonal $\Delta_\#(c) = \sum c' \otimes c'' \in C_\#(X) \otimes C_\#(X)$ has the property that $d(c', c'') < \eta_i$ for all tensor products appearing in $\Delta_\#(c)$.

(iii) $p_i\colon X_i \to X_{i-1}$ is an ξ_i–homotopy equivalence over X_{i-2}, for $i \geq 2$. In other words, there exist an inverse $p_i'\colon X_{i-1} \to X_i$ and homotopies $h_i\colon p_i' \circ p_i \simeq \mathrm{Id}_{X_i}$ and $h_i'\colon p_i \circ p_i' \simeq \mathrm{Id}_{X_{i-1}}$ such that the tracks

$$\big\{(p_{i-1} \circ p_i \circ h_i)(x, t) : t \in [0, 1]\big\} \quad \text{and} \quad \big\{(p_{i-1} \circ h_i')(x', t) : t \in [0, 1]\big\}$$

have diameter less than ξ_i, for each $x \in X_i$ (respectively, $x' \in X_{i-1}$). Note that p_0 need not be a homotopy equivalence.

(iv) There is a regular neighborhood $W_0 \subset \mathbb{R}^L$ of X_0 such that $X_i \subset W_0$, for $i = 0, 1, \ldots$, and retractions $r_i\colon W_0 \to X_i$, satisfying $d(r_i, r_{i-1}) < \xi_i$ in $\mathbb{R}^L$.

(v) There are "thin" regular neighborhoods $W_i \subset \mathbb{R}^L$ with $\pi_i\colon W_i \to X_i$, where $W_i \subset \mathring{W}_{i-1}$ such that $W_{i-1} \setminus \mathring{W}_i$ is an ξ_i–thin h–cobordism with respect to $r_i\colon W_0 \to X_i$.
Let $W = W_{i-1} \setminus \mathring{W}_i$. Then there exist deformation retractions $r_t^0\colon W \to \partial_0 W$ and $r_t^1\colon W \to \partial_1 W$ with tracks of size $< \xi_i$ over X_{i-1}, ie the diameters of

$\{(r_i \circ r_t^0)(w) : t \in [0,1]\}$ and $\{(r_i \circ r_t^1)(w) : t \in [0,1]\}$ are smaller than ξ_i. Moreover, we can choose η_i and ξ_i such that

(a) $\sum \eta_i < \underset{\circ}{\infty}$; and

(b) $W_{i-1} \setminus \mathring{W}_i$ has a δ_i–product structure with $\sum \delta_i < \infty$, ie there is a homeomorphism

$$W = W_{i-1} \setminus \mathring{W}_i \xleftarrow{H} \partial_0 W \times I$$

satisfying

$$\mathrm{diam}\{(r_i \circ H)(w,t) : t \in I\} < \delta_i,$$

for every $w \in \partial_0 W$.

The property (v)(b) above follows from the "thin h–cobordism" theorem (see the article [8] by Quinn). One can assume that $\sum \xi_i < \infty$. Let $X = \bigcap_i W_i$. We are going to show that X is a generalized manifold:

(1) The map $r = \varinjlim r_i \colon W_0 \to X$ is well–defined and is a retraction, hence X is an ANR.

(2) To show that X is a generalized manifold we shall apply the next two theorems. They also imply Theorem 1.1 above. The first one is due to Daverman and Husch [2], but it is already indicated in [8] (see the remark after Theorem 3.3.2).

Theorem 2.1 *Suppose that M^n is a closed topological n–manifold, B is an ANR, and $p\colon M \to B$ is proper and onto. Then B is a generalized manifold, provided that p is an approximate fibration.*

Approximate fibrations are characterized by the property that for every $\varepsilon > 0$ and every diagram

$$\begin{tikzcd}
K \times \{0\} \arrow[r, "H_0"] \arrow[d, hook] & M \arrow[d, "p"] \\
K \times I \arrow[r, "h"'] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

where K is a polyhedron, there exists a lifting H of h such that $d(p \circ H, h) < \varepsilon$. Here d is a metric on B. In other words, $p\colon M \to B$ has the ε–homotopy lifting property for all $\varepsilon > 0$.

We apply Theorem 2.1 to the map $\rho\colon \partial W_0 \to X$ defined as follows: Let $\rho\colon W_0 \to X$ be the map which associates to $w \in W_0$ the endpoint $\rho(x) \in X$ following the tracks defined by the "thin" product structures of the h–cobordism when decomposing

$$W_0 = (W_0 \setminus \mathring{W}_1) \cup (W_1 \setminus \mathring{W}_2) \cup \dots$$

The restriction to ∂W_0 will also be denoted by ρ. By (v)(b) above, the map ρ is well–defined and continuous. We will show that it is an ε–approximate fibration for all $\varepsilon > 0$.

The map $\rho\colon W_0 \to X$ is the limit of maps $\rho_i\colon W_0 \to X_i$, where ρ_i is the composition given by the tracks $(W_0 \setminus \overset{o}{W}_1) \cup (W_1 \setminus \overset{o}{W}_2) \cup \cdots \cup (W_{i-1} \setminus \overset{o}{W}_i)$ followed by $\pi_i\colon W_i \to X_i$. The second theorem is due to Bryant, Ferry, Mio and Weinberger [1, Proposition 4.5].

Theorem 2.2 *Given n and B, there exist $\varepsilon_0 > 0$ and $T > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ the following holds: If $X \xrightarrow{p} B$ is an ε–Poincaré complex with respect to the $\mathcal{U}V^1$–map p and $W \subset \mathbb{R}^L$ is a regular neighborhood of $X \subset \mathbb{R}^L$, ie $\pi\colon W \to X$ is a neighborhood retraction, then $\pi\big|_{\partial W}\colon \partial W \to X$ has the $T\varepsilon$–lifting property, provided that the codimension of X in $\mathbb{R}^L$ is ≥ 3.*

This is applied as follows: Let $B \subset \mathbb{R}^L$ be a (small) regular neighborhood of $X \subset \mathbb{R}^L$. Hence $X_k \subset W_k \subset B$ for sufficiently large k. It follows by property (ii) that X_i is an η_i–Poincaré complex over $X_i \xrightarrow{p_i} X_{i-1} \subset B$, hence (for i sufficiently large) we get the following:

Corollary 2.3 *$\rho_i\colon \partial W_0 \to X_i$ is a $T\eta_i$–approximate fibration over B.*

Proof By the theorem above, $\pi_i\colon \partial W_i \to X_i$ is a $T\eta_i$–approximate fibration over B, hence so is $\rho_i\colon \partial W_0 \cong \partial W_i \to X_i$. □

It follows by construction that $\lim_{\xleftarrow{p_i}} X_i = X \subset B$, and so we have, in the limit, an approximate fibration $\rho\colon \partial W_0 \to X$ over $\mathrm{Id}\colon X \to X$, ie X is a generalized manifold. We will show in Section 4 that $I(X)$ is determined by the $\mathbb{Z}$–sector of $\sigma \in H_n(M;\mathbb{L})$.

3 Construction of the sequence of controlled Poincaré complexes

Before we begin with the construction we need more fundamental results about controlled surgery and approximations.

3.1 ε–δ surgery theory

We recall the main theorem of the article [6] by Pedersen, Quinn and Ranicki. Let B be a finite–dimensional compact ANR, and N^n a compact n–manifold (possibly with nonempty boundary ∂N), where $n \geq 4$. Then there exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exist $\delta > 0$ with the following property:

If $p\colon N \to B$ is a $\mathcal{U}V^1(\delta)$ map, then there exists a controlled exact surgery sequence

(1) $$H_{n+1}(B;\mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(N,p) \to [N,\partial N; G/\mathrm{TOP},*] \xrightarrow{\Theta} H_n(B;\mathbb{L}).$$

The controlled structure set $\mathcal{S}_{\varepsilon,\delta}(N,p)$ is defined as follows. Elements of $\mathcal{S}_{\varepsilon,\delta}(N,p)$ are (equivalence) classes of (M,g), where M is an n–manifold, $g\colon M \to N$ is a δ–homotopy equivalence over B and $g\big|_{\partial M}\colon \partial M \to \partial N$ is a homeomorphism. The pair (M,g) is related to (M',g') if there is a homeomorphism $h\colon M \to M'$, such that the diagram

$$\begin{tikzcd}
\partial M \arrow[rr, "h"] \arrow[dr, "g"'] & & \partial M' \arrow[dl, "g'"] \\
& \partial N &
\end{tikzcd}$$

commutes, and $g' \circ h$ is ε–homotopic to g over B. Since ε is fixed, this relation is not transitive. It is part of the assertion that it is actually an equivalence relation. Then $\mathcal{S}_{\varepsilon,\delta}(N,p)$ is the set of equivalence classes of pairs (M,g).

As in the classical surgery theory, the map

(2) $$H_{n+1}(B;\mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(N,p)$$

is the controlled realization of surgery obstructions, and

(3) $$\mathcal{S}_{\varepsilon,\delta}(N,p) \to [N,\partial N; G/\mathrm{TOP},*] \overset{\Theta}{\to} H_n(B;\mathbb{L})$$

is the actual (controlled) surgery part. The following discussion will show that (3) also holds for controlled Poincaré spaces (see Theorem 3.1 below). Moreover, δ is also of (arbitrary) small size, provided that such is also ε.

To see this we will go through some of the main points of the proof of [6, Theorem 1]. For $\eta, \eta' > 0$ we denote by $L_n(B,\mathbb{Z},\eta,\eta')$ the set of highly η–connected n–dimensional quadratic Poincaré complexes modulo highly η'–connected algebraic cobordisms. Then there is a well–defined obstruction map

$$\Theta_\eta\colon [N, G/\mathrm{TOP}] \to L_n(B,\mathbb{Z},\eta,\eta')$$

(for simplicity we shall assume that $\partial N = \varnothing$). If $(f,b)\colon M^n \to N^n$ is a degree-one normal map one can do controlled surgery to obtain a highly η–connected normal map $(f',b')\colon M'^n \to N^n$ over B. If N^n is a manifold this can be done for every $\eta > 0$. If N^n is a Poincaré complex, it has to be η–controlled over B. By Theorem 1.1 above, this holds in particular for generalized manifolds.

Given $\eta > 0$ there is an $\eta' > 0$ such that if (f',b') and (f'',b'') are normally bordant, highly η–connected, degree-one, normal maps, there is then a highly η'–connected normal bordism between them. (Again this is true if N is an η–Poincaré complex over B.) This defines Θ_η.

To eventually complete surgeries in the middle dimension we assume that the map $p\colon N \to B$ is $\mathcal{U}V^1$. Then one has the following (see [6, p243]). Given $\delta > 0$ there exists $\eta > 0$ such that if $\Theta_\eta\big([f',b']\big) = 0$, then (f',b') is normally cobordant to a δ–homotopy equivalence. Moreover, if (f'',b'') and (f',b') are highly η–connected degree-one normal maps being normally cobordant, then there is a highly connected η'–bordism between them (ie for given η there is such an η'). Then controlled surgery produces a controlled h–cobordism which gives an ε–homotopy by the thin h–cobordism theorem. This defines an element of $\mathcal{S}_{\varepsilon,\delta}(N,p)$, and shows the semi–exactness of the sequence

$$\mathcal{S}_{\varepsilon,\delta}(N,p) \to [N, G/\mathrm{TOP}] \xrightarrow{\Theta_\eta} L_n(B,\mathbb{Z},\eta,\eta'), \tag{4}$$

ie that $\mathcal{S}_{\varepsilon,\delta}(N,p)$ maps onto the kernel of Θ_η. We note that semi–exactness also holds for η–controlled Poincaré complexes over B.

One cannot expect the sequence (4) to be exact, ie that the composition map is zero, since passing from topology to algebra one loses control. As it was noted by Pedersen, Quinn and Ranicki [6, p243], ε and δ are determined by the controlled Hurewicz and Whitehead theorems. Exactness of (4) will follow by the Squeezing Lemma of Pedersen and Yamasaki [7, Lemma 4].

The proof of (3) will be completed by showing that the assembly map

$$A\colon H_n(B;\mathbb{L}) \to L_n(B,\mathbb{Z},\eta,\eta')$$

is bijective for sufficiently small η. This follows by splitting the controlled quadratic Poincaré complexes (ie the elements of $L_n(B',\mathbb{Z},\eta,\eta')$) into small pieces over small simplices of B (we assume for simplicity that B is triangulated). If δ is given, and if we want a splitting where each piece is δ–controlled, we must start the subdivision with a sufficiently small η–controlled quadratic Poincaré complex (see the following Remark). This can be done by [6, Lemma 6] (see also Yamasaki [12, Lemma 2.5]). Since $A \circ \Theta = \Theta_\eta$, we get (3) from (4). The stability constant ε_0 is determined by the largest η for which A is bijective.

Remark Yamasaki has estimated the size of η in the Splitting Lemma. If one performs a splitting so that the two summands are δ–controlled, then one needs an η–controlled algebraic quadratic Poincaré complex with η of size $\delta/(an^k+b)$, where a, b, k depend on X (k is conjectured to be 1), and n is the length of the complex. Of course, squeezing also follows from the bijectivity of A for small η, but the result [7, Lemma 3] of Pedersen and Yamasaki is somehow a clean statement to apply (see Theorem 3.1 below). We also note that the bijectivity of A is of course, independent of whether N is a manifold or a Poincaré complex.

Theorem 3.1 *Suppose that* $N \xrightarrow{p} B$ *is a* $\mathcal{U}V^1$ *map. Let* $\delta > 0$ *be given (sufficiently small, ie* $\delta < \delta_0$ *for some* δ_0*). Then there is* $\eta > 0$ *(small with respect to* δ*), such that if* N *is an* η*–Poincaré complex over* B*, and* $(f, b)\colon M \to N$ *is a degree-one normal map, then* $\Theta(f, b) = 0 \in H_n(B; \mathbb{L})$ *if (and only if)* (f, b) *is normally bordant to a* δ*–equivalence.*

The "only if" part is more delicate and follows by [7, Lemma 3]. So let $f\colon M^n \to N^n$ be a δ–equivalence defining a quadratic η_1–Poincaré complex C in $L_n(B, \mathbb{Z}, \eta_1, \eta_1')$ which is η_1–cobordant to zero via $[N, G/\mathrm{TOP}] \to L_n(B, \mathbb{Z}, \eta_1, \eta_1')$.

Then C is $\kappa\eta_1$–cobordant to an arbitrary small quadratic Poincaré complex (ie to a quadratic η–complex) which is $\kappa\eta_1'$–cobordant to zero, with η_1 sufficiently small (ie η sufficiently small). In this case we can also assume that A is bijective. This proves the "only if" part.

Theorem 3.1 can also be stated as follows:

Theorem 3.1′ *Let* N *be a sufficiently fine* η*–Poincaré complex over a* $\mathcal{U}V^1$*–map* $p\colon N \to B$*. Then there exist* $\varepsilon > 0$ *and* $\delta > 0$*, both sufficiently small, such that the sequence*

$$\mathcal{S}_{\varepsilon,\delta}(N, p) \to [N, G/TOP] \to H_n(B; \mathbb{L})$$

is exact. In particular, it holds for generalized manifolds.

3.2 $\mathcal{U}V^1$ approximation

Here we recall the results [1, Proposition 4.3, Theorem 4.4] of Bryant, Ferry, Mio and Weinberger.

Theorem 3.2 *Suppose that* $f\colon (M^n, \partial M) \to B$ *is a continuous map from a compact* n*–manifold with boundary such that the homotopy fiber of* f *is simply connected. If* $n \geq 5$ *then* f *is homotopic to a* $\mathcal{U}V^1$*–map. In case that* $f|_{\partial M}$ *is already* $\mathcal{U}V^1$*, the homotopy is relative* ∂M*.*

We state the second theorem in the form which we will need.

Theorem 3.3 (Ferry [3, Theorem 10.1]) *Let $p\colon N^n \to B$ be a map from a compact n–manifold into a polyhedron, where $n \geq 5$. Then:*

- (i) *Given $\varepsilon > 0$, there is a $\delta > 0$, such that if p is a $\mathcal{U}V^1(\delta)$–map then p is ε–homotopic to a $\mathcal{U}V^1$–map.*
- (ii) *Suppose that $p\colon N \to B$ is a $\mathcal{U}V^1$ map. Then for each $\varepsilon > 0$ there is a $\delta > 0$ (depending on p and ε) such that if $f\colon M \to N$ is a $(\delta{-}1)$–connected map (over B) from a compact manifold M of dimension at least 5, then f is ε–close over B to a $\mathcal{U}V^1$–map $g\colon M \to N$.*

3.3 Controlled gluing

Theorem 3.4 (Bryant–Ferry–Mio–Weinberger [1, Proposition 4.6]) *Let $(M_1, \partial M_1)$ and $(M_2, \partial M_2)$ be (orientable) manifolds and $p_i\colon M_i \to B$ be $\mathcal{U}V^1$–maps. Then there exist $\varepsilon_0 > 0$ and $T > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$ and $h\colon \partial M_1 \to \partial M_2$ an (orientation preserving) ε–equivalence, $M_1 \cup_h M_2$ is a $T\varepsilon$–Poincaré complex over B.*

3.4 Approximation of retractions

Theorem 3.5 (Bryant–Ferry–Mio–Weinberger [1, Proposition 4.10]) *Let X and Y be finite polyhedra. Suppose that V is a regular neighborhood of X with $\dim V \geq 2\dim Y + 1$ and $r\colon V \to X$ is a retraction. If $f\colon Y \to X$ is an ε–equivalence with respect to $p\colon X \to B$, then there exists an embedding $i\colon Y \to V$ and a retraction $s\colon V \to i(Y)$ with $d(p\circ r, p\circ s) < 2\varepsilon$.*

We now begin with the construction. Let M^n be a closed oriented (topological) manifold of dimension $n \geq 6$. Let $\sigma \in H_n(M;\mathbb{L})$ be fixed. Moreover, we assume that M is equipped with a simplicial structure. Then let $M = B \cup_D C$ be such that B is a regular neighborhood of the 2–skeleton, $D = \partial B$ is its boundary and C is the closure of the complement of B. So $D = \partial C = B \cap C$ is of dimension ≥ 5.

By Theorem 3.2 above we can replace $(B, D) \subset M$ and $(C, D) \subset M$, by $\mathcal{U}V^1$–maps $j\colon (B, D) \to M$ and $j\colon (C, D) \to M$, and realize σ according to $H_n(M;\mathbb{L}) \to \mathcal{S}_{\varepsilon,\delta}(D, j)$ by a degree-one normal map $F_\sigma\colon V \to D \times I$ with $\partial_0 V = D$, $\partial_1 V = D'$, $F_\sigma\big|_{\partial_0 V} = \mathrm{Id}$ and $f_\sigma{=}F_\sigma\big|_{\partial_1 V}\colon D' \to D$ a δ–equivalence over M.

We then define $X_0 = B \cup_{f_\sigma} -V \cup_{\mathrm{Id}} C$, where $-V$ is the cobordism V turned upside down. We use the map $-F_\sigma \cup \mathrm{Id}\colon -V \cup_{\mathrm{Id}} C \to D \times I \cup C \cong C$ to extend j to a map $p_0\colon X_0 \to M$.

The Wall realization $V \to D \times I$ is such that V is a cobordism built from D by adding high–dimensional handles (similarly beginning with D'). Therefore p_0 is a $\mathcal{U}V^1$ map: If (K, L) is a simplicial pair with K a 2–complex, and if there is given a diagram

$$\begin{tikzcd}
L \arrow[r, "\alpha_0"] \arrow[d, hook] & X_0 \arrow[d, "p_0"] \\
K \arrow[r, "\alpha"'] & M
\end{tikzcd}$$

then we first move (by an arbitrary small approximation) α and α_0 into B by general position arguments. Then one uses the $\mathcal{U}V^1$–property of $j\colon B \to M$. By Theorem 3.4, X_0 is a $T\delta$–Poincaré complex over M. Note that we can choose δ as small as we want, hence we get an η_0–Poincaré complex for a prescribed η_0. This completes the first step.

To continue we define a manifold M_0^n and a degree-one normal map $g_0\colon M_0^n \to X_0$ by

$$M_0 = B \cup_{\mathrm{Id}} V \cup_{\mathrm{Id}} -V \cup_{\mathrm{Id}} C \to B \cup_{\mathrm{Id}} D \times I \cup_{f_\sigma} -V \cup_{\mathrm{Id}} C \cong X_0,$$

using $F_\sigma \cup \mathrm{Id}\colon V \cup_{\mathrm{Id}} -V \to D \times I \cup_{f_\sigma} -V$. By construction it has a controlled surgery obstruction $\sigma \in H_n(M; \mathbb{L})$.

Moreover, there is $\overline{\sigma} \in H_n(X_0; \mathbb{L})$ with $p_{0*}(\overline{\sigma}) = \sigma$. This can be seen from the diagram

$$\begin{tikzcd}
H_n(M_0; \mathbb{L}) \arrow[r, "g_{0*}"] & H_n(X_0; \mathbb{L}) \arrow[r, "p_{0*}"] & H_n(M; \mathbb{L}) \\
H^0(M_0; \mathbb{L}) \arrow[u, "\cong"] & H^0(X_0; \mathbb{L}) \arrow[l, "g_0^*"] & H^0(M; \mathbb{L}) \arrow[l, "p_0^*"] \arrow[u, "\cong"']
\end{tikzcd}$$

The vertical isomorphisms are Poincaré dualities. Since p_0 is a $\mathcal{U}V^1$ map, $\overline{\sigma}$ belongs to the same $\mathbb{Z}$–sector as σ. We will again denote $\overline{\sigma}$ by σ.

We construct $p_1\colon X_1 \to X_0$ as above: Let $M_0 = B_1 \cup_{D_1} C_1$, let B_1 be a regular neighborhood of the 2–skeleton (as fine as we want), let C_1 be the closure of the complement and let $D_1 = C_1 \cap B_1 = \partial C_1 = \partial B_1$, and $g_0\colon D_1 \to X_0$ be a $\mathcal{U}V^1$ map. Then we realize $\sigma \in H_n(X_0; \mathbb{L}) \to \mathcal{S}_{\varepsilon_1,\delta_1}(D_1, g_0)$ by $F_{1,\sigma}\colon V_1 \to D_1 \times I$ with $\partial_0 V_1 = D_1$, $\partial_1 V_1 = D_1'$, $F_{1,\sigma}\big|_{\partial_0 V_1} = \mathrm{Id}$ and $f_{1,\sigma} = F_{1,\sigma}\big|_{\partial_1 V_1}\colon D_1' \to D_1$ a δ_1–equivalence over X_0.

We define $p_1'\colon X_1' \to X_0$ by

$$X_1' = B_1 \underset{f_{1,\sigma}}{\cup} -V_1 \underset{\mathrm{Id}}{\cup} C_1 \xrightarrow{f_1'} M_0 \cong B_1 \underset{\mathrm{Id}}{\cup} D_1 \times I \underset{\mathrm{Id}}{\cup} C_1,$$

using $-F_{1,\sigma}\colon -V_1 \to D_1 \times I$, and then $p_1' = g_0 \circ f_1'\colon X_1' \to M_0 \to X_0$.

We now observe that

(i) by Theorem 3.4, X_1' is a $T_1\delta_1$–Poincaré complex over X_0; and

(ii) p_1' is a degree-one normal map with controlled surgery obstruction

$$-p_{0*}(\overline{\sigma}) + \sigma = 0 \in H_n(M;\mathbb{L}).$$

Let $\xi_1 > 0$ be given. We now apply Theorem 3.1 to produce a ξ_1–homotopy equivalence by surgeries outside the singular set (note that the surgeries which have to be done are in the manifold part of X_1'). For this we need a sufficiently small η_0–Poincaré structure on X_0. However, this can be achieved as noted above. This finishes the second step.

We now proceed by induction. What we need for the third step in order to produce $p_2\colon X_2 \to X_1$ is

(i) a degree-one normal map $g_1\colon M_1 \to X_1$ with controlled surgery obstruction $\sigma \in H_n(X_0;\mathbb{L})$; and

(ii) $\overline{\sigma} \in H_n(X_1;\mathbb{L})$ with $p_{1*}(\overline{\sigma}) = \sigma$, in the same $\mathbb{Z}$–sector as $\sigma \in H_n(X_0;\mathbb{L})$.

One can get $g_1\colon M_1 \to X_1$ as follows: Consider $g_1'\colon M_1' \to X_1'$, where

$$M_1' = B_1 \cup_{\mathrm{Id}} V_1 \cup_{\mathrm{Id}} -V_1 \cup_{\mathrm{Id}} C_1 \to B_1 \cup_{\mathrm{Id}} D_1 \times I \cup_{f_{1,\sigma}} -V_1 \cup_{\mathrm{Id}} C_1 \cong X_1'$$

is induced by $F_{1,\sigma}\colon V_1 \to D_1 \times I$ and the identity. The map g_1' is a degree-one normal map. Then one performs the same surgeries on g_1' as one has performed on $p_1'\colon X_1' \to X_0$ to obtain X_1. This produces the desired g_1. For (ii) we note that p_{1*} is a bijective map preserving the $\mathbb{Z}$–sectors (since p_1 is $\mathcal{U}V^1$).

So we have obtained the sequence of controlled Poincaré spaces $p_i\colon X_i \to X_{i-1}$ and $p_0\colon X_0 \to M$ with degree-one normal maps $g_i\colon M_i \to X_i$ and controlled surgery obstructions $\sigma \in H_n(X_{i-1};\mathbb{L})$. The properties (iv) and (v) of Section 2 now follow by the thin h–cobordism theorem and approximation of retraction.

4 Nonresolvability, the DDP property and existence of generalized manifolds

4.1 Nonresolvability

At the beginning of the construction we have $\sigma \in H_n(M;\mathbb{L})$, where M is a closed (oriented) n–manifold with $n \geq 6$. For each m we constructed degree-one normal maps $g_m\colon M_m \to X_m$ over $p_m\colon X_m \to X_{m-1}$, with controlled surgery obstructions $\sigma_m \in H_n(X_{m-1};\mathbb{L})$, $p_{0*}(\sigma_1) = \sigma$, $p_{m*}(\sigma_{m+1}) = \sigma_m$, and all σ_m belong to the same $\mathbb{Z}$–sector as σ. So we will call all of them σ.

We consider the normal map $g_m\colon M_m \to X_m$ as a controlled normal map over the identity map Id: $X_m \to X_m$, and over $q_m\colon X_m \subset W_m \xrightarrow{\rho} X$ (see Section 2). Since $\rho\big|_{\partial W_m}$ is an approximate fibration and $d(r_i, r_{i-1}) < \xi_i$ and $\sum_{i=m+1}^{\infty} \xi_i < \varepsilon$, for large m, we can assume that q_m is $\mathcal{U}V^1(\delta)$ for large m, so $(q_m)_*\colon H_n(X_m;\mathbb{L}) \to H_n(X;\mathbb{L})$ maps σ to $(q_m)_*(\sigma) = \sigma'$, being in the same $\mathbb{Z}$–sector as σ. The map $(q_m)_*$ is a bijective, and we denote σ' by σ. In other words, we have a surgery problem

$$\begin{tikzcd}
M_m \arrow[r, "g_m"] & X_m \arrow[d, "q_m"] \\
& X
\end{tikzcd}$$

over X, with controlled surgery obstruction $\sigma \in H_n(X;\mathbb{L})$. Our goal is to consider the surgery problem

$$\begin{tikzcd}
M_m \arrow[r, "q_m \circ g_m"] & X_m \arrow[d, "\mathrm{Id}"] \\
& X
\end{tikzcd}$$

over Id: $X \to X$, and prove that $\sigma \in H_n(X;\mathbb{L})$ is its controlled surgery obstruction.

Observe that q_m is a δ–homotopy equivalence over Id: $X \to X$ if m is sufficiently large (for a given δ).

Let $\mathcal{N}(X) \cong [X, G/\mathrm{TOP}]$ be the normal cobordism classes of degree-one normal maps of X, and let $HE_\delta(X)$ be the set of δ–homotopy equivalences of X over Id: $X \to X$. Our claim will follow from the following lemma.

Lemma 4.1 *Let $HE_{\delta'}(X) \times \mathcal{N}(X) \xrightarrow{\mu} \mathcal{N}(X)$ be the action map, ie $\mu(h, f) = h \circ f$. Then for sufficiently small $\delta' > 0$, the diagram*

$$\begin{tikzcd}
HE_{\delta'}(X) \times \mathcal{N}(X) \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}"'] & \mathcal{N}(X) \arrow[d, "\Theta"] \\
\mathcal{N}(X) \arrow[r, "\Theta"'] & H_n(X;\mathbb{L})
\end{tikzcd}$$

commutes.

Proof This follows from Theorem 3.1′ since $HE_{\delta'}(X)\times\mathcal{S}_{\varepsilon'',\delta''}(X,\mathrm{Id})\to\mathcal{S}_{\varepsilon,\delta}(X,\mathrm{Id})$ for sufficiently small δ' and δ''. □

We apply this lemma to the map $HE_\delta(X_m,X)\times\mathcal{N}(X_m)\to\mathcal{N}(X)$, which sends (h,g) to $h\circ g$, where $HE_\delta(X_m,X)$ are the δ–homotopy equivalences $X_m\to X$ over Id_X. Let $\psi_m\colon X\to X_m$ be a controlled inverse of q_m. Then ψ_m induces

$$\psi_{m*}\colon HE_\varepsilon(X_m,X)\to HE_\delta(X),$$

where δ is some multiple of ε. One can then write the following commutative diagram (for sufficiently small δ).

$$\begin{tikzcd}
HE_\varepsilon(X_m,X)\times H_n(X_m;\mathbb{L}) \arrow[drr] & & \\
HE_\varepsilon(X_m,X)\times\mathcal{N}(X_m) \arrow[u, "\mathrm{Id}\times\Theta"] \arrow[r] \arrow[d, "(\psi_m)_*\times(q_m)_*"'] & \mathcal{N}(X) \arrow[r, "\Theta"'] & H_n(X;\mathbb{L}) \\
HE_\delta(X)\times\mathcal{N}(X) \arrow[r, "\mathrm{pr}"'] \arrow[ur, "\mu"'] & \mathcal{N}(X) \arrow[ur, "\Theta"'] &
\end{tikzcd}$$

with $HE_\varepsilon(X_m,X)\times\mathcal{N}(X_m)\to H_n(X;\mathbb{L})$ given by $(h,\tau)\to h_*(\tau)$.

It follows from this that for large enough m, $q_m\circ g_m\colon M_m\to X$ has controlled surgery obstruction $\sigma\in H_n(X;\mathbb{L})$. Hence we get non–resolvable generalized manifolds if the $\mathbb{Z}$–sector of σ is $\neq 1$.

4.2 The DDP Property

The construction allows one to get the DDP property for X (see [1, Section 8]). Roughly speaking, this can be seen as follows. The first step in the construction is to glue a highly connected cobordism V into a manifold M of dimension $n\geq 6$, in between the regular neighborhood of the 2–skeleton.

The result is a space which has the DDP. The other constructions are surgery on middle–dimensional spheres, which also preserves the DDP. But since we have to take the limit of the X_m's, one must do it more carefully (see [1, Definition 8.1]):

Definition 4.2 Given $\varepsilon>0$ and $\delta>0$, we say that a space Y has the (ε,δ)–DDP if for each pair of maps $f,g\colon D^2\to Y$ there exist maps $\overline{f},\overline{g}\colon D^2\to Y$ such that $d(\overline{f}(D^2),\overline{g}(D^2))>\delta$, $d(f,\overline{f})<\varepsilon$ and $d(g,\overline{g})<\varepsilon$.

Lemma 4.3 *$\{X_m\}$ have the (ε,δ)–DDP for some $\varepsilon>\delta>0$.*

Proof The manifolds M^n_m, for $n \geq 6$, have the (ε, δ)–DDP for all ε and δ. In fact, one can choose a sufficiently fine triangulation, such that any $f\colon D^2 \to M$ can be placed by arbitrary small moves into the 2–skeleton or into the dual $(n-3)$–skeleton. Then δ is the distance between these skeleta. The remarks above show that the X_m have the (ε, δ)–DDP for some ε and δ. □

It can then be shown that $X = \varprojlim X_i$ has the $(2\varepsilon, \delta/2)$–DDP (see [1, Proposition 8.4]).

4.3 Special cases

(i) Let M^n and $\sigma \in H_n(M;\mathbb{L})$ be given as above. The first case which can occur is that σ goes to zero under the assembly map $A\colon H_n(M;\mathbb{L}) \to L_n(\pi_1 M)$. Then we can do surgery on the normal maps $F_\sigma\colon V \to D \times I$, $F_{1,\sigma}\colon V_1 \to D_1 \times I$ and so on, to replace them by products. In this case the generalized manifold X is homotopy equivalent to M.

(ii) Suppose that A is injective (or is an isomorphism). Then X cannot be homotopy equivalent to any manifold, if the $\mathbb{Z}$–sector of σ is $\neq 1$. Suppose that $N^n \to X$ were a homotopy equivalence. It determines an element in $[X, G/\mathrm{TOP}]$ which must map to $(1,0) \in H_n(X;\mathbb{L})$, because its surgery obstruction in $L_n(\pi_1 X)$ is zero and A is injective. This contradicts our assumption that the index of X is not equal to 1. Examples of this type are given by the n–torus $M^n = T^n$.

4.4 Acknowledgements

The second author was supported in part by the Ministry for Higher Education, Science and Technology of the Republic of Slovenia research grant. We thank M Yamasaki for helpful insights and the referee for the comments.

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Received: 7 July 2005