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**FOLIATIONS ON SUPERMANIFOLDS
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FOLIATIONS ON SUPERMANIFOLDS AND CHARACTERISTIC CLASSES

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We construct secondary characteristic classes of regular superfoliations on smooth supermanifolds. We interpret these secondary characteristic classes as characteristic classes of flat foliated connections.

1. Introduction

Smooth supermanifolds are becoming an increasingly important subject in mathematical physics [Leites 1980]; superfoliations should then be a central object. In this paper, we construct secondary characteristic classes of regular superfoliations with trivialized normal bundle, in the spirit of [Bernstein and Rosenfeld 1973; Bott and Haefliger 1972; Fuchs 1986]. We show that the role of $\text{Vect}(n)$ (the Lie algebra of formal vector fields with n variables) from the classical theory is played by the even part $\text{Vect}(n, m)_0$ of the super-Lie algebra $\text{Vect}(n, m)$ of formal supervector fields with n even variables and m odd variables. More precisely, when on a supermanifold \mathcal{M} we are given a superfoliation \mathcal{F} of codimension $n + \varepsilon m$ with trivialized normal bundle, we will associate to any element H in the Chevalley–Eilenberg cohomology $H^*(\text{Vect}(n, m)_0)$ a secondary characteristic class $\varphi_{\mathcal{M}, \mathcal{F}}(H) \in H^*(M)$, where M is the base manifold of the supermanifold \mathcal{M} .

Theorem A. *For any supermanifold \mathcal{M} foliated by a superfoliation \mathcal{F} of codimension $n + \varepsilon m$ with trivialized normal bundle, there exists a natural homomorphism $\varphi_{\mathcal{M}, \mathcal{F}}$ from $H^*(\text{Vect}(n, m)_0)$ to $H^*(M)$ such that:*

- (1) *If $\Phi : \mathcal{N} \rightarrow \mathcal{M}$ is a submersion of supermanifolds and $\varphi : N \rightarrow M$ is the map induced by Φ on their base manifolds M and N , then*

$$\varphi_{\mathcal{N}, \Phi^*\mathcal{F}} = \varphi^* \circ \varphi_{\mathcal{M}, \mathcal{F}},$$

where $\Phi^\mathcal{F}$ is the pull-back of the superfoliation \mathcal{F} via Φ and $\varphi^* : H^*(M) \rightarrow H^*(N)$ is the pull-back through φ .*

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- (2) If \mathcal{M} is an ordinary smooth manifold endowed with a foliation F of codimension n with trivialized normal bundle, then φ_F reduces to the well-known homomorphism of Bernstein, Bott, Fuchs, Haefliger and Rosenfeld [Bernstein and Rosenfeld 1973; Bott and Haefliger 1972; Fuchs 1973].

The reader should not be surprised that the de Rham cohomology of \mathcal{M} is not used: by a result of Batchelor [Leites 1980], this cohomology is indeed isomorphic to the cohomology of the base manifold. Note that the even part $\text{Vect}(n, m)_0$ appears instead of $\text{Vect}(n, m)$ itself. The theory is indeed more interesting like this, since, for example, $H^*(\text{Vect}(n, m))$ has only two nontrivial generators for $n < m$ (see [Fuchs 1973]). In [Koszul 1988], Godbillon–Vey classes for superfoliations of codimension $0 + \varepsilon m$ are constructed on supermanifolds of superdimension $n + \varepsilon m$ as maps from $H^*(\text{Vect}(0, m))$ to $H^*(\mathcal{M})$. We will see that these classes vanish for any class $H \in H^*(\text{Vect}(0, m))$ whose restriction to the cohomology $H^*(\text{Vect}(0, m)_0)$ of the even part is 0. Furthermore, we will show in Section 4D that the classes constructed in [Koszul 1988] are among those built in this article.

Given a vector bundle $E \rightarrow M$ and a foliation F , a *foliated connection* means a connection on E defined (smoothly) over each leaf of the foliation F . A foliated connection is said to be *flat* if it is flat on each leaf, and *trivial* if E is a trivial vector bundle and F a foliation with trivialized normal bundle. For simplicity, we say *flat trivial foliated connection* to name the entire collection of a foliation with trivialized normal bundle, a vector bundle, and a flat foliated connection. There is a canonical way to construct, from a flat foliated connection, a superfoliation with trivialized normal bundle on a supermanifold. Therefore, our theory of secondary characteristic classes of superfoliations also gives a theory of secondary characteristic classes of flat foliated connections.

Furthermore, to a superfoliation \mathcal{F} of codimension $n + \varepsilon m$ with trivialized normal bundle on a supermanifold \mathcal{M} we shall associate a flat trivial foliated connection, that is, a foliation $F_{\mathcal{F}}$ of codimension n with trivialized normal bundle on the base manifold M , a trivial vector bundle $E_{\mathcal{F}}$ of dimension m over M , and a flat foliated connection $\nabla^{\mathcal{F}}$ of this bundle. It should be noted that this construction is not the inverse of the previous one. However, we shall see that our theory of secondary characteristic classes, because of our preceding construction, gives in fact a theory of secondary characteristic classes of flat foliated connections. More precisely, we show:

Theorem B. *The homomorphism $\varphi_{\mathcal{F}, \mathcal{M}} : H^*(\text{Vect}(n, m)_0) \rightarrow H^*(M)$ is completely determined by the flat trivial foliated connection $(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}})$.*

The paper is organized as follows. Section 2 is devoted to constructions and properties of superfoliations on supermanifolds. More precisely, in Section 2A

we introduce several useful maps; in Section 2B we recall the definition of a superfoliation; in Section 2C we exhibit relations between superfoliations and flat foliated connections; in Section 2D we show how to replace a superfoliation with a superfoliation with trivialized normal bundle.

Section 3 is devoted to the study of the Chevalley–Eilenberg cohomology of the Lie algebra $\text{Vect}(n, m)_0$. We introduce this cohomology in Section 3A, then present in Section 3B technical results that will play a fundamental role in the construction of secondary characteristic classes. We show in Section 3C how to compute the cohomology of $\text{Vect}(n, m)_0$ with the help of the Weil algebra.

We construct secondary characteristic classes of superfoliations with trivialized normal bundle in Section 4. Secondary characteristic classes of a superfoliation are defined in Section 4A by constructing a homomorphism of differential graded algebras from the complex $\wedge \text{Vect}(n, m)_0$ of the Lie algebra $\text{Vect}(n, m)_0$ to the de Rham complex $\Omega(M)$ of the manifold M . We give examples in Section 4B, and prove Theorem A (in fact, a more precise statement) in Section 4D.

We relate in Section 5A the previously constructed secondary characteristic classes to the secondary characteristic classes of the (ordinary) foliation $F_{\mathcal{F}}$ on the base manifold. We prove Theorem B in Section 5B.

Most result of this paper have been announced, but not proved, in [Laurent-Gengoux 2004].

2. Superfoliations

2A. The algebra of superdifferential forms. For any supervector space \mathcal{V} , denote by \mathcal{V}_0 its even part. For convenience, write DGA for “differential graded algebra”. For a DGA (A, d_A) , denote by $[a] \in H^*(A)$ the cohomology class of an element $a \in \text{Ker } d_A$. Throughout this article, \mathcal{M} is a smooth supermanifold of superdimension $p + \varepsilon q$ (p and q being nonnegative integers), and M is its p -dimensional base manifold. In the sequel, we say “dimension” for “superdimension”, and “codimension” for “superdimension”

We now recall some basic results about supermanifolds, and introduce some maps that we will need in the sequel.

Denote by $\mathcal{O}(\mathcal{M})$ the superalgebra of superfunctions on \mathcal{M} , and by $\mathcal{I}(\mathcal{M})$ the superalgebra of nilpotent superfunctions; denote by $\text{Re}(f)$ the smooth function on M defined by the canonical projection $\mathcal{O}(\mathcal{M}) \rightarrow \mathcal{O}(\mathcal{M})/\mathcal{I}(\mathcal{M}) \simeq \mathcal{C}^\infty(M, \mathbb{R})$.

Let $(\Omega(\mathcal{M}), \wedge)$ be the superalgebra of differential forms on \mathcal{M} . The notion of parity of a superdifferential form can be ambiguous; in this paper the parity is the one that endows the superalgebra $\Omega(\mathcal{M})$ with a structure of a supercommutative superalgebra. More precisely, in a local chart $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$, the 1-form dx_i is odd, the 1-form $d\theta_j$ is even, the 0-form x_i is even, and the 0-form θ_j is odd.

The parity is then extended by multiplication: for any $\omega_1, \omega_2 \in \Omega(\mathcal{M})$, the parity of $\omega_1 \wedge \omega_2$ is the sum of the parity of ω_1 with the parity of ω_2 (modulo 2). For example, $\theta_j dx_i$ is even, $x_i^2 d\theta_j$ is even, and $dx_i \wedge d\theta_j$ is odd. Note that, with this definition, a 1-form on an ordinary differential manifold is an odd element. Let $d_{\mathcal{M}} : \Omega^*(\mathcal{M}) \rightarrow \Omega^{*+1}(\mathcal{M})$ be the de Rham differential.

Let $E_{\mathcal{M}}$ be a q -dimensional vector bundle over M whose sheaf of sections is $\mathcal{F}(\mathcal{M})/\mathcal{F}(\mathcal{M})^2$. The supermanifold $\mathcal{M}(M, E)$ whose superfunctions are the sections of $\bigwedge^* E_{\mathcal{M}}$ is isomorphic to \mathcal{M} (as a supermanifold, see [Leites 1980]). A local chart $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$ is said to be *attached* to $E_{\mathcal{M}}$ if the odd superfunctions $\theta_1, \dots, \theta_q$ are identified with sections s_1, \dots, s_q of $E_{\mathcal{M}}$. Note that in this case the supervector fields $\partial/\partial\theta_1, \dots, \partial/\partial\theta_q$ can be identified with sections of the dual vector bundle $E_{\mathcal{M}}^*$, and the 1-forms $d\theta_1, \dots, d\theta_q$ can be identified with sections of $E_{\mathcal{M}}$ again. More precisely, there is a natural bijection \mathcal{S} from the vector bundle whose sections are 1-forms in $\Omega^1(\mathcal{M})$ of the form $\sum_{j=1}^q f_j(x_1, \dots, x_p) d\theta_j$ to the vector bundle $E_{\mathcal{M}}$ given, in a local chart attached to $E_{\mathcal{M}}$, by

$$\mathcal{S}\left(\sum_{j=1}^q f_j(x_1, \dots, x_p) d\theta_j\right) = \sum_{j=1}^q f_j(x_1, \dots, x_p) s_j,$$

with $f_1(x_1, \dots, x_p), \dots, f_q(x_1, \dots, x_p)$ smooth functions on M . It is straightforward to check that \mathcal{S} does not depend on the local chart attached to $E_{\mathcal{M}}$.

We recall the definition of three maps:

- The first map is a well-known (see [Leites 1980; Tuynman 2004]) canonical DGA homomorphism p from $(\Omega^*(\mathcal{M}), \wedge, d_{\mathcal{M}})$ onto $(\Omega^*(M), \wedge, d_M)$, constructed in [Leites 1980]. In a local chart $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$, it is defined on the generators $dx_i, d\theta_j, f$ (f being a superfunction) of $\Omega(\mathcal{M})$ by

$$p(dx_i) = dx_i, \quad p(d\theta_i) = 0, \quad p(f) = \text{Re}(f).$$

A proof of the next lemma can be found in [Leites 1980; Tuynman 2004].

Lemma 2.1. *The map p from $(\Omega^*(\mathcal{M}), \wedge, d_{\mathcal{M}})$ onto $(\Omega^*(M), \wedge, d_M)$ is well defined (that is, independent of the chart) and a DGA homomorphism. The kernel of p contains odd $2k$ -forms and even $(2k+1)$ -forms.*

Since p is a DGA homomorphism, it induces a homomorphism \hat{p} from $H^*(\mathcal{M})$ to $H^*(M)$. According to a theorem of Batchelor (see [Tuynman 2004, Theorem 8.2]), $\hat{p} : H^*(\mathcal{M}) \rightarrow H^*(M)$ is indeed an isomorphism.

- The second map, ρ , is a map from $\Omega^1(\mathcal{M})$ to $\Gamma(E_{\mathcal{M}})$, the space of sections of $E_{\mathcal{M}}$. It is defined in a local chart by

$$(2-1) \quad \rho\left(\sum_{i=1}^p f_i dx_i + \sum_{j=1}^q g_j d\theta_j\right) = \mathcal{S}\left(\sum_{j=1}^q \text{Re}(g_j) d\theta_j\right) = \sum_{j=1}^q \text{Re}(g_j) s_j.$$

Lemma 2.2. *The map ρ is well defined (that is, it does not depend on the chart attached to $E_{\mathcal{M}}$). For any odd 1-form ω , we have $\rho(\omega) = 0$. For any nilpotent function $f \in \mathcal{F}(\mathcal{M})$ and any $\omega \in \Omega^1(M)$, we have $\rho(f\omega) = 0$.*

- The third map that we need, Π , is a map from the super-Lie algebra $\text{Vect}(\mathcal{M})$ of supervector fields on \mathcal{M} to the Lie algebra $\text{Vect}(M)$ of vector fields on M . For any odd supervector field $\mathcal{X} \in \text{Vect}(\mathcal{M})$, we simply set $\Pi(\mathcal{X}) = 0$. For any even supervector field $\mathcal{X} \in \text{Vect}(\mathcal{M})_0$, we set $\Pi(\mathcal{X}) = X$, where X is the derivation of $C^\infty(M, \mathbb{R}) \simeq \mathbb{C}(\mathcal{M})/\mathcal{F}(\mathcal{M})$ induced by the even derivation \mathcal{X} of $\mathbb{C}(\mathcal{M})$.

In a local chart attached to $E_{\mathcal{M}}$, Π is given by

$$\Pi\left(\sum_{i=1}^p f_i \frac{\partial}{\partial x_i} + \sum_{j=1}^q g_j \frac{\partial}{\partial x_j}\right) = \sum_{i=1}^p \text{Re}(f_i) \frac{\partial}{\partial x_i}.$$

Lemma 2.3. *The restriction of Π to the subalgebra $\text{Vect}(\mathcal{M})_0$ of even supervector fields is a Lie algebra homomorphism.*

The next properties can be checked locally in a chart; we leave the computations to the reader. Denote by $\iota_{\mathcal{Y}}\eta$ the contraction of a supervector field by a 1-form in $\Omega(\mathcal{M})$. For any 2-form $\eta \in \Omega^2(\mathcal{M})$ and even supervector field \mathcal{Y} such that $\Pi(\mathcal{Y}) = 0$,

$$(2-2) \quad \rho(\iota_{\mathcal{Y}}\eta) = 0.$$

For any even 1-form $\alpha \in \Omega^1(\mathcal{M})$ with $\rho(\alpha) = 0$ and any even supervector field \mathcal{Y} ,

$$(2-3) \quad \rho(\iota_{\mathcal{Y}}d\alpha) = 0.$$

For any even supervector field $\mathcal{X} \in \text{Vect}(\mathcal{M})_0$ and odd 1-form $\omega \in \Omega^1(\mathcal{M})$,

$$(2-4) \quad \text{Re}(\iota_{\mathcal{X}}\omega) = \iota_{\Pi(X)}p(\omega).$$

2B. Definition of a superfoliation. We recall the definition of a superfoliation of codimension $n + \varepsilon m$ [Leites 1980; Monterde et al. 1997; Tuynman 2004]. First, a *distribution* of codimension $n + \varepsilon m$ is a sub-supervector bundle of $T\mathcal{M}$ of dimension $(p - n) + \varepsilon(q - m)$.

Definition 2.4. A *superfoliation* \mathcal{F} of codimension $n + \varepsilon m$ is a distribution $\mathcal{D}_{\mathcal{F}}$ of dimension $(p - n) + \varepsilon(q - m)$ whose sections are closed under the bracket of supervector fields.

Remark 2.5. Note that in the literature the terminology is not fixed: in [Leites 1980] the name “foliation” is used, while in [Tuynman 2004] it is called “integrable distribution”. We choose “superfoliation” to avoid confusions when dealing simultaneously with superfoliations on a supermanifold and (ordinary) foliations on its base manifold.

A supermanifold is said to be *foliated* when it is endowed with a superfoliation of codimension $n + \varepsilon m$, for some $n, m \in \mathbb{N}$. A supervector field \mathcal{X} is said to be *tangent to the leaves* if and only if it is a section of the distribution that defines \mathcal{F} . We denote by $\mathcal{X}_{\mathcal{F}}$ the superalgebra of supervector fields tangent to the leaves of \mathcal{F} .

For a superfoliation \mathcal{F} , denote by $\Omega_{\mathcal{F}}^*$ the subalgebra of $\Omega(\mathcal{M})$ of k -forms ω such that $\iota_{\mathfrak{y}_1} \dots \iota_{\mathfrak{y}_k} \omega = 0$ for any supervector fields $\mathfrak{y}_1, \dots, \mathfrak{y}_k$ tangent to the leaves of \mathcal{F} . Of course,

$$(2-5) \quad d_{\mathcal{M}} \Omega_{\mathcal{F}}^* \subset \Omega_{\mathcal{F}}^*.$$

In particular, $\Omega_{\mathcal{F}}^1$ is the space of 1-forms $\omega \in \Omega^1(\mathcal{M})$ such that $\omega(\mathcal{X}) = 0$ for any supervector field \mathcal{X} tangent to the leaves.

A superfunction is said to be *constant on the leaves* if $\iota_{\mathcal{X}} d_{\mathcal{M}} f = 0$ for any supervector field \mathcal{X} tangent to the leaves. We denote by $\mathbb{C}_{\mathcal{F}}$ the superalgebra of superfunctions constant on the leaves of \mathcal{F} .

Here is a basic example: On the supermanifold $\mathbb{R}^{p,q}$, denote by x_1, \dots, x_p (respectively, $\theta_1, \dots, \theta_q$) the even (respectively, odd) variables. The *elementary superfoliation* $\mathbb{R}_{n,m}^{p,q}$ is the superfoliation given by the distribution generated by the supervector fields $\partial/\partial x_{n+1}, \dots, \partial/\partial x_p, \partial/\partial \theta_{m+1}, \dots, \partial/\partial \theta_q$.

An important particular case is when the superfoliation is defined by a distribution that admits a trivialized normal bundle.

Definition 2.6. A superfoliation \mathcal{F} of codimension $n + \varepsilon m$ is said to *have a trivialized normal bundle* if the distribution $\mathcal{D}_{\mathcal{F}}$ that defines \mathcal{F} is generated by a free family of n odd and m even 1-forms b_1, \dots, b_{n+m} ; that is,

$$\mathcal{D}_{\mathcal{F}} = \{ \mathcal{X} \in \text{Vect}(\mathcal{M}) \mid \iota_{\mathcal{X}} b_1 = \dots = \iota_{\mathcal{X}} b_{n+m} = 0 \}.$$

We say that the 1-forms b_1, \dots, b_{n+m} *define* the superfoliation \mathcal{F} .

Example 2.7. The elementary superfoliation $\mathbb{R}_{n,m}^{p,q}$ is defined by the odd 1-forms dx_1, \dots, dx_n together with the even 1-forms $d\theta_1, \dots, d\theta_m$.

If the superfoliation \mathcal{F} with trivialized normal bundle is defined by a free family b_1, \dots, b_{n+m} , then (see [Monterde et al. 1997]) there must exist 1-forms b_i^j , $i, j \in \{1, \dots, n+m\}$, satisfying

$$(2-6) \quad d_{\mathcal{M}} b_i = \sum_{j=1}^{n+m} b_j \wedge b_i^j.$$

Moreover, we can assume that the parity of b_i^j is the sum of the parity of b_i with the parity of b_j .

Remark 2.8. For any superfoliation \mathcal{F} with trivialized normal bundle, defined by 1-forms b_1, \dots, b_{n+m} (with b_1, \dots, b_n odd, and b_{n+1}, \dots, b_{n+m} even), there exist supervector fields X_1, \dots, X_{n+m} (with X_1, \dots, X_n even, and X_{n+1}, \dots, X_{n+m} odd) such that $b_i(X_j) = \delta_i^j$ (where δ_i^j is the Kronecker symbol). The family X_1, \dots, X_{n+m} is free (with respect to the $\mathbb{O}(\mathcal{M})$ -module structure). The supertangent bundle $T\mathcal{M}$ is the direct sum of the distribution generated by X_1, \dots, X_{n+m} with the distribution $\mathcal{D}_{\mathcal{F}}$ that defines \mathcal{F} . This justifies calling it “with trivialized normal bundle”.

We now define morphisms between foliated supermanifolds, and pull-backs of superfoliations.

Definition 2.9. Let \mathcal{M}_1 and \mathcal{M}_2 be supermanifolds, and let \mathcal{F}_1 and \mathcal{F}_2 be superfoliations of codimension $n + \varepsilon m$ on \mathcal{M}_1 and \mathcal{M}_2 respectively. A submersion $\Phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is said to be a *morphism of superfoliations* if and only if

$$(2-7) \quad \mathbb{O}(\mathcal{M}_1) \Phi^* \Omega_{\mathcal{F}_2}^1 = \Omega_{\mathcal{F}_1}^1,$$

where $\mathbb{O}(\mathcal{M}_1) \Phi^* \Omega_{\mathcal{F}_2}^1$ denotes the $\mathbb{O}(\mathcal{M}_1)$ -module generated by $\Phi^* \Omega_{\mathcal{F}_2}^1$

In other words, (2-7) means that any 1-form in $\Omega_{\mathcal{F}_1}^1$ is a linear combination, with coefficients in $\mathbb{O}(\mathcal{M}_1)$, of pull-backs by Φ of forms from $\Omega_{\mathcal{F}_2}^1$.

Note that, if \mathcal{F}_1 has a trivialized normal bundle, so has \mathcal{F}_2 . If b_1, \dots, b_{n+m} are the 1-forms defining \mathcal{F}_1 , then the superfoliation \mathcal{F}_2 is defined by the 1-forms $\Phi^* b_1, \dots, \Phi^* b_{n+m}$.

The *category of foliated supermanifolds* is the category whose objects are superfoliations and whose arrows are morphism of foliated manifolds. For any $n, m \in \mathbb{N}$, the superfoliations of codimension $n + \varepsilon m$ make up a subcategory, while the superfoliations of codimension $n + \varepsilon m$ with trivialized normal bundle form again a subcategory of the latter category.

An *isomorphism* of foliated manifolds is an invertible morphism of foliated manifolds.

The local structure of a superfoliation is always the same, as proved in [Hill and Simanca 1991] (see also [Tuynman 2004, Chapter V.4]):

Theorem 2.10 [Hill and Simanca 1991; Monterde et al. 1997]. *Any superfoliation of codimension $n + \varepsilon m$ on a supermanifold of dimension $p + \varepsilon q$ is locally isomorphic to the elementary superfoliation $\mathbb{R}_{n,m}^{p,q}$.*

Let \mathcal{N} and \mathcal{M} be supermanifolds, and let \mathcal{F} be a superfoliation of codimension $n + \varepsilon m$ on \mathcal{M} . Given any submersion $\Phi : \mathcal{N} \rightarrow \mathcal{M}$, the *pull-back* $\Phi^* \mathcal{F}$ is the unique superfoliation satisfying

$$\mathbb{O}(\mathcal{N}) \Phi^* \Omega_{\mathcal{F}}^1 = \Omega_{\Phi^* \mathcal{F}}^1.$$

This definition needs some justification. For any open set $U \subset M$, we denote by \mathcal{U} the supermanifold defined by $\mathcal{O}(M)|_U$ and we say that \mathcal{U} is an open subset of M . Let $r + \varepsilon s$, with $r \geq p$ and $s \geq q$, be the dimension of \mathcal{N} . By Theorem 2.10, in local coordinates $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$ the superfoliation \mathcal{F} restricted to \mathcal{U} is defined by $dx_1, \dots, dx_n, d\theta_1, \dots, d\theta_m$. Since Φ is a submersion, there are local coordinates $(x_1, \dots, x_r, \theta_1, \dots, \theta_s)$ on some open subset $\mathcal{V} \subset \mathcal{N}$ such that Φ is given by

$$\Phi(x_1, \dots, x_r, \theta_1, \dots, \theta_s) = (x_1, \dots, x_p, \theta_1, \dots, \theta_q).$$

The pull-back $\Phi^*\mathcal{F}$ of \mathcal{F} via Φ is the superfoliation whose restriction to \mathcal{V} is defined by the 1-forms $dx_1, \dots, dx_n, d\theta_1, \dots, d\theta_m \in \Omega^1(\mathcal{V})$. This justifies the existence and uniqueness of the pull-back of a superfoliation via a submersion.

2C. Geometric constructions associated to a superfoliation. Let N be an (ordinary) smooth manifold and F a foliation of codimension n on N , defined by a distribution D_F . Let $E \rightarrow N$ be a vector bundle over N , and let $X_F = \Gamma(D_F)$ be the algebra of vector fields tangent to F . A *foliated connection* is a bilinear map from $X_F \otimes \Gamma(E)$ to $\Gamma(E)$, denoted by $(X, s) \mapsto \nabla_X s$, that satisfies the usual axioms of a connection; that is, for all $X \in X_F$, $f \in C^\infty(M)$, $s \in \Gamma(E)$, we have

$$\nabla_{fX} s = f \nabla_X s \quad \text{and} \quad \nabla_X f s = f \nabla_X s + (X \cdot f) s.$$

A foliated connection is said to be *flat* if ∇ is flat on each leaf of F ; that is, if for all $X, Y \in X_F$, $s \in \Gamma(E)$ the identity

$$\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s = 0$$

holds. A foliated connection is said to be *trivial* if E is a trivial vector bundle on M and F is a foliation with trivialized normal bundle on M .

In the following, we will say simply “foliated connection” (respectively, “trivial flat foliated connection”) for the collection of a foliation F , a vector bundle E , and a foliated connection ∇ . We denote by (M, F, E, ∇) a foliated connection (respectively, a trivial flat foliated connection).

2C(a). From superfoliations to flat foliated connections. To a superfoliation on \mathcal{M} we associate a foliation $F_{\mathcal{F}}$ on the base manifold M , a vector bundle $E_{\mathcal{F}}$ on M , and a flat foliated connection $\nabla^{\mathcal{F}}$ on the latter. In the particular case of a superfoliation of codimension $0 + \varepsilon m$ on a supermanifold of dimension $n + \varepsilon m$, this construction is identical to the construction given in [Koszul 1988, Lemme 2.1].

Definition 2.11. Given a superfoliation \mathcal{F} of codimension $n + \varepsilon m$, we denote by $(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}})$ the flat foliated connection defined in what follows.

It is easy to check that $p(\Omega_{\mathbb{R}_n^p, q}^*) = \Omega_{\mathbb{R}_n^p}^*$, where by \mathbb{R}_n^p we mean the foliation $\mathbb{R}_{n,0}^{p,0}$ of codimension n on \mathbb{R}^p . In other words, \mathbb{R}_n^p is the foliation of codimension n on \mathbb{R}^p defined by the 1-forms dx_{n+1}, \dots, dx_p . Theorem 2.10 immediately implies:

Lemma 2.12. *The distribution whose sections are $X_{F_{\mathcal{F}}} = \Pi(\mathcal{X}_{\mathcal{F}}) \subset \Gamma(TM)$ defines a foliation $F_{\mathcal{F}}$ of codimension n on M . Moreover, $\Omega_{F_{\mathcal{F}}}^* = p(\Omega_{\mathcal{F}}^*)$.*

Remark 2.13. If \mathcal{F} is a superfoliation with trivialized normal bundle, defined by 1-forms $\omega_1, \dots, \omega_n, a_1, \dots, a_m$, then $F_{\mathcal{F}}$ is a foliation with trivialized normal bundle defined by the 1-forms $p(\omega_1), \dots, p(\omega_n) \in \Omega^1(M)$.

Now, we define $E_{\mathcal{F}}$ to be the vector subbundle of $E_{\mathcal{M}}$ whose sections $\Gamma(E_{\mathcal{F}})$ are the sections of $E_{\mathcal{M}}$ of the form $\rho(\omega)$ with $\omega \in \Omega_{\mathcal{F}}^1$, where $\Omega_{\mathcal{F}}^1$ is the space of 1-forms such that $\iota_{\mathcal{X}}\omega = 0$ for any supervector field $\mathcal{X} \in \mathcal{X}_{\mathcal{F}}$ tangent to the leaves of \mathcal{F} , and ρ was defined in (2-1). In the case of $\mathbb{R}_{n,m}^{p,q}$, $E_{\mathcal{F}}$ is generated by $\mathcal{S}(d_{\mathbb{R}^{p,q}}\theta_1), \dots, \mathcal{S}(d_{\mathbb{R}^{p,q}}\theta_m)$. Then, Theorem 2.10 implies that $E_{\mathcal{F}}$ is an m -dimensional vector bundle on M .

Remark 2.14. If \mathcal{F} is a superfoliation with trivialized normal bundle, defined by 1-forms $\omega_1, \dots, \omega_n, a_1, \dots, a_m$, then $E_{\mathcal{F}}$ is a trivial vector space, and a trivialization is given by the sections $\rho(a_1), \dots, \rho(a_m) \in \Gamma(E_{\mathcal{F}})$.

For any section s of $E_{\mathcal{F}}$, let $\omega \in \Omega_{\mathcal{F}}^1$ be an even element satisfying $\rho(\omega) = s$. For any vector field $Y \in X_{F_{\mathcal{F}}}$ tangent to the foliation $F_{\mathcal{F}}$, let $\mathcal{Y} \in (\mathcal{X}_{\mathcal{F}})_0$ be an even supervector field satisfying $\Pi(\mathcal{Y}) = Y$. We define a foliated connection by

$$(2-8) \quad \nabla_Y^{\mathcal{F}} s = \rho(L_{\mathcal{Y}}\omega).$$

Lemma 2.15. *The foliated connection $\nabla_Y^{\mathcal{F}} s$ is well defined and flat.*

Proof. Since $L_{\mathcal{Y}} = d_{\mathcal{M}}\iota_{\mathcal{Y}} + \iota_{\mathcal{Y}}d_{\mathcal{M}}$, the identity

$$\rho(L_{\mathcal{Y}}\omega) = \rho(\iota_{\mathcal{Y}}d_{\mathcal{M}}\omega)$$

holds. In order to check that $\nabla_Y^{\mathcal{F}} s = \rho(L_{\mathcal{Y}}\omega)$ is a well-defined section of $E_{\mathcal{F}}$, notice these facts:

- Let $\mathcal{X} \in \mathcal{X}_{\mathcal{F}}$ be a supervector field tangent to the leaves of \mathcal{F} such that $\Pi(\mathcal{X}) = 0$. Equation (2-2) implies that $\rho(L_{\mathcal{X}}\omega) = \rho(\iota_{\mathcal{X}}d_{\mathcal{M}}\omega)$ is equal to 0.
- Let $\alpha \in \Omega_{\mathcal{F}}^1$ be an even 1-form such that $\rho(\alpha) = 0$. Equation (2-3) implies that $\rho(L_Y\alpha) = 0$.

We now verify that indeed $\nabla^{\mathcal{F}}$ is a section of $E_{\mathcal{F}}$. If $\omega \in \Omega_{\mathcal{F}}^1$, then $L_{\mathcal{Y}}\omega$ is an element of $\Omega_{\mathcal{F}}^1$ as well, since

$$\iota_{\mathcal{X}}L_{\mathcal{Y}}\omega = \iota_{[\mathcal{X}, \mathcal{Y}]} \omega - \iota_{\mathcal{Y}}L_{\mathcal{X}}\omega = 0$$

for all $X \in \mathcal{X}_{\mathcal{F}}$. The section $\nabla_Y^{\mathcal{F}} s$ of $E_{\mathcal{M}}$ given by $\nabla_Y^{\mathcal{F}} s = \rho(L_{\mathcal{Y}}\omega)$ is thus a section of $E_{\mathcal{F}}$ again.

Now, we check that the curvature is zero. For this, we show that there exists a local trivialization of $E_{\mathcal{F}}$ by sections s_1, \dots, s_m that are horizontal (that is, $\nabla_Y^{\mathcal{F}} s_i = 0$

for all $Y \in X_{F_{\mathcal{F}}}$). For any $x \in M$, let U be a neighborhood of x and $\varphi_U : \mathcal{U} \rightarrow \mathbb{R}_{n,m}^{p,q}$ an isomorphism of foliated manifolds, where \mathcal{U} is the supermanifold over U defined by restricting \mathcal{M} to U . Since the superfunctions $\theta_1, \dots, \theta_m$ are constant on the leaves of $\mathbb{R}_{n,m}^{p,q}$, the odd superfunctions $f_1 = \varphi_U^* \theta_1, \dots, f_m = \varphi_U^* \theta_m \in \Omega(\mathcal{M})$ are constant on the leaves of \mathcal{F} . Moreover, the sections $s_1 = \rho(d_{\mathcal{M}} f_1), \dots, s_n = \rho(d_{\mathcal{M}} f_m)$ define a local trivialization of $E_{\mathcal{F}}$. For any vector field $Y \in X_{F_{\mathcal{F}}}$ and any even supervector field $\mathcal{Y} \in (\mathcal{X}_{\mathcal{F}})_0$ with $\Pi(\mathcal{Y}) = Y$, we have

$$\nabla_Y^{\mathcal{F}} s_i = \rho(L_{\mathcal{Y}} d_{\mathcal{M}} f_i) = \rho(d_{\mathcal{M}} t_{\mathcal{Y}} d_{\mathcal{M}} f_i) = \rho(0) = 0.$$

The existence of a local horizontal trivialization of $E_{\mathcal{F}}$ implies that the connection $\nabla^{\mathcal{F}}$ is flat. \square

We summarize:

Proposition 2.16. *Let \mathcal{F} be a superfoliation of codimension $n + \varepsilon m$.*

- (1) *$(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}})$ is a flat foliated connection.*
- (2) *If the superfoliation \mathcal{F} has a trivialized normal bundle, then the flat foliated connection $(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}})$ is trivial.*

Proof. Statement (1) is a consequence of Lemma 2.12, while (2) is a consequence of Remarks 2.13 and 2.14 \square

2C(b). *From flat foliated connections to superfoliations.* Let (N, F, E, ∇) be a flat foliated connection. Consider the supermanifold $\mathcal{M}(N, E)$ whose superfunctions are the sections of $\bigwedge E$. For any vector field $X \in \text{Vect}(N)$, ∇_X is an even derivation of the superalgebra of superfunctions $\Gamma(\bigwedge E)$. These even derivations can be considered as even vector fields of the supermanifold $\mathcal{M}(N, E)$. Moreover, since $[\nabla_X, \nabla_Y] = \nabla_{[X, Y]}$, these even derivations form a Lie algebra. The $\Gamma(\bigwedge E)$ -module generated by the supervector fields $\{\nabla_X \mid X \in X_F\}$ is a distribution that defines a superfoliation of codimension $n + \varepsilon m$ on the supermanifold $\mathcal{M}(N, E)$.

This provides a canonical way to associate a superfoliation of codimension $n + \varepsilon m$ to a manifold N , a foliation F of codimension n , a vector-bundle E of dimension m , and a flat foliated connection ∇ .

Definition 2.17. We denote by $\mathcal{F}(N, F, E, \nabla)$ the superfoliation of codimension $n + \varepsilon m$ on the supermanifold $\mathcal{M}(N, E)$, defined above.

A foliated connection (N, F, E, ∇) is said to be *trivial* if F has a trivialized normal bundle and the vector bundle E is trivial. The next proposition is straightforward and we leave it to the reader.

Proposition 2.18. (1) *If (N, F, E, ∇) is a trivial flat foliated connection, then the superfoliation $\mathcal{F}(N, F, E, \nabla)$ has a trivialized normal bundle.*

(2) For any foliated connection (M, F, E, ∇) , we have

$$(2-9) \quad (M, F, E, \nabla) = (M, F_{\mathcal{F}(M,F,E,\nabla)}, E_{\mathcal{F}(M,F,E,\nabla)}, \nabla^{\mathcal{F}(M,F,E,\nabla)}).$$

Remark 2.19. By (2-9), the construction of Section 2C(a) is the left inverse of the construction of Section 2C(b). However, it is not a right inverse. More precisely, we show now that the construction of Section 2C(a) is not injective.

Consider the supercircle $S^{1,3}$ defined by a trivial vector bundle $E_3 = S^1 \times \mathbb{R}^3 \rightarrow S^1$ of dimension 3 over S^1 . Denote by $x \in S^1$ the even parameter and by $\theta_1, \theta_2, \theta_3$ the odd parameters associated to three canonical sections s_1, s_2, s_3 of $E_3 = S^1 \times \mathbb{R}^3 \rightarrow S^1$.

Take the superfoliations \mathcal{F}_1 and \mathcal{F}_2 defined by the distributions of codimension $0 + \varepsilon 3$ generated by the distributions $\mathcal{O}(\mathcal{M}) \partial / \partial x$ and $\mathcal{O}(\mathcal{M}) (\partial / \partial x + \theta_1 \theta_2 \theta_3 \partial / \partial \theta_1)$. These two superfoliations define the same foliation on M , the foliation with only one leaf: S^1 itself. They define the same vector bundle: E_3 itself. They both also define the same connection, namely, the connection given by

$$\nabla_{a(x) \frac{\partial}{\partial x}} s = a(x) \frac{ds}{dx}$$

for all $a(x) \in C^\infty(S^1)$ and $s \in \Gamma(E_3)$. However, these superfoliations are not isomorphic, because their superalgebras of superfunctions constant on leaves are not isomorphic.

For the superfoliation \mathcal{F}_1 , the superfunctions $\theta_1, \theta_2, \theta_3$ are constant on leaves, and the algebra of superfunctions that are constant on the leaves of \mathcal{F}_1 is therefore isomorphic to $\bigwedge \mathbb{R}^3$. In particular, the vector space of odd superfunctions that are constant on the leaves of \mathcal{F}_1 has dimension 4.

For the superfoliation \mathcal{F}_2 , any odd superfunction f can be written

$$f = \sum_{i=1}^3 f_i(x) \theta_i + g(x) \theta_1 \theta_2 \theta_3$$

with $f_1, f_2, f_3, g \in \mathcal{C}^\infty(S^1, \mathbb{R})$. If f is an odd superfunction constant on the leaves of \mathcal{F}_2 , then

$$\left(\frac{\partial}{\partial x} + \theta_1 \theta_2 \theta_3 \frac{\partial}{\partial \theta_1} \right) \cdot f = 0 \quad \text{and} \quad \sum_{i=1}^3 \frac{df_i(x)}{dx} \theta_i + \left(\frac{dg(x)}{dx} + f_1(x) \right) \theta_1 \theta_2 \theta_3 = 0.$$

This implies that $df_1/dx = df_2/dx = df_3/dx = 0$ and $dg/dx + f_1(x) = 0$. Hence, for any $i \in \{1, 2, 3\}$, we obtain $f_i(x) = a_i$ for some constant $a_i \in \mathbb{R}$. But the equation $dg/dx + a_1 = 0$ has no periodic solution unless $a_1 = 0$. As a consequence, the vector space of odd superfunctions that are constant on the leaves of \mathcal{F}_2 has only dimension 3. The superfoliations \mathcal{F}_1 and \mathcal{F}_2 are therefore not isomorphic.

2D. Superfoliations with trivialized normal bundle. We describe a canonical process to replace a superfoliation on \mathcal{M} by a superfoliation with trivialized normal bundle on a $GL(n, m)$ -bundle $\mathcal{P}_{\mathcal{F}}$ over \mathcal{M} . For a definition of the super-Lie group $GL(n, m)$, see [Leites 1980; Tuynman 2004].

The tangent bundle of the superfoliation \mathcal{F} is a supervector subbundle of $T\mathcal{M}$, of dimension $(p - n) + \varepsilon(q - m)$. The normal bundle \mathcal{G} of \mathcal{F} is a supervector subbundle of $T^*\mathcal{M}$ of dimension $n + \varepsilon m$. Let $\mathcal{P}_{\mathcal{F}}$ be the frame bundle of \mathcal{G} ; this is a $GL(n, m)$ -bundle over \mathcal{M} . Denote by $\pi : \mathcal{P}_{\mathcal{F}} \rightarrow \mathcal{M}$ the canonical projection on \mathcal{M} .

By the construction of a frame bundle, there is a canonical inclusion of $\mathcal{P}_{\mathcal{F}}$ into $\mathcal{G} + \mathcal{G}[1] \subset (T^*\mathcal{M})^{\oplus n} \oplus T^*\mathcal{M}[1]^{\oplus m}$, where $\mathcal{G}[1]$ and $T^*\mathcal{M}[1]$ are the supervector bundles obtained by reversing parities on the fibers. Let α be the canonical odd 1-form of $T^*\mathcal{M}$ and β the canonical even 1-form of $T^*\mathcal{M}[1]$. Since we have not been able to locate these 1-forms in the existing literature, we introduce them now. Consider local coordinates on $T^*\mathcal{M}$

$$(x_1, \dots, x_p, \theta_1, \dots, \theta_q, y_1, \dots, y_p, \eta_1, \dots, \eta_q),$$

where $x_1, \dots, x_p, \theta_1, \dots, \theta_q$ are local coordinates on \mathcal{M} and $y_1, \dots, y_p, \eta_1, \dots, \eta_q$ are the even and odd coordinates on the cotangent bundle of \mathcal{M} corresponding to the basis dual to $\partial/\partial x_1, \dots, \partial/\partial x_p, \partial/\partial \theta_1, \dots, \partial/\partial \theta_q$ of $T\mathcal{M}$. We define α as

$$\alpha = \sum_{i=1}^p y_i dx_i - \sum_{j=1}^p \eta_j d\theta_j.$$

It is routine to check that α does not depend on the local coordinates $x_1, \dots, x_p, \theta_1, \dots, \theta_q$ on \mathcal{M} . The 1-form β is defined by the same formula, where y_1, \dots, y_p are now considered to be odd variables and η_1, \dots, η_q are considered to be even variables.

For $i = 1, \dots, n$, denote by $f_i : \mathcal{P}_{\mathcal{F}} \rightarrow T^*\mathcal{M}$ the i -th projection on $T^*\mathcal{M}$ and, for $j = 1, \dots, m$, denote by $g_j : \mathcal{P}_{\mathcal{F}} \rightarrow T^*\mathcal{M}[1]$ the j -th projection on $T^*\mathcal{M}[1]$. Define a family of 1-forms on $\mathcal{P}_{\mathcal{F}}$ as

$$\omega_i = f_i^* \alpha, \quad i = 1, \dots, n, \quad \text{and} \quad a_j = g_j^* \beta, \quad j = 1, \dots, m.$$

Proposition 2.20. *The pull-back $\pi^*\mathcal{F}$ of the superfoliation \mathcal{F} by π is a superfoliation on $\mathcal{P}_{\mathcal{F}}$ of codimension $n + \varepsilon m$ with trivialized normal bundle, defined by the 1-forms $\omega_1, \dots, \omega_n, a_1, \dots, a_m$.*

Proof. According to Theorem 2.10, we just have to check that Proposition 2.20 holds in the case of the superfoliation $\mathbb{R}_{n,m}^{p,q}$. Let $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$ be the (global) coordinates of $\mathbb{R}_{n,m}^{p,q}$ and $y_i^j, \eta_k^j, \zeta_i^l, z_k^l$, for $i, j \in \{1, \dots, n\}$ and $k, l \in \{1, \dots, m\}$, and with $\det(y_i^k) \neq 0$ and $\det(z_k^l) \neq 0$, be the (global) coordinates of $GL(n, m)$. These coordinates define a system of coordinates on the supermanifold

$\mathcal{P}_{\mathbb{R}_{n,m}^{p,q}} \simeq \mathbb{R}_{n,m}^{p,q} \times GL(n, m)$. In the latter system, we have

$$\omega_i = \sum_{k=1}^n y_k^i dx_k - \sum_{l=1}^m \theta_l^i d\theta_l \quad \text{and} \quad a_j = \sum_{k=1}^n \zeta_k^j dx_k - \sum_{l=1}^m z_l^j d\theta_l.$$

Since $\det(y_i^k) \neq 0$ and $\det(z_l^j) \neq 0$, the 1-forms $\omega_1, \dots, \omega_n, a_1, \dots, a_m \in \Omega^1(\mathcal{P}_{\mathbb{R}_{n,m}^{p,q}})$ and the 1-forms $dx_1, \dots, dx_n, d\theta_1, \dots, d\theta_m \in \Omega^1(\mathbb{R}_{n,m}^{p,q})$ define the same superfoliation on $\mathcal{P}_{\mathbb{R}_{n,m}^{p,q}}$. This superfoliation is clearly $\pi^*\mathcal{F}$. This completes the proof. \square

Remark 2.21. Proposition 2.20 provides a canonical way to replace a superfoliation \mathcal{F} , which has no trivialized normal bundle, by a superfoliation $\pi^*\mathcal{F}$ that has a trivialized normal bundle. This construction is functorial, in the sense that, if \mathcal{F}_1 is a superfoliation on \mathcal{M}_1 , then \mathcal{F}_2 is a superfoliation on \mathcal{M}_2 , and, if $\Phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a morphism of foliated supermanifolds, then Φ induces a submersion $\Phi_\Phi : \mathcal{P}_{\mathcal{F}_1} \rightarrow \mathcal{P}_{\mathcal{F}_2}$ that is a morphism of foliated supermanifolds with respect to the superfoliations $\pi^*\mathcal{F}_1$ and $\pi^*\mathcal{F}_2$. In other words, the application $\mathcal{F} \mapsto \mathcal{P}_{\mathcal{F}}$ defines a functor from the category of foliated supermanifolds of codimension $n + \varepsilon m$ to the category of foliated supermanifolds of codimension $n + \varepsilon m$ with trivialized normal bundle.

3. The cohomology of $\text{Vect}(n, m)_0$

For any super-Lie algebra \mathfrak{g} , let X_1, \dots, X_g be a basis of \mathfrak{g} and a_1, \dots, a_g the dual basis, with $a_i(X_j) = (-1)^{|X_i||X_j|} \delta_{ij}$, where $|X_i|$ is the parity of X_i , and $a_i(X_j) = 0$ for $i \neq j$. If for any $i, j, k \in \{1, \dots, g\}$ there exists some $C_{i,j}^k \in \mathbb{R}$ with

$$[X_i, X_j] = \sum_{k=1}^g C_{i,j}^k X_k,$$

then the Chevalley–Eilenberg differential is given by (see [Fuchs 1986])

$$(3-1) \quad \delta a_k = \frac{1}{2} \sum_{i,j=1}^g (-1)^{|X_i||X_j|} C_{i,j}^k a_i \wedge a_j.$$

Since we work with infinite-dimensional Lie algebras, we have to replace the Chevalley–Eilenberg cohomology by an infinite-dimensional generalization, but the signs in Equation (3-1) will remain valid.

3A. The Chevalley–Eilenberg complex of $\text{Vect}(n, m)$ and $\text{Vect}(n, m)_0$. To construct the theory of secondary characteristic classes of superfoliations, we need some prerequisites about the cohomology of a certain infinite-dimensional Lie algebra of supervector fields; this construction can be found, for instance, in [Fuchs 1986].

Take the superalgebra of *superpolynomials*

$$\mathbb{R}[x_1, \dots, x_n, \theta_1, \dots, \theta_m] = \mathbb{R}[x_1, \dots, x_n] \otimes \bigwedge \mathbb{R}^m.$$

The degree of a superpolynomial $x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m}$, with $i_1, \dots, i_n \in \mathbb{N}$ and $j_1, \dots, j_m \in \{0, 1\}$, is defined to be $i_1 + \dots + i_n + j_1 + \dots + j_m$. This turns $\mathbb{R}[x_1, \dots, x_n, \theta_1, \dots, \theta_m]$ into an \mathbb{N} -graded superalgebra.

The graded super-Lie algebra

$$\text{Vect}(n, m) = \bigoplus_{i=-1}^{+\infty} \text{Vect}(n, m)^i$$

of *formal supervector fields* is defined to be the super-Lie algebra of superderivations of $\mathbb{R}[x_1, \dots, x_n, \theta_1, \dots, \theta_m]$. Elements of $\text{Vect}(n, m)^i$ are said to be of *weight* i . We denote $\text{Vect}(n, 0)$ simply by $\text{Vect}(n)$. We denote the even part of $\text{Vect}(n, m)$ by $\text{Vect}(n, m)_0 = \bigoplus_{i=-1}^{+\infty} \text{Vect}(n, m)_0^i$.

Let $\bigwedge \text{Vect}(n, m)^*$ be the DGA of multilinear superalternating forms for the projective topology of $\text{Vect}(n, m)$; let $\bigwedge \text{Vect}(n, m)_0^*$ be the DGA of multilinear superalternating forms for the projective topology of $\text{Vect}(n, m)_0^*$; see, for instance, [Fuchs 1986]. “Projective” means, in short, that an element of $\text{Vect}(n, m)^*$ (respectively, $\text{Vect}(n, m)_0^*$) is a linear form on $\text{Vect}(n, m)$ that vanishes on all the spaces $\text{Vect}(n, m)^i$ (respectively, $\text{Vect}(n, m)_0^i$) but for finitely many $i \in \mathbb{N} \cap \{-1\}$. The spaces $\bigwedge \text{Vect}(n, m)^*$ and $\bigwedge \text{Vect}(n, m)_0^*$ are the exterior products of $\text{Vect}(n, m)^*$ and $\text{Vect}(n, m)_0^*$, respectively.

Denote by ∂ be the Chevalley–Eilenberg differential on $\bigwedge \text{Vect}(n, m)^*$, and by ∂_0 the Chevalley–Eilenberg differential on $\bigwedge \text{Vect}(n, m)_0^*$. The cohomology of the DGA $(\bigwedge \text{Vect}(n, m)_0^*, \partial_0)$ is called the (*Chevalley–Eilenberg*) *cohomology of* $\text{Vect}(n, m)_0$, and is denoted by $H^*(\text{Vect}(n, m)_0)$. Maybe the name “Chevalley–Eilenberg” is not absolutely correct, since we do not consider the complex of all skew-symmetric forms, but there is no risk of confusion here.

Let \mathcal{E} be the ideal (with respect to the product \wedge) generated by the even elements of $\text{Vect}(n, m)^*$, that is, by the continuous linear forms on $\text{Vect}(n, m)$ that identically vanish on $\text{Vect}(n, m)_0$. The next lemma will be useful.

Lemma 3.1. *The DGA $\bigwedge \text{Vect}(n, m)^* / \mathcal{E}$ and $\bigwedge \text{Vect}(n, m)_0^*$ are isomorphic.*

Remark 3.2. Note that $\text{Vect}(n, m)$ is sometimes denoted by $W(n, m)$. We prefer the notation $\text{Vect}(n, m)$ in order to avoid confusion with the Weil algebra.

3B. Two technical lemmas about the complex $(\bigwedge \text{Vect}(n, m)^*, \delta)$. To prove Proposition 4.2, we need to fix some notation and give the two technical Lemmata 3.5 and 3.6 about the DGA $(\bigwedge \text{Vect}(n, m), \partial)$.

The family

$$(3-2) \quad \left\{ \begin{array}{l} \frac{1}{i_1! \dots i_n!} x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial x_i}, \\ \frac{1}{i_1! \dots i_n!} x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial \theta_j} \end{array} \right.$$

(with $i_1, \dots, i_n \in \mathbb{N}$, $j_1, \dots, j_m \in \{0, 1\}$, $i = 1, \dots, n$, and $j = 1, \dots, m$) is a basis of $\text{Vect}(n, m)$. Denote elements of its dual basis by $(1/(i_1! \dots i_n!) x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \partial/\partial x_i)^*$ and $(1/(i_1! \dots i_n!) x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \partial/\partial \theta_j)^*$.

Remark that, if we denote by $(x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \partial/\partial x_i)^*$ and $(x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \partial/\partial \theta_j)^*$ the elements of the basis dual to the basis given by $x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \partial/\partial x_i$ and $x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \partial/\partial \theta_j$, then we have

$$(3-3) \quad \left\{ \begin{array}{l} \left(\frac{1}{i_1! \dots i_n!} x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial x_i} \right)^* = i_1! \dots i_n! \left(x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial x_i} \right)^* \\ \left(\frac{1}{i_1! \dots i_n!} x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial \theta_j} \right)^* = i_1! \dots i_n! \left(x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial \theta_j} \right)^* \end{array} \right.$$

For convenience, we introduce two notations.

Definition 3.3. For all $i \in \{1, \dots, n+m\}$, we define $|i|$ to be 0 for $i \in \{1, \dots, n\}$ and be 1 for $i \in \{n+1, \dots, n+m\}$.

Definition 3.4. We define h_i , $i = 1, \dots, n+m$, to be x_i if $|i| = 0$ and be θ_{i-n} if $|i| = 1$.

For example, we denote by $h_{i_1} \dots h_{i_s} h_{n+j_1} \dots h_{n+j_t} \partial/\partial h_1 \in \text{Vect}(n, m)$ the element that we used to denote by $x_{i_1} \dots x_{i_s} \theta_{j_1} \dots \theta_{j_t} \partial/\partial x_1$, where $i_1, \dots, i_s \in \{1, \dots, n\}$ and $j_1, \dots, j_t \in \{1, \dots, m\}$.

Define the *weight* of $(h_{i_1} \dots h_{i_k} \partial/\partial h_i)^*$ to be $k-1$ for any $i_1, \dots, i_k \in \{1, \dots, n+m\}$. Define Λ_k to be the subalgebra of $\bigwedge \text{Vect}(n, m)_0^*$ generated (with respect to \wedge) by elements of weights $-1, 0, 1, \dots, k$. One has $\partial \Lambda_{k-1} \subset \Lambda_k$.

We now compute the Chevalley–Eilenberg differential in the basis dual to the basis in (3-2). For this purpose, we define $c_j^{i_1, \dots, i_k}$ for any $i_1, \dots, i_k, j \in \{1, \dots, n+m\}$ by

$$c_j^{i_1, \dots, i_k} = \prod_{i=1}^n K(i, [i_1, \dots, i_k])! \left(h_{i_1} \dots h_{i_k} \frac{\partial}{\partial h_j} \right)^*,$$

where $K(i, [i_1, \dots, i_k])$ is the number of integers equal to i in the list $[i_1, \dots, i_k]$. If $|j| = 1$ and j appears more than twice in the list $[i_1, \dots, i_k]$, then, of course,

$$(3-4) \quad c_j^{i_1, \dots, i_k} = 0.$$

Otherwise, we have

$$c_j^{i_1, \dots, i_k} = \varepsilon \left(\frac{1}{k_1! \dots k_n!} x_1^{k_1} \dots x_n^{k_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial h_j} \right)^*$$

for some $\varepsilon \in \{-1, 1\}$, where $k_1 = K(1, [i_1, \dots, i_k])$, \dots , $k_n = K(n, [i_1, \dots, i_k])$ and where, for any $i \in \{1, \dots, m\}$, $j_i = 1$ if and only if $n + i$ appears in the list $[i_1, \dots, i_k]$.

For any $i_1, \dots, i_k, j \in \{1, \dots, n+m\}$, we define $P_j^{i_1, \dots, i_k} \in \Lambda_{k-1}$ by:

$$(3-5) \quad \partial c_j^{i_1, \dots, i_k} = P_j^{i_1, \dots, i_k} + \sum_{l=1}^{n+m} (-1)^{|l|} \left(\frac{\partial}{\partial h_l} \right)^* \wedge c_j^{l, i_1, \dots, i_k}.$$

Lemma 3.5. *For any $i_1, \dots, i_k, j \in \{1, \dots, n+m\}$, $P_j^{i_1, \dots, i_k}$ is an element of Λ_{k-1} .*

Proof. Denote by $\Sigma_{l,k}$ the set of shuffles of the sets $\{1, \dots, l\}$ and $\{l+1, \dots, k\}$. We have

$$(3-6) \quad \partial c_j^{i_1, \dots, i_k} = \sum_{l=1}^{n+m} \sum_{\sigma \in \Sigma_{l,k}} D(i_1, \dots, i_k, l, j, \sigma) c_l^{i_{\sigma(1)}, \dots, i_{\sigma(k)}} \wedge c_j^{l, i_{\sigma(1)}, \dots, i_{\sigma(l)}},$$

for some $D(i_1, \dots, i_k, l, j, \sigma) \in \mathbb{R}$.

From the relations

$$(3-7) \quad \left[\frac{\partial}{\partial x_i}, \frac{1}{k_1! \dots (k_i+1)! \dots k_n!} x_1^{k_1} \dots x_i^{k_i+1} \dots x_n^{k_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial h_j} \right] \\ = \frac{1}{k_1! \dots k_i! \dots k_n!} x_1^{k_1} \dots x_i^{k_i} \dots x_n^{k_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial h_j},$$

$$(3-8) \quad \left[\frac{\partial}{\partial \theta_l}, \frac{\theta_l}{k_1! \dots k_n!} x_1^{k_1} \dots x_n^{k_n} \theta_1^{j_1} \dots \theta_{l-1}^{j_{l-1}} \theta_{l+1}^{j_{l+1}} \dots \theta_m^{j_m} \frac{\partial}{\partial h_j} \right] \\ = \frac{1}{k_1! \dots k_n!} x_1^{k_1} \dots x_n^{k_n} \theta_1^{j_1} \dots \theta_{l-1}^{j_{l-1}} \theta_{l+1}^{j_{l+1}} \dots \theta_m^{j_m} \frac{\partial}{\partial h_j},$$

we obtain, using (3-1), (3-4), and the definition of ∂ , that

$$\partial c_j^{i_1, \dots, i_k} - \sum_{l=1}^{n+m} (-1)^{|l|} \left(\frac{\partial}{\partial h_l} \right)^* \wedge c_j^{l, i_1, \dots, i_k}$$

is a linear combinations of products of elements of weight $0, \dots, k-1$ that do not involve any terms of weight -1 or k . In other words, it is an element of Λ_{k-1} . \square

Lemma 3.6. *For any $i_1, \dots, i_k, l \in \{1, \dots, n+m\}$, define 2-forms $Q_j^{l, i_1, \dots, i_k} \in \Lambda_k$ by*

$$(3-9) \quad Q_j^{l, i_1, \dots, i_k} = P_j^{l, i_1, \dots, i_k} - \sum_{l'=1}^{n+m} (-1)^{|l|+|l'|} \left(h_l \frac{\partial}{\partial h_{l'}} \right)^* \wedge c_j^{l', i_1, \dots, i_k}.$$

The following identity holds:

$$(3-10) \quad \partial P_j^{i_1, \dots, i_k} = \sum_{l=1}^{n+m} \left(\frac{\partial}{\partial h_l} \right)^* \wedge Q_j^{l, i_1, \dots, i_k}.$$

Proof. The identity

$$(3-11) \quad \partial c_i = \sum_{j=1}^{n+m} (-1)^{|j|} c_j \wedge c_i^j$$

holds for any $i \in \{1, \dots, n+m\}$. Applying ∂ to (3-5) and using (3-11), we obtain

$$\begin{aligned} \partial P_j^{i_1, \dots, i_k} + \partial \sum_{l=1}^{n+m} (-1)^{|l|} \left(\frac{\partial}{\partial h_l} \right)^* \wedge c_j^{l, i_1, \dots, i_k} &= 0, \\ \partial P_j^{i_1, \dots, i_k} + \sum_{l=1}^{n+m} \sum_{l'=1}^{n+m} (-1)^{|l'|+|l|} \left(\frac{\partial}{\partial h_{l'}} \right)^* \wedge \left(h_{l'} \frac{\partial}{\partial h_l} \right)^* \wedge c_j^{l, i_1, \dots, i_k}, \\ &\quad - \sum_{l=1}^{n+m} \left(\frac{\partial}{\partial h_l} \right)^* \wedge \partial c_j^{l, i_1, \dots, i_k} = 0, \\ \partial P_j^{i_1, \dots, i_k} + \sum_{l=1}^{n+m} \sum_{l'=1}^{n+m} (-1)^{|l'|+|l|} \left(\frac{\partial}{\partial h_l} \right)^* \wedge \left(h_l \frac{\partial}{\partial h_{l'}} \right)^* \wedge c_j^{l', i_1, \dots, i_k} - \sum_{l=1}^{n+m} \left(\frac{\partial}{\partial h_l} \right)^* \wedge P_j^{l, i_1, \dots, i_k} \\ &\quad - \sum_{l'=1}^{n+m} \sum_{l=1}^{n+m} (-1)^{|l'|} \left(\frac{\partial}{\partial h_{l'}} \right)^* \wedge \left(\frac{\partial}{\partial h_l} \right)^* \wedge c_j^{l', l, i_1, \dots, i_k} = 0. \end{aligned}$$

Now, for any $l', l \in \{1, \dots, n+m\}$, we have

$$\begin{cases} \left(\frac{\partial}{\partial h_l} \right)^* \wedge \left(\frac{\partial}{\partial h_{l'}} \right)^* = (-1)^{(|l|+1)(|l'|+1)} \left(\frac{\partial}{\partial h_{l'}} \right)^* \wedge \left(\frac{\partial}{\partial h_l} \right)^* \\ c_j^{l', l, i_1, \dots, i_k} = (-1)^{|l'| |l|} c_j^{l, l', i_1, \dots, i_k} \end{cases}$$

Therefore, we have

$$(-1)^{|l'|} \left(\frac{\partial}{\partial h_l} \right)^* \wedge \left(\frac{\partial}{\partial h_{l'}} \right)^* \wedge c_j^{l', l, i_1, \dots, i_k} = -(-1)^{|l|} \left(\frac{\partial}{\partial h_{l'}} \right)^* \wedge \left(\frac{\partial}{\partial h_l} \right)^* \wedge c_j^{l, l', i_1, \dots, i_k}$$

and

$$\sum_{l'=1}^{n+m} \sum_{l=1}^{n+m} (-1)^{|l'|} \left(\frac{\partial}{\partial h_l} \right)^* \wedge \left(\frac{\partial}{\partial h_{l'}} \right)^* \wedge c_j^{l', l, i_1, \dots, i_k} = 0.$$

The lemma follows immediately. \square

3C. Generators of the cohomology of $\text{Vect}(n, m)_0$. We introduce a DGA whose cohomology is $H^*(\text{Vect}(n, m)_0)$ but that is easier to work with than $\text{Vect}(n, m)_0$ itself.

Let gl_n (respectively, gl_m) be the Lie algebra of linear endomorphism of \mathbb{R}^n (respectively, \mathbb{R}^m). Let $(a_{i,j})_{i,j \in \{1, \dots, n\}}$ (respectively, $(d_{k,l})_{k,l \in \{1, \dots, m\}}$) be the canonical basis of gl_n (respectively, gl_m).

The elements $\text{Vect}(n, m)_0^0$ of weight 0 in $\text{Vect}(n, m)_0$ form a Lie subalgebra of $\text{Vect}(n, m)_0$, isomorphic to $gl_n \oplus gl_m$ through the isomorphism φ with

$$(3-12) \quad \varphi\left(x_i \frac{\partial}{\partial x_j}\right) = a_{i,j} \quad \text{and} \quad \varphi\left(\theta_k \frac{\partial}{\partial \theta_l}\right) = d_{k,l}.$$

Let p_0 be the projection from $\text{Vect}(n, m)_0$ to $\text{Vect}(n, m)_0^0$, with kernel

$$\text{Ker } p_0 = \text{Vect}(n, m)_0^{-1} \oplus \bigoplus_{i=1}^{\infty} \text{Vect}(n, m)_0^i.$$

Let $\alpha : (gl_n \oplus gl_m)^* \rightarrow \text{Vect}(n, m)_0^*$ be the linear map $\alpha = p_0^* \circ \varphi^*$, where p_0^* and φ^* are the dual maps of p_0 and φ .

The map α defines a $gl_n \oplus gl_m$ -connection (see [Guillemin and Sternberg 1999, Chapter 3]) of the DGA $\bigwedge \text{Vect}(n, m)_0^*$ that induces a DGA homomorphism (denoted again by α) from the Weil algebra $W(gl_n \oplus gl_m) = S((gl_n \oplus gl_m)^*) \otimes \bigwedge (gl_n \oplus gl_m)^*$ to $\bigwedge \text{Vect}(n, m)_0^*$.

Let \mathcal{K} be the kernel of α . The cohomology of the DGA $W(gl_n \oplus gl_m)/\mathcal{K}$ is denoted by $H^*(W(gl_n \oplus gl_m)/\mathcal{K})$. Write $\tilde{\alpha}$ for the DGA homomorphism from $W(gl_n \oplus gl_m)/\mathcal{K}$ to $\bigwedge \text{Vect}(n, m)_0^*$ induced by α . Let $\hat{\alpha} : H^*(W(gl_n \oplus gl_m)/\mathcal{K}) \rightarrow H^*(\text{Vect}(n, m)_0)$ be the map induced by $\tilde{\alpha}$ in cohomology.

Theorem 3.7. *The map $\hat{\alpha}$ is an isomorphism between $H^*(W(gl_n \oplus gl_m)/\mathcal{K})$ and $H^*(\text{Vect}(n, m)_0)$.*

The proof is provided in Appendix A.

4. Secondary characteristic classes of superfoliations

We construct, for any superfoliation \mathcal{F} on \mathcal{M} of codimension $n + \varepsilon m$ with trivialized normal bundle, a map from $H^*(\text{Vect}(n, m)_0)$ to the de Rham cohomology of the base manifold $H^*(\mathcal{M})$.

We use the conventions introduced in Definitions 3.3 and 3.4. Throughout this section, we often say ‘‘homomorphism’’ for ‘‘DGA homomorphism’’.

4A. DGA homomorphism defining a superfoliation.

Definition 4.1. We say that a DGA homomorphism

$$\omega : (\bigwedge \text{Vect}(n, m)^*, \partial) \rightarrow (\Omega^*(\mathcal{M}), d_{\mathcal{M}})$$

defines the superfoliation \mathcal{F} of codimension $n + \varepsilon m$ with trivialized normal bundle if the 1-forms $\omega((\partial/\partial h_i)^*)$, $i = 1, \dots, n+m$, define the superfoliation \mathcal{F} .

Consider a superfoliation \mathcal{F} with trivialized normal bundle on \mathcal{M} , defined by 1-forms b_1, \dots, b_{n+m} , where b_1, \dots, b_n are odd 1-forms and b_{n+1}, \dots, b_{n+m} are

even 1-forms. Take 1-forms $(b_i^j)_{i,j \in \{1, \dots, n+m\}}$ such that

$$(4-1) \quad d_{\mathcal{M}} b_i = \sum_{j=1}^n b_j \wedge b_i^j - \sum_{j=n+1}^{n+m} b_j \wedge b_i^j.$$

According to (2-6), such 1-forms b_i^j always exist. The choice of signs in (4-1) is explained by:

Proposition 4.2. *For any superfoliation \mathcal{F} with trivialized normal bundle on \mathcal{M} , there exists a DGA homomorphism ω from $(\wedge \text{Vect}(n, m)^*, \partial)$ to $(\Omega(\mathcal{M}), d_{\mathcal{M}})$ that defines the superfoliation \mathcal{F} . Moreover, we can assume that*

$$(4-2) \quad \omega((\partial/\partial h_i)^*) = b_i, \quad i = 1, \dots, n+m$$

$$(4-3) \quad \omega((h_i \partial/\partial h_j)^*) = b_j^i, \quad i, j = 1, \dots, n+m.$$

First, we need the following lemma, proved in Appendix B.

Lemma 4.3. *Let d_1, \dots, d_n be odd 1-forms and d_{n+1}, \dots, d_{n+m} even 1-forms, forming a free family. Let d^i , $i = 1, \dots, n+m$, be 2-forms on \mathcal{M} such that*

$$\sum_{i=1}^{n+m} d_i \wedge d^i = 0$$

There exist homogeneous 1-forms $d^{i,l}$ of parity $|i| + |l| + 1$ (modulo 2) such that

$$d^{i,l} = -(-1)^{(|i|+1)(|l|+1)} d^{l,i} \quad \text{and} \quad \sum_{i=1}^{n+m} d_i \wedge d^{i,l} = d^l.$$

Proof of Proposition 4.2. Define $\varepsilon(i_1, \dots, i_k, \sigma) \in \{-1, 1\}$, for any permutation σ of $\{1, \dots, k\}$, by

$$h_{i_{\sigma(1)}} \dots h_{i_{\sigma(k)}} = \varepsilon(i_1, \dots, i_k, \sigma) h_{i_1} \dots h_{i_k}.$$

We will construct by induction a homomorphism $\omega : \Lambda_k \rightarrow \Omega(\mathcal{M})$ such that, for any $\alpha \in \Lambda_{k-1}$,

$$(4-4) \quad d_{\mathcal{M}} \omega(\alpha) = \omega(\partial(\alpha)).$$

Equations (4-2) and (4-3) define such a map ω for $k = 0$. Equations (3-11) and (4-1) imply that the condition (4-4) holds for $k = 0$. We have therefore constructed ω for $k = 0$.

We now assume that ω can be constructed for some $k \in \mathbb{N}$ and construct it for $k + 1$. Applying ω to (3-5), we obtain

$$d_{\mathcal{M}} \omega(c_j^{i_1, \dots, i_k}) = \omega(P_j^{i_1, \dots, i_k}) + \sum_{l=1}^{n+m} (-1)^{|l|} \omega((\partial/\partial h_l)^*) \wedge \omega(c_j^{l, i_1, \dots, i_k}).$$

Denote $\omega(c_j^{i_1, \dots, i_n})$ by $b_j^{i_1, \dots, i_n}$. In particular, $b_l = \omega((\partial/\partial x_l)^*)$. By applying $d_{\mathcal{M}}$ to both sides of the previous expression and using Equation (4-2), we obtain

$$d_{\mathcal{M}}\omega(P_j^{i_1, \dots, i_k}) + d_{\mathcal{M}}\left(\sum_{l=1}^{n+m} (-1)^{|l|} b_l \wedge b_j^{l, i_1, \dots, i_k}\right) = 0.$$

By (4-1), we have

$$d_{\mathcal{M}}\omega(P_j^{i_1, \dots, i_k}) + \sum_{i=1}^{n+m} b_i \wedge \left(\sum_{l=1}^{n+m} (-1)^{|i|+|l|} b_l^i \wedge b_j^{l, i_1, \dots, i_k} - d_{\mathcal{M}}b_j^{i, i_1, \dots, i_k}\right) = 0.$$

Since $P_j^{i_1, \dots, i_k} \in \Lambda_{k-1}$, we have $\omega(\partial P_j^{i_1, \dots, i_k}) = d_{\mathcal{M}}\omega(P_j^{i_1, \dots, i_k})$. Equation (3-10) implies that

$$\sum_{i=1}^{n+m} b_i \wedge \left(\omega(Q_j^{i, i_1, \dots, i_k}) + \sum_{l=1}^{n+m} (-1)^{|l|+|i|} b_l^i \wedge b_j^{l, i_1, \dots, i_k} - d_{\mathcal{M}}b_j^{i, i_1, \dots, i_k}\right) = 0.$$

From (3-9) it follows that

$$\sum_{i=1}^{n+m} b_i \wedge (\omega(P_j^{i, i_1, \dots, i_k}) - d_{\mathcal{M}}c_j^{i, i_1, \dots, i_k}) = 0.$$

By Lemma 4.3, there exist 1-forms $b_j^{i, l, i_1, \dots, i_k}$ satisfying

$$(4-5) \quad \begin{cases} d_{\mathcal{M}}b_j^{i, i_1, \dots, i_k} - \omega(P_j^{i, i_1, \dots, i_k}) = \sum_{l=1}^{n+m} (-1)^{|l|} b_l \wedge b_j^{l, i, i_1, \dots, i_k} \\ (-1)^{|i|} b_j^{i, l, i_1, \dots, i_k} = -(-1)^{(|l|+1)(|i|+1)} (-1)^{|l|} b_j^{l, i, i_1, \dots, i_k}. \end{cases}$$

This last equation can be rewritten

$$(4-6) \quad b_j^{i, l, i_1, \dots, i_k} = (-1)^{|i||l|} b_j^{l, i, i_1, \dots, i_k}.$$

Equation (4-5) and the relations

$$\begin{cases} b_j^{i_{\sigma(1)}, \dots, i_{\sigma(k)}, i_{\sigma(k+1)}} = \varepsilon(i_1, \dots, i_{k+1}, \sigma) b_j^{i_1, \dots, i_k, i_{k+1}} \\ P_j^{i_{\sigma(1)}, \dots, i_{\sigma(k)}, i_{\sigma(k+1)}} = \varepsilon(i_1, \dots, i_{k+1}, \sigma) P_j^{i_1, \dots, i_k, i_{k+1}}. \end{cases}$$

imply that, for any permutation σ of $\{1, \dots, k+1\}$,

$$(4-7) \quad b_j^{i_{\sigma(1)}, \dots, i_{\sigma(k+1)}} = \varepsilon(i_1, \dots, i_{k+1}, \sigma) b_j^{i_1, \dots, i_{k+1}}.$$

Equations (4-6) and (4-7) imply that, for any permutation σ of $\{1, \dots, k+2\}$ and any $i_1, \dots, i_{k+1}, i_{k+2} \in \{1, \dots, n+m\}$, we have

$$(4-8) \quad b_j^{i_{\sigma(1)}, \dots, i_{\sigma(k+1)}, i_{\sigma(k+2)}} = \varepsilon(i_1, \dots, i_k, i_{k+2}, \sigma) b_j^{i_1, \dots, i_{k+1}, i_{k+2}}.$$

We can then define ω on elements of weight $k + 1$ by $\omega(c_j^{i_1, \dots, i_{k+2}}) = b_j^{i_1, \dots, i_{k+2}}$. Equation (4-8) implies that

$$\omega(c_j^{i_{\sigma(1)}, \dots, i_{\sigma(k+2)}}) = \varepsilon(i_1, \dots, i_{k+2}, \sigma) \omega(c_j^{i_1, \dots, i_{k+2}}),$$

and hence ω is well defined. The map ω can be uniquely extended to a DGA homomorphism from Λ_{k+1} to $\Omega(\mathcal{M})$. According to (3-5) and (4-5), this homomorphism satisfies $d_{\mathcal{M}}\omega(\alpha) = \omega(\partial(\alpha))$ for any $\alpha \in \Lambda_k$. This completes the proof. \square

4B. Construction of secondary characteristic classes. Assume that

$$\omega : \bigwedge \text{Vect}(n, m)_0^* \rightarrow \Omega(\mathcal{M})$$

is a DGA homomorphism defining a superfoliation \mathcal{F} of codimension $n + \varepsilon m$ with trivialized normal bundle. Set $\beta_\omega = p \circ \omega$, where p is the canonical projection from $\Omega^*(\mathcal{M})$ to $\Omega^*(M)$. By Lemma 2.1, the kernel of p contains all odd $2k$ -forms and even $(2k+1)$ -forms. As a consequence, $\omega(\mathcal{E})$ is contained in the kernel of β_ω . The homomorphism β_ω induces a homomorphism from $\bigwedge \text{Vect}(n, m)^* / \mathcal{E}$ to $\Omega^*(M)$. By Lemma 3.1, $\bigwedge \text{Vect}(n, m)^* / \mathcal{E} \simeq \bigwedge \text{Vect}(n, m)_0^*$, and therefore β_ω induces a homomorphism β'_ω from $\bigwedge \text{Vect}(n, m)_0^*$ to $\Omega^*(M)$.

Definition 4.4. Let $\varphi_{\mathcal{M}, \mathcal{F}}$ be the homomorphism from $H^*(\text{Vect}(n, m)_0)$ to $H^*(M)$ induced in cohomology by the DGA homomorphism

$$\beta'_\omega : \bigwedge \text{Vect}(n, m)_0^* \rightarrow \Omega^*(M).$$

This definition is justified by:

Proposition 4.5. *The homomorphism $\varphi_{\mathcal{M}, \mathcal{F}} : H^*(\text{Vect}(n, m)_0) \rightarrow H^*(M)$ is independent of the choice of homomorphism $\omega : \bigwedge \text{Vect}(n, m)^* \rightarrow \Omega(\mathcal{M})$ defining the superfoliation \mathcal{F} .*

First, we need a lemma. Let ω^1 and ω^2 be two homomorphisms defining the superfoliation \mathcal{F} , and let $\varphi_{\mathcal{M}, \mathcal{F}}^1, \varphi_{\mathcal{M}, \mathcal{F}}^2$ be the homomorphisms $\bigwedge \text{Vect}(n, m)^* \rightarrow \Omega(\mathcal{M})$ constructed from ω_1 and ω_2 as in Definition 4.4.

Lemma 4.6. *If $p \circ \omega^1((x_i \partial / \partial x_j)^*) = p \circ \omega^2((x_i \partial / \partial x_j)^*)$ and $p \circ \omega^1((\theta_k \partial / \partial \theta_l)^*) = p \circ \omega^2((\theta_k \partial / \partial \theta_l)^*) \in \Omega(M)$ for all $i, j \in \{1, \dots, n\}$ and $k, l \in \{1, \dots, m\}$, then*

$$\varphi_{\mathcal{M}, \mathcal{F}}^1 = \varphi_{\mathcal{M}, \mathcal{F}}^2.$$

Proof. A DGA homomorphism β from the Weil algebra $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \bigwedge \mathfrak{g}^*$ of a Lie algebra \mathfrak{g} to a given DGA (A, d_A) depends only on the restriction of β to $1 \otimes \bigwedge^1 \mathfrak{g}^*$, see [Guillemin and Sternberg 1999]. In particular, the homomorphisms $\beta_{\omega^a} \circ \alpha : W(\mathfrak{gl}_n \oplus \mathfrak{gl}_m) \rightarrow \Omega(M)$ for $a = 1, 2$ depend only on the

1-forms $p \circ \omega((x_i \partial / \partial x_j)^*)$ and $p \circ \omega((\theta_k \partial / \partial \theta_l)^*)$, where $i, j \in \{1, \dots, n\}$ and $k, l \in \{1, \dots, m\}$. Therefore,

$$\beta_{\omega^1} \circ \alpha = \beta_{\omega^2} \circ \alpha \quad \text{and} \quad \beta'_{\omega^1} \circ \tilde{\alpha} = \beta'_{\omega^2} \circ \tilde{\alpha}.$$

For $a = 1, 2$, the diagram

$$\begin{array}{ccc} \frac{W(gl_n + gl_m)}{\mathfrak{H}} & \xrightarrow{\tilde{\alpha}} & \bigwedge \text{Vect}(n, m)_0^* \\ & \searrow \beta'_{\omega^a} \circ \tilde{\alpha} & \downarrow \beta'_{\omega^a} \\ & & \Omega(M) \end{array}$$

is commutative. Moreover, according to Theorem 3.7, $\tilde{\alpha}$ induces an isomorphism in cohomology. This completes the proof. \square

Proof of Proposition 4.5. Set $a_i = \omega^1((\partial / \partial h_i)^*)$ and $b_i = \omega^2((\partial / \partial h_i)^*)$. Set also $a_i^j = \omega^1((h_j \partial / \partial h_i)^*)$ and $b_i^j = \omega^2((h_j \partial / \partial h_i)^*)$.

For any superfunction f on the supermanifold $\mathcal{M} \times \mathbb{R}$, denote by $f|_t$ its restriction to $M \times \{t\}$. Denote by i_t the maps from \mathcal{M} to $\mathcal{M} \times \mathbb{R}$ induced by $f \rightarrow f|_t$. Let

$$\text{pr} : \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M}$$

be the natural projection.

On the supermanifold $\mathcal{M} \times \mathbb{R}$, consider the superfoliation \mathcal{F}' that is the pull-back of \mathcal{F} by the projection pr . There exist 1-forms c_1, \dots, c_{n+m} defining the superfoliation \mathcal{F}' and 1-forms $(c_i^k)_{i,k \in \{1, \dots, n+m\}}$ satisfying the identity

$$d_{\mathcal{M}} c_i = \sum_{k=1}^{n+m} (-1)^{|k|} c_k \wedge c_i^k$$

and the following two properties:

- The 1-forms c_i , $i \in \{1, \dots, n+m\}$ (respectively, c_i^k , $i, k \in \{1, \dots, n+m\}$) restricted to $\mathcal{M} \times (-\infty, 1/4]$ are equal to $\text{pr}^* a_i$, $i = 1, \dots, n+m$ (respectively, $\text{pr}^* a_k^i$, $i, k \in \{1, \dots, n+m\}$).
- The 1-forms c_i , $i \in \{1, \dots, n+m\}$ (respectively, c_i^k , $i, k \in \{1, \dots, n+m\}$) restricted to $\mathcal{M} \times (3/4, \infty)$ are equal to $\text{pr}^* b_i$, $i = 1, \dots, n+m$ (respectively, $\text{pr}^* b_k^i$, $i, k \in \{1, \dots, n+m\}$).

By Proposition 4.2, there exists a homomorphism $\omega_3 : \bigwedge \text{Vect}(n, m)^* \rightarrow \Omega(\mathcal{M} \times \mathbb{R})$ defining \mathcal{F}' , such that $\omega_3((h_i \partial / \partial h_j)^*) = c_j^i$ and $\omega_3((\partial / \partial h_i)^*) = c_i$. Then, the

The editors acknowledge the use of the diagrams package by Paul Taylor.

following diagram is commutative:

$$\begin{array}{ccccc}
 & & \Omega(M) & & \\
 & & \nearrow p \circ \omega^1 \circ \tilde{\alpha} & & \nwarrow i_1^* \\
 \wedge(\text{Vect}(n, m)_0) & \xrightarrow{\tilde{\alpha}} & \frac{W(\mathfrak{gl}_n \oplus \mathfrak{gl}_m)}{\mathfrak{H}} & \xrightarrow{p \circ \omega_3 \circ \tilde{\alpha}} & \Omega(M \times \mathbb{R}). \\
 & & \searrow p \circ \omega^2 \circ \tilde{\alpha} & & \nearrow i_2^* \\
 & & \Omega(M) & &
 \end{array}$$

Therefore, we also have the commutative diagram

$$\begin{array}{ccccc}
 & & H^*(M) & & \\
 & & \nearrow \varphi_{\mathcal{M}, \mathcal{F}}^1 \circ \hat{\alpha} & & \nwarrow i_1^* \\
 H^*(\text{Vect}(n, m)_0) & \xrightarrow{\hat{\alpha}} & H^*\left(\frac{W(\mathfrak{gl}_n \oplus \mathfrak{gl}_m)}{\mathfrak{H}}\right) & \xrightarrow{\varphi_{\mathcal{M} \times \mathbb{R}, \mathcal{F}'} \circ \hat{\alpha}} & H^*(M \times \mathbb{R}), \\
 & & \searrow \varphi_{\mathcal{M}, \mathcal{F}}^2 \circ \hat{\alpha} & & \nearrow i_2^* \\
 & & H^*(M) & &
 \end{array}$$

where $\varphi_{\mathcal{M} \times \mathbb{R}, \mathcal{F}'} : H^*(\text{Vect}(n, m))$ is constructed as in Definition 4.4, with the help of the DGA homomorphism ω_3 defining \mathcal{F}' .

Since by Theorem 3.7 $\hat{\alpha}$ is a bijection, and $i_1^* : H^*(\mathcal{M} \times \mathbb{R}) \rightarrow H^*(\mathcal{M})$ and $i_2^* : H^*(\mathcal{M} \times \mathbb{R}) \rightarrow H^*(\mathcal{M})$ are isomorphisms with $i_1^* \circ (i_2^*)^{-1} = \text{Id}$, we have

$$\varphi_{\mathcal{M}, \mathcal{F}}^1 = \varphi_{\mathcal{M}, \mathcal{F}}^2. \quad \square$$

We now study how $\varphi_{\mathcal{M}, \mathcal{F}}$ behaves with respect to morphisms of superfoliations.

Proposition 4.7. *Let $\overline{\mathcal{F}}$ (respectively, \mathcal{G}) be a superfoliation with trivialized normal bundle on \mathcal{M} (respectively, \mathcal{N}). Let $\Phi : \mathcal{N} \rightarrow \mathcal{M}$ be a morphism of foliated supermanifolds, and $\varphi : \mathcal{N} \rightarrow \mathcal{M}$ the map induced by Φ , from the base manifold N of \mathcal{N} to the base manifold M of \mathcal{N} . Let $\varphi^* : H^*(M) \rightarrow H^*(N)$ be the pull-pack by φ . Under these hypotheses, we have*

$$(4-9) \quad \varphi_{\mathcal{N}, \mathcal{G}} = \varphi^* \circ \varphi_{\mathcal{M}, \overline{\mathcal{F}}}.$$

In other words, for any supermanifold \mathcal{N} and any submersion $\Phi : \mathcal{N} \rightarrow \mathcal{M}$, we have $\varphi_{\mathcal{N}, \Phi^* \overline{\mathcal{F}}} = \varphi^* \circ \varphi_{\mathcal{M}, \overline{\mathcal{F}}}$, where $\Phi^* \overline{\mathcal{F}}$ is the pull-back via Φ of the superfoliation $\overline{\mathcal{F}}$.

Proof. If $\omega : \wedge \text{Vect}(n, m)^* \rightarrow \Omega(\mathcal{M})$ is a DGA homomorphism defining $\overline{\mathcal{F}}$, then $\Phi^* \circ \omega : \wedge \text{Vect}(n, m)^* \rightarrow \Omega(\mathcal{N})$ is a homomorphism of DGAs defining \mathcal{G} . Let $p_N : \Omega(\mathcal{N}) \rightarrow \Omega(N)$ and $p_M : \Omega(\mathcal{M}) \rightarrow \Omega(M)$ be the canonical projections of the supermanifolds \mathcal{N} and \mathcal{M} to their base manifolds, as defined in Lemma 2.1. We

have $p_N \circ \Phi^* = \varphi^* \circ p_M$, which implies that

$$p_N \circ \Phi^* \circ \omega = \varphi^* \circ p_M \circ \omega.$$

Hence, $\beta_{\Phi^* \circ \omega} = \varphi^* \circ \beta_\omega$, and Equation (4-9) follows. \square

4C. Examples.

4C(a). *Superfoliations of codimension $0 + \varepsilon 1$.* If \mathcal{F} is a superfoliation of codimension $0 + \varepsilon 1$ defined by an even 1-form $a \in \Omega^1(\mathcal{M})$, then there exists an odd 1-form $b \in \Omega^1(\mathcal{M})$ with $d_{\mathcal{M}} a = -a \wedge b$.

By Proposition A.11, $H^k(\text{Vect}(0, 1)_0) = 0$ if $k \notin \{0, 1\}$, and $H^1(\text{Vect}(0, 1)_0) \simeq \mathbb{R}$. Moreover, $H^1(\text{Vect}(0, 1)_0)$ is generated by $H = [(\theta \partial / \partial \theta)^*]$. By Proposition 4.2, there exists a homomorphism $\omega : \wedge \text{Vect}(0, 1)^* \rightarrow \Omega(\mathcal{M})$ such that $\omega((\theta \partial / \partial \theta)^*) = b$. By construction, we have $\varphi_{\mathcal{M}, \mathcal{F}}(H) = [p(b)]$. See [Laurent-Gengoux 2004] for additional details.

Example 4.8. Let $E \rightarrow S^1$ be a trivial 1-dimensional vector bundle. The supermanifold $S^{1,1} = \mathcal{M}(S^1, E)$, with base manifold S^1 and superalgebra of superfunctions $\Omega(\mathcal{M}) = \Gamma(\wedge^* E)$, is called a *supercircle*. Let $x \in S^1$ be the even parameter and θ the odd parameter.

Let $t \in \mathbb{R}^*$ be a real number different from zero, and \mathcal{F}_t the superfoliation of codimension $0 + 1\varepsilon$ defined by the 1-form $b_t = d\theta + t\theta dx$. Note that

$$d_{S^{1,1}} b_t = t dx \wedge d\theta = (d\theta + t\theta dx) \wedge t dx.$$

By the preceding discussion, we have

$$\varphi_{S^{1,1}, \mathcal{F}_t}(H) = -[p(t dx)] = -t[dx].$$

For $t \neq 0$, we obtain a nonzero element of $H^1(S^1)$. We have thus constructed a supermanifold \mathcal{M} and a superfoliation \mathcal{F} with $\varphi_{\mathcal{M}, \mathcal{F}}$ not zero.

4C(b). *Superfoliations of codimension $1 + \varepsilon 1$.* Let \mathcal{F} be a superfoliation of codimension $1 + \varepsilon 1$ given by an odd 1-form $\omega \in \Omega^1(\mathcal{M})$ and an even 1-form $a \in \Omega^1(\mathcal{M})$.

There exist $b, c \in \Omega^1(\mathcal{M})$ with $d_{\mathcal{M}} a = \omega \wedge c - a \wedge b$. By Proposition 4.2, there is a DGA homomorphism $\omega_{\mathcal{F}} : \wedge \text{Vect}(1, 1)^* \rightarrow \Omega(\mathcal{M})$, defining the superfoliation \mathcal{F} , with $\omega_{\mathcal{F}}((\partial / \partial x)^*) = \omega$ and $\omega_{\mathcal{F}}((\theta \partial / \partial \theta)^*) = b$.

By Proposition A.12, $H^k(\text{Vect}(n, m)_0)$ vanishes if $k \neq 3$, and $H^3(\text{Vect}(n, m)_0)$ is generated the classes H_1, H_2, H_3 described by Equations (A-13), (A-14), and (A-15).

Proposition 4.9. *Let \mathcal{F} be a superfoliation of codimension $1 + \varepsilon 1$ defined by the odd 1-form ω and the even 1-form a . Suppose b is an odd 1-form and c an even 1-form on \mathcal{M} satisfying $d_{\mathcal{M}} a = \omega \wedge c - a \wedge b$.*

(1) There exist $\alpha, \beta, \xi \in \Omega^1(M)$ on the base manifold M such that

$$(4-10) \quad \begin{cases} d_M p(\omega) = p(\omega) \wedge \alpha, \\ d_M \alpha = p(\omega) \wedge \beta, \\ d_M p(b) = p(\omega) \wedge \xi. \end{cases}$$

(2) We have

$$(4-11) \quad \begin{cases} \varphi_{\mathcal{M}, \mathcal{F}}(H_1) = [\beta \wedge \alpha \wedge p(\omega)], \\ \varphi_{\mathcal{M}, \mathcal{F}}(H_2) = [\xi \wedge \alpha \wedge p(\omega)], \\ \varphi_{\mathcal{M}, \mathcal{F}}(H_3) = [\xi \wedge p(b) \wedge p(\omega)]. \end{cases}$$

Proof. (1) Define α, β, ξ by

$$(4-12) \quad \begin{cases} \alpha = p \circ \omega_{\mathcal{F}}((x \partial/\partial x)^*) \\ \beta = p \circ \omega_{\mathcal{F}}((\frac{1}{2}x^2 \partial/\partial x)^*) \\ \xi = p \circ \omega_{\mathcal{F}}((x\theta \partial/\partial \theta)^*) \end{cases}$$

Applying $p \circ \omega_{\mathcal{F}}$ to the identity

$$\partial \left(\frac{\partial}{\partial x} \right)^* = \left(\frac{\partial}{\partial x} \right)^* \wedge \left(x \frac{\partial}{\partial x} \right)^* - \left(\frac{\partial}{\partial \theta} \right)^* \wedge \left(\theta \frac{\partial}{\partial x} \right)^*,$$

and using Lemma 2.1, we obtain

$$d_M(p(\omega)) = p(\omega) \wedge \alpha.$$

Applying $p \circ \omega_{\mathcal{F}}$ to the identity

$$\partial \left(x \frac{\partial}{\partial x} \right)^* = \left(\frac{\partial}{\partial x} \right)^* \wedge \left(\frac{x^2}{2} \frac{\partial}{\partial x} \right)^* - \left(\frac{\partial}{\partial \theta} \right)^* \wedge \left(\theta x \frac{\partial}{\partial x} \right)^* - \left(x \frac{\partial}{\partial \theta} \right)^* \wedge \left(\theta \frac{\partial}{\partial x} \right)^*,$$

and using Lemma 2.1, we get

$$d_M(\alpha) = p(\omega) \wedge \beta.$$

Applying $p \circ \omega_{\mathcal{F}}$ to the identity

$$\partial \left(\theta \frac{\partial}{\partial \theta} \right)^* = \left(\frac{\partial}{\partial x} \right)^* \wedge \left(x\theta \frac{\partial}{\partial \theta} \right)^*$$

and using Lemma 2.1, we obtain

$$d_M(p(b)) = p(\omega) \wedge \xi.$$

Thus, there exist 1-forms α, β, ξ satisfying (4-10).

(2) We must now check that the cohomology classes

$$[\beta \wedge \alpha \wedge p(\omega)], [\xi \wedge \alpha \wedge p(\omega)], [\xi \wedge p(b) \wedge p(\omega)] \in H^3(M)$$

do not depend on the chosen 1-forms α, β, ξ that satisfy (4-10).

The 1-form $[\beta \wedge \alpha \wedge p(\omega)]$ is the Godbillon–Vey class of the codimension-1 foliation $F_{\mathcal{F}}$ constructed from \mathcal{F} on the base manifold in Section 2C(a). This $F_{\mathcal{F}}$ is defined by $p(\omega)$. The fact that $[\beta \wedge \alpha \wedge p(\omega)]$ does not depend on the choice of α and β was proved in [Godbillon and Vey 1971].

From the identity $\xi \wedge \alpha \wedge p(\omega) = -(d_M p(b)) \wedge \alpha = \xi \wedge d_M p(\omega)$ follows that $[\xi \wedge \alpha \wedge p(\omega)]$ does not depend on the 1-forms α, β, ξ ; while from the identity $\xi \wedge p(b) \wedge p(\omega) = -(d_M p(b)) \wedge p(b)$ follows that $[\xi \wedge \alpha \wedge p(\omega)]$ does not depend on the 1-forms α, β, ξ . This completes the proof. \square

Example 4.10. Let M be a manifold and F a foliation of codimension 1 on M , defined by $\omega \in \Omega^1(M)$ with $d_M \omega = \omega \wedge \alpha$. Assume moreover that the Godbillon–Vey class $-\alpha \wedge d_M \alpha \in H^3(M)$ is not zero.

Let $E \rightarrow M$ be the trivial 1-dimensional bundle $E = \mathbb{R} \times M$. Consider the supermanifold \mathcal{M} with $\mathcal{O}(\mathcal{M}) = \Gamma(\wedge E)$, and denote by θ the unique odd parameter corresponding to some constant section of E . There is a canonical embedding $I : \Omega^1(M) \rightarrow \Omega^1(\mathcal{M})$ given by the pull-back of the canonical projection $\mathcal{M} \rightarrow M$. Of course, $p \circ I = \text{Id}_{\Omega^1(M)}$.

Define a superfoliation of codimension $1 + \varepsilon 1$ by the odd 1-form $I(\omega)$ and the even 1-form $a = d\theta + \theta I(\alpha)$. We check that these 2-forms define a superfoliation. We have $d_{\mathcal{M}} I(\omega) = I(\omega) \wedge I(\alpha)$ and

$$(4-13) \quad d_{\mathcal{M}} a = d\theta \wedge I(\alpha) + \theta d_{\mathcal{M}} I(\alpha) = (d\theta + \theta I(\alpha)) \wedge I(\alpha) + \theta I(d_M \alpha).$$

Since $d_M \alpha = \omega \wedge \xi$ for some $\xi \in \Omega^1(M)$ [Godbillon and Vey 1971], we have

$$d_{\mathcal{M}} a = a \wedge I(\alpha) + I(\omega) \wedge \theta I(\xi).$$

Therefore, the pair $(a, I(\omega))$ defines a superfoliation of codimension $1 + \varepsilon 1$.

Now, from (4-13) we see that $b = \alpha$ and $p(b) = \alpha$. By (4-11), we obtain

$$\varphi_{\mathcal{M}, \mathcal{F}}(H_1) = \varphi_{\mathcal{M}, \mathcal{F}}(H_2) = \varphi_{\mathcal{M}, \mathcal{F}}(H_3) = -[\alpha \wedge d_M \alpha] \neq 0.$$

Example 4.11. We describe a superfoliation of codimension $1 + \varepsilon 1$ with trivialized normal bundle such that $\varphi_{\mathcal{M}, \mathcal{H}}(H_1) = 0$ and $\varphi_{\mathcal{M}, \mathcal{H}}(H_3) \neq 0$.

Consider the supermanifold given by the trivial 1-vector bundle E over the 3-dimensional torus $T^3 \simeq (S^1)^3$. Let $x, y, z \in S^1$ be the coordinates of T^3 , and let θ be the odd parameter corresponding to the constant section of E . Take $f(x), g(x)$ two smooth functions on S^1 with $\int_{S^1} W(f, g) \neq 0$, where W is the Wronskian.

We leave the reader to check that the odd 1-form $\omega = dx$ and the even 1-form

$$a = d\theta + \theta(f(x)dy + g(x)dz)$$

define a superfoliation of codimension $1 + \varepsilon 1$. In this case, we can choose $b = f(x)dy + g(x)dz$, and it is routine to check that

$$\varphi_{\mathcal{M}, \mathcal{H}}(H_3) = [W(f, g) dx \wedge dy \wedge dz]$$

Since $[W(f, g) dx \wedge dy \wedge dz] = (\int_{S^1} W(f, g)) [dx \wedge dy \wedge dz]$, it follows that $\varphi_{\mathcal{M}, \mathcal{H}}(H_3)$ is a nonzero class in de Rham cohomology. Since $p(\omega) = dx$ is a closed 1-form, $\varphi_{\mathcal{M}, \mathcal{H}}(H_1) = 0$.

4C(c). *Superfoliations of codimension $0 + \varepsilon m$.* According to Proposition A.10, we have $H^*(\text{Vect}(0, m)_0) \simeq H^*(gl_m)$, where $H^*(gl_m)$ is the Chevalley–Eilenberg cohomology of the Lie algebra gl_m . In such a case, therefore, $\varphi_{\mathcal{M}, \mathcal{F}}$ is a map from $H^*(gl_m)$ to $H^*(M)$. Moreover, if ω is a DGA homomorphism defining the superfoliation \mathcal{F} , then $\varphi_{\mathcal{M}, \mathcal{F}}$ is given by

$$(4-14) \quad d_k^l \rightarrow p(\omega((\theta_k \partial/\partial\theta_l)^*)) \quad \text{for any } k, l = 1, \dots, m.$$

Example 4.12. We compute this homomorphism in a particular case. The semidirect product $gl_m \ltimes \mathbb{R}^m$ of the Lie algebra gl_m with \mathbb{R}^m can be considered as a super-Lie algebra with even part gl_m and odd part \mathbb{R}^m . Let \mathcal{G} be the super-Lie group associated to this Lie algebra by Lie’s third theorem (which is true for super-Lie algebras [Tuynman 2004]). Since $gl_m \subset gl_m \ltimes \mathbb{R}^m$, the Lie group GL_n acts on the left on \mathcal{G} , and a superfoliation of codimension $0 + \varepsilon m$ is given on \mathcal{G} by this left action.

This superfoliation is defined by the left-invariant forms $\bar{a}_1, \dots, \bar{a}_m$ associated to the canonical basis $a_1, \dots, a_m \in (\mathbb{R}^m)^*$. Moreover, for any $k \in \{1, \dots, m\}$, we have

$$d_{\mathcal{G}} \bar{a}_k = \sum_{l=1}^m \bar{a}_l \wedge \bar{d}_k^l$$

where $\bar{d}_k^l \in T^*(\mathcal{G})$ are the left invariant 1-forms on \mathcal{G} associated to the canonical basis d_k^l of $gl(m)^*$.

By construction, therefore, there exists a homomorphism ω defining the superfoliation, such that

$$\omega((\theta_k \partial/\partial\theta_l)^*) = \bar{d}_k^l \quad \text{for any } k, l \in \{1, \dots, m\}.$$

By Equation (4-14), we obtain that, for any $H \in H^*(gl_m)$,

$$\varphi_{\mathcal{M}, \mathcal{F}}(H) = \bar{H},$$

where \bar{H} is the class of the left-invariant form on GL_m that corresponds to H . In other words, $\varphi_{\mathcal{M}, \mathcal{F}}$ is equal to the natural homomorphism $H^*(gl_m) \rightarrow H^*(GL_m)$.

It is well known that $H^*(GL_m) \simeq H^*(O(m))$, where $O(m)$ is the orthogonal group, and that $H^3(O(m)) \simeq \mathbb{R}$ and $H^3(gl_m) \simeq \mathbb{R}$ for $m \geq 3$. We leave the reader to check that the homomorphism $H^3(gl_m) \rightarrow H^3(GL_m)$ is not trivial. Therefore, we have proved the existence of nontrivial secondary characteristic classes for superfoliations of codimension $0 + \varepsilon m$ with $m \geq 3$.

4D. Conclusion. We summarize the results of the preceding sections.

Theorem 4.13. *For any supermanifold \mathcal{M} foliated by a superfoliation \mathcal{F} of codimension $n + \varepsilon m$ with trivialized normal bundle, there exists a map $\varphi_{\mathcal{M}, \mathcal{F}}$ from $H^*(\text{Vect}(n, m)_0)$ to $H^*(M)$ such that:*

- (1) $\varphi_{\mathcal{M}, \mathcal{F}}$ is a functor from the category of supermanifolds endowed with a superfoliation with trivialized normal bundle to the category of algebra homomorphisms from $H^*(\text{Vect}(n, m)_0)$ to an algebra A ;
- (2) for $(n, m) = (0, 1)$, or $(n, m) = (1, 1)$, or $n = 0$ and $m \geq 3$, there exists a supermanifold \mathcal{M} and a superfoliation \mathcal{F} of codimension $n + \varepsilon m$ such that $\varphi_{\mathcal{M}, \mathcal{F}}$ is not the zero map;
- (3) if \mathcal{M} is an ordinary smooth manifold endowed with a foliation F of codimension n with trivialized normal bundle, then $\varphi_{\mathcal{F}}$ reduces to the usual homomorphism of Bernstein, Bott, Fuchs, Haefliger and Rosenfeld [Bernstein and Rosenfeld 1973; Bott and Haefliger 1972; Fuchs 1986].

Proof. By “category of algebra homomorphisms from $H^*(\text{Vect}(n, m)_0)$ to an algebra A ” we mean the category whose objects are algebra homomorphism from $H^*(\text{Vect}(n, m)_0)$ to an algebra A , and whose arrows between objects

$$\varphi_A : H^*(\text{Vect}(n, m)_0) \rightarrow A \quad \text{and} \quad \varphi_B : H^*(\text{Vect}(n, m)_0) \rightarrow B$$

are homomorphisms $\varphi : A \rightarrow B$ such that $\varphi_B = \varphi \circ \varphi_A$. Conclusion (1) is now a paraphrase of Proposition 4.7.

Statement (2) follows from Examples 4.8, 4.10, and 4.12. Note that a more precise statement will be given in Remark 5.3.

It is proved in [Fuchs 1986, Section 3.2.B (page 231)] that, when we are given a foliation F of codimension n with trivialized normal bundle, the “classical” map of Bernstein, Bott, Fuchs, Haefliger and Rosenfeld is constructed by the passing to cohomology of a DGA homomorphism ω from $(\wedge \text{Vect}(n)^*, \partial)$ to $(\Omega(M), d_M)$, with the foliation F being defined by the 1-forms $\omega((\partial/\partial x_i)^*)$. This proves (3). \square

The functoriality of this construction allows us to say that the assignment $\mathcal{F} \mapsto \varphi_{\mathcal{F}}(H)$, for any $H \in H^*(\text{Vect}(n, m)_0)$, defines a *secondary characteristic class of superfoliations with trivialized normal bundle*.

We have defined in Proposition 2.18 a way to construct from any flat trivial foliated connection (M, F, E, ∇) a supermanifold $\mathcal{M}(M, E)$ whose base manifold is M , and a superfoliation $\mathcal{F}(M, F, E, \nabla)$ on \mathcal{M} with trivialized normal bundle. We call *secondary characteristic classes of flat trivial foliated connections* the secondary characteristic classes of this superfoliation. More precisely, we have associated a homomorphism from $H^*(\text{Vect}(n, m)_0)$ to $H^*(M)$ to any flat trivial foliated connection on M .

Remark 4.14. One may ask whether, for any $H \in H^*(\text{Vect}(n, m)_0)$, there is a superfoliation \mathcal{F} and a supermanifold \mathcal{M} with $\varphi_{\mathcal{M}, \mathcal{F}}(H) \neq 0$. This question remains open even in the case of foliations on smooth manifolds, and therefore we cannot hope to find a simple answer.

Remark 4.15. According to Section 2D, if the superfoliation \mathcal{F} of codimension $n + \varepsilon m$ does not have a trivialized normal bundle, then we can replace it by a superfoliation $\text{pr}^*(\mathcal{F})$ with trivialized normal bundle, over some $GL(n, m)$ -principal bundle $\mathcal{P}_{\mathcal{F}} \rightarrow \mathcal{M}$. The map $\varphi_{\mathcal{P}_{\mathcal{F}}, \text{pr}^*\mathcal{F}}$ takes values in the cohomology of the base manifold of $\mathcal{P}_{\mathcal{F}}$, which is a $GL(n) \times GL(m)$ -principal bundle over M . Therefore, for any $H \in H^*(\text{Vect}(n, m))$, we can construct secondary characteristic classes of superfoliation of codimension $n + \varepsilon m$ (not necessarily with trivialized normal bundle), but these characteristic classes have values in the cohomology of some $GL(n) \times GL(m)$ -principal bundle over the base manifold M of \mathcal{M} . By Remark 2.21, this construction is functorial, that is, it behaves well with respect to pull-backs of superfoliations.

We now link this construction to the cohomology of $H(\text{Vect}(n, m))$ and the Godbillon–Vey classes constructed in [Koszul 1988].

The homomorphism ω constructed in Proposition 4.2 induces a homomorphism $\varphi_\omega : H^*(\text{Vect}(n, m)) \rightarrow H^*(\mathcal{M})$, where $H^*(\text{Vect}(n, m))$ is the cohomology of the superalgebra $\text{Vect}(n, m)$. The following diagram is commutative:

$$\begin{array}{ccc} H^*(\text{Vect}(n, m)) & \xrightarrow{\quad\quad} & H^*(\mathcal{M}) \\ \downarrow J & \searrow \varphi_\omega & \downarrow \hat{p} \\ H^*(\text{Vect}(n, m)_0) & \xrightarrow{\varphi_{\mathcal{M}, \mathcal{F}}} & H^*(M), \end{array}$$

with $J : H^*(\text{Vect}(n, m)_0) \rightarrow H^*(\text{Vect}(n, m))$ given by the inclusion $\text{Vect}(n, m)_0 \rightarrow \text{Vect}(n, m)$, and $\hat{p} : H^*(\mathcal{M}) \rightarrow H^*(M)$ induced by p .

According to Batchelor’s theorem, \hat{p} is indeed an isomorphism, and thus $\varphi_\omega = \hat{p}^{-1} \circ \varphi_{\mathcal{M}, \mathcal{F}} \circ J$ is independent of ω . To emphasize that this homomorphism does not depend on ω , we denote it by $\psi_{\mathcal{M}, \mathcal{F}}$. We cannot say that $\psi_{\mathcal{M}, \mathcal{F}}$ gives new secondary

characteristic classes, since

$$(4-15) \quad \psi_{\mathcal{M}, \mathcal{F}} = \hat{p} \circ \varphi_{\mathcal{M}, \mathcal{F}} \circ J.$$

In [Koszul 1988], a homomorphism from $H^*(\text{Vect}(0, m))$ to $H^*(M)$ is associated to any superfoliation of codimension $n + \varepsilon m$ with trivialized normal bundle, on a supermanifold of dimension $n + \varepsilon m$. It is easy to check that, by construction, this homomorphism coincides with $\psi_{\mathcal{M}, \mathcal{F}}$. It defines classes of superfoliations of dimension $0 + \varepsilon m$, called ‘‘Godbillon–Vey classes’’ by the author. By Equation (4-15), these Godbillon–Vey classes are among the classes we built in this article.

From Proposition A.11 and [Koszul 1988, Corollaire 1.2], it follows that $J : H^1(\text{Vect}(0, 1)) \rightarrow H^1(\text{Vect}(0, 1)_0)$ is an isomorphism. This implies that the class constructed in Section 4C(a) is equal to the class constructed in [Koszul 1988, Exemple 1].

We summarize:

Proposition 4.16. (1) *For any superfoliation \mathcal{F} of codimension $n + \varepsilon m$ with trivialized normal bundle, the following diagram is commutative:*

$$\begin{array}{ccc} H^*(\text{Vect}(n, m)) & \xrightarrow{\quad\quad\quad} & H^*(M) \\ \downarrow J & \searrow \psi_{\mathcal{M}, \mathcal{F}} & \downarrow \hat{p} \\ H^*(\text{Vect}(n, m)_0) & \xrightarrow{\quad\quad\quad} & H^*(M). \end{array}$$

- (2) *The Godbillon–Vey classes constructed in [Koszul 1988] are among the secondary characteristic classes of superfoliation constructed above.*
- (3) *In particular, for foliations of codimension $0 + \varepsilon 1$, the secondary class constructed in Section 4C(a) coincides with the class constructed in [Koszul 1988, Exemple 1].*

5. Foliated flat vector bundles

5A. Secondary characteristic classes on the base manifold. Let $F_{\mathcal{F}}$ be the codimension- n foliation induced by \mathcal{F} on M , as in Lemma 2.12. Since F has a trivialized normal bundle, according to Theorem 4.13 to any $H \in H^*(\text{Vect}(n))$ is associated an element $K \in H^*(M)$ by the ‘‘classical’’ construction of Bernstein, Bott, Fuchs, Haefliger and Rosenfeld [Bernstein and Rosenfeld 1973; Bott and Haefliger 1972; Fuchs 1986]. We would like to investigate the relation between this construction and our construction.

There is a natural inclusion i from $\text{Vect}(n)^*$ into $\text{Vect}(n, m)_0^*$, obtained by considering an element $(x_1^{i_1} \dots x_n^{i_n} \partial/\partial x_a)^* \in \text{Vect}(n)^*$ as an element of $\text{Vect}(n, m)_0^*$.

Lemma 5.1. *The inclusion i is a DGA homomorphism, and induces a map from $H^*(\text{Vect}(n))$ to $H^*(\text{Vect}(n, m)_0)$.*

Proof. The family

$$(5-1) \quad \left\{ \begin{array}{l} x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial x_i}, \\ \text{and} \\ x_1^{i_1} \dots x_n^{i_n} \theta_1^{j_1} \dots \theta_m^{j_m} \frac{\partial}{\partial \theta_j}, \end{array} \right. \begin{array}{l} i_1, \dots, i_n \in \mathbb{N}, \\ j_1, \dots, j_m \in \{0, 1\} \text{ with } \sum_{k=1}^n j_k \text{ even,} \\ \text{and } i \in \{1, \dots, n\}; \\ \\ i_1, \dots, i_n \in \mathbb{N}, \\ j_1, \dots, j_m \in \{0, 1\} \text{ with } \sum_{k=1}^m j_k \text{ odd,} \\ \text{and } j \in \{1, \dots, m\}, \end{array}$$

forms a basis of $\text{Vect}(n, m)_0$.

The θ -degree $(h_{i_1} \dots h_{i_k} \partial/\partial h_i) \in \text{Vect}(n, m)_0$ is the number of integers greater or equal to $n + 1$ in the list i_1, \dots, i_k, i . For example, the θ -degree of $x_1 \theta_1 \partial/\partial \theta_2$ or $x_3 \theta_2 \theta_1 \partial/\partial x_2$ is 2, and the θ -degree of $x_1^3 \partial/\partial x_1$ is 0.

If we enumerate the basis as described in (5-1), then the structure constant $\Gamma_{i,j}^k$ is equal to zero if the index k corresponds to an element of θ -degree 0 and one of the indices i or j corresponds to an element of nonzero θ -degree. Moreover, if i, j, k correspond to elements of the basis (5-1) with a vanishing θ -degree, then the structure constant $\Gamma_{i,j}^k$ is equal to the corresponding structure constant in $\text{Vect}(n)$. By the definition of the Chevalley–Eilenberg differential, this implies that $i(\wedge \text{Vect}(n)^*)$ is stable under ∂_0 , and that i is a DGA homomorphism. \square

Proposition 5.2. *Take $H \in H^*(\text{Vect}(n)^*)$ and $H' = i(H) \in H^*(\text{Vect}(n, m)_0)$. One has*

$$\varphi_{M,F}(H) = \varphi_{M,\mathcal{F}}(H').$$

Proof. If $\omega : \wedge \text{Vect}(n, m) \rightarrow \Omega(M)$ is a DGA homomorphism that defines \mathcal{F} , then $p \circ \omega \circ i : \wedge \text{Vect}(n) \rightarrow \Omega(M)$ is a DGA homomorphism that defines F . The diagram

$$\begin{array}{ccc} \wedge(\text{Vect}(n))^* & \xrightarrow{i} & \wedge(\text{Vect}(n, m)_0)^* \\ & \searrow p \circ \omega \circ i & \downarrow p \circ \omega \\ & & \Omega(M) \end{array}$$

is commutative. As a consequence, we have the commutative diagram

$$\begin{array}{ccc} H^*(\text{Vect}(n)) & \longrightarrow & H^*(\text{Vect}(n, m)_0) \\ & \searrow \varphi_{M,F} & \downarrow \varphi_{M,\mathcal{F}} \\ & & H^*(M). \end{array} \quad \square$$

Remark 5.3. The nontrivial secondary classes in Section 4C are not secondary classes of the induced foliation on the base manifold. We could therefore replace (2) in Theorem 4.13 by a more precise statement:

Let S be a supplement of $H^*(\text{Vect}(n))$ in $H^*(\text{Vect}(n, m)_0)$. For $(n, m) = (0, 1)$, or $(n, m) = (1, 1)$, or $n = 0$ and $m \geq 3$, there exists a supermanifold \mathcal{M} and a superfoliation \mathcal{F} of codimension $n + \varepsilon m$ such that the restriction of $\varphi_{\mathcal{M}, \mathcal{F}}$ to S is not the zero map.

5B. Secondary characteristic classes of foliated connections. Let \mathcal{F} be a superfoliation of codimension $n + \varepsilon m$ with trivialized normal bundle, defined by odd 1-forms $\omega_1, \dots, \omega_n$ and even 1-forms a_1, \dots, a_m . It is convenient to rename these forms b_1, \dots, b_{n+m} , where $b_i = \omega_i$ for all $i \in \{1, \dots, n\}$ and $b_{n+j} = a_j$ for all $j \in \{1, \dots, m\}$. In fact, we will in general use both notations at the same time. Let $(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}})$ be the trivial flat foliated connection associated to the superfoliation \mathcal{F} , as constructed in Section 2C.

Theorem 5.4. *The homomorphism $\varphi_{\mathcal{M}, \mathcal{F}} : H^*(\text{Vect}(n, m)_0) \rightarrow H^*(M)$ is completely determined by the flat trivial foliated connection $(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}})$.*

In other words, the theory of secondary characteristic classes of superfoliations can indeed be considered as a theory of secondary characteristic classes of flat foliated connections.

Proof. Let $(b_i^j)_{i, j \in \{1, \dots, n+m\}}$ be homogeneous 1-forms satisfying the relation

$$(5-2) \quad d_{\mathcal{M}} b_i = (-1)^{|j|} \sum_{j=1}^{n+m} b_j \wedge b_i^j$$

We divide the proof into four steps.

Step 1: $\varphi_{\mathcal{M}, \mathcal{F}}$ depends only on $(p(b_i^j))_{i, j \in \{1, \dots, n\}}$ and $(p(b_k^l))_{k, l \in \{n+1, \dots, n+m\}}$.

By Lemma 4.6, the secondary characteristic classes depend only on the 1-forms $(p(b_i^j))_{i, j \in \{1, \dots, n\}}$ and $(p(b_k^l))_{k, l \in \{n+1, \dots, n+m\}}$. Moreover, any family of 1-forms $(\tilde{b}_i^j)_{i, j \in \{1, \dots, n+m\}}$ such that (5-2) holds for some 1-forms $\tilde{b}_1, \dots, \tilde{b}_{n+m}$ that define the same superfoliation \mathcal{F} will define the same homomorphism $\varphi_{\mathcal{M}, \mathcal{F}}$ by Proposition 4.5.

Step 2: $\varphi_{\mathcal{M}, \mathcal{F}}$ depends only on $F_{\mathcal{F}}$ and $(p(b_k^l))_{k, l \in \{n+1, \dots, n+m\}}$.

After applying p to (5-2) and taking $i \in \{1, \dots, n\}$, we see that the 1-forms $(p(b_i^j))_{i, j \in \{1, \dots, n\}}$ satisfy

$$d_M(p(\omega_i)) = \sum_{j=1}^n p(\omega_j) \wedge p(b_i^j).$$

By Remark 2.13, the 1-forms $p(\omega_i)$, $i = 1, \dots, n$, define the foliation $F_{\mathcal{F}}$. For any other choice of 1-forms $\tilde{\omega}_1, \dots, \tilde{\omega}_n \in \Omega^1(M)$ and any 1-forms $\tilde{c}_i^j \in \Omega^1(M)$, $i, j \in \{1, \dots, n\}$, with $d_M \tilde{\omega}_i = \sum_{j=1}^n \tilde{\omega}_j \wedge \tilde{c}_i^j$, there exist 1-forms $\tilde{b}_1, \dots, \tilde{b}_{n+m} \in \Omega^1(\mathcal{M})$

defining \mathcal{F} , and 1-forms $\tilde{b}_i^j \in \Omega_1(\mathcal{M})$, $i, j \in \{1, \dots, n+m\}$, satisfying (5-2) such that, for all $i, j \in \{1, \dots, n\}$, the identity $p(b_i^j) = c_i^j$ holds. As a consequence, $\varphi_{\mathcal{M}, \mathcal{F}}$ is entirely determined by $F_{\mathcal{F}}$ and $(p(b_k^l))_{k,l \in \{n+1, \dots, n+m\}}$

Step 3: $\varphi_{\mathcal{M}, \mathcal{F}}$ depends only on $F_{\mathcal{F}}$ and on the restriction of $(p(b_k^l))_{k,l \in \{n+1, \dots, n+m\}}$ to the tangent space of the leaves of $F_{\mathcal{F}}$.

For $k \in \{n+1, \dots, n+m\}$, Equation (5-2) can be rewritten

$$d_{\mathcal{M}} a_{k-n} = \sum_{i=1}^n \omega_i \wedge b_k^i - \sum_{l=n+1}^{n+m} a_l \wedge b_k^l.$$

As consequence, for any even superfunctions $f_{k,j}^l$ with $k, j \in \{1, \dots, m\}$ and $l \in \{1, \dots, n\}$,

$$d_{\mathcal{M}} a_{k-n} = \sum_{i=1}^n \omega_i \wedge \left(b_k^i - \sum_{c=1}^m f_i^{k,c} a_c \right) - \sum_{l=n+1}^{n+m} a_l \wedge \left(b_k^l - \sum_{c=1}^n f_c^{k,l} \omega_c \right).$$

We have therefore

$$d_{\mathcal{M}} a_{k-n} = \sum_{i=1}^n \omega_i \wedge \tilde{b}_k^i - \sum_{l=n+1}^{n+m} a_l \wedge \tilde{b}_k^l,$$

where

$$(5-3) \quad \tilde{b}_k^i = b_k^i - \sum_{c=n+1}^{n+m} f_i^{k,c} a_c \quad \text{and} \quad \tilde{b}_k^l = b_k^l - \sum_{c=1}^n f_c^{k,l} \omega_c.$$

Consequently, we can add to the 1-forms $(b_k^l)_{k,l \in \{n+1, \dots, n+m\}}$ any linear combination of the 1-forms $\omega_1, \dots, \omega_n$ without modifying $\varphi_{\mathcal{M}, \mathcal{F}}$.

Applying p to (5-3), we obtain

$$(5-4) \quad p(\tilde{b}_k^i) = p(b_k^i) - \sum_{c=1}^n \text{Re}(f_c^{k,i}) p(\omega_c)$$

Therefore, we can add to the 1-forms $(p(b_k^l))_{k,l \in \{n+1, \dots, n+m\}}$ any linear combination of the 1-forms $p(\omega_1), \dots, p(\omega_n)$ without modifying $\varphi_{\mathcal{M}, \mathcal{F}}$. This implies that $\varphi_{\mathcal{M}, \mathcal{F}}$ depends only on the restriction of $(p(b_k^l))_{k,l \in \{n+1, \dots, n+m\}}$ to the tangent spaces of the leaves of $F_{\mathcal{F}}$.

Step 4: The restriction of $(p(b_k^l))_{k,l \in \{n+1, \dots, n+m\}}$ to the tangent space of the leaves of $F_{\mathcal{F}}$ depends only on $E_{\mathcal{F}}$ and $\nabla^{\mathcal{F}}$.

Let X be a vector field tangent to the leaves of $F_{\mathcal{F}}$, and $\mathcal{X} \in (\mathcal{X}_{\mathcal{F}})_0$ an even supervector field tangent to \mathcal{F} such that $\Pi(\mathcal{X}) = X$. One has, for any $k \in \{n+1, \dots, n+m\}$,

$$\begin{aligned} \nabla_X^{\mathcal{F}}(\rho(a_{k-n})) &= \rho(L_{\mathcal{X}} a_{k-n}) = \rho(\iota_{\mathcal{X}} d_{\mathcal{M}} a_{k-n}) = \rho\left(\iota_{\mathcal{X}} \sum_{j=1}^{n+m} (-1)^{|j|} b_j \wedge b_k^j\right) \\ &= \rho\left(\iota_{\mathcal{X}} \sum_{j=1}^n \omega_j \wedge b_k^j - \sum_{l=n+1}^{n+m} a_j \wedge b_k^l\right). \end{aligned}$$

By the definition of the map ρ , one has $\rho((\iota_{\mathcal{X}}\omega_j)b_{n+i}^j) = 0$ for any $j \in \{1, \dots, n+m\}$ and $\rho(\iota_{\mathcal{X}}(\omega_j \wedge b_{n+i}^j)) = 0$ for any $j \in \{1, \dots, n\}$. Therefore,

$$\nabla_X^{\mathcal{F}}(\rho(a_{k-n})) = -\rho\left(\sum_{j=1}^m (\iota_{\mathcal{X}}a_j \wedge b_{n+i}^{n+j})\right).$$

By Equation (2-1), we obtain

$$\nabla_X^{\mathcal{F}}(\rho(a_{k-n})) = -\sum_{l=n+1}^{n+m} (\operatorname{Re}(\iota_{\mathcal{X}}b_k^l))\rho(a_j).$$

By (2-4), the identity $\operatorname{Re}(\iota_{\mathcal{X}}b_k^l) = \iota_X p(b_k^l)$ holds, and thus

$$\nabla_X^{\mathcal{F}}(\rho(a_{k-n})) = -\sum_{l=n+1}^{n+m} (\iota_X p(b_k^l))\rho(a_l).$$

The restrictions of the 1-forms $(p(b_{n+i}^{n+j}))_{i,j \in \{1, \dots, m\}}$ to the tangent space of each leaf of $F_{\mathcal{F}}$ are therefore completely determined by the connection $\nabla^{\mathcal{F}}$. This completes the proof. \square

Remark 5.5. We have associated secondary characteristic classes to a flat trivial foliated vector bundle. In particular, we have associated characteristic classes to any $SO(n)$ -bundle over a manifold M , endowed with a flat foliated connection in the sense of [Kamber and Tondeur 1975]. In [Kamber and Tondeur 1974] or [Kamber and Tondeur 1975], characteristic classes are also associated to such objects. It should be interesting to investigate the relation with this construction, in particular with the map (4.4) from [Kamber and Tondeur 1974].

Remark 5.6. Before ending this section, we have to point out the relation between this approach and the theory of Γ -structures [Bott and Haefliger 1972]. A foliation on a supermanifold \mathcal{M} with base manifold M can be defined by a family $\{f_i\}_{i \in I}$ of local submersions onto $\mathbb{R}^{n,m}$, defined on an open covering $\{U_i\}_{i \in I}$ of M . For any two submersions f_i and f_j , there exists φ_i^j in $\operatorname{Diff}(\mathbb{R}^{n,m})$ such that $f_i = \varphi_i^j \circ f_j$, where $\operatorname{Diff}(\mathbb{R}^{n,m})$ is the pseudogroup of local diffeomorphisms of the supermanifold $\mathbb{R}^{n,m}$. This is (almost) the definition of a $\operatorname{Vect}(n, m)$ -structure from [Bott and Haefliger 1972]; the only difference is that Γ is not a subspace of local diffeomorphisms of some vector space. As a consequence, it should be possible to obtain again most constructions of the present paper by generalizing the results of [Bott and Haefliger 1972] to this case; note that the Lie algebra of $\operatorname{Diff}(\mathbb{R}^{n,m})$ is precisely $\operatorname{Vect}(n, m)_0$.

Appendix A. The cohomology of $\operatorname{Vect}(n, m)_0$

We prove Theorem 3.7 and, as an application, compute $H^*(\operatorname{Vect}(n, m)_0)$ in some particular cases. First, we will need some technical results about representations

of the Lie algebra $gl_n \oplus gl_m$. The methods of this section are mainly inspired by [Astashkevich and Fuchs 1993; Fuchs 1986].

For any vector space V and any $k \in \mathbb{N}$, we denote by $V^{\otimes k}$ the tensor-product of k copies of V . For any family of vector spaces V_1, \dots, V_k , we denote by $\bigotimes_{i=1}^k V_i$ the tensor product $V_1 \otimes V_2 \otimes \dots \otimes V_k$. We denote by $S^k(V)$ the space of elements of degree k in the symmetric algebra of V .

A1. Some results about representations of $gl_n \oplus gl_m$. Let V and W be vector spaces of dimension n and m , respectively. Let gl_n and gl_m be the Lie algebras of linear endomorphism of V and W , respectively. The Lie algebras gl_n and gl_m act on V and W , respectively, by $g \cdot v = g(v)$ and $h \cdot w = h(w)$, where $v \in V$, $g \in gl_n$, $w \in W$, $h \in gl_m$. Moreover, the dual spaces V^* and W^* are gl_n - and gl_m -modules, respectively, with actions

$$g \cdot v' = (-g)^*(v') \quad \text{and} \quad h \cdot w' = (-h)^*(w'),$$

where g^* and h^* are the dual endomorphisms and $v' \in V^*$, $w' \in W^*$. These actions extend naturally to an action of the Lie algebra $gl_n \oplus gl_m$ on

$$E_{p,q}^{k,l} = V^{\otimes k} \otimes (V^*)^{\otimes l} \otimes W^{\otimes p} \otimes (W^*)^{\otimes q}.$$

We give some lemmas about the representations of $gl_n \oplus gl_m$. It is well known (see [Fuchs 1986, Theorem 2.1.4] or [Howe 1989]) that, for any vector space V ,

$$\begin{cases} (V^{\otimes k} \otimes (V^*)^{\otimes l})^{gl_n} = 0 & \text{if } k \neq l, \\ (V^{\otimes k} \otimes (V^*)^{\otimes k})^{gl_n} = \bigoplus_{\sigma \in \Sigma_k} a_\sigma \sum_{i_1=1}^n \dots \sum_{i_k=1}^n x_{i_1} \otimes \dots \otimes x_{i_n} \otimes y_{i_{\sigma(1)}} \otimes \dots \otimes y_{i_{\sigma(n)}}, \end{cases}$$

where $a_\sigma \in \mathbb{R}$, $x_1, \dots, x_n \in V$ is a basis of V , and $y_1, \dots, y_n \in V^*$ is the dual basis. The following is an obvious generalization of this result:

Lemma A.1. *Let $\{x_i \mid i = 1, \dots, n\}$ be a basis of V and $\{y_i \in V^* \mid i = 1, \dots, n\}$ the dual basis. Let $\{\zeta_i \mid i = 1, \dots, m\}$ be a basis of W and $\{\eta_i \in W^* \mid i = 1, \dots, m\}$ the dual basis.*

(1) *If $k \neq l$ or $p \neq q$, then $(E_{p,q}^{k,l})^{gl_n \oplus gl_m} = 0$*

(2) *If $k = l$ and $p = q$, the space $(E_{p,p}^{k,k})^{gl_n \oplus gl_m}$ is generated by the elements*

$$g_{\sigma,\tau} = \sum_{s_1=1}^n \dots \sum_{s_k=1}^n \sum_{s'_1=1}^m \dots \sum_{s'_p=1}^m x_{s_1} \otimes \dots \otimes x_{s_k} \otimes y_{s_{\sigma(1)}} \otimes \dots \otimes y_{s_{\sigma(k)}} \otimes \zeta_{s'_1} \otimes \dots \otimes \zeta_{s'_p} \otimes \eta_{s'_{\tau(1)}} \otimes \dots \otimes \eta_{s'_{\tau(p)}},$$

where σ is a permutation of $\{1, \dots, k\}$ and τ a permutation of $\{1, \dots, p\}$.

Let H be a $gl_n \oplus gl_m$ -submodule of $V^{\otimes k} \otimes (V^*)^{\otimes m} \otimes W^{\otimes p} \otimes (W^*)^{\otimes l}$. Let G be the $gl_n \oplus gl_m$ -module

$$G := E_{p,q}^{k,l} / H.$$

Such a $gl_n \oplus gl_m$ -module is said to be of *type I*.

Lemma A.2. *Let π be the projection of $E_{p,q}^{k,l}$ onto G . One has:*

$$(A-1) \quad G^{gl_n \oplus gl_m} = \pi(E_{p,q}^{k,l})^{gl_n \oplus gl_m}.$$

Proof. Both spaces reduce to 0 if $k \neq l$ or if $p \neq q$. If $k = l$ and $p = q$, the center of $gl_n \oplus gl_m$ acts trivially, and then both G and $E_{p,q}^{k,l}$ are $sl_n \oplus sl_m$ -modules satisfying

$$(A-2) \quad G^{sl_n \oplus sl_m} = G^{gl_n \oplus gl_m}, \quad (E_{p,p}^{k,k})^{sl_n \oplus sl_m} = (E_{p,p}^{k,k})^{gl_n \oplus gl_m}.$$

Since $sl_n \oplus sl_m$ is a semisimple Lie algebra, any finite-dimensional $sl_n \oplus sl_m$ -module is the direct sum of the $sl_n \oplus sl_m$ -submodule of invariants with the $sl_n \oplus sl_m$ -submodule of coinvariants. In particular, we have the decompositions

$$\begin{cases} G = (sl_n \oplus sl_m) \cdot G \oplus G^{sl_n \oplus sl_m} \\ E_{p,p}^{k,k} = (sl_n \oplus sl_m) \cdot E_{p,p}^{k,k} \oplus (E_{p,p}^{k,k})^{sl_n \oplus sl_m} \end{cases}$$

Since π is a Lie algebra morphism,

$$\begin{cases} \pi((E_{p,p}^{k,k})^{sl_n \oplus sl_m}) \subset G^{sl_n \oplus sl_m} \\ \pi((sl_n \oplus sl_m) \cdot E_{p,p}^{k,k}) \subset (sl_n \oplus sl_m) \cdot G \end{cases}$$

Since π is onto, we must have

$$\begin{cases} \pi((E_{p,p}^{k,k})^{sl_n \oplus sl_m}) = G^{sl_n \oplus sl_m} \\ (sl_n \oplus sl_m) \cdot \pi(E_{p,p}^{k,k}) = (sl_n \oplus sl_m) \cdot G \end{cases}$$

The conclusion thus follows from (A-2). \square

We now introduce the $gl_n \oplus gl_m$ -modules that we will study. (In the sequel, we will often simply say “module” for a $gl_n \oplus gl_m$ -module.)

Definition A.3. Consider a family $L_1 = \{(\mu_a, \nu_a) \mid a = 1, \dots, K\}$ of pairs of nonnegative integers such that $\mu_a + \nu_a \geq 2$ and ν_a is even. Consider a family $L_2 = \{(p_b, q_b) \mid b = 1, \dots, K'\}$ of pairs of nonnegative integers such that $p_b + q_b \geq 2$ and q_b is odd. Let $L \in \mathbb{N}$ and let $\mathcal{L} = L_1 \sqcup L_2 \sqcup \{L\}$. We associate to \mathcal{L} the $gl_n \oplus gl_m$ -module

$$G_{\mathcal{L}} = \bigwedge_{a=1}^K (S^{\mu_a}(V) \otimes \bigwedge^{\nu_a} W \otimes V^*) \wedge \bigwedge_{b=1}^{K'} (S^{p_b}(V) \otimes \bigwedge^{q_b} W \otimes W^*) \wedge \bigwedge^L V^*.$$

Such a module is called a *module of type II*.

Our goal for now is to describe $(G_{\mathcal{F}})^{gl_n \oplus gl_m}$; see Section A2 for motivation. We describe a natural projection

$$\pi_{\mathcal{F}} : Q \rightarrow G_F$$

where $Q = E_{n, K'}^{m, K+L}$, with $m = \sum_{a=1}^K \mu_a + \sum_{b=1}^{K'} p_b$ and $n = \sum_{a=1}^K \nu_a + \sum_{b=1}^{K'} q_b$.

First, we can construct an isomorphism ψ of $gl_n \oplus gl_m$ -module from Q to the module

$$Q_{\mathcal{F}} = \bigotimes_{a=1}^K (V^{\otimes \mu_a} \otimes W^{\otimes \mu_a} \otimes V^*) \otimes \bigotimes_{b=1}^{K'} (V^{\otimes p_b} \otimes W^{\otimes q_b} \otimes W^*) \otimes (V^*)^{\otimes L}$$

obtained by permuting terms in the tensor product defining Q , according to the rule: allow permutation of the tensor product of two elements if and only if they do not belong to the same vector space. With this restriction, there is only one natural isomorphism ψ from Q to $Q_{\mathcal{F}}$.

Second, there is a natural projection

$$\pi' : \bigotimes_{a=1}^K (V^{\otimes \mu_a} \otimes W^{\otimes \nu_a} \otimes V^*) \otimes \left(\bigotimes_{b=1}^{K'} V^{\otimes p_b} \otimes W^{\otimes q_b} \otimes W^* \right) \otimes (V^*)^{\otimes L} \longrightarrow G_{\mathcal{F}}$$

constructed by taking the appropriate symmerizations and skew-symmetrizations. Define $\pi_{\mathcal{F}}$ as $\pi_{\mathcal{F}} = \pi' \circ \psi$.

Proposition A.4. (1) If $\sum_{a=1}^K \mu_a + \sum_{b=1}^{K'} p_b \neq K + L$ or if $\sum_{a=1}^K \nu_a + \sum_{b=1}^{K'} q_b \neq K'$, then $G_{\mathcal{F}}^{gl_n \oplus gl_m} = 0$.

(2) If there exists $a \in \{1, \dots, K\}$ with $\nu_a \neq 0$, or if there exists $b \in \{1, \dots, K'\}$ with $q_b \neq 1$, then $G_{\mathcal{F}}^{gl_n \oplus gl_m} = 0$.

Proof. Since $G_{\mathcal{F}}$ is a module of type I, Lemma A.1(1) and Lemma A.2 immediately imply statement (1). The identity $\sum_{a=1}^K \nu_a + \sum_{b=1}^{K'} q_b = K'$ implies statement (2), since q_b is odd for $b = 1, \dots, K$, and ν_a is even for $a = 1, \dots, K$, and since all these numbers are nonnegative. \square

From now on we assume that $q_1 = \dots = q_{K'} = 1$, $\nu_1 = \dots = \nu_K = 0$, and

$$(A-3) \quad \sum_{a=1}^K \mu_a + \sum_{b=1}^{K'} p_b = K + L.$$

Write $H = \sum_{a=1}^K \mu_a$.

The map $\pi_{\mathcal{M}}$ can now be easily described: for all $s_1, \dots, s_{K+L} \in \{1, \dots, n\}$, $t_1, \dots, t_{K'} \in \{1, \dots, m\}$, $s'_1, \dots, s'_{K'} \in \{1, \dots, m\}$, and $t'_1, \dots, t'_{K'} \in \{1, \dots, m\}$, we have

$$\begin{aligned}
\text{(A-4)} \quad \pi_{\mathcal{L}}(x_{s_1} \otimes \cdots \otimes x_{s_{K+L}} \cdot t_{s_1} \otimes \cdots \otimes y_{t'_{K'}} \zeta_{s'_1} \cdots \zeta_{s'_{K'}} \cdot \eta_{t'_1} \cdots \eta_{t'_{K'}}) \\
= \bigwedge_{i=0}^K R(s_{\mu_1+\cdots+\mu_{i-1}+1}, \dots, s_{\mu_1+\cdots+\mu_i}) \\
\quad (x_{s_{\mu_1+\cdots+\mu_i+1}} \cdots x_{s_{\mu_1+\cdots+\mu_{i+1}}} \cdot y_{t'_i}) \\
\wedge \bigwedge_{j=0}^{K'} R(s_{H+p_1+\cdots+p_{j-1}+1}, \dots, s_{H+p_1+\cdots+p_j}) \\
\quad (x_{s_{H+p_1+\cdots+p_{j-1}+1}} \cdots x_{s_{H+p_1+\cdots+p_j}} \cdot \zeta_{s'_j} \cdot \eta_{t'_j}) \\
\wedge \bigwedge_{k=K+1}^{K+L} y_{t'_k},
\end{aligned}$$

where, for all $h, a_1, \dots, a_h \in \mathbb{N}$, $R(a_1, \dots, a_h)$ is an integer that appears in the symmetrization map $\bigotimes^h V \rightarrow S^*(V)$. We leave the reader to check that indeed

$$\text{(A-5)} \quad R(a_1, \dots, a_h) = \prod_{i=1}^n K(i, [a_1, \dots, a_h])!.$$

Two indices $i, j \in \{1, \dots, K+L\}$ are said *symmetric with respect to \mathcal{L}* if there exists $k \in \{0, \dots, K-1\}$ such that $i, j \in \{\sum_{a=1}^k \nu_a, \dots, \sum_{a=1}^{k+1} \nu_a\}$, or if there exists $k \in \{0, \dots, K'-1\}$ such that $i, j \in \{\sum_{a=1}^k p_b, \dots, \sum_{a=1}^{k+1} p_b\}$. Intuitively, i and j are symmetric with respect to \mathcal{L} if the projection $\pi_{\mathcal{L}}$ maps the i -th and the j -th terms of the tensor product $V^{\otimes(K+L)}$ involved in the definition of Q to the same symmetric algebra $S^{\mu_a}(V)$ or to the same symmetric algebra $S^{p_b}(V)$.

Two indices $i, j \in \{1, \dots, K+L\}$ are said *antisymmetric with respect to \mathcal{L}* if $i \geq K+1$ and $j \geq K+1$. Intuitively, i and j are antisymmetric with respect to \mathcal{L} if the projection $\pi_{\mathcal{L}}$ maps the i -th and the j -th terms of the tensor product $(V^*)^{\otimes(K+L)}$ involved in the definition of Q to the exterior algebra $\bigwedge^L V^*$.

Lemma A.5. *Let σ be a permutation of $\{1, \dots, K+L\}$. If there are two indices $i, j \in \{1, \dots, K+L\}$ symmetric with respect to \mathcal{L} such that $\sigma(i), \sigma(j)$ are antisymmetric with respect to \mathcal{L} , then $g_{\sigma, \tau} = 0$.*

Proof. Consider two indices $i, j \in \{1, \dots, K+L\}$ symmetric with respect to \mathcal{L} such that $\sigma(i), \sigma(j)$ are antisymmetric with respect to \mathcal{L} . For any $k, l \in \{1, \dots, n\}$, the terms in (A-4) corresponding to $s_i = k$ and $s_j = l$ and the terms corresponding to $s_i = l$ and $s_j = k$ appear with opposite signs. By Equation (A-4), $\pi_{\mathcal{L}}(g_{\sigma, \tau})$ must therefore vanish. \square

Lemma A.6. *If $\mu_a \neq 2$ for some $a \in \{1, \dots, K\}$, or if $p_b \neq 1$ for some $b \in \{1, \dots, K\}$, then $G_{\mathcal{L}}^{g^l_m \oplus g^l_m} = 0$.*

Proof. If $\mu_a \geq 3$ for some a in $\{1, \dots, K\}$ or if $p_b \geq 2$ for some b in $\{1, \dots, K'\}$, then by (A-3) we must have $L > K + K'$. By a simple argument of cardinality, this implies that, for any permutation σ of $\{1, \dots, K+L\}$, there exist $i', j' \in \{K+1, \dots, K+L\}$ such that $\sigma^{-1}(i'), \sigma^{-1}(j')$ are symmetric with respect to \mathcal{L} . Therefore, by Lemma A.5, $\pi_{\mathcal{L}}(g_{\sigma, \tau}) = 0$. \square

Assume now that $\mu_1 = \dots = \mu_K = 2$ and $p_1 = \dots = p_{K'} = 1$. In this case,

$$G_{\mathcal{F}} = \bigwedge^K (S^2(V) \otimes V^*) \wedge \bigwedge^{K'} (V \otimes W \otimes W^*) \wedge \bigwedge^{K+K'} V^*.$$

Let \mathcal{A} be the exterior algebra over the space $S^2(V) \otimes V^* \oplus V \otimes W \otimes W^* \oplus V^*$.

Lemma A.7. *If σ and τ are permutations of $\{1, \dots, K+L\}$ and $\{1, \dots, K'\}$, respectively, then $\pi_{\mathcal{F}}(g_{\sigma, \tau})$ is an element of the algebra generated (with respect to the product \wedge) by elements of the form*

$$\begin{aligned} \sum_{c=1}^n \gamma_c^i y_c \wedge x_c x_i \cdot y_j &\in (S^2(V) \otimes V^*) \otimes V^*, & i, j \in \{1, \dots, n\}, \\ \sum_{c=1}^n y_c \wedge x_c \cdot \zeta_k \cdot \eta_l &\in (V \otimes W \otimes W^*) \wedge V^*, & k, l \in \{1, \dots, m\}, \end{aligned}$$

where, for any $i, c = 1, \dots, n$, γ_c^i is defined by $\gamma_c^i = 1$ for $i \neq c$ and by $\gamma_c^c = 2$.

Proof. Let σ be a permutation of $\{1, \dots, 2K+K'\}$. If there is an $i \in \{1, \dots, K\}$ such that $\sigma(2i-1) \geq K$ and $\sigma(2i+1) \geq K$, then Lemma A.5 implies that $\pi_{\mathcal{F}}(g_{\sigma, \tau}) = 0$. Assume now that such an $i \in \{1, \dots, K\}$ does not exist for σ ; then, for all $i \in \{1, \dots, K\}$, one of the two integers $\sigma(2i-1), \sigma(2i)$ is greater than K , and one is smaller or equal to K . We define a permutation σ' of $\{1, \dots, K\}$ by $\sigma'(i) = \min\{\sigma(2i-1), \sigma(2i)\}$.

By Lemma A.1(2), we have

$$\begin{aligned} \pi_{\mathcal{F}}(g_{\sigma, \tau}) = \sum_{s_1, \dots, s_{2K+K'}=1}^n \sum_{s'_1, \dots, s'_{K'}=1}^m \bigwedge_{i=1}^K R(s_{2i-1}, s_{2i})(x_{s_{2i-1}} x_{s_{2i}} \cdot y_{s_{\sigma(i)}}) \\ \wedge \bigwedge_{j=1}^{K'} x_{s_{2K+j}} \zeta_{s'_j} \cdot \eta_{s'_{\tau(j)}} \wedge \bigwedge_{k=K+1}^{2K+K'} y_{s_{\sigma(k)}}. \end{aligned}$$

From Equation (A-5), it follows that $R(s_{2i-1}, s_{2i}) = \gamma_{s_{2i-1}}^{s_{2i}}$. Therefore,

$$\begin{aligned} \pi_{\mathcal{F}}(g_{\sigma, \tau}) = \varepsilon \sum_{s_1, \dots, s_K=1}^n \sum_{s'_1, \dots, s'_{K'}=1}^m \bigwedge_{i=1}^K \left(\sum_{c=1}^n y_c \wedge \gamma_c^{s_i} x_c x_{s_i} \cdot y_{s_{\sigma'(i)}} \right) \\ \wedge \bigwedge_{j=1}^{K'} \left(y_c \wedge \sum_{c=1}^n x_c \cdot \zeta_{t_i} \cdot \eta_{t_{\tau(i)}} \right) \end{aligned}$$

for some $\varepsilon \in \{-1, +1\}$. \square

We recapitulate:

Theorem A.8. *Let $G_{\mathcal{F}}$ be a $gl_n \oplus gl_m$ -module of type II (as in Definition A.3).*

- *If $\sum_{a=1}^K \mu_a + \sum_{b=1}^{K'} p_b \neq K + L$, then $G_{\mathcal{F}}^{gl_n \oplus gl_m} = 0$.*
- *If one of the even integers $\{v_a\}_{a=1, \dots, K}$ is not 0, then $G_{\mathcal{F}}^{gl_n \oplus gl_m} = 0$.*
- *If one of the odd integers $\{q_b\}_{b=1, \dots, K'}$ is not 1, then $G_{\mathcal{F}}^{gl_n \oplus gl_m} = 0$.*
- *If one of the integers $\{\mu_a\}_{a=1, \dots, K}$ is not 2, then $G_{\mathcal{F}}^{gl_n \oplus gl_m} = 0$.*

- If one of the integers $\{p_b\}_{b=1,\dots,K'}$ is not 1, then $G_{\mathcal{L}}^{gl_n \oplus gl_m} = 0$.
- If $\mu_a = 2$ and $\nu_a = 0$ for $a = 1, \dots, n$, and if $p_b = q_b = 1$ for $b = 1, \dots, m$, then the space $G_{\mathcal{L}}^{gl_n \oplus gl_m}$ is contained in the subalgebra of \mathcal{A} generated by the elements

$$\begin{cases} \sum_{c=1}^n y_c \wedge (\gamma_c^i x_c \cdot x_i \cdot y_j), & i, j \in \{1, \dots, n\}, \\ \sum_{c=1}^n y_c \wedge x_c \cdot \zeta_k \cdot \eta_l, & k, l \in \{1, \dots, m\}. \end{cases}$$

Later, we will also need the following:

Lemma A.9 [Fuchs 1986]. *For any $gl_n \oplus gl_m$ -module E of finite dimension,*

$$H^*(gl_n \oplus gl_m, E) = H^*(gl_n \oplus gl_m, E^{gl_n \oplus gl_m}).$$

Actually, $H^*(gl_n \oplus gl_m, E) = H^*(gl_n \oplus gl_m, E^{gl_n \oplus gl_m}) = H^*(gl_n \oplus gl_m, \mathbb{R}) \otimes E^{gl_n \oplus gl_m}$.

A2. The cohomology of $\mathbf{Vect}(n, m)_0$ and the Weil algebra. Now we are able to prove Theorem 3.7. Let \mathcal{H} be the kernel of the DGA homomorphism $\tilde{\alpha} : W(gl_n \oplus gl_m) \rightarrow \bigwedge \mathbf{Vect}(n, m)_0^*$, and denote again by $\tilde{\alpha}$ the induced DGA homomorphism from $W(gl_n \oplus gl_m)/\mathcal{H}$ to $\bigwedge \mathbf{Vect}(n, m)_0^*$. We restate the theorem:

Theorem 3.7. *Let $\hat{\alpha}$ be the map from $H^*(W(gl_n \oplus gl_m)/\mathcal{H})$ to $H^*(\mathbf{Vect}(n, m)_0)$ induced by $\tilde{\alpha}$. The map $\hat{\alpha}$ is an isomorphism.*

Proof. We first describe $\tilde{\alpha} : W(gl_n \oplus gl_m) \rightarrow \bigwedge \mathbf{Vect}(n, m)_0^*$ precisely. By (3-12), the DGA homomorphism $\tilde{\alpha}$ from $W(gl_n \oplus gl_m)$ to $\bigwedge \mathbf{Vect}(n, m)_0^*$ is given by:

$$(A-6) \quad 1 \otimes a_{i,j} \mapsto (x_i \partial / \partial x_j)^*, \quad 1 \otimes d_{i,j} \mapsto (\theta_i \partial / \partial \theta_j)^*,$$

where $a_{i,j}$, for $i, j \in \{1, \dots, n\}$, and $d_{i,j}$, for $i, j \in \{1, \dots, m\}$, are bases that are dual to the canonical bases of gl_n and gl_m , respectively.

One can easily check that

$$(A-7) \quad \begin{cases} \partial \left(x_i \frac{\partial}{\partial x_j} \right)^* - \sum_{k=1}^n \left(x_i \frac{\partial}{\partial x_k} \right)^* \wedge \left(x_k \frac{\partial}{\partial x_j} \right)^* = \sum_{c=1}^n \left(\frac{\partial}{\partial x_c} \right)^* \wedge \left(\frac{x_c x_i}{\gamma_c^i} \frac{\partial}{\partial x_j} \right)^*, \\ \partial \left(\theta_i \frac{\partial}{\partial \theta_j} \right)^* - \sum_{k=1}^m \left(\theta_i \frac{\partial}{\partial \theta_k} \right)^* \wedge \left(\theta_k \frac{\partial}{\partial \theta_j} \right)^* = \sum_{c=1}^n \left(\frac{\partial}{\partial x_c} \right)^* \wedge \left(x_c \theta_i \frac{\partial}{\partial \theta_j} \right)^*. \end{cases}$$

Since $\tilde{\alpha}$ is a DGA homomorphism, by the definition of the differential of a Weil algebra [Guillemin and Sternberg 1999], we have

$$(A-8) \quad \begin{aligned} \tilde{\alpha}(a_{i,j} \otimes 1) &= \sum_{c=1}^n \left(\frac{\partial}{\partial x_c} \right)^* \wedge \left(\frac{x_c x_i}{\gamma_c^i} \frac{\partial}{\partial x_j} \right)^*, \\ \tilde{\alpha}(d_{i,j} \otimes 1) &= \sum_{c=1}^n \left(\frac{\partial}{\partial x_c} \right)^* \wedge \left(x_c \theta_i \frac{\partial}{\partial \theta_j} \right)^*. \end{aligned}$$

We compute the cohomology of $\text{Vect}(n, m)_0$ with the help of the Hochschild–Serre spectral sequence [Fuchs 1986] associated to the sub-Lie algebra of elements of weight zero. Its second term $E_{i,j}^2$ is

$$E_{i,j}^2 = H^i(\mathfrak{gl}_n \oplus \mathfrak{gl}_m, \bigwedge^j \bigoplus_{k \neq 0} F_k),$$

where F_k is the space of elements of weight k in $(\text{Vect}(n, m)_0)^*$. By Lemma A.9, $E_{i,j}^2$ reduces to

$$E_{i,j}^2 = H^i\left(\mathfrak{gl}_n \oplus \mathfrak{gl}_m, \left(\bigwedge^j \bigoplus_{k \neq 0} F_k\right)^{\mathfrak{gl}(n) \oplus \mathfrak{gl}_m}\right).$$

The $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ -module $\bigwedge^j \bigoplus_{k \neq 0} F_k$ is a direct sum of modules of type II. For example,

$$F_{-1} \simeq W^* \quad \text{and} \quad F_1 = (S^2(V) \otimes V^*) \oplus (V \otimes W \otimes W^*).$$

More generally, denoting by $[x]$ the integer part of $x \in \mathbb{R}$,

$$F_k \simeq \bigoplus_{i=0}^{\lfloor (k+1)/2 \rfloor} (V^{\otimes k+1-2i} \otimes W^{\otimes 2i} \otimes V^*) \oplus \bigoplus_{j=0}^{\lfloor k/2 \rfloor} (V^{\otimes k+1-(2j+1)} \otimes W^{\otimes (2j+1)} \otimes W^*).$$

Therefore, the modules $\bigwedge^{j_1} F_{i_1} \wedge \cdots \wedge \bigwedge^{j_k} F_{i_k}$, where $i_1, \dots, i_k \in \{-1, 1, 2, 3, \dots\}$ and $j_1, \dots, j_k \in \mathbb{N}$, are direct sums of modules of type II.

The $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ -module

$$\bigwedge^j F_{-1} \otimes \bigwedge^j F_1$$

is a module of type II with $\mu_a = 2$ and $\nu_a = 0$ for $a = 1, \dots, j$, with $p_b = 1$ and $q_b = 1$ for $b = 1, \dots, j$, and with $L = j$. By Theorem A.8, the modules $\bigwedge^{j_1} F_{i_1} \wedge \cdots \wedge \bigwedge^{j_k} F_{i_k}$ have no nontrivial space of invariants except those of the form $\bigwedge^j F_{-1} \wedge \bigwedge^j F_1$. Therefore,

$$\begin{cases} \left(\bigwedge_{k \neq 0}^{2j} F_k\right)^{\mathfrak{gl}_n \oplus \mathfrak{gl}_m} = \left(\bigwedge^j F_{-1} \otimes \bigwedge^j F_1\right)^{\mathfrak{gl}_n \oplus \mathfrak{gl}_m}, \\ \left(\bigwedge_{k \neq 0}^{2j+1} F_k\right)^{\mathfrak{gl}_n \oplus \mathfrak{gl}_m} = 0. \end{cases}$$

By Theorem A.8(6), we obtain that the $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ -invariant elements of $\bigwedge^j \bigoplus_{k \neq 0} F_k$ are elements of the subalgebra of $\bigwedge \text{Vect}(n, m)_0^*$ generated by

$$\begin{cases} \sum_{c=1}^n \gamma_c^i \left(\frac{\partial}{\partial x_c}\right)^* \wedge \left(x_c x_i \frac{\partial}{\partial x_j}\right)^*, & i, j \in \{1, \dots, n\}, \\ \sum_{c=1}^m \left(\frac{\partial}{\partial x_c}\right)^* \wedge \left(x_c \theta_k \frac{\partial}{\partial \theta_l}\right)^*, & k, l \in \{1, \dots, m\}. \end{cases}$$

From (3-3), the identity $\gamma_c^i(x_c x_i \partial/\partial x_j)^* = ((x_c x_i/\gamma_c^i) \partial/\partial x_j)^*$ holds. Hence, the generators can be written

$$(A-9) \quad \begin{cases} \sum_{c=1}^n \left(\frac{\partial}{\partial x_c} \right)^* \wedge \left(\frac{x_c x_i}{\gamma_c^i} \frac{\partial}{\partial x_j} \right)^*, & i, j \in \{1, \dots, n\}, \\ \sum_{c=1}^m \left(\frac{\partial}{\partial x_c} \right)^* \wedge \left(x_c \theta_k \frac{\partial}{\partial \theta_l} \right)^*, & k, l \in \{1, \dots, m\}. \end{cases}$$

Let $\mathcal{F} = \tilde{\alpha}(W(gl_n \oplus gl_m))$ be the sub-DGA of $\bigwedge^*(\text{Vect}(n, m)_0)$ that is the image of the complex $W(gl_n \oplus gl_m)$ through $\tilde{\alpha}$. By (A-8) and (A-9), we have

$$(A-10) \quad \left(\bigwedge^j \bigoplus_{k \neq 0} F_k \right)^{gl_n \oplus gl_m} \subset \mathcal{F}.$$

Moreover, by Equation (A-6) we have the identity

$$(A-11) \quad \mathcal{F} \cap F_0 = F_0 \simeq gl_n \oplus gl_m.$$

The cohomology of \mathcal{F} can be computed with the help of a spectral sequence whose second term is

$$\tilde{E}_{i,j}^2 = H^i \left(\mathcal{F} \cap F_0, \mathcal{F} \cap \left(\bigwedge^j \bigoplus_{k \neq 0} F_k \right) \right).$$

By (A-11), $\tilde{E}_{i,j}^2 = H^i(gl_n \oplus gl_m, \mathcal{F} \cap (\bigwedge^j \bigoplus_{k \neq 0} F_k))$. From Lemma A.9 we obtain $\tilde{E}_{i,j}^2 = H^i(gl_n \oplus gl_m, (\mathcal{F} \cap (\bigwedge^j \bigoplus_{k \neq 0} F_k))^{gl_n \oplus gl_m})$. By (A-10), the identity $\tilde{E}_{i,j}^2 = E_{i,j}^2$ holds.

Therefore, the cohomology of $\bigwedge \text{Vect}(n, m)_0^*$ is equal to the cohomology of the subcomplex $\mathcal{F} = \tilde{\alpha}(W(gl_n \oplus gl_m)) \simeq W(gl_n \oplus gl_m)/\mathcal{K}$. This completes the proof of Theorem 3.7. \square

As an application, we compute the cohomology of $H^*(\text{Vect}(n, m)_0)$ in some particular cases:

Proposition A.10. *The cohomology of $\text{Vect}(0, m)_0$ is isomorphic to the Chevalley–Eilenberg cohomology of the Lie algebra gl_m .*

Proof. The DGA homomorphism from $W(gl_m)$ to $\bigwedge \text{Vect}(0, m)_0^*$ is:

$$1 \otimes d_{i,j} \mapsto (\theta_i \partial/\partial \theta_j)^* \quad \text{and} \quad d_{i,j} \otimes 1 \mapsto 0.$$

The kernel of this homomorphism is $S(gl(m)) \otimes 1$. The cohomology of

$$W(gl_m)/S(gl(m)) \otimes 1 = \bigwedge gl_m$$

is the Chevalley–Eilenberg cohomology of the Lie algebra gl_m . \square

In particular:

Proposition A.11. *We have isomorphisms $H^1(\text{Vect}(0, 1)_0) \simeq H^0(\text{Vect}(0, 1)_0) \simeq \mathbb{R}$, and $H^1(\text{Vect}(0, 1)_0) = 0$. Moreover, $H^1(\text{Vect}(0, 1)_0)$ is generated by $(\theta \partial/\partial\theta)^*$.*

We also determine the cohomology of $\text{Vect}(1, 1)_0$:

Proposition A.12. *The cohomology of $\text{Vect}(1, 1)_0$ is given by $H^n(\text{Vect}(1, 1)_0) = 0$ if $n \neq 0, 3$, and by $H^0(\text{Vect}(1, 1)_0) = \mathbb{R}$ and $H^3(\text{Vect}(1, 1)_0) = \mathbb{R}^3$. Moreover, the latter is generated by the classes described in (A-13), (A-14), and (A-15).*

Proof. We denote by a and d the generators $a_{1,1}$ and $d_{1,1}$ of $gl_1 \oplus gl_1$. The DGA homomorphism from $W(gl_1 \oplus gl_1)$ to $\wedge \text{Vect}(1, 1)_0^*$ is given by

$$(A-12) \quad \begin{cases} 1 \otimes a \mapsto \left(x \frac{\partial}{\partial x}\right)^*, \\ 1 \otimes d \mapsto \left(\theta \frac{\partial}{\partial \theta}\right)^*, \\ a \otimes 1 \mapsto 2 \left(\frac{\partial}{\partial x}\right)^* \wedge \left(x^2 \frac{\partial}{\partial x}\right)^* = \left(\frac{\partial}{\partial x}\right)^* \wedge \left(\frac{x^2}{2} \frac{\partial}{\partial x}\right)^*, \\ d \otimes 1 \mapsto \left(\frac{\partial}{\partial x}\right)^* \wedge \left(x\theta \frac{\partial}{\partial \theta}\right)^*. \end{cases}$$

The kernel \mathcal{K} of this application is generated by $S^2(gl_1 \oplus gl_1) \otimes 1$. This implies that the cohomology vanishes in all degrees different from 3. We can now compute the cohomology of $W(gl_1 \oplus gl_1)/\mathcal{K}$. It is easy to check that $H^3(\text{Vect}(1, 1)_0) = \mathbb{R}^3$. Generators of $H^3(\text{Vect}(1, 1)_0)$ are given by

$$(A-13) \quad H_1 = \left[\left(\frac{x^2}{2} \frac{\partial}{\partial x}\right)^* \wedge \left(x \frac{\partial}{\partial x}\right)^* \wedge \left(\frac{\partial}{\partial x}\right)^* \right],$$

$$(A-14) \quad H_2 = \left[\left(x\theta \frac{\partial}{\partial \theta}\right)^* \wedge \left(x \frac{\partial}{\partial x}\right)^* \wedge \left(\frac{\partial}{\partial x}\right)^* \right],$$

$$(A-15) \quad H_3 = \left[\left(x\theta \frac{\partial}{\partial \theta}\right)^* \wedge \left(\theta \frac{\partial}{\partial \theta}\right)^* \wedge \left(\frac{\partial}{\partial x}\right)^* \right]. \quad \square$$

Appendix B. Proof of Lemma 4.3

Lemma 4.3. *Let d_1, \dots, d_n be odd 1-forms and d_{n+1}, \dots, d_{n+m} even 1-forms, forming a free family. If d^i , $i \in \{1, \dots, n+m\}$, are 2-forms on \mathcal{M} such that, for any $j \in \{1, \dots, n+m\}$,*

$$(B-1) \quad \sum_{i=1}^{n+m} d_i \wedge d^i = 0,$$

then there exist homogeneous 1-forms $d^{i,j}$, of parity $|i| + |l| + 1$, with

$$(B-2) \quad d^{i,l} = -(-1)^{(|i|+1)(|l|+1)} d^{l,i} \quad \text{and} \quad \sum_{i=1}^{n+m} d_i \wedge d^{i,l} = d^l.$$

Step 1: Checking that $d^{i;l} = (-1)^{(l+1)(l+1)} d^{l;i}$.

From $G^{t,u;i} = (-1)^{(l+1)(l+1)} G^{i,u;l}$, we obtain

$$(B-6) \quad \sum_{u=n+m+1}^{p+q} d_u \wedge G^{i,u;l} = (-1)^{(l+1)(l+1)} \sum_{u=n+m+1}^{p+q} d_t \wedge G^{l,u;i}.$$

Moreover, from (B-3) and (B-4), we obtain

$$(B-7) \quad F^{l,s;i} + (-1)^{(s+1)(l+|i|)} F^{i,l;s} + (-1)^{(l+1)(l+1)} F^{i,s;l} = 0.$$

It is then straightforward to check that

$$\begin{aligned} F^{l,s;i} + \frac{1}{2}(-1)^{(s+1)(l+|i|)} F^{l,i;s} \\ = -(-1)^{(l+1)(l+1)} \left(F^{i,s;l} + \frac{1}{2}(-1)^{(s+1)(l+|i|)} F^{i,l;s} \right). \end{aligned}$$

This and (B-6) imply that

$$(B-8) \quad d^{i;l} = -(-1)^{(l+1)(l+1)} d^{l;i}.$$

Step 2: Checking that $\sum_{i=1}^{n+m} d_i \wedge d^{i;l} = d^l$.

We compute $\sum_{i,s=1}^{n+m} (-1)^{(s+1)(l+|i|)} d_i \wedge d_s F^{i,l;s}$. It is equal to

$$\frac{1}{2} \sum_{i,s=1}^{n+m} (-1)^{(s+1)(l+|i|)} d_i \wedge d_s F^{i,l;s} + \frac{1}{2} \sum_{i,s=1}^{n+m} (-1)^{(s+1)(l+|i|)} d_s \wedge d_i \wedge F^{s,l;i}.$$

From the identity $(-1)^{(s+1)(l+|i|)} (-1)^{(s+1)(l+1)} = (-1)^{(l+1)(s+1)}$, we deduce

$$\begin{aligned} \sum_{i,s=1}^{n+m} (-1)^{|s|(l+1)+|i|(l+1)} d_i \wedge d_s \wedge F^{i,l;s} \\ = \frac{1}{2} d_i \wedge d_s \wedge \left((-1)^{(s+1)(l+|i|)} F^{i,l;s} + \frac{1}{2} (-1)^{(l+1)(s+1)} F^{s,l;i} \right) \\ = \frac{1}{2} d_i \wedge d_s \wedge \left((-1)^{(s+1)(l+|i|)} F^{i,l;s} + (-1)^{(l+1)(l+1)} F^{l,s;i} \right). \end{aligned}$$

From (B-7), we obtain

$$(B-9) \quad \sum_{i,s=1}^{n+m} (-1)^{(s+1)(l+|i|)} d_i \wedge d_s F^{i,l;s} = -\frac{1}{4} \sum_{i,s=1}^{n+m} d_i \wedge d_s \wedge F^{i,s;l}.$$

The result now follows immediately from (B-5) and (B-9). \square

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