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This paper initiates a systematic study of the relation of commensurability of surface automorphisms, or equivalently, fibered commensurability of 3-manifolds fibering over S^1. We show that every hyperbolic fibered commensurability class contains a unique minimal element. The situation for toroidal manifolds is more complicated, and we illustrate a range of phenomena that can occur in this context.

1. Introduction

The main purpose of this paper is to study the equivalence relation of *commensurability* of surface automorphisms. Informally, two surface automorphisms are commensurable if they lift to automorphisms of a finite covering surface that have nontrivial common powers. Equivalently, a surface automorphism determines a foliation of a 3-manifold by closed surfaces, and two automorphisms are commensurable if their corresponding 3-manifolds admit common finite covers for which the pulled-back foliations are isotopic. Thus commensurability of surface automorphisms is a special case of the study of commensurability of 3-manifolds equipped with a certain kind of geometric structure; again informally, we call this commensurability relation *fibered commensurability*.

The relation of commensurability of 3-manifolds is well-studied; see, for example, [Thurston 1979, Chapter 6; Borel 1981; Macbeath 1983; Neumann 1997; Behrstock and Neumann 2010]. When studying commensurability in a given context, the most important distinction to make is between those commensurability classes that admit *finitely many* minimal elements, and those that admit *infinitely* many. For example, amongst hyperbolic 3-manifolds, this is precisely the distinction between nonarithmetic and arithmetic commensurability classes; see [Margulis 1991; Borel 1981], for instance. This distinction has a cleaner statement if one is prepared to work in the category of orbifolds: each commensurability class of nonarithmetic hyperbolic 3-manifolds contains a unique minimal element.

Fibered commensurability is more rigid than ordinary commensurability. However, a given 3-manifold can fiber in infinitely many different ways. For Seifert

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manifolds, there is exactly one fibered commensurability class of surface bundles of all closed (resp. with torus boundary) Seifert fibered manifolds whose fiber has negative Euler characteristic, and this class contains infinitely many minimal elements. On the other hand, in the hyperbolic world we obtain:

Theorem 3.1 (Hyperbolic Theorem). *Every commensurability class of hyperbolic fibered pairs contains a unique* (*orbifold*) *minimal element.*

An immediate corollary is that for a fibered hyperbolic 3-manifold M, each fibered commensurability class contains at most finitely many fibrations of M; hence M has either one fibered commensurability class, or infinitely many fibered commensurability classes.

The reducible case is more complicated:

Examples 5.3 and 5.5 (Toroidal cases). *There are examples of graph manifolds with infinitely many fibered commensurability classes, and a single graph manifold can fiber in infinitely many ways in a single commensurability class.*

As these results suggest, obstructions to commensurability of surface automorphisms arise from their behavior on pseudo-Anosov orbits, and near their reducing systems. We describe such obstructions in detail.

In Section 2, we give basic definitions and illustrate their meaning, in the special case of commensurability of spherical and toral automorphisms. We recall the Nielsen–Thurston classification of surface automorphisms, and discuss a "normal form" for automorphisms. This standard material may be skipped by the expert.

In Section 3, we study fibered commensurability of hyperbolic manifolds, and prove Theorem 3.1. We also list some commensurability invariants of pseudo-Anosov automorphisms (Lemma 3.10 and Proposition 3.15), and describe examples that illustrate their use.

Finally, Section 4 and Section 5 are devoted to the case of reducible automorphisms, especially of graph manifolds. In Section 4 we define certain numerical commensurability invariants for reducible maps (Theorem 4.3, as well as Proposition 4.11), and give many examples. In Section 5 we give examples of graph manifolds with infinitely many incommensurable fibrations, including one with boundary (Example 5.3) that also admits infinitely many *commensurable* (but nonisomorphic) fibrations, and a closed one (Example 5.5) that admits incommensurable fibrations of the same genus.

2. Fibered commensurability

Basic definitions. Let F be a compact surface. An *automorphism* ϕ of F is an isotopy class of self-homeomorphisms of F. We use the notation (F, ϕ) where ϕ is an automorphism of F.

Remark 2.1. When F has boundary, it is more usual to study isotopy classes of self-homeomorphisms fixed pointwise on the boundary. However, since we are interested in automorphisms which might permute boundary components, we adhere to this nonstandard convention.

One surface automorphism can "cover" another in two distinct ways: either topologically (in the sense that one surface covers the other) or dynamically (in the sense that one automorphism is a power of another). We consider covering in both senses in the sequel. More formally, we make the following definition.

Definition 2.2. A pair $(\tilde{F}, \tilde{\phi})$ *covers* (F, ϕ) if there is a finite cover $\pi : \tilde{F} \to F$ and representative homeomorphisms $\tilde{f}$ and f of $\tilde{\phi}$ and ϕ respectively so that $\pi \circ \tilde{f} = f \circ \pi$ as maps $\tilde{F} \to F$.

Remark 2.3. The relation of covering is transitive: if (F_1, ϕ_1) covers (F_2, ϕ_2), and (F_2, ϕ_2) covers (F_3, ϕ_3), then (F_1, ϕ_1) covers (F_3, ϕ_3). This follows by appealing to a "normal form" for representative homeomorphisms which is compatible with finite covers. This normal form is well-known, and summarized in Theorem 2.14 and Proposition 2.15 below.

An automorphism ϕ of F determines an outer automorphism ϕ_* of $\pi_1(F)$ preserving peripheral subgroups, and by the well-known theorem of Dehn and Nielsen [Nielsen 1927], this correspondence is a bijection. A cover $\tilde{F}$ determines a conjugacy class of subgroups G of $\pi_1(F)$, and an automorphism ϕ of F lifts to an automorphism $\tilde{\phi}$ of $\tilde{F}$ if and only if G and $\phi_*(G)$ are conjugate in $\pi_1(F)$. However, a particular lift $\tilde{\phi}$ depends on a choice of conjugating element. Thus a finite cover of surfaces $\tilde{F} \to F$ might determine zero, one, or many covers of automorphisms $(\tilde{F}, \tilde{\phi}) \to (F, \phi)$ (even if $\tilde{\phi}$ is primitive).

Example 2.4. If $\tilde{F} \to F$ is any finite cover, then (F, id) is covered by $(\tilde{F}, \psi)$ where ψ is any element of the deck group of the cover.

Definition 2.5. Two automorphisms (F_1, ϕ_1) and (F_2, ϕ_2) are *commensurable* if there is a surface $\tilde{F}$, automorphisms $\tilde{\phi}_1$ and $\tilde{\phi}_2$ of $\tilde{F}$, and nonzero integers k_1 and k_2, so that $(\tilde{F}, \tilde{\phi}_i)$ covers (F_i, ϕ_i) for $i = 1, 2$, and if $\tilde{\phi}_1^{k_1} = \tilde{\phi}_2^{k_2}$ as automorphisms of $\tilde{F}$. Moreover say (F_1, ϕ_1) and (F_2, ϕ_2) are *topologically commensurable* if $|k_1| = |k_2| = 1$, and *dynamically commensurable* if $\tilde{F} = F_1 = F_2$.

Commensurability of automorphisms is readily seen to be an equivalence relation, and is the main object of study in this paper.

Statements about surfaces and automorphisms can usefully be translated into statements about 3-manifolds with certain types of foliations. These objects — "fibered pairs", to be defined below — admit natural generalizations to objects called *orbifold* fibered pairs, that are awkward to discuss in the language of surfaces

and automorphisms. Certain theorems in this paper are more elegantly stated and proved in this category. A basic reference for the theory of orbifolds is [Thurston 1979, Chapter 13].

Definition 2.6. A *fibered pair* is a pair $(M, \mathcal{F})$ where M is a compact 3-manifold with boundary a union of tori and Klein bottles, and $\mathcal{F}$ is a foliation by compact surfaces. More generally, an *orbifold fibered pair* is a pair $(O, \mathcal{G})$ where O is a compact 3-orbifold, and $\mathcal{G}$ is a foliation of O by compact 2-orbifolds.

At interior points (resp. boundary points) an orbifold fibered pair $(O, \mathcal{G})$ looks locally like the quotient of an open ball in $\mathbb{R}^3$ (resp. a relatively open ball in a vertical half-space) foliated by horizontal planes by a finite group of smooth foliation-preserving homeomorphisms.

A surface automorphism (F, ϕ) determines a fibered pair whose underlying manifold is an F bundle over S^1 with monodromy ϕ, and whose foliation is the foliation by surface fibers (which are all homeomorphic to F). If we want to emphasize its dynamical origin, we use the notation $[F, \phi]$ in the sequel to denote the fibered pair associated to the automorphism (F, ϕ).

If the underlying orbifold O is *good* (i.e., it admits a finite manifold cover) then $(O, \mathcal{G})$ is finitely covered by a pair $(M, \mathcal{F})$ where M is a manifold, and every leaf of $\mathcal{F}$ is a compact surface. After passing to a further 2-fold cover if necessary, we can assume $\mathcal{F}$ is co-orientable, in which case M fibers over S^1 in such a way that the leaves of $\mathcal{F}$ are the fibers.

Definition 2.7. A fibered pair $(\tilde{M}, \tilde{\mathcal{F}})$ *covers* $(M, \mathcal{F})$ if there is a finite covering of manifolds $\pi : \tilde{M} \to M$ such that $\pi^{-1}(\mathcal{F})$ is isotopic to $\tilde{\mathcal{F}}$. Two fibered pairs $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are *commensurable* if there is a third fibered pair $(\tilde{M}, \tilde{\mathcal{F}})$ that covers both.

If $(M_i, \mathcal{F}_i)$ for $i = 1, 2$ are fibered pairs with co-orientable foliations, then they are commensurable in the sense of Definition 2.7 if and only if the associated surface automorphisms are commensurable. Thus, the category of fibered pairs enlarges the category of surface automorphisms in such a way that the definition of commensurability of a surface automorphism is the same, whichever category we use.

To stress that the definition of commensurability of fibered pairs depends on both the underlying 3-manifold and the foliation, we call this equivalence relation *fibered commensurability*.

The relation of covering is transitive, but it is not yet a partial order because of the existence of automorphisms of finite order. We must take such examples into account in order to define minimal elements with respect to commensurability.

Definition 2.8. We say that two fibered pairs $(M, \mathcal{F})$ and $(N, \mathcal{G})$ are *covering equivalent* if each covers the other. Call a covering equivalence class *minimal* if no representative covers any element of another covering equivalence class.

The relation of covering descends to a transitive relation on covering equivalence classes, and defines a partial order on such classes. Minimal classes are minimal with respect to this partial order.

Remark 2.9. Each covering equivalence class of fibered pairs $[F, \phi]$ contains exactly one fibered pair unless ϕ is periodic. In the periodic case, (F, ϕ) and (G, ψ) are in the same covering equivalent class if and only if $F = G$ and both ϕ and ψ generate the same finite cyclic group. With this understood, in the sequel we are relaxed in our terminology, and use the word "minimal element" when we really mean "minimal class".

Simple cases. For simplicity, we usually restrict attention to the case that F (and therefore M) is closed. However, because of the nature of the theory of surface automorphisms, to really understand this case we are forced to consider surfaces (and 3-manifolds) with boundary, associated to the restrictions of automorphisms to invariant subsurfaces.

Evidently, the sign of $\chi(F)$ is a commensurability invariant of (F, ϕ). In the case of fibered pairs (of good orbifolds), all leaves have the same sign, so we can speak unambiguously about fibered pairs with spherical, Euclidean, or hyperbolic leaves. We first discuss the situation when $\chi(F) \geq 0$.

Example 2.10 (spherical automorphisms). There is only one commensurability class, consisting of the bundles $S^2 \times S^1$ and $S^2 \tilde{\times} S^1$, each foliated by spheres, and $\mathbb{RP}^3 \# \mathbb{RP}^3$, which can be thought of as an S^2 bundle over a mirror orbifold. The elements $S^2 \tilde{\times} S^1$ and $\mathbb{RP}^3 \# \mathbb{RP}^3$ are minimal.

Example 2.11 (toral automorphisms). The mapping class group of a torus is isomorphic to $\mathrm{GL}(2, \mathbb{Z})$, and every automorphism has a linear representative. An automorphism can be periodic, reducible, or Anosov. From elementary linear algebra, automorphisms in different classes are not commensurable. We discuss each case in turn.

(1) Periodic case: there is only one commensurability class; moreover there are exactly two minimal elements, corresponding to the periodic automorphisms of order 4 and 6 on a square and hexagonal torus respectively.

(2) Reducible case: as automorphisms, each map (T, ϕ) is represented by a matrix which can be conjugated into the form

$$\phi \sim \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

where $n \neq 0$. So there is only one commensurability class and two minimal elements, corresponding to the conjugacy classes of matrices

$$\phi \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

(3) Anosov case: the resulting Sol manifolds are commensurable if and only if they are fibered commensurable, which occurs if and only if the logarithms of the dilatations of the automorphisms are commensurable as real numbers. Hence there are infinitely many fibered commensurability classes.

Standard form for surface automorphisms. In the remainder of the paper therefore we concentrate on the case of surfaces F with $\chi(F) < 0$. Furthermore, unless we explicitly say to the contrary, all surfaces F are assumed to be compact and connected.

A commensurability between automorphisms restricts to a commensurability between the underlying surfaces. A complete set of commensurability invariants of compact surfaces are the sign of Euler characteristic, and the property of possessing (or not possessing) a nonempty boundary.

Lemma 2.12. *Let F_1 and F_2 be compact surfaces with $\chi < 0$. If both or neither have nonempty boundary, they are commensurable. Otherwise they are incommensurable.*

The proof is elementary; see [Massey 1974], for example. Since every compact surface orbifold with $\chi < 0$ is good, the lemma extends to orbifolds.

Notation 2.13. Suppose Γ (resp. F') is a union of circles (resp. a compact subsurface) in F. Let $F \setminus \Gamma$ (resp. $F \setminus F'$) denote the compact surface obtained by splitting F along Γ (resp. removing $\mathrm{int}F'$, the interior of F').

Recall the Nielsen–Thurston classification of surface automorphisms.

Theorem 2.14 [Thurston 1988; Fathi et al. 1979]. *Let ϕ be an automorphism of a compact surface F. Then the isotopy class of ϕ has a representative* (*which by abuse of notation we continue to denote by ϕ*) *so that either*

(1) *ϕ has finite order, and $[F, \phi]$ is a Seifert manifold with $\mathbb{H}^2 \times \mathbb{R}$ geometry; or*

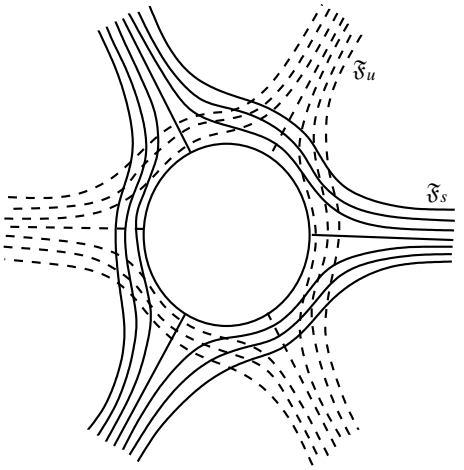
(2) *ϕ is pseudo-Anosov — i.e., F admits a pair of transversely measured singular foliations $\mathfrak{F}_s$ and $\mathfrak{F}_u$ with measures μ_s, μ_u, and there is a real number $\lambda > 1$ called the* dilatation *so that ϕ takes each foliation to itself, stretching μ_u by λ and compressing μ_s by $1/\lambda$ — and the interior of $[F, \phi]$ admits a complete hyperbolic structure of finite volume; or*

(3) *ϕ is reducible — i.e., there is a minimal nonempty embedded 1-manifold Γ in F with a ϕ-invariant tubular neighborhood $N(\Gamma)$ such that on each ϕ-orbit*

of $F \setminus N(\Gamma)$ the restriction of ϕ is either finite order or pseudo-Anosov, and $[F, \phi]$ is a 3-manifold with a JSJ decomposition (whose tori correspond to the ϕ orbits of Γ) into Seifert fibered and hyperbolic pieces.

In the sequel, we will need more precise control over the normal form of ϕ near the boundary of a subsurface on which ϕ is pseudo-Anosov. We say a representative pseudo-Anosov map ϕ on F with boundary is in *standard form* if it satisfies the following two conditions:

(1) Near each boundary circle, two p-pronged measured transverse foliations $(\mathfrak{F}^s, \mu^s)$ and $(\mathfrak{F}^u, \mu^u)$ have the form illustrated here (for the case p=3).



(2) On each ϕ-orbit on ∂F, the restriction of ϕ is periodic.

Proposition 2.15 [Jiang and Guo 1993]. *Each reducible map ϕ as in case* (3) *of Theorem 2.14 can be isotoped into a* standard form; *i.e.*:

(1) *The restriction of ϕ to each pseudo-Anosov orbit of $F \setminus N(\Gamma)$ is in standard form as above.*

(2) *The restriction of ϕ to each periodic orbit of $F \setminus N(\Gamma)$ is periodic.*

This completely fixes the behavior of ϕ on the complement of the regions $N(\Gamma)$. In the sequel we assume that each reducible map ϕ has been isotoped to its standard form in Proposition 2.15. Then for any such ϕ, there is some positive integer l so that ϕ^l is the identity on $\partial(F \setminus N(\Gamma(\phi)))$ and ϕ on $N(\Gamma)$ are Dehn twists along each $\gamma \in \Gamma(\phi)$ relative to $\partial(F \setminus N(\Gamma(\phi)))$.

Definition 2.16. Let ϕ be a reducible map. Say ϕ is *D-type* if it is generated by Dehn twists along components of $\Gamma(\phi)$; say ϕ is *D-type along* $\Gamma(\phi)$ if ϕ restricts to the identity along $\partial N(\Gamma(\phi))$.

Remark 2.17. Note that every ϕ has a power ϕ^l which is D-type along $\Gamma(\phi)$. Moreover, ϕ is a root of D-type, i.e., some power ϕ^l is D-type, if and only if ϕ is periodic on each ϕ-orbit of $F \setminus N(\Gamma)$. Alternatively, every ϕ is either a root of D-type or has pseudo-Anosov ϕ-orbits.

Finally we make the following notational convention. We denote surfaces in general by F, F_i, G and so on, and use $\Sigma_{g,n}$ to denote the surface of genus g with n boundary components. We sometimes abbreviate $\Sigma_{g,0}$ to Σ_g.

Seifert fibered case. Finite order automorphisms are very easy to understand. Suppose (F_1, ϕ_1) and (F_2, ϕ_2) have finite order, so that the manifolds $[F_1, \phi_1]$ and $[F_2, \phi_2]$ are Seifert manifolds with a product geometry. Each $[F_i, \phi_i]$ is finitely covered by a product $F_i \times S^1$. From Lemma 2.12 we can deduce:

Proposition 2.18. *There is exactly one fibered commensurability class of surface bundles of all closed* (*resp. with torus boundary*) *Seifert fibered manifolds whose fiber has negative Euler characteristic. This class contains infinitely many minimal elements.*

Proof. All that needs to be proved is that the class contains infinitely many minimal elements. A key observation is that if $\tilde{\phi}$ is primitive in $\mathrm{MCG}(\tilde{F})$ and has a fixed point near which it acts as a rotation through order p, the same is true of any $\phi \in \mathrm{MCG}(F)$ that it covers. This observation lets us construct infinitely many minimal elements, as follows.

For each genus $g > 1$, let ϕ_g be a maximum-order orientation-preserving periodic map on Σ_g. Then (see [Steiger 1935]) ϕ_g has order $4g+2$ (indeed there is a unique $\mathbb{Z}/(4g+2)\mathbb{Z}$ subgroup of $\mathrm{MCG}(\Sigma_g)$ up to conjugacy) and has exactly one fixed point, one periodic orbit of length 2 and one periodic orbit of length $2g+1$. Clearly (Σ_g, ϕ_g) is primitive, and (Σ_g, ϕ_g) and (Σ_g, ψ) cover each other if and only if $\psi = \phi_g^q$ for q coprime with $4g+2$. Now suppose (Σ_g, ϕ_g) covers (Σ_l, ψ) with $l \neq g$. Of course, we must have $l < g$. On the other hand by the observation above, ψ must have a fixed point near which it acts as a rotation through order $4g+2$, which implies that ψ is a periodic map on Σ_l of order at least $4g+2$, which is impossible. This completes the proof. □

3. Pseudo-Anosov automorphisms

Minimal elements. The most important fact we prove about commensurability of pseudo-Anosov automorphisms — equivalently, of fibered commensurability of hyperbolic fibered pairs — is the existence of finitely many minimal elements in each commensurability class. In fact, working in the orbifold category, the statement is as clean as it could be:

Theorem 3.1. *Every commensurability class of hyperbolic fibered pairs contains a unique* (*orbifold*) *minimal element.*

Remark 3.2. If M is not arithmetic, then the commensurability class of M (in the usual sense) contains a unique minimal element which is some orbifold O. However, if M is arithmetic, no such unique minimal element exists, and the commensurator of $\pi_1(M)$ is dense in $\mathrm{PSL}(2, \mathbb{C})$; see [Borel 1981; Margulis 1991].

Remark 3.3. Compare with Proposition 2.18 to see that the hypothesis of "hyperbolic" is essential here (in fact, the hyperbolic world is essentially the only context in which there are unique minimal elements in a commensurability class).

Proof of Theorem 3.1. Let $(M, \mathcal{F})$ be a fibered pair, and after passing to a 2-fold cover if necessary, assume that M fibers over S^1 with fibers the leaves of $\mathcal{F}$. Thus M has the structure of an F-bundle over S^1 with monodromy ϕ, for some compact surface F, and some pseudo-Anosov homeomorphism $\phi : F \to F$. The suspension of the product structure gives a pseudo-Anosov flow X transverse to $\mathcal{F}$, with finitely many closed singular orbits corresponding to the singular points of ϕ. The interior of the manifold M admits a unique complete singular Sol metric for which the leaves of $\mathcal{F}$ are Euclidean surfaces with cone singularities on the singular orbits of X; see [Thurston 1997] or [Fathi et al. 1979] for details.

Pulling back the singular Sol metric on M gives the interior of the universal cover $(\tilde{M}, \tilde{\mathcal{F}})$ the structure of a complete simply connected singular Sol manifold, for which the leaves of $\tilde{\mathcal{F}}$ are singular Euclidean planes, and on which $\pi_1(M)$ acts as a discrete finite covolume group of isometries. Let Λ denote the full group of isometries of $\tilde{M}$ with its singular Sol metric.

Claim: Λ *is itself a lattice, and it preserves the foliation* $\tilde{\mathcal{F}}$.

We show how the theorem follows from this Claim. Since $\pi_1(M) \subset \Lambda$ we have the foliation-preserving covering $p : (M, \mathcal{F}) = (\tilde{M}, \tilde{\mathcal{F}})/\pi_1(M) \to (\tilde{M}, \tilde{\mathcal{F}})/\Lambda$. Since $(M, \mathcal{F})$ is a hyperbolic surface bundle of finite volume, we conclude that $(\tilde{M}, \tilde{\mathcal{F}})/\Lambda$ is an orbifold fiber pair $(O, \mathcal{G})$. Notice that any covering map of fibered pairs $(\tilde{M}, \tilde{\mathcal{F}}) \to (M, \mathcal{F})$ is isotopic to an isometric covering of the interiors in the singular Sol metrics. Then it is easy to see that for any pair $(M', \mathcal{F}')$ commensurable with $(M, \mathcal{F})$ the group $\pi_1(M')$ embeds into Λ in such a way that $(M', \mathcal{F}')$ covers $(O, \mathcal{G})$.

Now we prove the Claim. First, it is evident that Λ preserves the stratification of $\tilde{M}$ into "ordinary" points (those with a neighborhood isometric to an open set in Sol) and singular points (those on the lifts of the singular flowlines of X). Moreover, any isometry between open subsets of Sol must preserve the foliation by Euclidean planes, as can be seen by appealing to the well-known structure of the point stabilizers in Isom(Sol); see [Thurston 1997, Chapter 3], for instance.

Since Λ is equal to the group of isometries of the nonsingular part of $\tilde{M}$, it follows that Λ is a Lie group, by the well-known theorem of Myers and Steenrod [1939]. Hence if Λ is not discrete, it must contain a continuous family of nontrivial isometries. Such isometries can only act on the singular flowlines as translations. Let $\ell(t)$ and $\ell'(t)$ be two such flowlines, parametrized by length in such a way that $\ell(t)$ and $\ell'(t)$ are contained in the same singular Euclidean leaf of $\tilde{M}$, for each t. Assume furthermore that for $|t|$ sufficiently small, the points $\ell(t)$ and $\ell'(t)$ can be joined by a unique (nonsingular) Euclidean geodesic in the singular Euclidean leaf containing them. Then for small t, the length of this Euclidean geodesic as a function of t has the form $\sqrt{e^{2t}x^2 + e^{-2t}y^2}$ for fixed x and y; in particular, the length of this Euclidean geodesic is not locally constant, and therefore (since elements of Λ preserve the foliation by singular Euclidean planes) a continuous family of isometries must fix ℓ and ℓ' pointwise. But this implies that $\tilde{M}$ admits no continuous family of nontrivial isometries, and Λ is discrete. Since it contains $\pi_1(M)$, it is therefore a lattice, as claimed. □

Remark 3.4. If F is closed, $\tilde{M}$ with its singular Sol metric and with its hyperbolic metric are quasi-isometric. Consequently if ℓ, ℓ' are two flowlines, the distance function $d(\cdot, \cdot)$ is proper on $\ell \times \ell'$ and therefore one obtains another proof that Λ contains no nontrivial continuous family.

Remark 3.5. A fibration of M over a circle is uniquely determined by an element of $H^1(M; \mathbb{Z})$, which is represented by a unique harmonic 1-form α in the hyperbolic metric on M. A cover $(\tilde{M}, \tilde{\mathcal{F}}) \to (M, \mathcal{F})$ pulls back the harmonic 1-form on M to the corresponding harmonic 1-form on $\tilde{M}$ (up to scale), so one can give a slightly different proof of Theorem 3.1 by using the pullback of this 1-form to $\mathbb{H}^3$ and arguing that its set of (projective) symmetries is discrete. Compare with the proof of Theorem 0.1 in [Agol 2006].

The following two corollaries are immediate:

Corollary 3.6. *For any positive constant C, the set of hyperbolic fibered pairs in a commensurability class whose underlying 3-manifold has volume bounded above by C contains only finitely many elements.*

Proof. Such a pair corresponds to a finite index subgroup of the orbifold fundamental group of $(O, \mathcal{G})$ (with notation as in Theorem 3.1) where the index is bounded by $C/\mathrm{vol}(O)$. Since $\pi_1(O)$ is finitely generated, the number of such subgroups is bounded. □

Corollary 3.7. *Suppose M is hyperbolic and fibers over S^1, and $\mathrm{rank}(H_1(M)) > 1$. Then M fibers over S^1 in ways representing infinitely many fibered commensurability classes.*

Example 3.8. Suppose (F, ϕ) is pseudo-Anosov. Let c be an essential simple closed curve on F, and let τ_c be a Dehn twist along c. Then the automorphisms $(F, \tau_c^l \circ \phi)$ are hyperbolic for all large l, while the volumes of $[F, \tau_c^l \circ \phi]$ are all bounded by the volume of the cusped manifold $[F, \phi] \setminus (c \times \{0\})$. By Corollary 3.6, there are infinitely many commensurability classes among the $(F, \tau_c^l \circ \phi)$ for large l. Of course, it is easy to see directly in this case that the underlying manifolds fall into infinitely many commensurability classes (in the usual sense); see [Anderson 2002], for instance. We give more substantial examples of incommensurable pseudo-Anosov automorphisms in the next subsection and after.

Remark 3.9. One trivial way to produce a hyperbolic 3-manifold M with many nonisotopic but commensurable fibrations is just to choose a 3-manifold with a large isometry group. We do not know explicit examples of two commensurable fibrations of a single hyperbolic 3-manifold with different genus.

Commensurability invariants. The following is an incomplete list of elementary commensurability invariants for pseudo-Anosov automorphisms:

(1) whether the underlying surface is closed or bounded;

(2) the commensurability class of the underlying 3-manifold of $[F, \phi]$.

(3) the commensurability class of $\log(K)$ where K is the dilatation;

(4) the set of orders of the singular points of the invariant foliations;

For later use we say a few words about (3) and (4). First we make some definitions. For a pseudo-Anosov automorphism (F, ϕ) with a pair of transversely measured singular foliations $\mathfrak{F}_{s,u}$, we use $\lambda(\phi) > 1$ to denote the dilatation of ϕ, and $\delta_n(\phi)$ to denote the number of singularities of degree n, then define $\Delta(\phi)$ to be the (infinite) vector whose coordinates are the $\delta_n(\phi)$.

The first observation to make is that for pseudo-Anosov automorphisms, $\lambda(*)$ is only affected by dynamical coverings, and $\Delta(*)$ is only affected by topological coverings.

Lemma 3.10. *Suppose (F_1, ϕ_1), (F_2, ϕ_2) are two commensurable pseudo-Anosov maps. Then for some $s, s' \in \mathbb{Q}_+$,*

(1) $\log \lambda(\phi_1) = s \log \lambda(\phi_2)$, *and moreover* $\log \lambda(\phi_1) = \log \lambda(\phi_2)$ *if they are topologically commensurable*; *and*

(2) $\Delta(\phi_1) = s' \Delta(\phi_2)$, *and moreover* $\Delta(\phi_1) = \Delta(\phi_2)$ *if they are dynamically commensurable.*

Proof. These facts follow immediately from the definitions (recall Definition 2.5; also, (1) follows from the proof of Proposition 4.11). □

Example 3.11 (bounded–unbounded). Remark 4.3 of [Hironaka 2009] gives an example of a pair of automorphisms $\phi_{(1,3)}$ defined on a genus 2 surface with four boundary components, and $\phi_{(3,4)}$ defined on a closed genus 3 surface with the same dilatation. The commensurability classes of these examples are also distinguished by the orders of the singular points.

Example 3.12. Explicit examples of incommensurable fibrations of the same hyperbolic 3-manifold are straightforward to construct and distinguish by means of Lemma 3.10. For example, in page 4 of [Hironaka 2009], fibrations of the complement of the link 6_2^2 in Rolfsen's tables [1976] are listed, and their singularity sets do not satisfy the commensurability condition in bullet (2) of Lemma 3.10.

Example 3.13. Incommensurable examples may be obtained by branched covers. Start with an Anosov automorphism ϕ of a torus T with dilatation K, and let P be a finite subset of T permuted by ϕ. Let F be obtained as a branched cover of T, branched over P. Then some power of ϕ lifts to an automorphism of F with dilatation a power of K. Different choices of branch orders give rise to incommensurable automorphisms of closed surfaces with the same dilatations, but usually incommensurable singular sets.

One may define a more subtle invariant of commensurability as follows. Let ϕ be a pseudo-Anosov automorphism of F, with measured foliations $\mathfrak{F}_{s,u}$ and projectively invariant transverse measures $\mu_{s,u}$, and singular set S (note that S is finite). For any pair of points p and q (possibly $p = q$) in the singular set, and any homotopy class of paths γ from p to q in the complement $F \setminus S$ we define a number $\ell(\gamma)$ to be the infimum, over all paths γ' from p to q which are homotopic to γ in $F \setminus S$ rel. endpoints, of the product

$$\ell(\gamma) = \inf_{\gamma'} \mu_s(\gamma')\mu_u(\gamma')$$

This number depends on the choice of measures μ_s, μ_u in their projective class, but is well-defined if we normalize the product of measures so that $\int_F d\mu_s d\mu_u = -\chi(F)$.

Definition 3.14. Define the *spectrum* of (F, ϕ) to be the set of numbers $\ell(\gamma)$ as γ varies over nontrivial homotopy classes of paths in $F \setminus S$ as above.

Proposition 3.15. *With the normalization of the product of measures as above, the spectrum is a commensurability invariant. Furthermore, it is strictly positive, and discrete as a subset of* $\mathbb{R}$ (*and is therefore bounded away from zero*).

Proof. By multiplicativity of Euler characteristic, the normalization of the product of measures is compatible under finite covers. Each homotopy class of arcs joining singular points on F lifts to an arc joining singular points in any cover $\tilde{F}$, so the spectrum as defined is a commensurability invariant.

It remains to show that the spectrum is discrete. By the properties of a pseudo-Anosov, we have $\ell(\gamma) = \ell(\phi^i(\gamma))$ for any homotopy class γ and any integer i. To show that the spectrum is discrete, it suffices to show that there are only finitely many ϕ-orbits of homotopy classes γ with $\ell(\gamma) \le C$.

Suppose $K > 1$ is the dilatation of ϕ, and γ' is any path between singular points on F. By the definition of $\mathfrak{F}_{s,u}$, we have $\mu_s(\phi(\gamma')) = K\mu_s(\gamma')$ and $\mu_u(\phi(\gamma')) = K^{-1}\mu_u(\gamma')$. So under the automorphism ϕ, the difference of their logs changes by $2\log K$. It follows that whatever the difference of logs is initially, after a suitable power of ϕ the absolute value of the difference can be taken to be at most $\log(K)$. In other words, there is some integer i so that

$$|\log(\mu_s(\phi^i(\gamma'))) - \log(\mu_u(\phi^i(\gamma')))| \le \log(K).$$

If A and B are positive numbers, a bound on AB and one on $|\log(A) - \log(B)|$ let us bound both A and B. It follows that if $\ell(\gamma) \le C$ then for some i, the homotopy class $\phi^i(\gamma)$ is represented by an arc $\beta = \phi^i(\gamma')$ for which both $\mu_s(\beta)$ and $\mu_u(\beta)$ are bounded, by a constant depending only on C and K. By the discreteness of S, there are only finitely many such relative homotopy classes $\phi^i(\gamma)$, and each of them has a positive ℓ length. So $\ell(\gamma)$ takes only finitely many values in $[0, C]$ (all of them positive). □

Remark 3.16. If Σ is a Riemann surface, any quadratic holomorphic differential α on Σ defines a pair of singular measured foliations, and we can define a spectrum as above for a pair (Σ, α). Multiplying α by a constant also multiplies the spectrum by a constant, so we can normalize to quadratic differentials with $\int_\Sigma |\alpha| = 1$. The set of such pairs (Σ, α) can be identified with the unit cotangent bundle in moduli space. The spectrum (defined as above) is constant on orbits of the Teichmüller flow (see, e.g., [Masur and Tabachnikov 2002] for a definition), and is discrete (by Proposition 3.15) for points on closed orbits of the flow. For general quadratic differentials the spectrum can have accumulation points, or its closure can contain a perfect set, or it can even be dense.

This invariant gives rise to a new way to distinguish commensurability classes of automorphisms.

Example 3.17 (different spectrum). As above, let ϕ be an Anosov automorphism of a torus T (with a flat metric on the torus of total area 1). The set of periodic points is dense, so we can choose two periodic points O, P. The stable and unstable foliations of ϕ give coordinates on T, at least in a neighborhood of O, so that $O = (0, 0)$ and $P = (x, y)$.

In a suitable cover of T branched over O and P we obtain an automorphism with dilatation a power of K for which the smallest term in the spectrum is at most $|xy|$ times a constant depending only on the combinatorics of the cover. By choosing

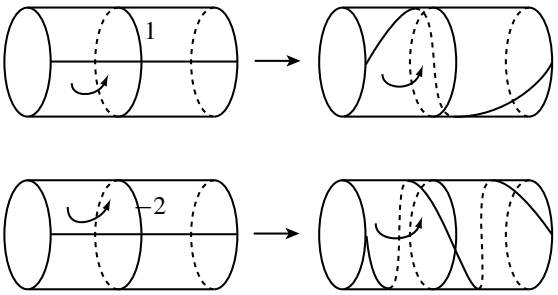
the periodic point P so that $|xy|$ is sufficiently small, we can ensure that the first term in the spectrum is as close to 0 as we desire, while at the same time fixing the orders of the singular points. By Proposition 3.15, this construction gives rise to infinitely many commensurability classes with commensurable log dilatation and the same combinatorial invariants.

Remark 3.18. Example 3.13 also produces examples of infinitely many (incommensurable) pseudo-Anosov maps with different singular orders but the same spectrum. It is not clear if there exists a pair of pseudo-Anosov maps with incommensurable log dilatations but the same spectrum.

4. Reducible automorphisms

Commensurability invariants of reducible automorphisms. We have assumed that each reducible map is in its standard form as described in Proposition 2.15. We also use the notation from that proposition without comment.

Let A be an oriented annulus A. The mapping class group of A rel. boundary is isomorphic to $\mathbb{Z}$, generated by a positive Dehn twist τ along the core circle. We denote the n-th power of such a Dehn twist by τ_n. Here is an illustration of the cases $n = 1$ and $n = -2$.



Remark 4.1. In this and later figures, the orientation of the surface is indicated by a "cup" shaped arrow, and the numbered circles on the surface indicate the power of a positive Dehn twist (with respect to the given orientation).

For a reducible map ϕ, choose l so that ϕ^l is the identity on $\partial(F \setminus N(\Gamma(\phi)))$. For each component $N(\gamma)$ of $N(\Gamma(\phi))$, where $\gamma \in \Gamma(\phi)$, $N(\gamma)$ has the induced orientation and $\phi^l | \partial N(\gamma)$ is the identity. Then the restriction of ϕ^l to $N(\gamma)$ is the n-th power of a Dehn twist for some integer n. Now define

$$I(\phi^l, \gamma); \quad I(\phi, \gamma) = I(\phi^l, \gamma)/l; \quad a_k(\phi) = \#\{\gamma \in \Gamma(\phi) |\ I(\phi, \gamma) = k\},\ k \in \mathbb{Q}$$

Further, define

$$S(\phi) = \{S \mid S \text{ a component of } F \setminus N(\Gamma(\phi))\}$$

and

$$\Omega(S) = \{\gamma \mid \gamma \text{ a component of } \partial S \setminus \partial F\}.$$

For every $S \in S(\phi)$, define

$$a_{S,k}(\phi) = \#\{\gamma \in \Omega(S) \mid I(\phi, \gamma) = k\}; \quad A(\phi, S) = \left(\sum_{k \in \mathbb{Q}_+} \frac{a_{S,k}(\phi)}{k}, \sum_{k \in \mathbb{Q}_-} \frac{a_{S,k}(\phi)}{-k} \right).$$

The following two numerical invariants are easy to compute:

$$A(\phi) = \frac{1}{2} \sum_{S \in S(\phi)} A(\phi, S) = \left(\sum_{k \in \mathbb{Q}_+} \frac{a_k(\phi)}{k}, \sum_{k \in \mathbb{Q}_-} \frac{a_k(\phi)}{-k} \right),$$

$$\Pi(\phi) = \left\{ \frac{1}{-\chi(S)} A(\phi, S) \;\middle|\; S \in S(\phi) \right\}.$$

We say that two sets of ordered pairs of rational numbers $\{(p_i, q_i)\}$ and $\{(p'_j, q'_j)\}$ are equal *up to a flip*, denoted $\{(p_i, q_i)\} \sim \{(p'_j, q'_j)\}$, if either they are equal, or $\{(p_i, q_i)\} = \{(q'_j, p'_j)\}$. Immediately we have:

Lemma 4.2. *Reversing the orientation of F preserves $A(\phi, S)$, and therefore also $A(\phi)$ and $\Pi(\phi)$, up to a flip.*

We can derive commensurability invariants from $A(\,\cdot\,)$ and $\Pi(\,\cdot\,)$ as follows:

Theorem 4.3. *Suppose (F_1, ϕ_1), (F_2, ϕ_2) are two reducible maps. If they are commensurable, then for some $s \in \mathbb{Q}_+$,*

$$A(\phi_1) \sim s A(\phi_2) \quad \textit{and} \quad \Pi(\phi_1) \sim s \Pi(\phi_2).$$

We postpone the proof of Theorem 4.3 until page 302.

Remark 4.4. The invariant $\Pi(\,\cdot\,)$ is typically better than $A(\,\cdot\,)$ at distinguishing commensurability classes (though not always; see Example 4.13). We say that a D-type map is *definite* if it is a product of Dehn twists in the components of $\Gamma(\phi)$ of the same sign. Note that the property of having a power which is definite (along $\Gamma(\phi)$) is a commensurability invariant. The invariant $A(\,\cdot\,)$ can distinguish between definite and indefinite maps, but can never distinguish different commensurability classes of definite maps, whereas $\Pi(\,\cdot\,)$ can.

Remark 4.5. Both $A(\phi)$ and $\Pi(\phi)$ can be encoded as a polynomial (with fractional exponents), as follows. For any pair of nonnegative rational numbers (p, q), define

$$S(\phi)(p, q) = \left\{ S \in S(\phi) \;\middle|\; \frac{A(\phi, S)}{-\chi(S)} = (p, q) \right\}, \quad \lambda(\phi)_{(p,q)} = \frac{\sum_{S \in S(\phi)(p,q)} \chi(S)}{\chi(F)}.$$

Now define a polynomial pair

$$P(\phi)(x, y) = (P_1(\phi)(x, y), P_2(\phi)(x, y)) = \sum_{(p,q) \in \mathbb{Q}^2} (p, q) \lambda(\phi)_{(p,q)} x^p y^q.$$

One can recover $A(\cdot)$ and $\Pi(\cdot)$ from this polynomial by the formulae

$$\frac{2}{-\chi(F)}A(\phi) = \sum_{(p,q)\in\mathbb{Q}^2} (p,q)\lambda(\phi)_{(p,q)} = P(\phi)(1,1),$$
$$\Pi(\phi) = \{(p,q) \mid \lambda(\phi)_{(p,q)} \neq 0\}.$$

One can show along lines similar to the proof of Theorem 4.3 (in the next subsection) that if two reducible maps (F_1, ϕ_1), (F_2, ϕ_2) are commensurable, then for some $s \in \mathbb{Q}_+$, we have

$$P(\phi_1)(x, y) \sim sP(\phi_2)(x^s, y^s).$$

Proof of Theorem 4.3. We need some lemmas, which can be verified immediately from the definitions.

Lemma 4.6. *If ϕ is a reducible map, we have, then for any positive integer k,*

$$I(\phi^k, \gamma) = kI(\phi, \gamma), \quad a_{S,n}(\phi^k) = a_{S,n/k}(\phi), \quad A(\phi^k, S) = \frac{1}{k}A(\phi, S). \tag{4-1}$$

Lemma 4.7. *Suppose two automorphisms ϕ_1 and ϕ_2 on F are isotopic, and two circles γ_1 and γ_2 on F are isotopic. If ϕ_i is D-type along γ_i, $i = 1, 2$, then $I(\phi_1, \gamma_1) = I(\phi_2, \gamma_2)$.*

Lemma 4.8. *$\Pi(\phi)$ and $A(\phi)$ are isotopy invariants.*

Proof of Lemma 4.8. This follows from the definitions, from Lemma 4.7 and from the fact that the reducible system Γ is unique up to isotopy; see Theorem 1 in [Wu 1987], for example. □

Now turning to the proof of Theorem 4.3 proper, suppose (F_1, ϕ_1) and (F_2, ϕ_2) are commensurable. Then there is a surface $\tilde{F}$, automorphisms $\tilde{\phi}_1$ and $\tilde{\phi}_2$ of $\tilde{F}$, and nonzero integers k_1 and k_2, so that $(\tilde{F}, \tilde{\phi}_i)$ covers (F_i, ϕ_i) for $i = 1, 2$, and $\tilde{\phi}_1^{k_1} = \tilde{\phi}_2^{k_2}$ as automorphisms of $\tilde{F}$. Denote the covering $\tilde{F} \to F_i$ by p_i, $i = 1, 2$. By Lemma 4.2, we may assume that the orientations of $\tilde{F}$, F_1 and F_2 have been chosen so that both p_1, p_2 are orientation-preserving.

Assume that $k_1 = k_2 = 1$ for the moment. By Lemma 4.8, we may assume that $\tilde{\phi}_1 = \tilde{\phi}_2$ as maps in usual sense (rather than in their isotopy class).

Consider the commutative diagram

$$\begin{array}{ccc} \partial p_1^{-1}(N(\Gamma(\phi_1))) & \xrightarrow{\tilde{\phi}_1^{k}|} & \partial p_1^{-1}(N(\Gamma(\phi_1))) \\ \downarrow{\scriptstyle p_1|} & & \downarrow{\scriptstyle p_1|} \\ \partial N(\Gamma(\phi_1)) & \xrightarrow{\phi_1^{k}|} & \partial N(\Gamma(\phi_1)) \end{array}$$

where k is chosen so that $\phi_1^k|_{\partial N(\Gamma(\phi_1))} = \mathrm{id}|_{\partial N(\Gamma(\phi_1))}$. It follows that the restriction of $\tilde{\phi_1}^k$ to $\partial p_1^{-1}(N(\Gamma(\phi_1)))$ is a deck transformation of the covering $p_1|$. Since $p_1|$ is a finite covering, by replacing k by a power if necessary, we can assume that $\tilde{\phi_1}^k$ agrees with id on $\partial p_1^{-1}(N(\Gamma(\phi_1)))$ and consequently maps every component of $p_1^{-1}(N(\Gamma(\phi_1)))$ to itself. For such a k, each ϕ_i^k, $\tilde{\phi_i}^k$, $i = 1, 2$ are D-type along their respective reducible systems, where $\Gamma(\tilde{\phi_i}^k) = p_i^{-1}(\Gamma(\phi_i))$.

For each $S_1 \in S(\phi_1)$ and each component $\tilde{S}$ of $p_1^{-1}(S_1)$, there exists a component $S_2 \in S(\phi_2)$, such that $\tilde{S}$ is a component of $p_2^{-1}(S_2)$. Assume $p_i| : \tilde{S} \to S_i$ are l_i-sheeted coverings, for $i = 1, 2$.

Pick a component $\gamma \in \Omega(S_1)$. Suppose that $\{\delta_1, \ldots, \delta_t\} = (p_1|\tilde{S})^{-1}(\gamma)$ and that $p_1 : \delta_i \to \gamma$ is a d_i-sheeted covering. Then $\sum_{i=1}^{l} d_i = l_1$.

Under an m-fold covering of annuli, a Dehn twist on the covering annulus projects to the m-th power of a Dehn twist on the image annulus. Consequently $d_i I(\tilde{\phi_1}^k, \delta_i) = I(\phi_1^k, \gamma)$, and by (4-1) we have

$$I(\tilde{\phi_1}^k, \delta_i) = \frac{kI(\phi_1, \gamma)}{d_i} \tag{4-2}$$

and moreover the $I(\tilde{\phi_1}^k, \delta_i)$ all have the same sign as the $I(\phi_1, \gamma)$, $i = 1, \ldots, t$ (because p_1 preserves orientation and $k > 0$). Suppose $I(\phi_1, \gamma) \neq 0$. Then by (4-2),

$$\sum_{i=1}^{t} \frac{1}{I(\tilde{\phi_1}^k, \delta_i)} = \sum_{i=1}^{t} \frac{d_i}{kI(\phi_1, \gamma)} = \frac{l_1}{kI(\phi_1, \gamma)} \tag{4-3}$$

Now we sum over circles $\delta \in \Omega(\tilde{S})$ with positive $I(\tilde{\phi_1}, \delta)$:

$$\begin{aligned}
\sum_{l>0} \frac{a_{\tilde{S},l}(\tilde{\phi_1}^k)}{l} &= \sum_{l>0} \frac{\#\{\delta \in \Omega(S_1') | I(\tilde{\phi}^k, \delta) = l\}}{l} \\
&= \sum_{\substack{\delta \in \Omega(\tilde{S}) \\ I(\tilde{\phi_1}, \delta) > 0}} \frac{1}{I(\tilde{\phi_1}^k, \delta)} = \sum_{\substack{\gamma_i \in \Omega(S_1) \\ I(\phi_1, \gamma_i) > 0}} \sum_{\delta \in (p_1|\tilde{S})^{-1}(\gamma_i)} \frac{1}{I(\tilde{\phi_1}^k, \delta)} \\
&= \frac{l_1}{k} \sum_{\substack{\gamma_i \in \Omega(S_1) \\ I(\phi_1, \gamma_i) > 0}} \frac{1}{I(\phi_1, \gamma_i)} = \frac{l_1}{k} \sum_{l>0} \frac{a_{S_1,l}(\phi_1)}{l},
\end{aligned}$$

where the penultimate equality follows from (4-3).

By a similar computation, we have

$$\sum_{l<0} \frac{a_{\tilde{S},l}(\tilde{\phi_1}^k)}{l} = \frac{l_1}{k} \sum_{l<0} \frac{a_{S_1,l}(\phi_1)}{l},$$

and therefore

$$A(\tilde{\phi}_i^{\,k}, \tilde{S}) = \frac{l_i}{k} A(\phi_i, S_i), \quad i = 1, 2. \tag{4-4}$$

By (4-1) we have

$$A(\tilde{\phi}_1, \tilde{S}) = kA(\tilde{\phi}_1^{\,k}, \tilde{S}) = A(\tilde{\phi}_2, S_2'). \tag{4-5}$$

Since $l_i = \chi(\tilde{S})/\chi(S_i)$, by (4-4) and (4-5), we get

$$\frac{A(\phi_1, S_1)}{-\chi(S_1)} = \frac{A(\tilde{\phi}_1, \tilde{S})}{-\chi(\tilde{S})} = \frac{A(\phi_2, S_2)}{-\chi(S_2)}. \tag{4-6}$$

From the definition of $\Pi(\cdot)$ we have $\Pi(\phi_2) \subset \Pi(\phi_1)$. By symmetry we have $\Pi(\phi_2) = \Pi(\phi_1)$. Summing over all Γ in the argument above in place of $\Omega(S_1)$, we get similarly

$$\frac{A(\phi_1)}{\chi(F_1)} = \frac{A(\phi_2)}{\chi(F_2)}.$$

From (4-1) we have $\Pi(\phi^k) = \Pi(\phi)/k$ and $A(\phi^k) = A(\phi)/k$ and the proof is complete. □

From the proof above immediately we have:

Corollary 4.9. *If (F_1, ϕ_1) and (F_2, ϕ_2) are topologically commensurable, then*

$$\frac{A(\phi_1)}{\chi(F_1)} \sim \frac{A(\phi_2)}{\chi(F_2)} \quad \textit{and} \quad \Pi(\phi_1) \sim \Pi(\phi_2).$$

Remark 4.10. We remind the reader that our invariants are defined for all reducible maps (and not just D-type examples and their roots). When reducible maps are not the roots of the D-type maps, then they have pseudo-Anosov orbits, and we can combine the invariants defined in Sections 3 and 4. For example, see the proposition below and Example 4.18.

Proposition 4.11. *Suppose (F_1, ϕ_1), (F_2, ϕ_2) are two commensurable reducible maps. Then for some $s \in \mathbb{Q}_+$,*

$$\log \lambda(\phi_1) = s \log \lambda(\phi_2) \quad \textit{and} \quad \Pi(\phi_1) \sim s^{-1}\Pi(\phi_2).$$

Here we think of $\lambda(\phi)$ for a reducible map ϕ as a (possibly empty) *set* of dilatations of the set of restrictions of ϕ to its pseudo-Anosov orbits.

Proof. From the definition of commensurability, there are positive integers k_1 and k_2 such that $(F_1, \phi_1^{k_1})$ and $(F_2, \phi_2^{k_2})$ are topologically commensurable, both covered by $(\tilde{F}, \tilde{\phi})$. Evidently we have $\lambda(\phi_1^{k_1}) = \lambda(\tilde{\phi}) = \lambda(\phi_2^{k_2})$, and therefore $k_1 \log \lambda(\phi_1) = \log \lambda(\phi_1^{k_1}) = \log \lambda(\phi_2^{k_2}) = k_2 \log \lambda(\phi_2)$ and then

$$\log \lambda(\phi_1) = \frac{k_2}{k_1} \log \lambda(\phi_2).$$

On the other hand, by Corollary 4.9 and (4-1), we have

$$\frac{\Pi(\phi_1)}{k_1} = \Pi(\phi_1^{k_1}) \sim \Pi(\phi_2^{k_2}) = \frac{\Pi(\phi_1)}{k_2}$$

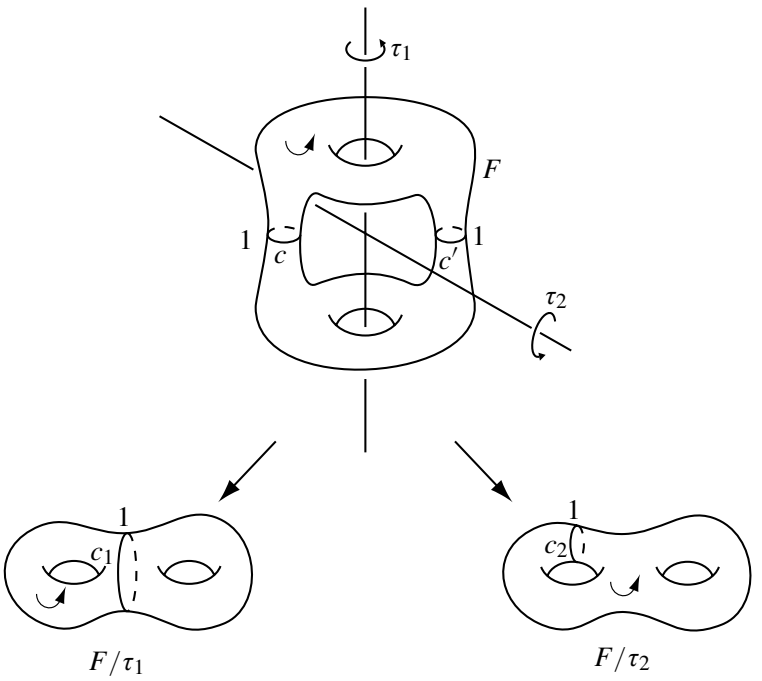
and therefore

$$\Pi(\phi_1) \sim \frac{k_1}{k_2}\Pi(\phi_1).$$

The proposition is proved by setting $s = k_2/k_1$. □

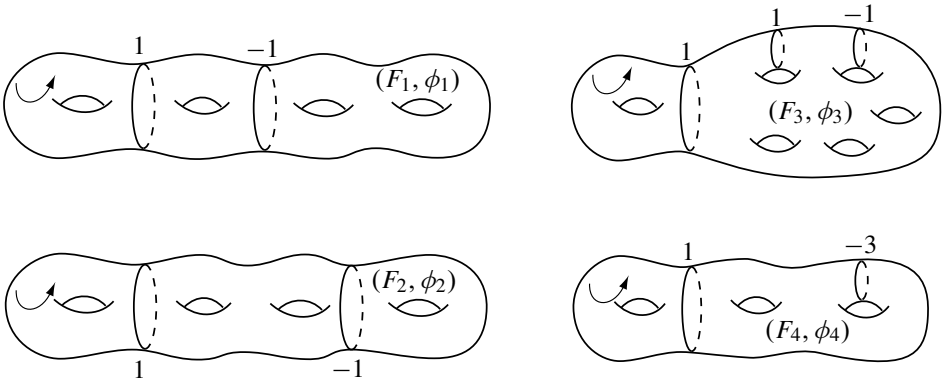
Examples of reducible automorphisms. In this section we give several examples, which illuminate the meaning of the invariants defined above. A D-type map on an oriented F can be indicated pictorially by assigning integers to disjoint essential simple closed curves on a surface; we use this convention in what follows.

Example 4.12. Dehn twists in separating and nonseparating curves (on the same surface) are commensurable. In the figure below, let $\tilde{\phi}$ be a D-type automorphism on a surface F of genus 3 generated by full Dehn twists on circles c and c' as indicated in the figure.



Then $\tilde{\phi}$ is invariant under both π-rotations along τ_1 and τ_2. Hence $\tilde{\phi}$ induces ϕ_i on F/τ_i, where ϕ_i is the Dehn twist along the circle c_i. Since c_1 is separating while c_2 not, ϕ_1 and ϕ_2 are not conjugate. But from the construction they are commensurable.

Example 4.13. This example show that $\Pi(\phi)$ is not always finer than $A(\phi)$. Four automorphisms are depicted here:



By computing $A(\phi)$ and $\Pi(\phi)$, it can be seen that no pair of them are commensurable. Notice that on one hand $A(\phi_1) = A(\phi_2) = (1, 1)$ and $\{(1, 0), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{3})\} = \Pi(\phi_1) \neq \Pi(\phi_2) = \{(1, 0), (\frac{1}{4}, \frac{1}{4}), (0, 1)\}$, and on the other hand $(2, 1) = A(\phi_3) \neq A(\phi_4) = (1, \frac{1}{3})$ and $\Pi(\phi_3) = \Pi(\phi_4) = \{(1, 0), (\frac{1}{3}, \frac{2}{9})\}$.

Example 4.14 (minimal elements). Let ϕ_g be a orientation-preserving periodic map on Σ_g of order $4g+2$ which rotates by $\pi/(2g+1)$ around its unique fixed point x_g (see the proof of Proposition 2.18). Remove a ϕ_g-invariant disc at x_g from Σ_g to get $\Sigma_{g,1}$. Connect $\Sigma_{2,1}$ and $\Sigma_{3,1}$ along their boundaries via an annulus A to form a closed surface Σ_5 and define ϕ on F_5 by $\phi|\Sigma_{2,1} = \phi_2|\Sigma_{2,1}$ and $\phi|\Sigma_{3,1} = \phi_3^{-1}|\Sigma_{3,1}$, and then extend to A by a continuous family of rotations through angles from $\frac{\pi}{5}$ to $\frac{\pi}{7}$. The difference in speeds on the boundary components is $\frac{2\pi}{35}$, and it follows that ϕ^{35} is a Dehn twist D_c. By the uniqueness of the reducible system and the argument similar in the proof of Proposition 2.18, one can verify (Σ_5, ϕ) is a minimal element. One can construct infinitely many minimal elements in such a way.

Remark 4.15. One can verify that 35 is the largest order of a root of a Dehn twists on Σ_5. It is amazing that the maximal order of roots of Dehn twist along *nonseparating* curves, which is 11 on Σ_5 (and in general is $2g + 1$ in Σ_g), was determined only very recently by several papers; see [Margalit and Schleimer 2009; McCullough and Rajeevsarathy 2009; Monden 2009].

Example 4.16. This example will be used in Section 5. Σ_{kn+1} can be presented as the union of $\Sigma_{1,n}$ and n copies of $\Sigma_{k,1}$ in a in symmetric way so that there is an action $\tau_{n,k}$ of order n which acts freely on the triple $\big(\Sigma_{kn+1}, \Sigma_{1,n}, \bigcup_1^n \Sigma_{k,1}\big)$.

Let D_c be the positive Dehn twist along one component c of $\partial\Sigma_{1,n}$ and let $\phi_{n,k}$ be the composition of $D_c \circ \tau_{n,k}$. Then one can verify that $D_{n,k} = \phi_{n,k}^n$ is D-type,

and is given by the product of a positive Dehn twist along each component of $\partial\Sigma_{1,n}$. For fixed k, the automorphisms $(\Sigma_{kn+1,0}, D_{n,k})$ and $(\Sigma_{km+1}, D_{m,k})$ have a common cover $(\Sigma_{kmn+1}, D_{mn,k})$. Therefore for fixed k, $(\Sigma_{kn+1}, \phi_{n,k})$ are in the same commensurability class for all n.

On the other hand one can verify by inspection that $\Pi(D_{n,k}) = \{(1, 0), (1/(2k-1), 0)\}$. So $(\Sigma_{kn+1}, D_{n,k})$ and $(\Sigma_{k'm+1}, D_{m,k'})$ are not commensurable for $k \neq k'$ by Theorem 4.3.

Example 4.17. Each D-type map (F, ϕ) is commensurable with a D-type map (F', ψ) so that the Dehn twist on each $\gamma \in \Gamma(\psi)$ is a single positive or negative Dehn twist. We can argue as below:

For simplicity, assume F is closed, $S(\phi) = \{S_i, i = 1, \ldots, k\}$, and set

$$d_\gamma = |I(\phi, \gamma)|.$$

By replacing ϕ by a power if necessary, we may assume that d_γ is an integer > 1 for each $\gamma \in \Gamma(\phi)$. Then for each i there is a covering $q_i : \tilde{S}_i \to S_i$ such that $q_i| : \tilde{\gamma} \to \gamma$ is of degree d_γ for each component $\gamma \in \partial S$ and each component $\tilde{\gamma}$ in $q_i^{-1}(\gamma)$. One quick way to see this is to attach an orbifold disk D_γ of index d_γ to each $\gamma \in \partial S_i$. The result is 2-dimensional orbifold which is good, since $\chi(S_i) < 0$ and each $d_\gamma > 1$. This orbifold has a manifold cover (see [Thurston 1979, Chapter 13]), and the restriction to S_i gives the required covering $q_i : \tilde{S}_i \to S_i$.

If P is a planar surface of negative Euler characteristic, then for every $n \geq 2$ coprime with the number of components of ∂P, there is a cover $\hat{P} \to P$ of degree n, which restricts to a cover of degree n on each boundary component of $\hat{P}$, and such that $\hat{P}$ is nonplanar. Moreover, every nonplanar surface with negative Euler characteristic has a covering of any given degree which is a covering of degree 1 on each boundary component. So after replacing ϕ by ϕ^n, we can find covers

$$\hat{q}_i : \hat{S}_i \to \tilde{S}_i$$

and a covering of degree $n \prod_{k \neq i} \deg(q_k)$ so that the restriction on each component of $\partial\hat{S}_i$ is a covering of degree exactly n. The coverings $p_i = q_i \circ \hat{q}_i : \hat{S}_i \to S_i$ match compatibly to produce a covering $p : \tilde{F} = \cup\hat{S}_i \to F$ such that $p| : \tilde{\gamma} \to \gamma$ is of degree nd_γ for each $\gamma \in \Gamma(\phi)$ and each component $\tilde{\gamma}$ in $p^{-1}(\gamma)$. Define a D-type map $\tilde{\phi}$ on $\tilde{F}$ with $I(\tilde{\phi}, \tilde{\gamma}) = 1$ if $I(\phi, \gamma) > 0$, and $I(\tilde{\phi}, \tilde{\gamma}) = -1$ otherwise, then $\tilde{\phi}$ covers ϕ (see the paragraph before (4-2) in the proof of Theorem 4.3.)

Now we give an application of Proposition 4.11 to reducible maps which are not roots of D-type maps.

Example 4.18. Let F be a closed oriented surface of genus 2, and c a nonseparating circle in F. Let ϕ be any pseudo Anosov map on $F \setminus c$ with dilatation

$\lambda(\phi) = K$ and twist angle $2\pi r$ near c, $r \in \mathbb{Q}$, and let τ_c be a positive Dehn twist along c. Then:

(1) $\tau^{k_1} \circ \phi$ and $\tau^{k_2} \circ \phi$ are commensurable if and only if $k_1 = k_2$.

(2) $\tau \circ \phi^{k_1}$ and $\tau \circ \phi^{k_2}$ are commensurable if and only if $k_1 = k_2$.

The proofs of (1) and (2) are similar; we only give a proof of (1). Note that

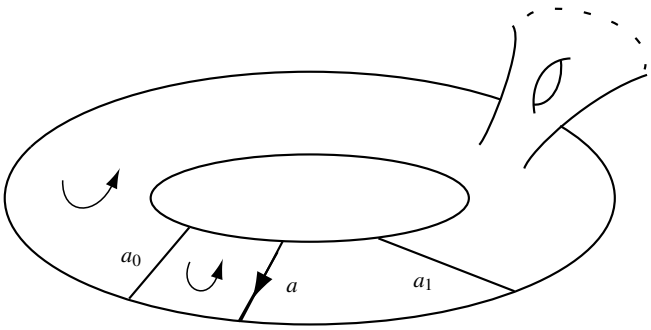
$$\Pi(\tau^k \circ \phi) = \left(\frac{1}{(k-r)}, 0\right) \quad \text{and} \quad \lambda(\tau^k \circ \phi) = \lambda(\phi) = K > 1,$$

where r and K depend only on ϕ. If $\tau^{k_1} \circ \phi$ and $\tau^{k_2} \circ \phi$ are commensurable, by Proposition 4.11 and the fact we are considering the automorphism in the same oriented surface F, we should have $\log K = s \log K$ and $1/(k_1 - r) = s^{-1}/(k_2 - r)$ for some $s \in \mathbb{Q}_+$. The first equality implies that $s = 1$, and the second implies $k_1 = k_2$.

5. Commensurable and incommensurable bundles in graph manifolds

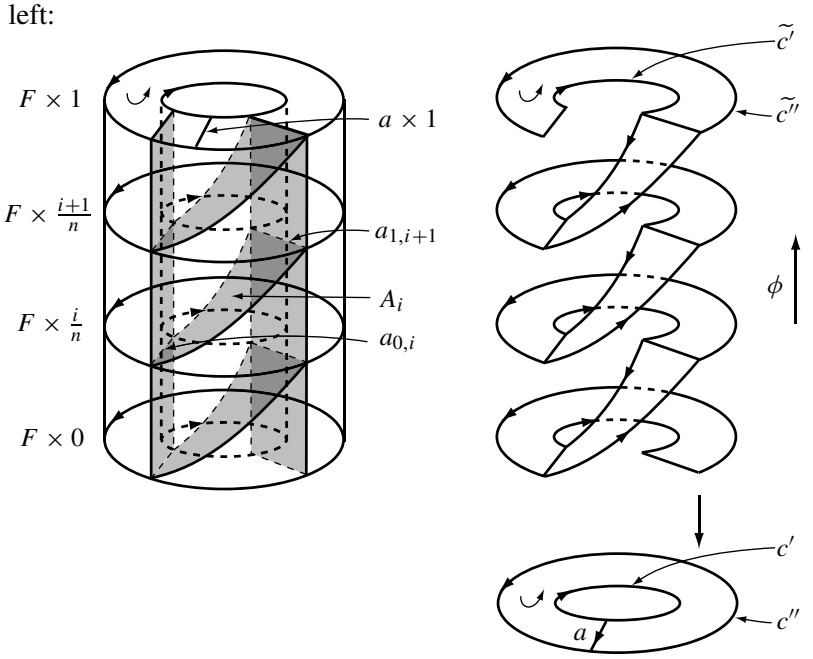
In this section we give two more complicated examples. The first (Example 5.3) is an example of a graph manifold that is the total space of infinitely many incommensurable fibrations, and at the same time fibers in infinitely many ways in the *same* commensurability class. The second (Example 5.5) is an example of a graph manifold that is the total space of infinitely many incommensurable fibrations, including two incommensurable fibrations with the same genus. Both examples depend on a construction that we turn to now.

Primary construction. Let F be a compact oriented surface with the induced orientation on ∂F. Let a be an essential oriented arc on F connecting two different components of ∂F. Let a_0 and a_1 be the two components of the quadrilateral $\partial N(a) \setminus \partial F$ such that the direction on a_0 induced from the orientation on $\partial N(a)$ is parallel to that on a:



Then in $F \times [0, 1]$, the surface $F \times \{\frac{i}{n}\}$ intersects the quadrilateral $a_j \times [0, 1]$ in the arc $a_{j,i} = a_j \times \{\frac{i}{n}\}$ for each integer $n \geq 2$, where $j = 0, 1$ and $i = 0, 1, \ldots, n$.

Let $A_1, \ldots, A_n$ be n pairwise disjoint quadrilaterals that are properly embedded in $N(a) \times [0, 1]$ so that A_i is a stair connecting $a_{0,i}$ and $a_{1,i+1}$, as shows in the figure on the left:



Let $F_i = (F \times \frac{i}{n}) \setminus (N(a) \times [0, 1])$ and build a surface $R(a, n) = \bigcup_{i=0}^{n} F_i \cup \bigcup_{l=1}^{n} A_l$ in $F \times [0, 1]$; see right diagram above. A similar surface $R(\alpha, n)$ in $F \times [0, 1]$ can be constructed if we replace a by a disjoint union of essential arcs α on F.

We call the quotient of $R(\alpha, n)$ in $F \times S^1 = [F, \mathrm{id}]$ the *n-floor staircase* along α in $F \times S^1$, or just *n-floor along* α for short, and denote it as $F(\alpha, n)$. Note that the surface $F(\alpha, n)$ is transverse to the S^1 fibers. If α is empty, then $F(\varnothing, n)$ is just n disjoint copies of F in $F \times S^1$.

Let S^1 have the orientation induced from $[0, 1]$. Then both $F \times S^1$ and $\partial F \times S^1$ are oriented. For each component $c \in \partial F$, the torus $c \times S^1$ has product coordinates (c, t). The proof of the following lemma is a routine verification:

Lemma 5.1. *Let $p : F \times S^1 \to F$ be the projection. Suppose that $\alpha \cap c \leq 1$ for each component $c \in \partial F$. Then*:

(1) *$p : F(\alpha, n) \to F$ is a cyclic covering of degree n. Moreover $F(\alpha, n)$ is a surface of genus $1 - k + n(k - 1 + g)$ with $n(\#\partial F - 2k) + 2k$ boundary components, where $k = \#\alpha$.*

(2) *$p^{-1}(c)$ is either connected or has n components for each component c of ∂F, and $p^{-1}(c)$ is connected if and only if $\alpha \cap c \neq \varnothing$. Moreover suppose a is an arc in α with tail in c' and head in c'', then $\tilde{c}' = p^{-1}(c')$ has slope $(n, -1)$ and $\tilde{c}'' = p^{-1}(c'')$ has slope $(n, 1)$.*

(3) *Let $\tilde{\tau}$ be the $2\pi/n$-rotation of $F \times S^1$ along the oriented S^1 factor, and let $\tilde{c}'$ and $\tilde{c}''$ be as in* (2)*. Then τ, the restriction $\tilde{\tau}$ on $F(\alpha, n)$ is a generator of the deck group of the covering in* (1)*, which rotates $\tilde{c}'$ and $\tilde{c}''$ through $2\pi/n$ in negative and positive directions respectively*; *see right diagram on the previous page.*

(4) $F \times S^1 = [F, id] = [F(\alpha, n), \tau]$, *and*

$$p_{\alpha,n} : F(\alpha, n) \times S^1 = [F(\alpha, n), \tau^n] \to F \times S^1 = [F, id]$$

is a cyclic covering of degree n.

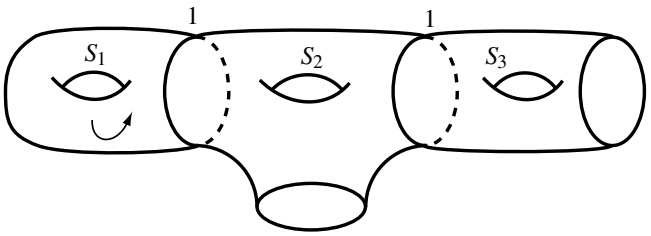
Remark 5.2. We can perform a similar construction for a nonseparating circle γ in F, in which case the description of the boundary is much simpler: each component of ∂F gives rise to precisely n copies of $\partial F(\gamma, n)$.

Example 5.3. We describe a graph manifold that

(1) admits fibrations representing infinitely many fibered commensurability classes, and

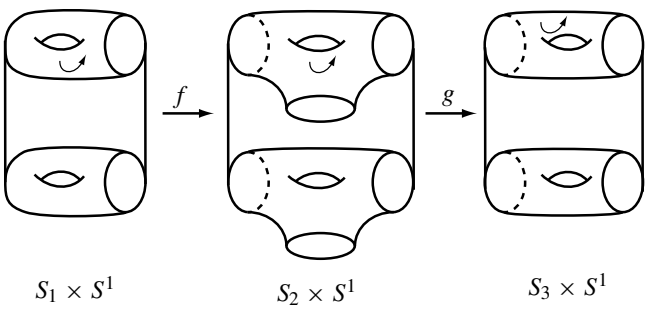
(2) admits infinitely many fibrations representing the *same* fibered commensurability class.

First take $M = [F_1, \phi_1]$, where the oriented surface F_1 and the monodromy ϕ_1 are as shown here:



Note that M has two boundary components and ϕ is D-type and definite.

Another view of M is this:

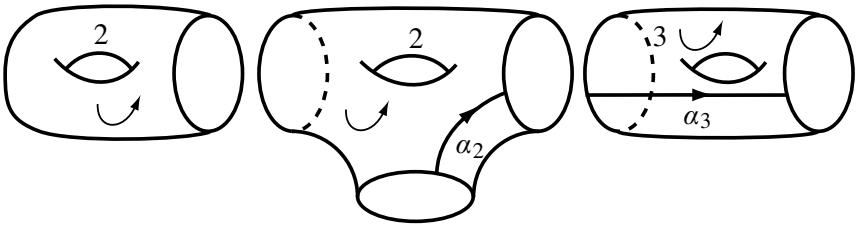


Here every component is of the form $S_i \times S^1$ (depicted in the figure as an $S_i \times I$) for $i = 1, 2, 3$, and two pairs of boundary tori are identified by maps f and g expressed in terms of coordinates by the maps

$$f(1, 0) = (-1, 0), \quad f(0, 1) = (-1, 1);$$
$$g(1, 0) = (-1, 0), \quad g(0, 1) = (-1, 1).$$

Recall that this notation means that each $(1, 0)$ denotes the homotopy class of some component of some ∂S_i, and each $(0, 1)$ denotes an $S^1 \times *$.

Now we construct another surface fibration of the same underlying manifold $M = [F_2, \phi_2]$ as follows. Pick oriented arcs $\alpha_i \in S_i$, $i = 2, 3$ as follows:



Then construct

$$S_1' = S_1(\varnothing, 2), \quad S_2' = S_2(\alpha_2, 2), \quad S_3' = S_3(\alpha_3, 3)$$

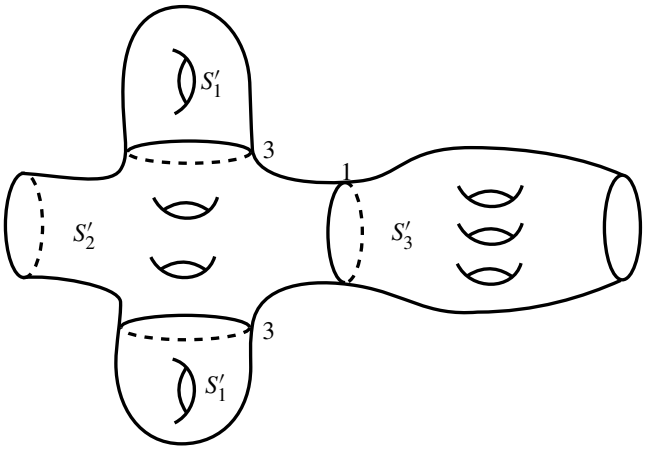
in $S_i \times S^1$, $i = 1, 2, 3$.

By Lemma 5.1(1), it is easy to see that S_1' is two copies of S_1, that S_2' is a surface of genus 2 with 4 boundary components, and that S_3' is a surface of genus 3 with 2 boundary components. By Lemma 5.1(2), we see that $\tilde{c}_2'$ is of slope $(2, 1)$ in $c_2' \times S^1$, and $\tilde{c}_3''$ is of slope $(-3, 1)$ in $c_3'' \times S^1$,

Since g sends $(2, 1)$ to $(-3, 1)$, the maps f and g match S_1', S_2' and S_3' together to produce a new surface F_2 in M. Let τ_i be the generator of the (cyclic) deck group for the covering $p_i : S_i' \to S_i$ given by Lemma 5.1(3). Then τ_1, τ_2, τ_3 have periods 2, 2, 3 respectively. Now the new surface bundle structures $[S_i, \tau_i]$ in $S_i \times S^1$ given by Lemma 5.1(4), $i = 1, 2, 3$, match to produce a new surface bundle structure of M, which we denote by $[F_2, \phi_2]$.

The monodromy map ϕ_2 is a virtual D-type automorphism whose restriction on each S_i' is τ_i. Hence ϕ_2 permutes the two copies of S_1 in F_2. Moreover under this permutation, each copy also undergoes a half-twist relative to S_2'. By Lemma 5.1(3), τ_2 rotates $\tilde{c}_2''$ by π and τ_3 rotate $\tilde{c}_3'$ by $-\frac{2}{3}\pi$ respective along the directions shown in the first figure on page 310. So the relative twist at $S_2' \cap S_3'$ is

$\pi - \frac{2\pi}{3} = \frac{1}{3}\pi$. Now ϕ_2^6 is a D-type automorphism, as shown here:



A direct computation gives

$$\Pi(\phi_1) = \{(1,0), (\tfrac{2}{3},0), (\tfrac{1}{2},0)\} \quad \text{and} \quad \Pi(\phi_2) = \{(2,0), (\tfrac{5}{3},0), (1,0)\}$$

Consequently there is no $s \in \mathbb{Q}$ so that $\Pi(\phi_1) \sim s\Pi(\phi_2)$. By Theorem 4.3, (F_1, ϕ_1) and (F_2, ϕ_2) are not commensurable.

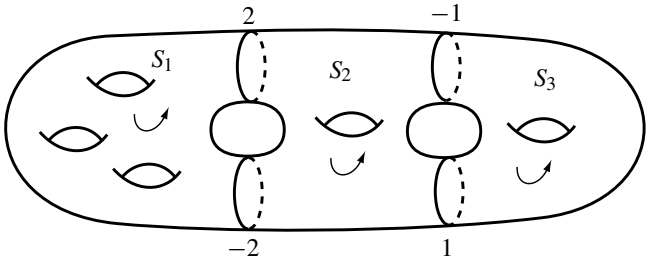
If we perform a similar construction starting from $S_1(\varnothing, n)$, $S_2(\alpha_2, n)$, and $S_3(\alpha_3, n+1)$ in $S_i \times S^1$, $i = 1, 2, 3$, we will get a surface bundle structure $[F_n, \phi_n]$ on M, where ϕ_n is a virtual D-type automorphism and $\phi_n^{n(n+1)}$ is a D-type automorphism, and $\Pi(\phi_n) = \{(n,0), (\frac{2n+1}{3},0), (\frac{n}{2},0)\}$. So for any positive integers $i \neq j$, the automorphisms (F_i, ϕ_i), (F_j, ϕ_j) are not commensurable. We have verified that M fibers in infinitely many incommensurable ways.

On the other hand if we start from $S_1(\gamma, n)$, $S_2(\varnothing, n)$ and $S_3(\varnothing, n)$, where γ is a nonseparating circle in S_1, then by Remark 5.2 and the argument above, we can produce a fibration of M with monodromy $(\Sigma_{2n+1,2n}, \phi_{2,n})$, where we adapt the notations in Example 4.16, and use $\Sigma_{2,3} = S_2 \cup S_3$ in place of $\Sigma_{2,1}$. As observed in Example 4.16, the automorphisms $(\Sigma_{2n+1,2n}, \phi_{2,n})$ are commensurable for all n. So M admits infinitely many distinct but commensurable fibrations, as claimed.

Remark 5.4. One can modify the construction in Example 5.3 to a more general setting where the arc connecting two boundary components of F passes through the cores of more than one Dehn twist. For simplicity, consider a D-type map which is either a single positive or negative Dehn twist on each $\gamma \in \Gamma(\phi)$ (compare with Example 4.17). Then one always gets infinitely many fibered commensurability classes unless the $\chi(S_i)$ satisfy a certain linear equation so that the invariants in Section 4 fail to distinguish them, where S_i's are pieces of $F \setminus \Gamma(\phi)$ meeting the arc.

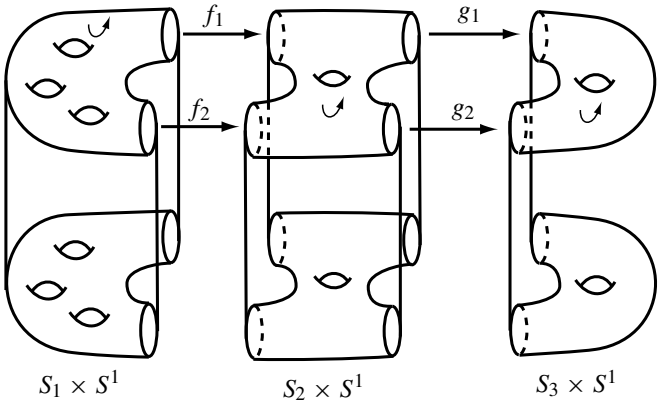
Example 5.5. We now give an example of a *closed* graph manifold which fibers in infinitely many incommensurable ways, including two incommensurable fibrations with fibers of the same genus.

Let $M = [F, \phi]$ be the graph manifold with ϕ as indicated here:

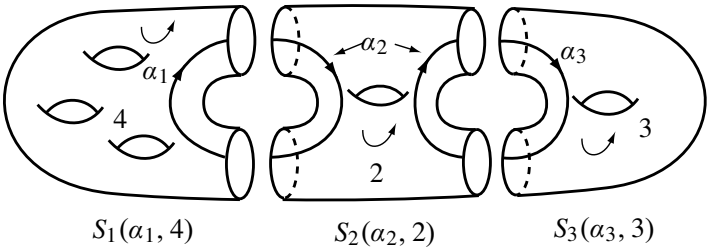


Our discussion of the bottom figure on page 310 applies mutatis mutandis in this case, leading to the following diagram, with gluings given by

$$f_1(1,0)=(-1,0),\quad f_1(0,1)=(2,1);\qquad f_2(1,0)=(-1,0),\quad f_2(0,1)=(-2,1);$$
$$g_1(1,0)=(-1,0),\quad g_1(0,1)=(-1,1);\qquad g_2(1,0)=(-1,0),\quad g_2(0,1)=(1,1).$$



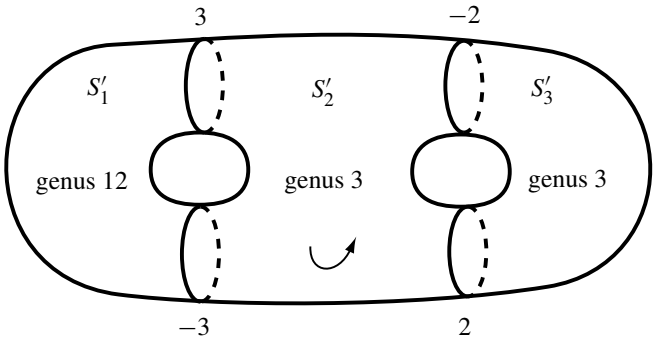
First we construct infinitely many commensurability classes of fibrations of M. Pick oriented arcs $\alpha_i \in S_i$, $i = 1, 2, 3$ as follows:



Construct $S_1' = S_1(\alpha_1, 4)$, $S_2' = S_2(\alpha_2, 2)$, $S_3' = S_3(\alpha_3, 3)$ in $S_i \times S^1$, $i = 1, 2, 3$, respectively. Then f_i and g_i, $i = 1, 2$ paste the boundary of S_i' together to produce another bundle structure on M; i.e., we have $M = [\Sigma_{20}, \phi_2]$, where ϕ_2^{12} is a D-type automorphism on the surface of genus 20. We can check that $(\Sigma_{20}, \phi_2^{12})$ has invariant

$$\Pi(\phi_2) = \{(\tfrac{1}{6}, \tfrac{1}{6}), (\tfrac{5}{4}, \tfrac{5}{4}), (1, 1)\}$$

and is as follows:



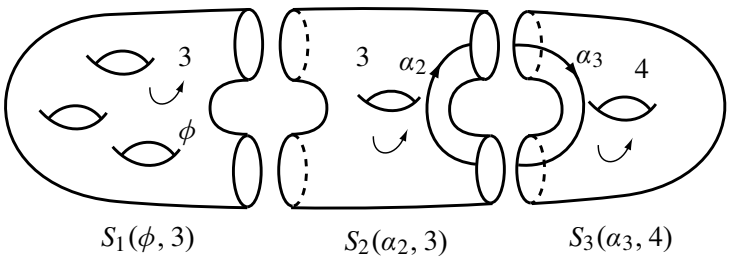
We can perform a similar construction starting from $S_1(\alpha_1, n+2)$, $S_2(\alpha_2, n)$ and $S_3(\alpha_3, n+1)$ in $S_i \times S^1$, $i = 1, 2, 3$, and obtain a surface bundle structure $[\Sigma_{6n+8}, \phi_n]$ on M, where $\phi_n^{n(n+1)(n+2)}$ is a D-type automorphism of a surface of genus $6n + 8$ and

$$\Pi(\phi_n) = \{(\tfrac{n}{12}, \tfrac{n}{12}), (\tfrac{3n+4}{8}, \tfrac{3n+4}{8}), (\tfrac{n}{2}, \tfrac{n}{2})\}.$$

So for any positive integers $i \neq j$, (Σ_{6i+8}, ϕ_i), (Σ_{6j+8}, ϕ_j) are incommensurable.

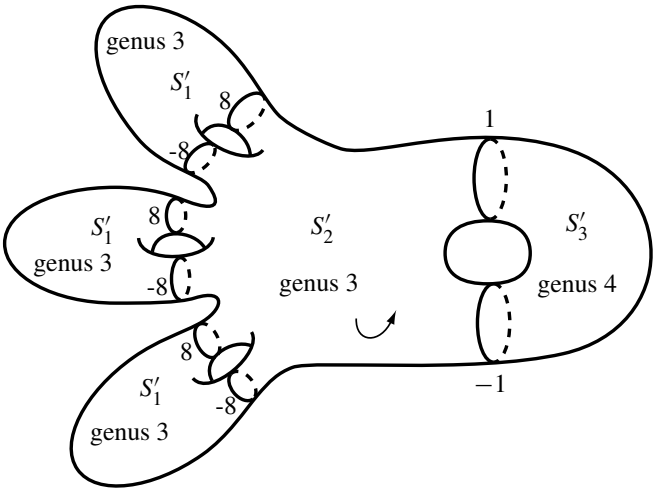
Now we construct another surface bundle structure $[\Sigma_{20}, \psi]$ on M, which is not commensurable with (Σ_{20}, ϕ_2), where ϕ_2 is the automorphism above.

Pick oriented arcs $\alpha_i \in S_i$, $i = 1, 2, 3$ as follows:



and construct $S_1' = S_1(\varnothing, 3)$, $S_2' = S_2(\alpha_2, 3)$, $S_3' = S_3(\alpha_3, 4)$ in $S_i \times S^1$, $i = 1, 2, 3$, respectively. Then f_i and g_i, $i = 1, 2$ glue the boundary of S_i' together to provide M another structure of surface bundle: $M = [\Sigma_{20}, \psi]$, where ψ^{12} is a D-type automorphism on Σ_{20} of genus 20. We can check that (Σ_{20}, ψ^{12}) has invariants

$\Pi(\psi) = \{(\frac{1}{4}, \frac{1}{4}), (\frac{11}{8}, \frac{11}{8}), (\frac{3}{2}, \frac{3}{2})\}$ and is as follows:



By Theorem 4.3 we deduce that (F_2, ψ) and (F_2, ϕ_2) are not commensurable, as claimed.

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