Chapter 2

SOLUTION OF FREDHOLM INTEGRAL EQUATIONS

OUTLINE

- Solution of Homogeneous Fredholm Integral Equation
- Solution of General Fredholm Integral Equation
- Eigen Values and Eigen Functions
- Orthogonality of Eigen Functions

2.1 SOLUTION OF HOMOGENEOUS FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND WITH SEPARABLE (OR DEGENERATE) KERNEL

[MEERUT-2001]

Consider a homogeneous Fredholm integral equation of the second kind

$$u(x) = \lambda \int_a^b k(x, t)\, u(t)\, dt \qquad \text{......(1)}$$

Here, we have assumed that $k(x, t)$ is separable, therefore, we can take

$$k(x, t) = \sum_{i=1}^{n} f_i(x)\, g_i(t) \qquad \text{......(2)}$$

Put this value of $k(x, t)$ in equation (1), we get

$$u(x) = \lambda \int_a^b \left[\sum_{i=1}^{n} f_i(x)\, g_i(t) \right] u(t)\, dt = \lambda \sum_{i=1}^{n} \int_a^b f_i(x)\, g_i(t)\, u(t)\, dt$$

$$= \lambda \sum_{i=1}^{n} f_i(x) \int_a^b g_i(t)\, u(t)\, dt \qquad \text{......(3)}$$

Assume that

$$\int_a^b g_i(t)\, u(t)\, dt = C_i \qquad (i = 1, 2, 3, ..., n) \qquad \text{......(4)}$$

Then equation (3) gives

$$u(x) = \lambda \sum_{i=1}^{n} C_i f_i(x) \qquad \text{.....(5)}$$

Now, we want to evaluate the constants C_i

Multiplying both sides of (5) successively by $g_1(x), g_2(x), \ldots g_n(x)$ and integrating over the range of [a, b], we get

$$\int_a^b g_1(x)\, u(x)\, dx = \lambda \sum_{i=1}^{n} C_i \int_a^b g_1(x)\, f_i(x)\, dx \qquad \text{.....(6)}$$

$$\int_a^b g_2(x)\, u(x)\, dx = \lambda \sum_{i=1}^{n} C_i \int_a^b g_2(x)\, f_i(x)\, dx \qquad \text{.....(7)}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\int_a^b g_n(x)\, u(x)\, dx = \lambda \sum_{i=1}^{n} C_i \int_a^b g_n(x)\, f_i(x)\, dx \qquad \text{.....(8)}$$

Let us define

$$\alpha_{ji} = \int_a^b g_j(x)\, f_i(x) \qquad [i, j \in \mathbb{N}] \qquad \text{.....(9)}$$

Using the value of α_{ji} in (6) and using (4), we get

$$C_1 = \lambda \sum_{i=1}^{n} C_i\, \alpha_{1i}$$

$$= \lambda(C_1\alpha_{11} + C_2\alpha_{12} + \ldots\ldots + C_n\alpha_{1n})$$

$$\Rightarrow \quad (1 - \lambda\alpha_{11})C_1 - \lambda\alpha_{12}C_2 - \ldots\ldots - \lambda\alpha_{1n}C_n = 0$$

Similarly, we can simplify (7), ... (8). Then, we get a system of homogeneous linear equation to find the value of C_i.

$$\left.\begin{aligned}
(1 - \lambda\alpha_{11})C_1 - \lambda\alpha_{12}C_2 - \ldots\ldots - \lambda\alpha_{1n}C_n &= 0 \\
-\lambda\alpha_{21}C_1 + (1 - \lambda\alpha_{22})C_2 - \ldots\ldots - \lambda\alpha_{2n}C_n &= 0 \\
\ldots \quad \ldots \quad \ldots \quad \ldots & \\
-\lambda\alpha_{n1}C_1 - \lambda\alpha_{n2}C_2 \ldots\ldots + (1 - \lambda\alpha_{nn})C_n &= 0
\end{aligned}\right\} \qquad \text{.....(10)}$$

The determinant of (10), i.e., $D(\lambda)$, can be written as

$$D(\lambda) = \begin{vmatrix} (1 - \lambda\alpha_{11}) & -\lambda\alpha_{12} & \cdots & -\lambda\alpha_{1n} \\ -\lambda_{21} & (1 - \lambda_{22}) & \cdots & -\lambda_{2n} \\ \vdots & & \vdots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & -(1 - \lambda_{nn}) \end{vmatrix} = 0 \qquad \text{.....(11)}$$

If $D(\lambda) \neq 0$, the system of equations (10) has only trivial solution given by $C_1 = C_2 = \ldots\ldots = C_n = 0$. Therefore, using (5), we observed that (1) has only zero or 2 trivial solutions, i.e., $u(x) = 0$.

If $D(\lambda) = 0$ at least one of the C_is can be assigned arbitrarily and the remaining C_i's can be determined accordingly. Hence, when $D(\lambda) = 0$, then exists infinitely many of integral equation (1)

Definition

The value(s) of λ for which $D(\lambda) = 0$ are called the eigen values and only non-trivial solution of integral equation (1) is called a corresponding eigen function of (1).

Therefore, the eigen values of (1) can be obtained by $D(\lambda) = 0$.

i.e.,
$$\begin{vmatrix} (1-\lambda\alpha_{11}) & -\lambda\alpha_{12} & \cdots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & (1-\lambda\alpha_{22}) & \cdots & -\lambda\alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda\alpha_{n1} & \lambda\alpha_{n2} & \cdots & -(1-\lambda\alpha_{nn}) \end{vmatrix} = 0$$

Remark

❶ The degree of equation (11) in λ is $m \le n$. Therefore, if integral equation (1) has separate kernel (2), then integral equation (1) has at most n eigen values.

2.2 ORTHOGONALITY AND REALITY OF EIGEN FUNCTIONS

Two functions $f_1(x)$ and $f_2(x)$, continuous on the interval [a, b] are said to be orthogonal on [a, b], if

$$\int_a^b f_1(x)\, f_2(x)\, dx = 0$$

THEOREM-1

If $k(x, t)$ is symmetric and $f_0(x)$ and $f_1(x)$ are eigen functions of $k(x, t)$ corresponding to eigen values λ_0 and λ_1 respectively $[\lambda_0 \neq \lambda_1]$. Then $f_0(x)$ and $f_1(x)$ are orthogonal on [a, b], i.e.,

$$\int_a^b f_0(x)\, f_1(x)\, dx = 0$$ **[MEERUT-1995,9699,2001,03,04,08 GARHWAL-2001]**

Proof : We have that $f_0(x)$ and $f_1(x)$ are eigen functions corresponding to eigen values λ_0 and λ_1 $[\lambda_0 \neq \lambda_1]$ respectively of homogeneous Fredholm integral equation of second kind

$$u(x) = \lambda \int_a^b k(x, t)\, u(t)\, dt \qquad \text{......(1)}$$

Since λ_0 and λ_1 are the eigen values of the corresponding eigen functions $f_0(x)$ and $f_1(x)$, therefore, from (1), we can write

$$f_0(x) = \lambda_0 \int_a^b k(x, t)\, f_0(t)\, dt \qquad \text{......(2)}$$

and
$$f_1(x) = \lambda_1 \int_a^b k(x, t)\, f_1(t)\, dt \qquad \text{......(3)}$$

We know that the kernel $k(x, t)$ is symmetric, therefore

$$k(x, t) = k(t, x) \qquad \text{......(4)}$$

Multiplying both sides of (2) by $f_1(x)$, we get

$$f_1(x)\, f_0(x) = \lambda_0 f_1(x) \int_a^b k(x, t)\, f_0(t)\, dt$$

Integrating w.r.t. x from a to b, we get

$$\int_a^b f_1(x)\, f_0(x)\, dx = \lambda_0 \int_a^b f_1(x) \left\{ \int_a^b k(x, t)\, f_0(t)\, dt \right\} dx$$

$$= \lambda_0 \int_a^b f_0(t) \left\{ \int_a^b k(x, t)\, f_1(x)\, dx \right\} dt$$

$$= \lambda_0 \int_a^b f_0(t) \left\{ \int_a^b k(t, x)\, f_1(x)\, dx \right\} dt \qquad \text{......(5)}$$

Equation (3) can be rewritten as

$$f_1(x) = \lambda_1 \int_a^b k(x, s)\, f_1(s)\, ds$$

$$\Rightarrow \qquad f_1(t) = \lambda_1 \int_a^b k(t, s)\, f_1(s)\, ds = \lambda_1 \int_a^b k(t, x)\, f_1(x)\, dx$$

i.e., $$\int_a^b k(t, x)\, f_1(x)\, dx = \frac{1}{\lambda_1} f_1(t) \qquad \text{......(6)}$$

Putting this value in (5), we get

$$\int_a^b f_1(x)\, f_0(x)\, dx = \lambda_0 \int_a^b f_0(t) \left\{ \frac{1}{\lambda_1} f_1(t) \right\} dt \qquad = \frac{\lambda_0}{\lambda_1} \int_a^b f_0(t)\, f_1(t)\, dt$$

i.e., $$\lambda_1 \int_a^b f_0(x)\, f_1(x)\, dx = \lambda_0 \int_a^b f_0(t)\, f_1(t)\, dt = \lambda_0 \int_a^b f_0(x)\, f_1(x)\, dx$$

$$\Rightarrow \qquad (\lambda_1 - \lambda_0) \int_a^b f_0(x)\, f_1(x)\, dx = 0$$

Since $\lambda_1 \neq \lambda_0$, i.e., $\lambda_1 - \lambda_0 \neq 0$. Therefore, $\int_a^b f_0(x)\, f_1(x)\, dx = 0$

Hence, the eigen functions $f_0(x)$ and $f_1(x)$ are orthogonal on [a, b].

THEOREM-2

The eigen values of a symmetric kernel are real.

[MEERUT-1994,95,2005BP, GARHWAL-1999]

Proof : Let

$$u(x) = \lambda \int_a^b k(x, t)\, u(t)\, dt \qquad \text{......(1)}$$

be a homogeneous Fredholm integral equation of second kind.

We shall show that the eigen values of λ are real.

Let if possible equation (1) has an eigen value λ_0, which is not real.

Therefore, we can write

$$\lambda_0 = \alpha + i\beta \qquad \text{......(2)}$$

Now, let $\phi_0(x) = u + iv$ ......(3)

be the corresponding eigen function of λ_0. Then, complex conjugate of λ_0 would necessarily be an eigen value corresponding to the eigen function $\bar{\phi}_0(x)$ (Complex conjugate of $\phi_0(x)$).

Therefore, we have

$$\bar{\lambda}_0 = \alpha - i\beta \qquad \text{......(4)}$$

and $$\bar{\phi}_0(x) = u - iv \qquad \text{......(5)}$$

Now, from (1), we can deduce that

$$\phi_0(x) = \lambda_0 \int_a^b k(x, t)\, \phi_0(t)\, dt \qquad \text{......(6)}$$

and $$\bar{\phi}_0(x) = \bar{\lambda}_0 \int_a^b k(x, t)\, \bar{\phi}_0(t)\, dt \qquad \text{......(7)}$$

Also, $k(x, t) = k(t, x)$ [$\because k(x, t)$ is symmetric] ......(8)

Multiplying both sides of (6) by $\bar{\phi}_0(x)$ and then integrating w.r.t. x over the interval [a, b], we get

$$\int_a^b \bar{\phi}_0(x)\, \phi_0(x)\, dx = \lambda_0 \int_a^b \bar{\phi}_0(x) \left\{ \int_a^b k(x, t)\, \phi_0(t)\, dt \right\} dx$$

$$= \lambda_0 \int_a^b \phi_0(t) \left\{ \int_a^b k(x, t)\, \bar{\phi}_0(x)\, dx \right\} dt$$

$$= \lambda_0 \int_a^b \phi_0(t) \left\{ \int_a^b k(t, x)\, \bar{\phi}_0(x)\, dx \right\} dt \qquad \text{......(9)}$$

Equation (7) can be written as

$$\bar{\phi}_0(x) = \bar{\lambda}_0 \int_a^b k(x, s)\, \bar{\phi}_0(s)\, ds = \bar{\lambda}_0 \int_a^b k(t, s)\, \bar{\phi}_0(s)\, ds = \bar{\lambda}_0 \int_a^b k(t, x)\, \bar{\phi}_0(x)\, dx$$

i.e., $$\int_a^b k(t, x)\, \bar{\phi}_0(x)\, dx = \frac{1}{\bar{\lambda}_0}\, \bar{\phi}_0(t) \qquad \text{......(10)}$$

Putting this value of (10) in (9), we get

$$\int_a^b \bar{\phi}_0(x)\, \phi_0(x)\, dx = \lambda_0 \int_a^b \phi_0(t) \left\{ \frac{1}{\bar{\lambda}_0}\, \bar{\phi}_0(t) \right\} dt$$

$$\Rightarrow \quad \bar{\lambda}_0 \int_a^b \bar{\phi}_0(x)\, \phi_0(x)\, dx = \lambda \int_a^b \phi_0(t)\, \bar{\phi}_0(t)\, dt$$

$$\Rightarrow \quad \bar{\lambda}_0 \int_a^b \bar{\phi}_0(x)\, \phi_0(x)\, dx = \lambda_0 \int_a^b \phi_0(x)\, \bar{\phi}_0(x)\, dx$$

$$\Rightarrow \quad (\lambda_0 - \bar{\lambda}_0) \int_a^b \bar{\phi}_0(x)\, \phi_0(x)\, dx = 0 \qquad \text{......(11)}$$

Using (2), (3), (4) and (5), we get

$$\left. \begin{aligned} \lambda_0 - \bar{\lambda}_0 &= (\alpha + i\beta) - (\alpha - i\beta) = 2i\beta \\ \text{and} \quad \bar{\phi}_0(x)\, \phi_0(x) &= (u - iv)(u + iv) \\ &= u^2 + v^2 \end{aligned} \right\} \qquad \text{......(12)}$$

Putting the values of (12) in (11), we get

$$2i\beta \int_a^b (u^2 + v^2)\, dx = 0 \qquad \text{......(13)}$$

Since $\phi_0(x)$ is an eigen function corresponding to eigen value λ_0, therefore $\phi_0(x) \neq 0$.

Thus, $\int_a^b (u^2 + v^2)\, dx \neq 0$

Therefore, from (13), we can say that $\beta = 0$.

i.e., imaginary part of the eigen value $\lambda_0 (= \alpha + i\beta)$ is zero.

Hence, $\lambda_0 = \alpha$, which is purely real. Hence, all eigen values of symmetric kernel are real.

SOLVED EXAMPLES

EXAMPLE 1

Find the eigen values and the corresponding eigen functions of the homogeneous integral equation

$$u(x) = \lambda \int_0^1 \sin \pi x \cos \pi t \; u(t)\, dt$$

Solution : Here, we have

$$u(x) = \lambda \int_0^1 \sin \pi x \cos \pi t \; u(t)\, dt$$

which can be written as

$$u(x) = \lambda \sin \pi x \int_0^1 \cos \pi t \; u(t)\, dt \qquad \text{......(1)}$$

Assume that

$$C = \int_0^1 \cos \pi t \; u(t)\, dt \qquad \text{(2)}$$

Then, from (1), we get

$$u(x) = C\lambda \sin \pi x \qquad \text{(3)}$$

$$\Rightarrow \quad u(t) = C\lambda \sin \pi t \qquad \text{(4)}$$

Putting the value of u(t) from (4) in (2), we get

$$C = \int_0^1 \cos \pi t (\lambda C \sin \pi t)\, dt = \frac{\lambda C}{2} \int_0^1 \sin 2\pi t \, dt$$

$$\Rightarrow \quad C = \frac{\lambda C}{2}\left[-\frac{\cos 2\pi t}{2\pi}\right]_0^1 = \frac{\lambda C}{2}\left[-\frac{1}{2\pi} + \frac{1}{2\pi}\right]$$

Therefore $C = 0$ and hence $u(x) = 0$ [from (3)]

$\Rightarrow$ For any λ, equation (1) has only zero solution $u(x) = 0$.

Hence, (1) does not have any eigen values or eigen functions.

EXAMPLE 2

Find the eigen values and eigen functions of the homogeneous integral equation

$$u(x) = \lambda \int_0^1 e^x e^t \; u(t)\, dt$$

Solution : The given equation can be written as

$$u(x) = \lambda e^x \int_0^1 e^t\, u(t)\, dt \qquad \text{......(1)}$$

Assume that $C = \int_0^1 e^t\, u(t)\, dt$ ......(2)

Then from (1), we have

$$u(x) = \lambda C e^x \qquad \text{......(3)}$$

$$\Rightarrow \quad u(t) = \lambda C e^t \qquad \text{......(4)}$$

Putting this value in (2), we get

$$C = \int_0^1 e^t (\lambda C e^t)\, dt = \lambda C \left[\frac{e^{2t}}{2}\right]_0^1 = \frac{\lambda C}{2}(e^2 - 1)$$

$$\Rightarrow \quad C\left[1 - \frac{\lambda}{2}(e^2 - 1)\right] = 0 \qquad \text{......(5)}$$

If $C = 0$, then (3) gives $u(x) = 0$. We, therefore, assume that for non-zero solution of (1), $C \neq 0$, then (5) gives

$$1 - \frac{\lambda}{2}(e^2 - 1) = 0 \quad \Rightarrow \quad \lambda = \frac{2}{e^2 - 1} \qquad \text{......(6)}$$

which is an eigen value of (1).

To find the corresponding eigen function, putting the value of λ [from (6)] in (3), we get

$$u(x) = \frac{2C}{e^2 - 1} . e^x$$

Hence, the eigen function, corresponding to the eigen value $\frac{2}{e^2 - 1}$ is e^x [$\because$ the constant $\frac{2C}{e^2 - 1}$ is taken as unity].

EXAMPLE 3

Solve the homogeneous Fredholm integral equation of the second kind

$$u(x) = \lambda \int_0^{2\pi} \sin(x + t)\, u(t)\, dt$$ **[MEERUT-2002]**

Solution : Here, the given integral equation can be written as

$$u(x) = \lambda \int_0^{2\pi} (\sin x \cos t + \cos x \sin t)\, u(t)\, dt$$

$$= \lambda \sin x \int_0^{2\pi} \cos t\, u(t)\, dt + \lambda \cos x \int_0^{2\pi} \sin t\, u(t)\, dt \qquad \text{......(1)}$$

Let us assume

$$C_1 = \int_0^{2\pi} \cos t\, u(t)\, dt \qquad \text{......(2)}$$

and $$C_2 = \int_0^{2\pi} \sin t\, u(t)\, dt \qquad \text{......(3)}$$

Using (2) and (3), (1) reduces to

$$u(x) = \lambda C_1 \sin x + \lambda C_2 \cos x \qquad \text{......(4)}$$

$$\Rightarrow \quad u(t) = \lambda C_1 \sin t + \lambda C_2 \cos t \qquad \text{......(5)}$$

Putting the value of $u(t)$, from (5) in (2), we get

$$C_1 = \int_0^{2\pi} \cos t\,(\lambda C_1 \sin t + \lambda C_2 \cos t)\,dt$$

$$= \frac{\lambda C_1}{2}\int_0^{2\pi} \sin 2t\,dt + \frac{\lambda C_2}{2}\int_0^{2\pi}(1+\cos 2t)\,dt$$

$$= \frac{\lambda C_1}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{\lambda C_2}{2}\left[t+\frac{\sin 2t}{2}\right]_0^{2\pi}$$

or $\quad C_1 = 0 + \lambda C_2 \pi \quad$ or $\quad C_1 - \lambda\pi C_2 = 0 \qquad$ ......(6)

Similarly, from (3), we get

$$C_2 = \int_0^{2\pi} \sin t\,(\lambda C_1 \sin t + \lambda C_2 \cos t)\,dt$$

$$= \frac{\lambda C_1}{2}\int_0^{2\pi}(1-\cos 2t)\,dt + \frac{\lambda C_2}{2}\int_0^{2\pi}\sin 2t\,dt$$

$$= \frac{\lambda C_1}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{\lambda C_2}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

or $\quad C_2 = \lambda C_1 \pi \quad$ or $\quad \lambda\pi C_1 - C_2 = 0 \qquad$ ......(7)

(6) and (7) gives the system of linear homogeneous equations for determination of C_1 and C_2. For non-zero solution, we must have

$$\begin{vmatrix} 1 & -\lambda\pi \\ \lambda\pi & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad -1 + \lambda^2\pi^2 = 0 \Rightarrow \lambda = \pm\frac{1}{\pi}$$

Therefore, the required eigen values are given by $\lambda_1 = \frac{1}{\pi}$ and $\lambda_2 = -\frac{1}{\pi}$

Determination of Eigen Function

(1) For $\lambda = \lambda_1 = \frac{1}{\pi}$

Putting $\lambda = \frac{1}{\pi}$ in (6) and (7), we get

$$C_1 - C_2 = 0 \qquad \text{......(9)}$$

$$C_1 - C_2 = 0 \qquad \text{......(10)}$$

which implies, $C_1 = C_2$. Therefore, from (4), we have

$$u(x) = \frac{1}{\pi}C_1 \sin x + \frac{1}{\pi}C_1 \cos x = \frac{C_1}{\pi}(\sin x + \cos x)$$

Let $\frac{C_1}{\pi} = 1$. Hence, the required eigen function is given by

$$u_1(x) = (\sin x + \cos x) \qquad \text{......(11)}$$

(2) For $\lambda = \lambda_2 = -\frac{1}{\pi}$

Putting $\lambda = -\frac{1}{\pi}$ in (6) and (7), we get

$$C_1 + C_2 = 0 \quad \text{......(12)}$$

and $$C_1 + C_2 = 0 \quad \text{......(13)}$$

which implies, $C_1 = -C_2$. Therefore, from (4), we have

$$u(x) = -\frac{1}{\pi}C_1 \sin x + \left(-\frac{1}{\pi}\right)(-C_1)\cos x = \left(-\frac{C_1}{\pi}\right)(\sin x - \cos x)$$

Taking $\left(-\frac{C_1}{\pi}\right) = 1$, the required eigen function is given by

$$u_2(x) = \sin x - \cos x$$

EXAMPLE 4

Find the eigen values and eigen functions of the homogeneous integral equation

$$u(x) = \lambda \int_0^{\pi} (\cos^2 x \cos 2t + \cos 3x \cos^3 t)\, u(t)\, dt$$

[MEERUT-1998, 2000, 01, 03,06(BP), 07(BP),08, GARHWAL-2004, KANPUR-2005]

Solution : Here, the given equation can be written as

$$u(x) = \lambda \cos^2 x \int_0^{\pi} (\cos 2t u(t)\, dt + \lambda \cos 3x \int_0^{\pi} \cos^3 t\, u(t)\, dt \quad \text{......(1)}$$

Let us assume

$$C_1 = \int_0^{\pi} \cos 2t\, u(t)\, dt \quad \text{......(2)}$$

$$C_2 = \int_0^{\pi} \cos^3 t\, u(t)\, dt \quad \text{......(3)}$$

Using (2) and (3), (1) gives

$$u(x) = \lambda C_1 \cos^2 x + \lambda C_2 \cos 3x \quad \text{......(4)}$$

$$\Rightarrow \quad u(t) = \lambda C_1 \cos^2 t + \lambda C_2 \cos 3t \quad \text{......(5)}$$

Putting this value of u(t) in (2), we get

$$C_1 = \int_0^{\pi} \cos 2t\, (\lambda C_1 \cos^2 t + \lambda C_2 \cos 3t)\, dt$$

$$\Rightarrow \quad C_1\Big[1 - \lambda \underbrace{\int_0^{\pi} \cos 2t \cos^2 t\, dt}_{=I_1}\Big] - \lambda C_2 \underbrace{\int_0^{\pi} \cos 2t \cos 3t\, dt}_{=I_2} = 0 \quad \text{......(6)}$$

Let $$I_1 = \int_0^{\pi} \cos 2t \cos^2 t\, dt = \int_0^{\pi} \cos 2t\left[\frac{1+\cos 2t}{2}\right] dt$$

$$= \frac{1}{2}\int_0^{\pi} \cos 2t\, dt + \frac{1}{2}\int_0^{\pi}\left(\frac{1+\cos 4t}{2}\right) dt$$

$$= 0 + \frac{1}{4}\left[t + \frac{\sin 4t}{4}\right]_0^{\pi} = \frac{\pi}{4}$$

Also, $\quad I_2 = \int_0^{\pi} \cos 2t \cos 3t \, dt = \frac{1}{2}\int_0^{\pi} [\cos 5t + \cos t] \, dt$

$$= \frac{1}{2}\left[\frac{\sin 5t}{5} + \sin t\right]_0^{\pi} = 0$$

Putting the values of I_1 and I_2 in (6), we get

$$C_1\left(1 - \frac{\lambda\pi}{4}\right) - 0.C_2 = 0 \qquad \text{......(7)}$$

Similarly, using (5), (3) gives $C_2 = \int_0^{\pi} \cos^3 t(\lambda C_1 \cos^2 t + \lambda C_2 \cos 3t) \, dt$

$$\Rightarrow \qquad \lambda C_1 \int_0^{\pi} \cos^5 t \, dt + C_2\left[\lambda \int_0^{\pi} \cos^3 t \cos 3t \, dt - 1\right] = 0 \qquad \text{......(8)}$$

Now, since $\cos^5(\pi - t) = -\cos^5 t$, therefore $\int_0^{\pi} \cos^5 t \, dt = 0$ ......(9)

Also, $\quad \int_0^{\pi} \cos^3 t \cos 3t \, dt = \frac{1}{4}\int_0^{\pi} \cos 3t \, (\cos 3t + 3\cos t) \, dt$

$$= \frac{1}{4}\int_0^{\pi} \cos^2 3t \, dt + \frac{3}{4}\int_0^{\pi} \cos 3t \cos t \, dt$$

$$= \frac{1}{4}\int_0^{\pi} \frac{1 + \cos 6t}{2} dt + \frac{3}{4}\int_0^{\pi} \frac{\cos 4t + \cos t}{2} dt$$

$$= \frac{1}{8}\left[t + \frac{\sin 6t}{6}\right]_0^{\pi} + \frac{3}{8}\left[\frac{\sin 4t}{4} + \sin t\right]_0^{\pi} = \frac{\pi}{8}$$

Therefore, $\int_0^{\pi} \cos^3 t \cos 3t \, dt = \frac{\pi}{8}$ ......(10)

Using (9) and (10), (8) gives

$$0.C_1 + C_2\left(\frac{\lambda\pi}{8} - 1\right) = 0$$

or

$$0.C_1 + C_2\left(1 - \frac{\lambda\pi}{8}\right) = 0 \qquad \text{......(11)}$$

For non-zero solution of the system of equations (7) and (11), we must have

$$\begin{vmatrix} 1 - \frac{\lambda\pi}{4} & 0 \\ 0 & 1 - \frac{\lambda\pi}{8} \end{vmatrix} = 0$$

or

$$\left(1 - \frac{\lambda\pi}{4}\right)\left(1 - \frac{\lambda\pi}{8}\right) = 0 \quad \Rightarrow \quad \lambda = \frac{4}{\pi} \text{ or } \frac{8}{\pi}$$

Hence, the eigen values of (1) are given by

$$\lambda_1 = 4/\pi \quad \text{and} \quad \lambda_2 = 8/\pi \qquad \text{......(12)}$$

Determination of Eigen function

(1) For $\lambda = \lambda_1 = 4/\pi$

Putting $\lambda = \lambda_1 = \frac{4}{\pi}$ in (7) and (11), we have

$$0.C_1 + 0.C_2 = 0 \quad \text{......(13)}$$

and $$0.C_1 - \frac{1}{2}C_2 = 0 \quad \text{......(14)}$$

On solving, we get $C_2 = 0$ and C_1 is arbitrary.

Putting these values in (4), we get

$$u(x) = \lambda C_1 \cos^2 x = \frac{4}{\pi} C_1 \cos^2 x$$

Setting $\frac{4}{\pi}C_1 = 1$, the required eigen function corresponding to $\lambda = \frac{4}{\pi}$ is given by

$$u_1(x) = \cos^2 x$$

(2) For $\lambda = \lambda_2 = 8/\pi$

Putting $\lambda = \lambda_2 = \frac{8}{\pi}$ in (7) and (11), we have

$$-C_1 + 0.C_2 = 0 \quad \text{......(15)}$$

and $$0.C_1 + 0.C_2 = 0 \quad \text{......(16)}$$

On solving, we get

$C_1 = 0$ and C_2 is arbitrary.

Therefore, from (4), we have

$$u(x) = \lambda C_2 \cos 3x = \frac{8}{\pi} C_2 \cos 3x$$

Setting $\frac{8}{\pi}C_2 = 1$, the required eigen function corresponding to $\lambda = \frac{8}{\pi}$ is given by

$$u_2(x) = \cos 3x$$

EXAMPLE 5

Find the eigen values and eigen functions of the homogeneous integral equation

$$u(x) = \lambda \int_{-1}^{1} (5xt^3 + 4x^2 t + 3tx)\, u(t)\, dt$$ **[MEERUT-1999, GARHWAL-1999]**

Solution : Here, the given equation can be written as

$$u(x) = 5\lambda x \int_{-1}^{1} t^3 u(t)\, dt + 4x^2 \lambda \int_{-1}^{1} t\, u(t)\, dt + 3x\lambda \int_{-1}^{1} t\, u(t)\, dt$$

$$= 5x\lambda \int_{-1}^{1} t^3 u(t)\, dt + (4x^2 + 3x)\, \lambda \int_{-1}^{1} t.u(t)\, dt \quad \text{......(1)}$$

Let us assume

$$C_1 = \int_{-1}^{1} t^3\, u(t)\, dt \qquad \text{......(2)}$$

and $$C_2 = \int_{-1}^{1} t\, u(t)\, dt \qquad \text{......(3)}$$

Then, (1) gives

$$u(x) = 5\lambda x C_1 + \lambda C_2(4x^2 + 3x) \qquad \text{......(4)}$$

$$\Rightarrow \quad u(t) = 5\lambda t C_1 + \lambda C_2(4t^2 + 3t) \qquad \text{......(5)}$$

Putting this value of u(t) in (2), we get

$$C_1 = \int_{-1}^{1} t^3 \left[5\lambda C_1 t + \lambda C_2(4t^2 + 3t) \right] dt$$

$$= 5\lambda C_1 \left[\frac{t^5}{5} \right]_{-1}^{1} + \lambda C_2 \left[4\left(\frac{t^6}{6}\right) + 3\left(\frac{t^5}{5}\right) \right]_{-1}^{1} = 2\lambda C_1 + \frac{6}{5}\lambda C_2$$

$$\Rightarrow \quad C_1(1 - 2\lambda) - \frac{6}{5}\lambda C_2 = 0 \qquad \text{......(6)}$$

Similarly, putting the value of u(t) in (3), we get

$$C_2 = \int_{-1}^{1} t \left[5\lambda C_1 t + \lambda C_2(4t^2 + 3t) \right] dt$$

$$= 5\lambda C_1 \left[\frac{t^3}{3} \right]_{-1}^{1} + \lambda C_2 \left[4\left(\frac{t^4}{4}\right) + 3\left(\frac{t^3}{3}\right) \right]_{-1}^{1} = \frac{10}{3}\lambda C_1 + 2\lambda C_2$$

$$\Rightarrow \quad -\frac{10}{3}\lambda C_1 + C_2(1 - 2\lambda) = 0 \qquad \text{......(7)}$$

For non-zero solution of (6) and (7), we must have

$$\begin{vmatrix} 1 - 2\lambda & -\frac{6}{5}\lambda \\ -\frac{10}{3}\lambda & 1 - 2\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1 - 2\lambda)^2 - 4\lambda^2 = 0 \Rightarrow \lambda = \frac{1}{4}$$

Determination of Eigen function

Putting $\lambda = \frac{1}{4}$ in (6) and (7), we have

$$\frac{1}{2}C_1 - \frac{3}{10}C_2 = 0 \qquad \text{......(8)}$$

and $$-\frac{5}{6}C_1 + \frac{1}{2}C_2 = 0 \qquad \text{......(9)}$$

On solving, we get

$$C_1 = \frac{3}{5}C_2 \qquad \text{......(10)}$$

Putting $\lambda = 1/4$ and using (10), equation (4) becomes

$$u(x) = 5.\frac{1}{4}\left(\frac{3}{5}C_2\right)x + \frac{1}{4}C_2(4x^2 + 3x) \;=\; C_2\left(x^2 + \frac{3}{2}x\right)$$

Setting $C_2 = 1$, we have $u(x) = x^2 + \frac{3}{2}x$ which is the required eigen function.

EXAMPLE 6

QP term *Determine the eigen values and eigen functions of the homogeneous integral equation*

$$u(x) = \lambda \int_0^1 k(x, t)\, u(t)\, dt$$

where $k(x, t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1), & t \le x \le 1 \end{cases}$ [GARHWAL-1999]

Solution : Here, we have

$$u(x) = \lambda \int_0^1 k(x, t)\, u(t)\, dt \qquad \text{......(1)}$$

where $k(x, t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1), & t \le x \le 1 \end{cases}$ ......(2)

Equation (1) can be written as

$$u(x) = \lambda\left[\int_0^x k(x, t)\, u(t)\, dt + \int_x^1 k(x, t)\, u(t)\, dt\right]$$

$$= \int_0^x \lambda t(x-1)\, u(t)\, dt + \int_x^1 \lambda x(t-1)\, u(t)\, dt \qquad \text{......(3)}$$

Differentiating (3) w.r.t. x and using Leibnitz's rule, we get

$$u'(x) = \int_0^x \lambda t\, u(t)\, dt + \lambda x(x-1)\, u(x) - 0 + \int_x^1 \lambda(t-1)\, u(t)\, dt + 0 - \lambda x(x-1)\, u(x)$$

$$\Rightarrow \qquad u'(x) = \int_0^x \lambda t\, u(t)\, dt + \int_x^1 \lambda(t-1)\, u(t)\, dt \qquad \text{......(4)}$$

Now, differentiating (4) w.r.t. x and using Leibnitz's rule as above, we get

$$u''(x) = 0 + \lambda x\, u(x) - 0 + 0 + 0 - \lambda(x-1)\, u(x)$$

$$\Rightarrow \qquad u''(x) - \lambda\, u(x) = 0 \qquad \text{......(5)}$$

Putting x = 0 and x = 1 successively in (3), we get

$$u(0) = 0 \;\text{ and }\; u(1) = 0 \qquad \text{......(6)}$$

Now, we shall solve (5) with boundary conditions (6) to find the eigen values and eigen functions.

Now, there are following cases.

(i) $\lambda = 0$

Put $\lambda = 0$ in (5), we get $u''(x) = 0$

The general solution of the above equation is

$$u(x) = Ax + B \qquad \ldots\ldots(7)$$

Now using (6), we get

$$0 = B \qquad \ldots\ldots(8)$$

and $\quad 0 = A + B \qquad \ldots\ldots(9)$

$\Rightarrow \quad A = B = 0$

Therefore (7) implies $u(x) = 0$ which is not an eigen function and therefore $\lambda = 0$ is not an eigen value.

(ii) $\lambda = \mu^2$ $(\mu \neq 0)$

Put $\lambda = \mu^2$ in (5), we get $u''(x) - \mu^2 u(x) = 0$.

whose general solution is given by

$$u(x) = Ae^{\mu x} + Be^{-\mu x} \qquad \ldots\ldots(10)$$

Now putting x = 0 and x = 1 in (10) and using (6), we get

$$0 = A + B \qquad \ldots\ldots(11)$$

and $\quad 0 = Ae^{\mu} + Be^{-\mu} \qquad \ldots\ldots(12)$

On solving, we get A = B = 0. Therefore, (7) gives u(x) = 0, which is not an eigen function and therefore, $\lambda = \mu^2$ does not give eigen values.

(iii) $\lambda = -\mu^2$ $(\mu \neq 0)$

Put $\lambda = -\mu^2$ in (5), we get $u''(x) + \mu^2 u(x) = 0$.

whose general solution is given by

$$u(x) = A\cos\mu x + B\sin\mu x \qquad \ldots\ldots(13)$$

Now putting x = 0 and x = 1 in (13) and using (6), we get

$$0 = A \qquad \ldots\ldots(14)$$

and $\quad 0 = A\cos\mu + B\sin\mu \qquad \ldots\ldots(15)$

On solving (14) and (15), we get $A = 0$ and $B\sin\mu = 0$.

But $B \neq 0$ [$\because$ B = 0 and A = 0, again we shall not get an eigen function].

Therefore, (15) gives

$$\sin\mu = 0 \Rightarrow \mu = n\pi, \quad n \in \mathrm{N}$$

$\Rightarrow$ The required eigen values are given by

$$\lambda_n = \lambda = -\mu^2 = -n^2\pi^2, \qquad n \in \mathrm{N}$$

Also, from (13), the corresponding eigen functions are given by

$$u_n(x) = B\sin n\pi x \qquad [\because A = 0, \mu = n\pi]$$

or $\quad u_n(x) = \sin n\pi x \qquad$ [Setting B = 1]

which is the required eigen function.

EXAMPLE 7

Determine the eigen values and eigen functions of the homogeneous equation

$$u(x) = \lambda \int_0^\pi k(x, t)\, u(t)\, dt$$

where $k(x, t) = \begin{cases} \cos x \sin t\,; & 0 \le x \le t \\ \cos t \sin x\,; & t \le x \le \pi \end{cases}$ **[MEERUT-1999, GARHWAL-2002]**

Solution : Here, given that

$$u(x) = \lambda \int_0^\pi k(x, t)\, u(t)\, dt \qquad \text{......(1)}$$

where $k(x, t) = \begin{cases} \cos x \sin t\,; & 0 \le x \le t \\ \cos t \sin x\,; & t \le x \le \pi \end{cases}$ ......(2)

Equation (1) can be written as

$$u(x) = \lambda\left[\int_0^x k(x, t)\, u(t)\, dt + \int_x^\pi k(x, t)\, u(t)\, dt\right]$$

$$\Rightarrow \quad u(x) = \int_0^x (\lambda \cos t \sin x)\, u(t)\, dt + \int_x^\pi (\lambda \cos x \sin t)\, u(t)\, dt \qquad \text{......(3)}$$

Differentiating (3) w.r.t. x, we get

$$u'(x) = \frac{d}{dx}\int_0^x (\lambda \cos t \sin x)\, u(t)\, dt + \frac{d}{dx}\int_x^\pi (\lambda \cos x \sin t)\, u(t)\, dt$$

$$= \int_0^x \frac{\partial}{\partial x}\{\lambda \cos t \sin x\, u(t)\}\, dt + \lambda \cos x \sin x\, u(x)\frac{dx}{dx}$$

$$-\lambda \cos 0 \sin x\, u(0)\frac{d0}{dx} + \int_x^\pi \frac{\partial}{\partial x}\{\lambda \cos x \sin t\, u(t)\}\, dt$$

$$+ \lambda \cos x \sin \pi\, u(\pi)\frac{d\pi}{dx} - \lambda \cos x \sin x\, u(x)\frac{d\pi}{dx} \qquad \text{[By Leibnitz's rule]}$$

$$= \int_0^\pi (\lambda \cos t \cos x)\, u(t)\, dt + \lambda \cos x \sin x\, u(x)$$

$$+ \int_x^\pi (-\lambda \sin x \sin t)\, u(t)\, dt - \lambda \cos x \sin x\, u(x)$$

$$\Rightarrow \quad u'(x) = \int_0^x (\lambda \cos t \cos x)\, u(t)\, dt - \int_x^\pi (\lambda \sin x \sin t)\, u(t)\, dt \qquad \text{......(4)}$$

Differentiating (4) w.r.t. x, we get

$$u''(x) = \frac{d}{dx}\int_0^x (\lambda \cos t \cos x)\, u(t)\, dt - \frac{d}{dx}\int_x^\pi (\lambda \sin x \sin t)\, u(t)\, dt$$

$$= \int_0^x \frac{\partial}{\partial x}\{\lambda \cos t \cos x\, u(t)\}\, dt + \lambda \cos^2 x\, u(x)\frac{dx}{dx} - \lambda \cos 0 \cos x\, u(0)\frac{d0}{dx}$$

$$-\left[\int_x^\pi \frac{\partial}{\partial x}\{\lambda \sin x \sin t\, u(t)\}\, dt + \lambda \sin x \sin \pi\, u(\pi)\frac{d\pi}{dx} - \lambda \sin^2 x\, u(x)\frac{dx}{dx}\right]$$

$$= -\int_0^x (\lambda \cos t \sin x)\, u(t)\, dt + \lambda \cos^2 x\, u(x)$$

$$-\int_x^{\pi} (\lambda \cos x \sin t)\, u(t)\, dt + \lambda \sin^2 x\, u(x)$$

$$= \lambda\, u(x) - \left[\int_0^x (\lambda \cos t \sin x)\, u(t)\, dt + \int_x^{\pi} (\lambda \cos x \sin t)\, u(t)\, dt \right]$$

$$= \lambda\, u(x) - u(x) \qquad \text{[Using (3)]}$$

$\Rightarrow \quad u''(x) - (\lambda - 1)\, u(x) = 0$ ......(5)

Now, putting $x = \pi$ in (3) and $x = 0$ in (4), we get

$u(\pi) = 0$ ......(6)

and $u'(0) = 0$ ......(7)

Now, we find the eigen values and corresponding functions. There are following three cases that arise :

(i) $\lambda - 1 = 0$, i.e., $\lambda = 1$

Put $\lambda = 1$ in (5), we get $u''(x) = 0$.

whose general solution is given by

$u(x) = Ax + B$ ......(8)

$\Rightarrow \quad u'(x) = A$ ......(9)

Putting $x = \pi$ in (8) and using (6), we get

$0 = A\pi + B$ ......(10)

Similarly, putting $x = 0$ in (9) and using (7), we get

$0 = A$ ......(11)

On solving (10) and (11), we get $A = 0, B = 0$.

$\Rightarrow \quad u(x) = 0$, which is not an eigen function and hence $\lambda = 1$ is not an eigen value.

(ii) $\lambda - 1 = \mu^2$ $(\mu \neq 0)$

Put $\lambda - 1 = \mu^2$ in (5), we get $u''(x) - \mu^2\, u(x) = 0$.

whose general solution is given by

$u(x) = Ae^{\mu x} + Be^{-\mu x}$ ......(12)

$\Rightarrow \quad u'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$ ......(13)

Now,

$u(\pi) = 0 \Rightarrow \quad 0 = Ae^{\mu\pi} + Be^{-\mu\pi}$ ......(14)

$u'(0) = 0 \Rightarrow \quad 0 = A\mu - B\mu$ ......(15)

Solving (14) and (15), we get A = B = 0.

$\Rightarrow$ $u(x) = 0$, which is again not an eigen function and hence $\lambda = 1 + \mu^2$ is not an eigen value.

(iii) $\lambda - 1 = -\mu^2$ $(\mu \neq 0)$

Put $\lambda - 1 = -\mu^2$ in (5), we get $u''(x) + \mu^2 u(x) = 0$.

whose general solution is given by

$$u(x) = A\cos\mu x + B\sin\mu x \quad \ldots\ldots(16)$$

$$\Rightarrow \quad u'(x) = -A\mu\sin\mu x + B\mu\cos\mu x \quad \ldots\ldots(17)$$

Now, $u(\pi) = 0 \Rightarrow \quad A\cos\mu\pi + B\sin\mu\pi = 0 \quad \ldots\ldots(18)$

$u'(\pi) = 0 \Rightarrow \quad B\mu = 0 \quad \ldots\ldots(19)$

Now, we must take $A \neq 0$, otherwise A = 0 = B gives u(x) = 0 as before.

Hence, (19) gives $\cos\mu\pi = 0$.

$$\Rightarrow \quad \mu\pi = (2n+1)\frac{\pi}{2} \quad \Rightarrow \quad \mu = \left(n + \frac{1}{2}\right)$$

But $\quad \lambda - 1 = -\mu^2 \Rightarrow \lambda = 1 - \mu^2$

Therefore, the eigen values are given by

$$\lambda_n = 1 - \mu^2 = 1 - \left(n + \frac{1}{2}\right)^2$$

Now, putting B = 0 and $\mu = \left(n + \frac{1}{2}\right)$ in (16), the corresponding eigen functions $u_n(x)$ are given by

$$u_n(x) = A\cos\left(n + \frac{1}{2}\right)x \quad \text{or} \quad u_n(x) = \cos\left(n + \frac{1}{2}\right)x \quad \text{[Setting A = 1]}$$

which is the required eigen function.

EXAMPLE 8

Determine the eigen values and eigen functions of the homogeneous integral equation

$$u(x) = \lambda\int_0^1 k(x,t)\,u(t)\,dt$$

where, $k(x,t) = \begin{cases} t(x+1); & 0 \le x \le t \\ x(t+1); & t \le x \le 1 \end{cases}$ **[GARHWAL-2000]**

Solution : Here, the given integral equation is

$$u(x) = \lambda\int_0^1 k(x,t)\,u(t)\,dt \quad \ldots\ldots(1)$$

where, $k(x,t) = \begin{cases} t(x+1); & 0 \le x \le t \\ x(t+1); & t \le x \le 1 \end{cases} \quad \ldots\ldots(2)$

Equation (1) can be written as

$$u(x) = \lambda\left[\int_0^x k(x,t)\,u(t)\,dt + \int_x^1 k(x,t)\,u(t)\,dt\right]$$

$$\Rightarrow \quad u(x) = \int_0^x (\lambda x(t+1)\,u(t)\,dt + \int_x^1 \lambda t(x+1)\,u(t)\,dt \quad \ldots\ldots(3)$$

Differentiating (3) w.r.t. x, we get

$$u'(x) = \int_0^x \lambda(t-1)\,u(t)\,dt + \lambda x(x-1)\,u(x)\cdot 1 - 0 + \int_x^1 \lambda t\,u(t)\,dt$$

$$+\, 0 - \lambda x(x-1)\,u(x) \qquad \text{[By Leibnitz's rule]}$$

$$= \int_0^x \lambda(t-1)\,u(t)\,dt + \int_x^1 \lambda t\,u(t)\,dt \qquad (4)$$

Now, differentiating (4) w.r.t. x (as before), we get

$$u''(x) = 0 + \lambda(x-1)\,u(x) - 0 + 0 - \lambda x\,u(x)$$

$$\Rightarrow \quad u''(x) + \lambda u(x) = 0 \qquad (5)$$

Putting $x = 0$ in (3) and (4), we get

$$u(0) = -\int_0^1 \lambda t\,u(t)\,dt \qquad (6)$$

and $$u'(0) = \int_0^1 \lambda t\,u(t)\,dt \qquad (7)$$

Putting $x = 1$ in (3) and (4), we get

$$u(1) = \int_0^1 \lambda(t-1)\,u(t)\,dt \qquad (8)$$

and $$u'(1) = \int_0^1 \lambda(t-1)\,u(t)\,dt \qquad (9)$$

From (6) and (7), we have

$$u(0) = -u'(0) \qquad (10)$$

From (8) and (9), we have

$$u(1) = u'(1) \qquad (11)$$

Now there are following three cases

(i) $\lambda = 0$

Put $\lambda = 0$ in (5), we get $u''(x) = 0$

$$\Rightarrow \quad u(x) = Ax + B \qquad (12)$$

$$\Rightarrow \quad u'(x) = A \qquad (13)$$

Putting $x = 0$ in (12) and (13), we get $u(0) = B$ and $u'(0) = A$

Therefore, (10) gives

$$B = -A \qquad (14)$$

Putting $x = 1$ in (12) and (13), we get $u(1) = A + B$

and $$u'(1) = A \qquad (15)$$

Therefore, (11) gives

$$A + B = A \quad \Rightarrow \quad B = 0 \qquad (16)$$

Also, from (14), we have $A = 0$

Hence, (12) gives $u(x) = 0$, which is not an eigen function and so $\lambda = 0$ is not an eigen value.

(ii) $\lambda = \mu^2 \ (\mu \neq 0)$

Put $\lambda = \mu^2$ in (5), we get $\quad u''(x) - \mu^2 u(x) = 0$.

The general solution of the above equation is

$$u(x) = Ae^{\mu x} + Be^{-\mu x} \qquad \text{......(17)}$$

$$\Rightarrow \quad u'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x} \qquad \text{......(18)}$$

$$\Rightarrow \quad u(0) = A + B \ \text{ and } \ u'(0) = A\mu - B\mu$$

Therefore, from (10), we get $A + B = A\mu - B\mu$.

$$\Rightarrow \quad A(1-\mu) + B(1+\mu) = 0 \qquad \text{......(19)}$$

Putting x = 1 in (17) and (18), we get

$$u(1) = Ae^{\mu} + Be^{-\mu} \ \text{ and } \ u'(1) = A\mu e^{\mu} - B\mu e^{-\mu}$$

Therefore, from (11), we get

$$Ae^{\mu} + Be^{-\mu} = A\mu e^{\mu} - B\mu e^{-\mu}$$

$$\Rightarrow \quad Ae^{\mu}(1-\mu) + Be^{-\mu}(1+\mu) = 0 \qquad \text{......(20)}$$

Now, for non-trivial solution of (19) and (20), we must have

$$\begin{vmatrix} 1-\mu & 1+\mu \\ e^{\mu}(1-\mu) & e^{-\mu}(1+\mu) \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\mu)(1+\mu)e^{-\mu} - (1-\mu)(1+\mu)\,e^{\mu} = 0$$

$$\Rightarrow \quad (1-\mu)(1+\mu)\,(e^{\mu} - e^{-\mu}) = 0$$

$$\Rightarrow \quad 2(1-\mu)(1+\mu)\sinh\mu = 0 \qquad \text{......(21)}$$

$\mu \neq 0 \Rightarrow \sinh\mu \neq 0$. Therefore, from (21), we have $(1-\mu)\,(1+\mu) = 0$.

$$\Rightarrow \quad \mu = 1 \ \text{ and } \ \mu = -1$$

When $\mu = 1$, (19) and (20) gives

$$A.0 + 2.B = 0 \quad \text{and} \quad A.0 + 2Be^{-1} = 0 \qquad \text{......(22)}$$

$\Rightarrow \quad B = 0$ and A is an arbitrary constant.

Therefore, (17) reduces to

$$u(x) = Ae^{x} \qquad \text{......(23)}$$

When $\mu = -1$, (19) and (20) gives

$$\left.\begin{aligned} &2A + B.0 = 0 \\ \text{and} \quad &2Ae + B.0 = 0 \end{aligned}\right\} \qquad \text{......(24)}$$

$\Rightarrow \quad$ A = 0 and B is an arbitrary constant.

Therefore, (17) reduces to

$$u(x) = Be^{x} \qquad \text{......(25)}$$

Now setting A = 1 in (23) and B = 1 in (25), the required eigen function is e^{x} which correspond to eigen value $\lambda = \mu^2 = (1)^2 = (-1)^2 = 1$.

(iii) $\lambda = -\mu^2$ $(\mu \neq 0)$

Put $\lambda = -\mu^2$ in (5), we get $u''(x) + \mu^2 u(x) = 0$.

The general solution of the above equation is

$$u(x) = A\cos\mu x + B\sin\mu x \qquad \text{......(26)}$$

$$\Rightarrow \quad u'(x) = -A\mu\sin\mu x + B\mu\cos\mu x \qquad \text{......(27)}$$

Putting x = 0 in (26) and (27), we get $u(0) = A$ and $u'(0) = B\mu$.

Therefore, from (10), we get

$$A = B\mu \qquad \text{......(28)}$$

Putting x = 1 in (26) and (27), we get

$$u(1) = A\cos\mu + B\sin\mu \quad \text{and} \quad u'(1) = -A\mu\sin\mu + B\mu\cos\mu$$

Therefore, (11) gives

$$A\cos\mu + B\sin\mu = -A\mu\sin\mu + B\mu\cos\mu \qquad \text{......(29)}$$

Using (28) in (29), we get

$$B\mu\cos\mu + B\sin\mu = -B\mu^2\sin\mu + B\mu\cos\mu$$

$$\Rightarrow \quad B(1+\mu^2)\sin\mu = 0 \qquad \text{......(30)}$$

But $B \neq 0$.

Again $(1+\mu^2) \neq 0$, for otherwise $1+\mu^2 = 0$ would give $\mu^2 = -1$ which is not possible as it is real and therefore μ^2 can not be negative.

Now (30) gives $\sin\mu = 0$.

$$\Rightarrow \quad \mu = n\pi; \quad n \in \mathbf{N}$$

Therefore, $\lambda = -\mu^2 = -n^2\pi^2, \; n \in \mathbf{N}$

Putting $\mu = n\pi$ and $A = B\mu$ in (26), we get

$$u(x) = B\mu\cos n\pi x + B\sin n\pi x = B(\mu\cos n\pi x + \sin n\pi x)$$

Setting B = 1, we get $u(x) = \mu\cos n\pi x + \sin n\pi x$.

which is the required eigen function.

EXAMPLE 9

Determine the eigen values and corresponding eigen functions of the homogeneous integral equation

$$u(x) = \lambda \int_1^2 \left(xt + \frac{1}{xt} \right) u(t)\, dt$$

[MEERUT-1997, 98, 2005BP, 06, GARHWAL-1999]

Solution : Here, we have

$$u(x) = \lambda \int_1^2 \left(xt + \frac{1}{xt} \right) u(t)\, dt$$

which can be written as

$$u(x) = \lambda x \int_1^2 t\, u(t)\, dt + \frac{\lambda}{x} \int_1^2 \frac{1}{t} u(t)\, dt \qquad \text{......(1)}$$

Let us assume

$$C_1 = \int_1^2 t\, u(t)\, dt \qquad \text{......(2)}$$

and $$C_2 = \int_1^2 \frac{1}{t} u(t)\, dt \qquad \text{......(3)}$$

Putting these values in (1), we get

$$u(x) = \lambda C_1 x + \frac{\lambda C_2}{x} \qquad \text{......(4)}$$

$$\Rightarrow \quad u(t) = \lambda C_1 t + \frac{\lambda C_2}{t} \qquad \text{......(5)}$$

Putting this value of u(t) in (2), we get

$$C_1 = \int_1^2 t \left(\lambda C_1 t + \frac{\lambda C_2}{t} \right) dt$$

$$= \lambda C_1 \left[\frac{t^3}{3} \right]_1^2 + \lambda C_2 [t]_1^2 = \lambda C_1 \left(\frac{8}{3} - \frac{1}{3} \right) + \lambda C_2 (2-1)$$

or $$\left(1 - \frac{7\lambda}{3}\right) C_1 - \lambda C_2 = 0 \qquad \text{......(6)}$$

Similarly, putting the value of u(t), from (5), in (3), we get

$$C_2 = \int_1^2 \frac{1}{t} \left(\lambda C_1 t + \frac{\lambda C_2}{t} \right) dt$$

$$\Rightarrow \quad C_2 = \lambda C_1 [\, t \,]_1^2 + \lambda C_2 \left[\frac{t^{-1}}{-1} \right]_1^2 = \lambda C_1 (2-1) + \lambda C_2 \left(-\frac{1}{2} + 1 \right)$$

$$\Rightarrow \quad -\lambda C_1 + \left(1 - \frac{1}{2}\lambda \right) C_2 = 0 \qquad \text{......(7)}$$

For non-zero solution of (6) and (7), we must have

$$\begin{vmatrix} 1 - \frac{7}{3}\lambda & -\lambda \\ -\lambda & 1 - \frac{1}{2}\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \left(1 - \frac{7}{3}\lambda\right)\left(1 - \frac{1}{2}\lambda\right) - \lambda^2 = 0 \Rightarrow \lambda^2 - 17\lambda + 6 = 0$$

i.e., $$\lambda = \frac{17 \pm \sqrt{(17)^2 - 24}}{2} = \frac{1}{2}(17 \pm \sqrt{265})$$

Hence, the required eigen values are given by

$$\left.\begin{aligned} \lambda_1 &= \frac{1}{2}(17+\sqrt{265}) = 16.6394 \\ \text{and} \quad \lambda_2 &= \frac{1}{2}(17-\sqrt{265}) = 0.3606 \end{aligned}\right\} \qquad \text{......(8)}$$

Determination of Eigen Function

(i) For $\lambda = \lambda_1 = 16.6394$

Putting $\lambda = \lambda_1 = 16.6394$ in (6) and (7), we get

$$\left[1-\frac{7}{3}(16.6394)\right]C_1 - 16.6394C_2 = 0 \qquad \text{......(9)}$$

$$-16.6394C_1 + \left(1-\frac{1}{2}(16.6394)\right)C_2 = 0 \qquad \text{......(10)}$$

On solving (9) and (10), we get

$$C_2 = -2.2732\ C_1 \qquad \text{......(11)}$$

Therefore, from (4), the eigen function $u_1(x)$ corresponding to the eigen value $\lambda = \lambda_1 = 16.6394$ is given by

$$u_1(x) = \lambda C_1 x + \frac{\lambda}{x}(-2.2732C_1)$$

$$\Rightarrow \quad u_1(x) = \lambda C_1\left[(x - 2.2732\frac{1}{x}\right] = \left[x - 2.2732\left(\frac{1}{x}\right)\right] \quad \text{(Setting } 16.6394\,C_1 = 1)$$

which is the required eigen function.

(ii) For $\lambda = \lambda_2 = 0.3606$

Putting $\lambda = \lambda_2 = 0.3606$ in (6) and (7), we get

$$\left[1-\frac{7}{3}(0.3606)\right]C_1 - 0.3606C_2 = 0 \qquad \text{......(12)}$$

and $$-0.3606C_1 + \left(1-\frac{1}{2}(0.3606)\right)C_1 = 0 \qquad \text{......(13)}$$

On solving (12) and (13), we get

$$C_2 = 0.4399\ C_1 \qquad \text{......(14)}$$

Therefore, from (4), the eigen function $u_2(x)$ corresponding to the eigen value $\lambda = \lambda_2 = 0.3606$ is given by

$$u_2(x) = \lambda C_1 x + \frac{\lambda}{x}(0.4399\,C_1) = \lambda\,C_1\left[x + 0.4399\left(\frac{1}{x}\right)\right]$$

$$= \left[x + 0.4399\left(\frac{1}{x}\right)\right] \qquad \text{(Setting } 0.3606\,C_1 = 1)$$

which is the required eigen function.

EXAMPLE 10

Show that the homogeneous integral equation

$u(x) = \lambda\int_0^1 (t\sqrt{x} - x\sqrt{t})\,u(t)\,dt$ *does not have real eigen values and eigen functions.*

Solution : Here, the given equation can be written as

$$u(x) = \lambda\sqrt{x}\int_0^1 t\,u(t)\,dt - \lambda x\int_0^1 \sqrt{t}\,u(t)\,dt \quad \text{......(1)}$$

Let us assume

$$C_1 = \int_0^1 t\,u(t)\,dt \quad \text{......(2)}$$

and $$C_2 = \int_0^1 \sqrt{t}\,u(t)\,dt \quad \text{......(3)}$$

Putting these values in (1), we get

$$u(x) = \lambda C_1\sqrt{x} - \lambda C_2 x \quad \text{......(4)}$$

$$\Rightarrow \quad u(t) = \lambda C_1\sqrt{t} - \lambda C_2 t \quad \text{......(5)}$$

Putting this value of $u(t)$ in (2), we get

$$C_1 = \int_0^1 t(\lambda C_1\sqrt{t} - \lambda C_2 t)\,dt = \lambda C_1\left[\frac{t^{5/2}}{5/2}\right]_0^1 - \lambda C_2\left[\frac{t^3}{3}\right]_0^1$$

$$\Rightarrow \quad \left(1-\frac{2\lambda}{5}\right)C_1 + \frac{\lambda}{3}C_2 = 0 \quad \text{......(6)}$$

Now, putting the values of u(t) from (5) in (3), we get

$$C_2 = \int_0^1 \sqrt{t}(\lambda C_1\sqrt{t} - \lambda C_2 t)\,dt$$

$$= \lambda C_1\left[\frac{t^2}{2}\right]_0^1 - \lambda C_2\left[\frac{t^{5/2}}{5/2}\right]_0^1 = \lambda C_1\left[\frac{t^2}{2}\right]_0^1 - \lambda C_2\left[\frac{t^{5/2}}{5/2}\right]_0^1$$

$$\Rightarrow \quad -\frac{\lambda}{2}C_1 + \left(1+\frac{2\lambda}{5}\right)C_2 = 0 \quad \text{......(7)}$$

For non-zero solution of (6) and (7), we must have

$$D(\lambda) = \begin{vmatrix} 1-\frac{2}{5}\lambda & \frac{\lambda}{3} \\ -\frac{\lambda}{2} & 1+\frac{2}{5}\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \left(1-\frac{2}{5}\lambda\right)\left(1+\frac{2}{5}\lambda\right) + \frac{\lambda^2}{6} = 0 \quad \Rightarrow \quad 1 + \frac{\lambda^2}{150} = 0$$

$$\Rightarrow \quad \lambda^2 + 150 = 0 \quad \Rightarrow \quad \lambda = \pm i\sqrt{150}$$

showing that $D(\lambda) \neq 0$ for any real value of λ. Hence, the system of equation (6) and (7) has unique solution $C_1 = C_2 = 0$, $\forall$ real λ. Therefore, from (4), u(x) = 0 is the only solution. Hence, the given equation does not have real eigen values and eigen functions.

(iv) Eigen values and eigen functions do not exist

(v) There are no real number and real eigen functions

(vi) $\lambda = \frac{1}{\pi}, u(x) = \sin x$

(4) (i) $\lambda = -3, -3, u(x) = x - 2x^2$ (ii) $\lambda = \frac{8}{\pi - 2}, u(x) = \sin^2 x$

(iii) $\lambda = 1/2, u(x) = \left(\frac{5}{2}\right)x + \left(\frac{10}{3}\right)x^2$

(iv) Eigen values and eigen functions do not exist

(5) (i) $\lambda_n = 4n^2 - 1, u_n(x) = \sin 2nx, n = 1, 2, 3, \ldots$

(ii) $\lambda_n = -\left(\frac{1}{3}\right)\mu_n^2; u_n(x) = \sin \mu_n x + \mu_n \cos \mu_n x$

(iii) $\lambda_n = \left(n + \frac{1}{2}\right)^2 . \cosh 1; u_n(x) = \sin \mu_n(\pi + x), n = 1, 2, 3, \ldots.$

(iv) $\lambda_n = \left(n + \frac{1}{2}\right)^2 - 1, u_n(x) = \sin\left(n + \frac{1}{2}\right)x, n = 1, 2, 3, \ldots$

(6) $\lambda_n^2 = -1 - \mu_n^2, \; u_n(x) = \sin \mu_n, \; n = 1, 2, 3, \ldots.$

2.3 FREDHOLM INTEGRAL EQUATION WITH SEPARABLE KERNEL

[MEERUT-1997, 2000, 01, 05]

Consider the Fredholm integral equation of second kind is given by

$$u(x) = F(x) + \lambda \int_a^b k(x, t)\, u(t)\, dt \qquad \ldots\ldots(1)$$

The separable kernel $k(x, t)$ can be written as

$$k(x, t) = \sum_{i=1}^{n} f_i(x)\, g_i(t) \qquad \ldots\ldots(2)$$

Put this value in (1), we get

$$u(x) = F(x) + \lambda \int_a^b \left[\sum_{i=1}^{n} f_i(x)\, g_i(t)\right] u(t)\, dt$$

$$= F(x) + \lambda \sum_{i=1}^{n} f_i(x) \int_a^b g_i(t)\, u(t)\, dt \qquad \ldots\ldots(3)$$

Let us assume

$$\int_a^b g_i(t)\, u(t)\, dt = C_i \qquad i \in N \qquad \ldots\ldots(4)$$

Then from (3), we have

$$u(x) = F(x) + \lambda \sum_{i=1}^{n} C_i f_i(x) \qquad \ldots\ldots(5)$$

To find the solution of the given equation (1), in the form of (5), we should find the value of constants C_i.

Now, multiplying (5) successively by $g_1(x), g_2(x), \ldots\ldots, g_n(x)$ and integrating over (a, b), we get

$$\int_a^b g_1(x)\, u(x)\, dx = \int_a^b g_1(x)\, F(x)\, dx + \lambda \sum_{i=1}^{n} C_i \int_a^b g_1(x)\, f_i(x)\, dx \qquad \ldots\ldots(6)$$

$$\int_a^b g_2(x)\, u(x)\, dx = \int_a^b g_2(x)\, F(x)\, dx + \lambda \sum_{i=1}^{n} C_i \int_a^b g_2(x)\, f_i(x)\, dx \qquad \ldots\ldots(7)$$

... ...

... ...

$$\int_a^b g_n(x)\, u(x)\, dx = \int_a^b g_n(x)\, F(x)\, dx + \lambda \sum_{i=1}^{n} C_i \int_a^b g_n(x)\, f_i(x)\, dx \qquad \ldots\ldots(8)$$

Define

$$\alpha_{ji} = \int_a^b g_j(x)\, f_i(x)\, dx \qquad (i, j \in N) \qquad \ldots\ldots(9)$$

$$\beta_j = \int_a^b g_j(x)\, F(x)\, dx \qquad (j \in N) \qquad \ldots\ldots(10)$$

Putting these values in (6), we get

$$C_1 = \beta_1 + \lambda \sum_{i=1}^{n} C_i \alpha_{1i} = \beta_1 + \lambda [C_1\alpha_{11} + C_2\alpha_{12} + \ldots\ldots + C_n\alpha_{1n}]$$

$$\Rightarrow \quad (1 - \lambda\alpha_{11})\, C_1 - \lambda\alpha_{12}\, C_2 - \ldots\ldots - \lambda\alpha_{1n}C_n = \beta_1$$

Similarly, we can solve (7) and (8).

Therefore, we get the following system of linear equations.

$$(1 - \lambda\alpha_{11})C_1 - \lambda\alpha_{12}C_2 - \ldots\ldots - \lambda\alpha_{1n}C_n = \beta_1 \qquad \ldots\ldots(11)$$

$$-\lambda\alpha_{21}C_1 + (1 - \lambda\alpha_{22})C_2 - \ldots\ldots - \lambda\alpha_{2n}C_n = \beta_2 \qquad \ldots\ldots(12)$$

... ...

... ...

$$-\lambda\alpha_{n1}C_1 - \lambda\alpha_{n2}C_2 - \ldots\ldots (1 - \lambda\alpha_{nn})C_n = \beta_n \qquad \ldots\ldots(13)$$

Therefore, the determinant of $D(\lambda)$ of this system is given by

$$D(\lambda) = \begin{vmatrix} 1 - \lambda\alpha_{11} & -\lambda\alpha_{12} & \cdots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & 1 - \lambda\alpha_{22} & \cdots & -\lambda\alpha_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \cdots & (1 - \lambda\alpha_{nn}) \end{vmatrix} \qquad \ldots\ldots(14)$$

which is a polynomial in λ of degree n.

Now, there are following cases :

Case I : When at least one right member of the system (11)......(13) is non-zero.

Here, we have the following two situations :

(i) **If $D(\lambda) \neq 0$: In this case, a unique non-zero solution of the system (11)......(13) exists and therefore (1) has a unique non-zero solution given**

by (3)

(ii) If $D(\lambda) = 0$. Then the above equations have either no solution or they possess infinite solution and hence (1) has no solution or infinite solutions.

Case II : When $f(x) = 0$

Here, we have the following two situations

(i) If $D(\lambda) \neq 0$: In this case, a unique zero solution $C_1 = C_2 = C_3 = \dots = C_n = 0$ of the system (11), ... (12) exists and therefore (1) has only unique zero solution $u(x) = 0$.

(ii) If $D(\lambda) = 0$: In this case the above system possesses infinite non-zero solutions and therefore (1) has infinite non-zero solutions. [Proceed same as in [illegible]]

Case III : When $f(x) \neq 0$ and $f(x)$ is orthogonal to all $g_i(t)$

In this case (7) gives

$$f_1 = 0,\ f_2 = 0,\ \dots,\ f_n = 0$$

and therefore the equation (11), ..., (12) reduce to a system of homogeneous linear equations.

Now following situations may arise

(i) If $D(\lambda) \neq 0$: In this case, a unique zero solution $C_1 = C_2 = C_3 = \dots = C_n = 0$ exists and therefore (1) has only one solution $u(x) = f(x)$.

(ii) If $D(\lambda) = 0$: In this case the above system have infinite non-zero solutions and therefore (1) has infinite non-zero solutions.

Remarks

- The determinant $D(\lambda)$ given by (8) is not identically zero, since when $\lambda = 0$ then $D(\lambda) = 1$.
- In case III situation (ii), the solution corresponding to the eigen values of λ are now expressed as the sum of $f(x)$ and arbitrary multiple of eigen functions.

SOLVED EXAMPLES

EXAMPLE 1

Solve the following integral equation

$$u(x) = x + \lambda \int_0^1 (1 + x + t)\, u(t)\, dt$$

[MEERUT-1998, 99, 2006, KANPUR-1999, GARHWAL-2003]

Solution : Here, the given equation is

$$u(x) = x + \lambda \int_0^1 (1 + x + t)\, u(t)\, dt \qquad \text{......(1)}$$

Equation (1) can be written as

$$u(x) = x + \lambda\left[(1 + x)\int_0^1 u(t)\, dt + \int_0^1 t\, u(t)\, dt \right]$$

$$= x + \lambda[(1 + x)C_1 + C_2] \qquad \text{......(2)}$$

where $C_1 = \int_0^1 u(t)\, dt$ and $C_2 = \int_0^1 t\, u(t)\, dt$ ......(3)

Using (2) and (3), we get

$$C_1 = \int_0^1 [t + \lambda[(1 + t)\, C_1 + C_2]\,]\, dt$$

and $$C_2 = \int_0^1 t\,[t + \lambda(1 + t)C_1 + \lambda C_2]\, dt$$

$$\Rightarrow \quad \left[1 - \lambda \int_0^1 (1 + t)\, dt\right] C_1 + \left[1 - \lambda \int_0^1 t\, dt\right] C_2 = \int_0^1 t^2\, dt$$

By evaluating the integrals, we obtain a system of algebraic equation.

$$\left(1 - \frac{3\lambda}{2}\right) C_1 - \lambda C_2 = \frac{1}{2}$$

$$-\frac{5\lambda}{6} C_1 + \left(1 - \frac{1}{2}\lambda\right) C_2 = \frac{1}{3} \qquad \text{......(4)}$$

The determinant $D(\lambda)$ of the system (4) is given by

$$\begin{vmatrix} 1 - \frac{3\lambda}{2} & -\lambda \\ -\frac{5\lambda}{6} & 1 - \frac{\lambda}{2} \end{vmatrix} = \left(1 - \frac{3\lambda}{2}\right)\left(1 - \frac{\lambda}{2}\right) - \frac{5\lambda^2}{6} \neq 0$$

Since $D(\lambda) \neq 0$, therefore, system (4) has a unique solution.

and $$C_1 = \frac{6 + \lambda}{12 - 24\lambda - \lambda^2}, \quad C_2 = \frac{4 - \lambda}{12 - 24\lambda - \lambda^2} \qquad \text{......(5)}$$

Hence, the solution of integral equation (1) is given by (2) and (5) such that

$$u(x) = x + \frac{\lambda}{(12 - 24\lambda - \lambda^2)}[10 + (6 + \lambda)x]$$

EXAMPLE 2

Solve $u(x) = e^x + \lambda \int_0^1 2e^x e^t\, u(t)\, dt$. **[MEERUT-2002,06, 08, GARHWAL-2000,02]**

Solution : Here, the given equation can be written as

$$u(x) = e^x + 2\lambda e^x \int_0^1 e^t \,.\, u(t)\, dt \qquad \text{......(1)}$$

Let $$C = \int_0^1 e^t\, u(t)\, dt \qquad \text{......(2)}$$

From (1) and (2), we have

$$u(x) = e^x + 2\lambda C e^x = e^x(1 + 2C\lambda) \qquad \text{......(3)}$$

$\Rightarrow \quad u(t) = e^t\,(1 + 2C\lambda)$ ......(4)

Therefore, from (2), we have

$$C = \int_0^1 [e^t \cdot e^t\,(1 + 2C\lambda)]\,dt$$

$$= (1 + 2C\lambda)\left[\frac{e^{2t}}{2}\right]_0^1 = (1 + 2\lambda C)\frac{1}{2}(e^2 - 1)$$

i.e., $\quad C[1 - \lambda(e^2 - 1)] = \frac{1}{2}(e^2 - 1) \;\Rightarrow\; C = \frac{e^2 - 1}{2[1 - \lambda(e^2 - 1)]}$ where $\lambda \neq \frac{1}{e^2 - 1}$

Putting this value in (3), we obtained the solution of integral equation (1) given by

$$u(x) = e^x\left[1 + 2\lambda.\frac{e^2 - 1}{2[1 - \lambda(e^2 - 1)]}\right] = e^x\left[\frac{1 - \lambda(e^2 - 1) + \lambda(e^2 - 1)}{1 - \lambda(e^2 - 1)}\right]$$

$\Rightarrow \quad u(x) = \dfrac{e^x}{1 - \lambda(e^2 - 1)}$, where $\lambda \neq \dfrac{1}{e^2 - 1}$

EXAMPLE 3

Solve the following integral equation

$$u(x) = x + \lambda\int_0^\pi (1 + \sin x \sin t)\,u(t)\,dt$$ **[KANPUR-1999]**

Solution : The given integral equation can be written as

$$u(x) = x + \lambda\left[\int_0^\pi u(t)\,dt + \sin x\int_0^\pi \sin t\,u(t)\,dt\right]$$

$$= x + \lambda[C_1 + C_2 \sin x] \quad ......(1)$$

where $\quad C_1 = \int_0^\pi u(t)\,dt,\; C_2 = \int_0^\pi \sin t\,u(t)\,dt$ ......(2)

From (1) and (2), we have

$$C_1 = \int_0^\pi [t + \lambda C_1 + \lambda C_2 \sin t]\,dt$$

$$C_2 = \int_0^\pi \sin t\,[t + \lambda C_1 + \lambda C_2 \sin t]\,dt$$

$$\Rightarrow \quad C_1\left[1 - \lambda\int_0^\pi dt\right] - C_2\lambda\int_0^\pi \sin t\,dt = \int_0^\pi t\,dt$$

$$\Rightarrow \quad -C_1\lambda\int_0^\pi \sin t\,dt + C_2\left[1 - \lambda\int_0^\pi \sin^2 t\,dt\right] = \int_0^\pi t \sin t\,dt$$

Therefore, we obtain a system of algebraic equation by evaluating the above integrals.

$$\left.\begin{aligned}(1 - \lambda\pi)C_1 - 2\lambda C_2 &= \frac{\pi^2}{2}\\ -2\lambda C_1 + \left(1 - \frac{\pi}{2}\lambda\right)C_2 &= \pi\end{aligned}\right\} \quad ......(3)$$

The determinant $D(\lambda)$ of the system (3) is given by

$$D(\lambda) = \begin{vmatrix} 1-\lambda\pi & -2\lambda \\ -2\lambda & 1-\dfrac{\lambda\pi}{2} \end{vmatrix} = (1-\lambda\pi)\left(1-\frac{\lambda\pi}{2}\right) - 4\lambda^2 \neq 0$$

Since, $D(\lambda) \neq 0$, therefore the system (3) has a unique solution given by

$$C_1 = \frac{2\lambda\pi + \frac{1}{2}\pi^2\left(1-\frac{1}{2}\lambda\pi\right)}{(1-\lambda\pi)\left(1-\frac{1}{2}\lambda\pi\right) - 4\lambda^2} \quad \text{and} \quad C_2 = \frac{\pi(1-2\lambda\pi)}{(1-\lambda\pi)\left((1-\frac{1}{2}\lambda\pi) - 4\lambda^2\right)}$$

Putting these values of C_1 and C_2 in (1), we obtain the required solution of the integral equation (1), given by

$$u(x) = x + \lambda\left[\frac{2\lambda\pi + \frac{1}{2}\pi^2\left(1-\frac{1}{2}\lambda\pi\right)}{(1-\lambda\pi)\left(1-\frac{1}{2}\lambda\pi\right) + 4\lambda^2} + \frac{\pi(1-2\lambda\pi)}{(1-\lambda\pi)\left((1-\frac{1}{2}\lambda\pi) + 4\lambda^2\right)}.\sin x\right]$$

EXAMPLE 4

Find the solution of the integral equation

$$u(x) = f(x) + \lambda\int_0^{2\pi} (\sin x \cos t)\, u(t)\, dt$$

Solution : Here, the given equation can be written as

$$u(x) = f(x) + \lambda \sin x . \int_0^{2\pi} \cos t\, u(t)\, dt \qquad \text{......(1)}$$

i.e., $$u(x) = f(x) + \lambda C \sin x \qquad \text{......(2)}$$

where $$C = \int_0^{2\pi} \cos t\, u(t)\, dt \qquad \text{......(3)}$$

$\Rightarrow$ $$u(t) = f(t) + \lambda C \sin t \qquad \text{......(4)}$$

Put this value in (3), we get

$$C = \int_0^{2\pi} \cos t\,[f(t) + \lambda C \sin t]\, dt = \int_0^{2\pi} \cos t\, f(t)\, dt + \frac{\lambda C}{2}\int_0^{2\pi} \sin 2t\, dt$$

$$= \int_0^{2\pi} \cos t\, f(t)\, dt + \frac{\lambda C}{2}\left[\frac{-\cos 2t}{2}\right]_0^{2\pi}$$

$$= \int_0^{2\pi} \cos t\, f(t)\, dt + \frac{\lambda C}{2}\left[-\frac{1}{2} + \frac{1}{2}\right] = \int_0^{2\pi} \cos t\, f(t)\, dt$$

Putting this value in (2), we obtain the required solution, given by

$$u(x) = f(x) + \lambda \sin x \int_0^{2\pi} f(t) \cos t\, dt$$

i.e., $$u(x) = f(x) + \lambda \int_0^{2\pi} (\sin x \cos t)\, f(t)\, dt$$

EXAMPLE 5

Solve $u(x) = 1 + \int_0^1 (1 + e^{x+t})\, u(t)\, dt$ [MEERUT-2005, 06(BP)]

Solution : Here, the given equation can be written as

$$u(x) = 1 + \int_0^1 u(t)\, dt + e^x \int_0^1 e^t\, u(t)\, dt \qquad \ldots\ldots(1)$$

Let $C_1 = \int_0^1 u(t)\, dt \qquad \ldots\ldots(2)$

and $C_2 = \int_0^1 e^t\, u(t)\, dt \qquad \ldots\ldots(3)$

Therefore, (1) gives

$$u(x) = 1 + C_1 + C_2 e^x \qquad \ldots\ldots(4)$$

$$\Rightarrow \quad u(t) = 1 + C_1 + C_2 e^t \qquad \ldots\ldots(5)$$

Put this value in (2), we get

$$C_1 = \int_0^1 (1 + C_1 + C_2 e^t)\, dt = [t + C_1 t + C_2 e^t]_0^1 = 1 + C_1 + C_2(e-1)$$

$$\Rightarrow \quad C_2 = -\frac{1}{(e-1)} \qquad \ldots\ldots(6)$$

Similarly, from (3), we have

$$C_2 = \int_0^1 e^t [1 + C_1 + C_2 e^t]\, dt = \left[e^t + C_1 e^t + C_2 \frac{e^{2t}}{2} \right]_0^1$$

$$= e - 1 + C_1(e-1) + \frac{C_2}{2}(e^2 - 1)$$

$$\Rightarrow \quad -\frac{1}{(e-1)} = e - 1 + C_1(e-1) - \frac{e^2-1}{2(e-1)}$$

$$\Rightarrow \quad C_1(e-1) = -\frac{1}{e-1} - \frac{(e-1)}{2}$$

$$\Rightarrow \quad C_1 = \frac{-(e^2 - 4e + 5)}{2(e-1)^2} \qquad \ldots\ldots(7)$$

Using (6) and (7) in (4), the required solution is given by

$$u(x) = 1 - \frac{e^2 - 4e + 5}{2(e-1)^2} - \frac{e^x}{e-1}$$

$$\Rightarrow \quad u(x) = \frac{e^2 - 2e - 1}{2(e-1)^2} - \frac{e^x}{e-1} = \frac{e^2 - 3 - 2e^x(e-1)}{2(e-1)^2}$$

EXAMPLE 6

Solve $u(x) = x + \lambda \int_0^1 (xt^2 + tx^2)\, u(t)\, dt$

Solution : Here, the given equation can be written as

$$u(x) = x + \lambda \int_0^1 (xt^2 + tx^2)\, u(t)\, dt$$

$$= x + \lambda x \int_0^1 t^2\, u(t)\, dt + \lambda x^2 \int_0^1 t\, u(t)\, dt \qquad \text{......(1)}$$

Let $\quad C_1 = \int_0^1 t^2\, u(t)\, dt \qquad$ ......(2)

and $\quad C_2 = \int_0^1 t\, u(t)\, dt \qquad$ ......(3)

Putting these values in (1), we get

$$u(x) = x + \lambda C_1 x + \lambda C_2 x^2 \qquad \text{......(4)}$$

$$\Rightarrow \quad u(t) = t + \lambda C_1 t + \lambda C_2 t^2 \qquad \text{......(5)}$$

Putting this value in (2), we get

$$C_1 = \int_0^1 t^2 (t + \lambda C_1 t + \lambda C_2 t^2)\, dt = \left[\frac{t^4}{4} + \frac{\lambda C_1 t^4}{4} + \frac{\lambda C_2 t^5}{5} \right]_0^1$$

$$= \frac{1}{4} + \frac{\lambda C_1}{4} + \frac{\lambda C_2}{5} \qquad \text{......(6)}$$

Similarly, from (3), we get

$$C_2 = \int_0^1 t(t + \lambda C_1 t + \lambda C_2 t^2)\, dt = \left[\frac{t^3}{3} + \frac{\lambda C_1 t^3}{3} + \frac{\lambda C_2 t^4}{4} \right]_0^1$$

$$\Rightarrow \quad C_2 = \frac{1}{3} + \frac{\lambda C_1}{3} + \frac{\lambda C_2}{4}$$

i.e., $\quad -4\lambda C_1 + (12 - 3\lambda) C_2 = 4 \qquad$ ......(7)

On solving (6) and (7), we get

$$C_1 = \frac{60 + \lambda}{240 - 120\lambda - \lambda^2} \quad \text{and} \quad C_2 = \frac{80}{240 - 120\lambda - \lambda^2}$$

Putting these values in (4), we get the required solution, given by

$$u(x) = x + \frac{\lambda x(60 + \lambda)}{240 - 120\lambda - \lambda^2} + \frac{80\lambda x^2}{240 - 120\lambda - \lambda^2}$$

$$\Rightarrow \quad u(x) = \frac{(240 - 60\lambda)x + 80\lambda x^2}{240 - 120\lambda - \lambda^2}$$

EXAMPLE 7

Solve the following integral equation

$$u(x) = \frac{6}{5}(1 - 4x) + \lambda \int_0^1 (x \log t - t \log x)\, u(t)\, dt$$

Solution : The given equation can be written as

$$u(x) = \frac{6}{5}(1 - 4x) + \lambda \left[x \int_0^1 \log t\, u(t)\, dt - \log x \int_0^1 t\, u(t)\, dt \right] \qquad \text{......(1)}$$

$$= \frac{6}{5}(1 - 4x) + \lambda (C_1 x - C_2 \log x) \qquad \text{......(2)}$$

where $C_1 = \int_0^1 \log t\, u(t)\, dt,\ C_2 = \int_0^1 t\, u(t)\, dt$ ......(3)

Using (2) in (3), we have

$$C_1 = \int_0^1 \log t\left[\frac{6}{5}(1-4t) + \lambda(C_1 t - C_2 \log t)\right] dt$$

and $$C_2 = \int_0^1 t\left[\frac{6}{5}(1-4t) + \lambda(C_1 t - C_2 \log t)\right] dt$$

or $$C_1\left[1 - \lambda\int_0^1 t\log t\, dt\right] + \lambda C_2 \int_0^1 (\log t)^2\, dt = \frac{6}{5}\int_0^1 (1-4t)\log t\, dt$$

and $$-\lambda C_1 \int_0^1 t^2 dt + C_2[1 + \lambda \int_0^1 t\log t\, dt] = \frac{6}{5}\int_0^1 t(1-4t)\, dt$$

or $$C_1 = \left[1 - \lambda\left(\frac{1}{2}t^2\log t - \frac{1}{4}t^2\right)_0^1\right] + \lambda C_2\left[t(\log t)^2 - 2(t\log t - t)\right]_0^1$$

$$= \frac{6}{5}\left[(t-2t^2)\log t - (t-t^2)\right]_0^1$$

and $$-\lambda C_1\left(\frac{1}{3}t^3\right)_0^1 + C_2\left[1 + \lambda\left(\frac{1}{2}t^2\log t - \frac{1}{4}t^2\right)_0^1\right] = \frac{6}{5}\left[\frac{t^2}{2} - \frac{4}{3}t^3\right]_0^1$$

or $$C_1\left[1 + \frac{\lambda}{4}\right] + 2\lambda C_2 = 0$$

$$-C_1(\lambda/3) + (1 - \lambda/4)\, C_2 = -1 \qquad \text{......(4)}$$

On solving, we get

$$C_2 = -\frac{4+\lambda}{8\lambda}C_1 \Rightarrow \left(\frac{\lambda}{3} + \frac{16-\lambda^2}{32\lambda}\right)C_1 = 1$$

$$\Rightarrow \quad C_1 = \frac{96\lambda}{48 + 29\lambda^2} \quad \text{and} \quad C_2 = -\frac{4+\lambda}{8\lambda}\cdot\frac{96\lambda}{48+29\lambda^2} = -\frac{12(4+\lambda)}{48+29\lambda^2}$$

Now, putting the values of C_1 and C_2 in (2), the required solution of the equation is given by

$$u(x) = \frac{6}{5}(1-4x) + \lambda\left[\frac{96\lambda x}{48+29\lambda^2} + \frac{12(4+\lambda)\log x}{48+29\lambda^2}\right]$$

or $$u(x) = \frac{6}{5}(1-4x) + \frac{48}{48+29\lambda^2}\left[2\lambda^2 x + \left(\lambda + \frac{\lambda^2}{4}\right)\log x\right]$$

EXAMPLE 8

Solve the integral equation

$$u(x) = \cos x + \lambda\int_0^{\pi} \sin x\, u(t)\, dt$$ **[GARHWAL-2001]**

Solution : The given equation can be written as

$$u(x) = \cos x + \lambda \sin x \int_0^{\pi} u(t)\,dt \qquad \ldots\ldots(1)$$

Let $\quad C = \int_0^{\pi} u(t)\,dt \qquad \ldots\ldots(2)$

Then from (1), we have

$$u(x) = \cos x + \lambda C \sin x \qquad \ldots\ldots(3)$$

$$\Rightarrow \quad u(t) = \cos t + \lambda C \sin t \qquad \ldots\ldots(4)$$

Putting this value in (2), we get

$$C = \int_0^{\pi} (\cos t + \lambda C \sin t)\,dt$$

$$= [\sin t]_0^{\pi} + \lambda C[-\cos t]_0^{\pi}$$

$$= 0 + \lambda C[-\cos \pi + \cos 0]$$

$$\Rightarrow \quad C = 2\lambda C$$

$$\Rightarrow \quad C(1-2\lambda) = 0 \quad \Rightarrow \quad C = 0, \text{ if } \lambda \neq \frac{1}{2}.$$

Hence, the required solution is given by $u(x) = \cos x, \ if \ \lambda \neq \frac{1}{2}$.

EXAMPLE 9

Solve the integral equation

$$u(x) = \cos x + \lambda \int_0^{\pi} \sin(x-t)\, u(t)\,dt$$

[MEERUT-2007, 07(BP), KANPUR-2002]

Solution : The given equation can be written as

$$u(x) = \cos x + \lambda \int_0^{\pi} (\sin x \cos t - \cos x \sin t)\, u(t)\,dt$$

$$= \cos x + \lambda \sin x \int_0^{\pi} \cos t\, u(t)\,dt - \lambda \cos x \int_0^{\pi} \sin t\, u(t)\,dt \qquad \ldots\ldots(1)$$

Let $\quad C_1 = \int_0^{\pi} \cos t\, u(t)\,dt \qquad \ldots\ldots(2)$

and $\quad C_2 = \int_0^{\pi} \sin t\, u(t)\,dt \qquad \ldots\ldots(3)$

Putting these values in (1), we get

$$u(x) = \cos x + \lambda C_1 \sin x - \lambda C_2 \cos x \qquad \ldots\ldots(4)$$

$$\Rightarrow \quad u(t) = \cos t + \lambda C_1 \sin t - \lambda C_2 \cos t \qquad \ldots\ldots(5)$$

Putting this value in (2), we get

$$C_1 = \int_0^{\pi} \cos t\,(\cos t + \lambda C_1 \sin t - \lambda C_2 \cos t)\,dt$$

$$\Rightarrow \quad C_1 = \int_0^{\pi}\left[(1-\lambda C_2)\cos^2 t\, dt + \frac{1}{2}\lambda C_1 \sin 2t\right] dt$$

$$= (1-\lambda C_2)\int_0^{\pi}\frac{1+\cos 2t}{2}\, dt + \frac{\lambda C_1}{2}\int_0^{\pi}\sin 2t\, dt$$

$$= \frac{1-\lambda C_2}{2}\left[t+\frac{\sin 2t}{2}\right]_0^{\pi} + \frac{\lambda C_1}{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi}$$

$$\Rightarrow \quad C_1 = \frac{1-\lambda C_2}{2}\pi \quad \text{or} \quad 2C_1 + \lambda\pi C_2 = \pi \qquad \ldots\ldots(6)$$

Similarly, from (3), we may get

$$C_2 = \int_0^{\pi}\sin t(\cos t + \lambda C_1 \sin t - \lambda C_2 \cos t)\, dt$$

$$= \frac{1-\lambda C_2}{2}\int_0^{\pi}\sin 2t\, dt + \frac{\lambda C_1}{2}\int_0^{\pi}(1-\cos 2t)\, dt$$

$$= \frac{1-\lambda C_2}{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi} + \frac{\lambda C_1}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{\pi}$$

$$\Rightarrow \quad C_2 = \frac{\lambda C_1 \pi}{2} \qquad \ldots\ldots(7)$$

On solving (6) and (7) for C_1 and C_2, we get $C_1 = \frac{2\pi}{4+\lambda^2\pi^2}$, $C_2 = \frac{\lambda\pi^2}{4+\lambda^2\pi^2}$

Hence, the required solution of integral equation (1) is given by

$$u(x) = \cos x + \frac{2\pi\lambda \sin x}{4+\lambda^2\pi^2} - \frac{\lambda^2\pi^2\cos x}{4+\lambda^2\pi^2} = \cos x\left[1-\frac{\lambda^2\pi^2}{4+\lambda^2\pi^2}\right] + \frac{2\pi\lambda\sin x}{4+\lambda^2\pi^2}$$

$$\Rightarrow \quad u(x) = \frac{4\cos x + 2\pi\lambda\sin x}{4+\lambda^2\pi^2}$$

EXAMPLE 10

Solve the integral equation

$$u(x) - \lambda\int_{-\pi}^{\pi}(x\cos t + t^2\sin x + \cos x \sin t)\, u(t)\, dt = x \qquad \text{[GARHWAL-2000]}$$

Solution : The given equation can be written as

$$u(x) = x + \lambda\int_{-\pi}^{\pi}(x\cos t + t^2\sin x + \cos x\sin t)\, u(t)\, dt$$

$$= x + \lambda x\int_{-\pi}^{\pi}\cos t\, u(t)\, dt + \lambda\sin x\int_{-\pi}^{\pi}t^2\, u(t)\, dt + \lambda\cos x\int_{-\pi}^{\pi}\sin t\, u(t)\, dt \qquad \ldots\ldots(1)$$

Let us assume

$$C_1 = \int_{-\pi}^{\pi}\cos t\, u(t)\, dt \qquad \ldots\ldots(2)$$

$$C_2 = \int_{-\pi}^{\pi}t^2\, u(t)\, dt \qquad \ldots\ldots(3)$$

and $\qquad C_3 = \int_{-\pi}^{\pi} \sin t \, u(t)\, dt$ ......(4)

Putting all these values in (1), we get

$$u(x) = x + \lambda C_1 x + \lambda C_2 \sin x + \lambda \cos x . C_3 \qquad ......(5)$$

$$\Rightarrow \quad u(t) = t + \lambda C_1 t + \lambda C_2 \sin t + \lambda C_3 . \cos t \qquad ......(6)$$

Putting the value of u(t) in (2), we get

$$C_1 = \int_{-\pi}^{\pi} \cos t \, (t + \lambda C_1 t + \lambda C_2 \sin t + \lambda C_3 \cos t)\, dt$$

$$= (1 + \lambda C_1) \int_{-\pi}^{\pi} t \cos t \, dt + \lambda C_2 \int_{-\pi}^{\pi} \sin t \cos t \, dt + \lambda C_3 \int_{-\pi}^{\pi} \cos^2 t \, dt$$

$$= 0 + 0 + 2\lambda C_3 \int_0^{\pi} \cos^2 t \, dt \qquad [\because t \cos t \text{ and } \sin t \cos t \text{ are odd functions}]$$

$$\Rightarrow \quad C_1 = 2\lambda C_3 \int_0^{\pi} \frac{(1 + \cos 2t)}{2} dt = \lambda C_3 \left[t + \frac{\sin 2t}{2} \right]_0^{\pi}$$

$$\Rightarrow \quad C_1 - \lambda \pi C_3 = 0 \qquad ......(7)$$

Similarly, putting the value of $u(t)$ (from (6)) in (3), we get

$$C_2 = \int_{-\pi}^{\pi} t^2 \, (t + \lambda C_1 t + \lambda C_2 \sin t + \lambda C_3 \cos t)\, dt$$

$$= (1 + \lambda C_1) \int_{-\pi}^{\pi} t^3 \, dt + \lambda C_2 \int_{-\pi}^{\pi} t^2 \sin t \, dt + \lambda C_3 \int_{-\pi}^{\pi} t^2 \cos t \, dt$$

$$= 2\lambda C_3 \int_0^{\pi} t^2 \cos t \, dt$$

$$\Rightarrow \quad C_2 = 2\lambda C_3 \left[(t^2 \sin t) \right]_0^{\pi} - 2\lambda C_3 \int_0^{\pi} 2t \sin t \, dt = -4\lambda C_3 \int_0^{\pi} t \, \lambda \sin t \, dt$$

$$= -4\lambda C_3 \left[[t(-\cos t)]_0^{\pi} - \int_0^{\pi} (-\cos t)\, dt \right] = -4\lambda C_3 \left[\pi + \int_0^{\pi} \cos t \, dt \right]$$

$$= -4\lambda \pi C_3 - 4\lambda \pi C_3 [\sin t]_0^{\pi}$$

$$\Rightarrow \quad C_2 + 4\lambda \pi C_3 = 0 \qquad ......(8)$$

Again, putting the value of u(t) [from (6)] in (4), we get

$$C_3 = \int_{-\pi}^{\pi} \sin t \, (t + \lambda C_1 t + \lambda C_2 \sin t + \lambda C_3 \cos t)\, dt$$

$$= (1 + \lambda C_1) \int_{-\pi}^{\pi} t \sin t \, dt + \lambda C_2 \int_{-\pi}^{\pi} \sin^2 t \, dt + \lambda C_3 \int_{-\pi}^{\pi} \sin t \cos t \, dt$$

$$= 2(1 + \lambda C_1) \int_0^{\pi} t \sin t \, dt + 2\lambda C_2 \int_0^{\pi} \sin^2 t \, dt + 0$$

$$= 2(1 + \lambda C_1) \left[[t(-\cos t)]_0^{\pi} - \int_0^{\pi} (-\cos t)\, dt \right] + 2\lambda C_2 \int_0^{\pi} \frac{1 - \cos 2t}{2} dt$$

$$= 2(1 + \lambda C_1) \left[\pi + (\sin t)_0^{\pi} \right] + \lambda C_2 \left[t - \frac{\sin^2 t}{2} \right]_0^{\pi}$$

$$\Rightarrow \quad C_3 = 2(1+\lambda C_1)\pi + \lambda C_2 \pi$$

$$\Rightarrow \quad -2\lambda C_1 \pi - \lambda \pi C_2 + C_3 = 2\pi \qquad \text{......(9)}$$

On solving (7), (8) and (9), we get

$$C_1 = \frac{2\pi^2\lambda}{1+2\lambda^2\pi^2}, \quad C_2 = \frac{-8\pi^2\lambda}{1+2\lambda^2\pi^2}, \quad C_3 = \frac{2\pi}{1+2\lambda^2\pi^2}$$

Putting all these values of C_i $(i = 1, 2, 3)$ in (5), the required solution is given by

$$u(x) = x + \frac{2\pi^2\lambda^2 x}{1+2\lambda^2\pi^2} - \frac{8\pi^2\lambda^2 \sin x}{1+2\lambda^2\pi^2} + \frac{2\pi\lambda \cos x}{1+2\lambda^2\pi^2}$$

$$\Rightarrow \quad u(x) = x + \frac{2\pi\lambda}{1+2\lambda^2\pi^2}(\lambda\pi x - 4\lambda\pi \sin x + \cos x)$$

EXAMPLE 11

Solve the integral equation

$$u(x) = f(x) + \lambda \int_0^1 (x+t)\, u(t)\, dt$$

Solution : The given equation can be written as

$$u(x) = f(x) + \lambda x \int_0^1 u(t)\, dt + \lambda \int_0^1 t\, u(t)\, dt \qquad \text{......(1)}$$

Let us assume

$$C_1 = \int_0^1 u(t)\, dt \qquad \text{......(2)}$$

and $$C_2 = \int_0^1 t\, u(t)\, dt \qquad \text{......(3)}$$

Putting the value of C_1 and C_2 in (1), we get

$$u(x) = f(x) + \lambda x C_1 + \lambda C_2 \qquad \text{......(4)}$$

$$\Rightarrow \quad u(t) = f(t) + \lambda t C_1 + \lambda C_2 \qquad \text{......(5)}$$

Putting this value of u(t) in (2), we get

$$C_1 = \int_0^1 [f(t) + \lambda t C_1 + \lambda C_2]\, dt = \int_0^1 f(t)\, dt + \lambda C_1 \left[\frac{t^2}{2}\right]_0^1 + \lambda C_2 [t]_0^1$$

$$\Rightarrow \quad C_1 = f_1 + \frac{\lambda C_1}{2} + \lambda C_2 \qquad \text{......(6)}$$

where $$f_1 = \int_0^1 f(t)\, dt \qquad \text{......(7)}$$

Similarly, putting the value of u(t) in (3), we get

$$C_2 = \int_0^1 t[f(t) + \lambda t C_1 + \lambda C_2]\, dt$$

$$= \int_0^1 t\, f(t)\, dt + \lambda C_1 \left[\frac{t^3}{3}\right]_0^1 + \lambda C_2 \left[\frac{t^2}{2}\right]_0^1$$

$$= f_2 + \frac{\lambda C_1}{3} + \frac{\lambda C_2}{2} \quad \text{......(8)}$$

where $f_2 = \int_0^1 t\, f(t)\, dt$ ......(9)

From (6) and (8), we can write

$$(2-\lambda)C_1 - 2\lambda C_2 = 2f_1 \quad \text{......(10)}$$

$$-2\lambda C_1 + 3(2-\lambda)\, C_2 = 6f_2 \quad \text{......(11)}$$

On solving (10) and (11), we get

$$C_1 = \frac{6(\lambda-2)\, f_1 - 12\lambda f_2}{\lambda^2 + 12\lambda - 12}, \quad C_2 = \frac{-4\lambda f_1 + 6(\lambda-2)\, f_2}{\lambda^2 + 12\lambda - 12}$$

Putting the values of C_1 and C_2 in (4), we get

$$u(x) = f(x) + \frac{\lambda x\{6(\lambda-2)\, f_1 - 12\lambda f_2\}}{\lambda^2 + 12\lambda - 12} + \lambda \frac{-4\lambda f_1 + 6(\lambda-2)\, f_2}{\lambda^2 + 12\lambda - 12}$$

$$= f(x) + \lambda \frac{f_1\{6x(\lambda-2) - 4\lambda\} + f_2\{6(\lambda-2) - 12\lambda x\}}{\lambda^2 + 12\lambda - 12}$$

$$= f(x) + \frac{\lambda}{\lambda^2 + 12\lambda - 12}\left[\{6x(\lambda-2) - 4\lambda\} \int_0^1 f(t)\, dt + \{6(\lambda-2) - 12\lambda x\} \int_0^1 t\, f(t)\, dt \right]$$

$$= f(x) + \frac{\lambda}{\lambda^2 + 12\lambda - 12}\left[\int_0^1 \{6x(\lambda-2) - 4\lambda\}\, f(t)\, dt + \int_0^1 \{6(\lambda-2) - 12\lambda x\}\, t\, f(t)\, dt \right]$$

$$\Rightarrow \quad u(x) = f(x) + \frac{\lambda}{\lambda^2 + 12\lambda - 12}\left[\int_0^1 6(\lambda-2)(x+t) - 12\lambda xt - 4\lambda \right] f(t)\, dt$$

Hence $u(x) = f(x) + \lambda \int_0^1 \frac{6(\lambda-2)(x+t) - 12\lambda xt - 4\lambda}{\lambda^2 + 12\lambda - 12} f(t)\, dt$

which is the required solution of given integral equation.

EXAMPLE 12

Show that the integral equation

$$u(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+t)\, u(t)\, dt$$

possesses no solution for $f(x) = x$, but it possesses infinitely many solutions when $f(x)=1$.

Solution : The given equation can be written as

$$u(x) = f(x) + \frac{\sin x}{\pi} \int_0^{2\pi} \cos t\, u(t)\, dt + \frac{\cos x}{\pi} \int_0^{2\pi} \sin t\, u(t)\, dt \quad \text{......(1)}$$

Let $C_1 = \int_0^{2\pi} \cos t\, u(t)\, dt$ ......(2)

and $C_2 = \int_0^{2\pi} \sin t\, u(t)\, dt$ ......(3)

Putting these values in (1), we get

$$u(x) = f(x) + \frac{C_1 \sin x}{\pi} + \frac{C_2 \cos x}{\pi} \quad \text{......(4)}$$

Now, there are following cases that arise :

Case I : Let f(x) = x

Put f(x) = x in (4), we get

$$u(x) = x + \frac{C_1 \sin x}{\pi} + \frac{C_2 \cos x}{\pi} \qquad \text{......(5)}$$

$$\Rightarrow \quad u(t) = t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi} \qquad \text{......(6)}$$

Putting the values of u(t) in (2), we get

$$C_1 = \int_0^{2\pi} \cos t\left(t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right) dt$$

$$\Rightarrow \quad C_1 = \int_0^{2\pi} t \cos t \, dt + \frac{C_1}{2\pi}\int_0^{2\pi} \sin 2t \, dt + \frac{C_2}{2\pi}\int_0^{2\pi} (1 + \cos 2t) \, dt$$

$$= [\, t \sin t \,]_0^{2\pi} - \int_0^{2\pi} \sin t \, dt + \frac{C_1}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -[-\cos t]_0^{2\pi} + \frac{C_2}{2\pi}[2\pi - 0\,]$$

$$\Rightarrow \quad C_1 - C_2 = 0 \qquad \text{......(7)}$$

Similarly, putting the values of u(t) in (3), we get

$$C_2 = \int_0^{2\pi} \sin t\left[t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right] dt$$

$$= \int_0^{2\pi} t \sin t \, dt + \frac{C_1}{2\pi}\int_0^{2\pi} (1 - \cos 2t) \, dt + \frac{C_2}{2\pi}\int_0^{2\pi} \sin 2t \, dt$$

$$= [-t \cos t]_0^{2\pi} - \int_0^{2\pi} (-\cos t) \, dt + \frac{C_1}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= -2\pi + [\sin t]_0^{2\pi} + \frac{C_1}{2\pi}(2\pi + 0)$$

$$\Rightarrow \quad C_1 - C_2 = 2\pi \qquad \text{......(8)}$$

Clearly, the system of equation (7) and (8) is inconsistent and therfore, it possesses no solution.

Case II : If f(x) = 1

Putting f(x) = 1 in (4), we get

$$u(x) = 1 + \frac{C_1 \sin x}{\pi} + \frac{C_2 \cos x}{\pi} \qquad \text{......(9)}$$

$$\Rightarrow \quad u(t) = 1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi} \qquad \text{......(10)}$$

Putting the values of u(t) [From (10)] in (2), we get

$$C_1 = \int_0^{2\pi} \cos t\left(1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right) dt$$

$$\Rightarrow \quad C_1 = \int_0^{2\pi} \cos t \, dt + \frac{C_1}{2\pi}\int_0^{2\pi} \sin 2t \, dt + \frac{C_2}{2\pi}\int_0^{2\pi} (1 + \cos 2t) \, dt$$

$$= \left[\sin t\right]_0^{2\pi} + \frac{C_1}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = 0 + 0 + \frac{C_2}{2\pi}(2\pi + 0)$$

$\Rightarrow \quad C_1 = C_2$ ......(11)

Similarly, putting the values of $u(t)$ from (10) in (3), we get

$$C_2 = \int_0^{2\pi} \sin t\left[1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right] dt$$

$$= \int_0^{2\pi} \sin t \, dt + \frac{C_1}{2\pi}\int_0^{2\pi}(1 - \cos 2t)\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} \sin 2t \, dt$$

$$= \left[-\cos t\right]_0^{2\pi} + \frac{C_1}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} = 0 + \frac{C_1}{2\pi}(2\pi + 0)$$

$\Rightarrow \quad C_2 = C_1$ ......(12)

From (11) and (12), we conclude that $C_1 = C_2 = C'$ (say). Therefore, the system (11)-(12) has infinite number of solutions $C_1 = C'$, $C_2 = C'$. Hence, the required solution of (1) is given by

$$u(x) = 1 + \frac{C'}{\pi}(\sin x + \cos x)$$

or $\quad u(x) = 1 + C(\sin x + \cos x)$, where $C = \frac{C'}{\pi}$

EXAMPLE 13

Solve the integral equation

$$u(x) = f(x) + \lambda\int_{-1}^{1}(xt + x^2t^2)\, u(t)\, dt$$

Also, find its resolvent kernel.

Solution : The given equation can be written as

$$u(x) = f(x) + \lambda x\int_{-1}^{1} t\, u(t)\, dt + \lambda x^2\int_{-1}^{1} t^2\, u(t)\, dt \qquad \text{......(1)}$$

Let us assume

$$C_1 = \int_{-1}^{1} t\, u(t)\, dt \qquad \text{......(2)}$$

and

$$C_2 = \int_{-1}^{1} t^2\, u(t)\, dt \qquad \text{......(3)}$$

Putting these values in (1), we get

$$u(x) = f(x) + \lambda C_1 x + \lambda C_2 x^2 \qquad \text{......(4)}$$

or

$$u(t) = f(t) + \lambda C_1 t + \lambda C_2 t^2 \qquad \text{......(5)}$$

Putting the value of $u(t)$ in (2), we get

$$C_1 = \int_{-1}^{1} t\left[f(t) + \lambda C_1 t + \lambda C_2 t^2\right] dt$$

$$= \int_{-1}^{1} t\, f(t)\, dt + \lambda C_1\left[\frac{t^3}{3}\right]_{-1}^{1} + \lambda C_2\left[\frac{t^4}{4}\right]_{-1}^{1} = \int_{-1}^{1} t\, f(t)\, dt + \frac{2\lambda C_1}{3}$$

$$\Rightarrow \quad C_1\left(1-\frac{2\lambda}{3}\right)=\int_{-1}^{1} t\, f(t)\, dt$$

i.e., $$C_1=\frac{3}{3-2\lambda}\int_{-1}^{1} t\, f(t)\, dt \qquad \text{......(6)}$$

Similarly, putting the value of u(t) in (3), we get

$$C_2=\int_{-1}^{1} t^2\left[f(t)+\lambda C_1 t+\lambda C_2 t^2\right] dt =$$

$$\int_{-1}^{1} t^2 f(t)\, dt+\lambda C_1\left[\frac{t^4}{4}\right]_{-1}^{1}+\lambda C_2\left[\frac{t^5}{5}\right]_{-1}^{1}$$

$$=\int_{-1}^{1} t^2 f(t)\, dt+\frac{2\lambda C_2}{5}$$

$$\Rightarrow \quad C_2=\frac{5}{5-2\lambda}\int_{-1}^{1} t^2 f(t)\, dt \qquad \text{......(7)}$$

Putting the values of C_1 and C_2 in (4), we get

$$u(x)=f(x)+\frac{3\lambda x}{3-2\lambda}\int_{-1}^{1} t\, f(t)\, dt+\frac{5\lambda x^2}{5-2\lambda}\int_{-1}^{1} t^2\, f(t)\, dt$$

$$=f(x)+\lambda\int_{-1}^{1}\left\{\frac{3xt}{3-2\lambda}+\frac{5x^2t^2}{5-2\lambda}\right\} f(t)\, dt \qquad \text{......(8)}$$

which is the required solution of the given integral equation.

The resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda)=\frac{3xt}{3-2\lambda}+\frac{5x^2t^2}{5-2\lambda}$$

EXAMPLE 14

Solve the integral equation and discuss all possible cases with the method of degenerate kernels

$$u(x)=F(x)+\lambda\int_0^1 (1-3xt)\, u(t)\, dt$$

[MEERUT-1990, 92, 93, 94, 97, 2002, 04, 06, 08]

Solution : The given equation can be written as

$$u(x)=F(x)+\lambda[C_1-3C_2x] \qquad \text{......(1)}$$

where, $C_1=\int_0^1 u(t)\, dt,\quad C_2=\int_0^1 t\, u(t)\, dt$ ......(2)

Putting the value of u(t) from (1) in (2), we get

$$C_1=\int_0^1 [F(t)+\lambda C_1-3C_2\lambda t]\, dt$$

and $$C_2=\int_0^1 t[F(t)+\lambda C_1-3C_2\lambda t]\, dt$$

$$\Rightarrow \quad C_1\left[1-\lambda\int_0^1 dt\right]+3C_2\lambda\int_0^1 t\, dt=\int_0^1 F(t)\, dt$$

and $$-C_1\lambda\int_0^1 dt+C_2\left[1+3\lambda\int_0^1 t\, dt\right]=\int_0^1 t\, F(t)\, dt$$

$$\Rightarrow \quad C_1(1-\lambda) + \frac{3}{2}\lambda C_2 = \int_0^1 F(t)\,dt$$

and $$-\frac{1}{2}C_1\lambda + (1+\lambda)\,C_2 = \int_0^1 t\,F(t)\,dt \qquad \ldots\ldots(3)$$

The determinant of this system is given by

$$D(\lambda) = \begin{vmatrix} 1-\lambda & \frac{3}{2}\lambda \\ -\frac{1}{2}\lambda & 1+\lambda \end{vmatrix} = 1 - \frac{\lambda^2}{4}$$

$\Rightarrow$ A unique solution exists if and only if $\lambda \neq \pm 2$.

Particularly, if $F(x) = 0$ and, then only trivial solution exists and given by $u(x) = 0$.

If $\lambda = +2$, then (3) gives

$$-C_1 + 3C_2 = \int_0^1 F(t)\,dt \quad \text{and} \quad -C_1 + 3C_2 = \int_0^1 t\,F(t)\,dt \qquad \ldots\ldots(4)$$

If $\lambda = -2$, then (3) gives

$$C_1 - C_2 = \frac{1}{3}\int_0^1 F(t)\,dt \quad \text{and} \quad C_1 - C_2 = \int_0^1 t\,F(t)\,dt \qquad \ldots\ldots(5)$$

The equation (4) and (5) are incompatible unless the function F(t) satisfies the condition

$$\int_0^1 F(t)\,dt = \int_0^1 t\,F(t)\,dt$$

$$\Rightarrow \quad \int_0^1 (1-t)\,F(t)\,dt = 0 \qquad \ldots\ldots(6)$$

and $$\frac{1}{3}\int_0^1 F(t)\,dt = \int_0^1 tF(t)\,dt \Rightarrow \int_0^1 (1-3t)\,F(t)\,dt \qquad \ldots\ldots(7)$$

When F(x) = 0, then the given integral equation becomes the homogeneous integral equation. If $\lambda = +2$ and F(x) = 0, equations (4) are redundant and either equation gives the condition $C_1 = 3C_2$. Therefore, the solution becomes

$$u(x) = A(1-x);\ \lambda = +2 \qquad \ldots\ldots(8)$$

when $A = 6C_2$

Hence, the function (1 - x) is the eigen function corresponding to the eigen value $\lambda = +2$.

Similarly, equation (5) gives

$$u(x) = B(1-3x); \quad \lambda = -2, \text{ where } B = -2C_1.$$

Therefore, the function $(1-3x)$ is the eigen function corresponding to the eigen value $\lambda = -2$. The solution (1) shows that any solution of the given integral equation may be expressed as the sum of F(x) and some linear combination of the eigen function

$$u(x) = F(x) + C(1-x) + D(1-3x) \qquad \ldots\ldots(10)$$

where $C = \frac{3\lambda}{2}(C_1 - C_2)$ and $D = \frac{\lambda}{2}(3C_2 - C_1)$

In the non-homogeneous case $F(x) \neq 0$, a unique solution exists if $\lambda \neq \pm 2$. If $\lambda = +2$, the equation (6) shows that no solution exists unless F(t) is orthogonal to (1 − t) over the interval (0, 1). Then, from (4), we have

$$C_1 = 3C_2 - \int_0^1 F(t)\, dt$$

Therefore, the solution of the integral equation is given by

$$u(x) = F(x) - 2\int_0^1 F(t)\, dt + A(1 - x) \qquad \text{......(11)}$$

when $\lambda = 2$, then $\int_0^1 (1 - t)\, F(t)\, dt = 0$ and $A = 6C_2$.

Then, infinitely many solution exist, different from each other by a multiple of an eigen function.

Similarly, if $\lambda = -2$, there exists no solution unless F(x) is orthogonal to $(1 - 3x)$ over $(0, 1)$, in which case infinitely many solutions exist.

When $\lambda = -2$, $u(x) = F(x) - \frac{2}{3}\int_0^1 F(t)\, dt + B(1 - 3x)$

where $\int_0^1 (1 - 3t)\, F(t)\, dt = 0$ and $B = -2C_2$.

EXAMPLE 15

Solve the following integral equation, using method of degenerate kernels

$$u(x) = x + \lambda \int_0^{2\pi} |\pi - t| \sin x\, u(t)\, dt$$

Solution : The given equation can be written as

$$u(x) = x + \lambda \int_0^{\pi} (\pi - t) \sin x\, u(t)\, dt + \lambda \int_{\pi}^{2\pi} (t - \pi) \sin x\, u(t)\, dt \qquad \text{......(1)}$$

Let $C_1 = \int_0^{\pi} (\pi - t)\, u(t)\, dt$ ......(2)

and $C_2 = \int_{\pi}^{2\pi} (t - \pi)\, u(t)\, dt$ ......(3)

Therefore, from equation (1), we get

$$u(x) = x + \lambda C_1 \sin x + \lambda \sin x\, C_2$$

$$\therefore \quad u(t) = t + \lambda C_1 \sin t + \lambda \sin t . C_2 \qquad \text{......(4)}$$

From equation (2) and (4), we get

$$C_1 = \int_0^{\pi} (\pi - t)\, [t + \sin t\, (\lambda C_1 + \lambda C_2) dt$$

$$= \int_0^{\pi} \pi[t + \sin t\, (\lambda C_1 + \lambda C_2) dt - \int_0^{\pi} t[t + \sin t\, (\lambda C_1 + \lambda C_2) dt$$

Let $I_1 = \pi \int_0^{\pi} [t + \sin t\, (\lambda C_1 + \lambda C_2)\, dt$ and

$$I_2 = \int_0^{\pi} [t^2 + \lambda(C_1 + C_2).t \sin t]\, dt$$

$$\therefore \quad I_1 = \pi \int_0^{\pi} [t + \sin t\, (\lambda C_1 + \lambda C_2)\, dt \qquad = \pi \left[\frac{t^2}{2} - (\lambda C_1 + \lambda C_2) \cos t \right]_0^{\pi}$$

$$= \frac{\pi^3}{2} - \pi \lambda (C_1 + C_2) [\cos \pi - \cos 0]$$

$$\Rightarrow \quad I_1 = \frac{\pi^3}{2} - 2\pi \lambda (C_1 + C_2)$$

Again, $I_2 = \int_0^{\pi} [t^2 + \lambda (C_1 + C_2) t \sin t]\, dt = [\, t^3/3 \,]_0^{\pi} + \int_0^{\pi} t.\sin t.\lambda\, (C_1 + C_2)\, dt$

$$= \frac{\pi^3}{3} + \left[[\, -t \cos t \,]_0^{\pi} + \int_0^{\pi} \cos t\, dt \right] \lambda (C_1 + C_2)$$

$$= \frac{\pi^3}{3} - \left[\pi \cos \pi - 0 \cos 0 - [\, \sin t \,]_0^{\pi} \right] \lambda (C_1 + C_2)$$

$$= \frac{\pi^3}{3} + [\, \pi + \sin \pi - \sin 0 \,] \lambda (C_1 + C_2) = \frac{\pi^3}{3} + \lambda (C_1 + C_2) \pi$$

Now, $C_1 = \frac{\pi^3}{2} + 2\lambda \pi\, (C_1 + C_2) - \frac{\pi^3}{3} - \lambda \pi (C_1 + C_2)$

$$= \frac{\pi^3}{6} + \lambda \pi\, (C_1 + C_2) \qquad \ldots\ldots(5)$$

and $C_2 = \int_{\pi}^{2\pi} (t - \pi)\, [\, t + \lambda\, (C_1 + C_2) \sin t \,]\, dt$

$$= \int_{\pi}^{2\pi} t\, [\, t + \lambda (C_1 + C_2) \sin t \,]\, dt - \pi \int_{\pi}^{2\pi} t\, [\, t + \lambda\, (C_1 + C_2) \sin t \,]\, dt$$

$$= \int_{\pi}^{2\pi} t^2\, dt + \int_0^{2\pi} \lambda\, (C_1 + C_2) t \sin t\, dt - \pi \int_{\pi}^{2\pi} t\, dt - \pi.\lambda\, (C_1 + C_2) \int_{\pi}^{2\pi} \sin t\, dt$$

$$= \left[\frac{t^3}{3} \right]_{\pi}^{2\pi} + \lambda\, (C_1 + C_2) \left[(-t \cos t) - \int_{\pi}^{2\pi} 1 (-\cos t)\, dt \right]_{\pi}^{2\pi}$$

$$- \pi \left[\frac{t^2}{2} \right]_{\pi}^{2\pi} - \pi \lambda\, (C_1 + C_2)\, [\, -\cos t \,]_{\pi}^{2\pi}$$

$$= \frac{1}{3} [8\pi^3 - \pi^3] + \lambda\, (C_1 + C_2) \left[-2\pi.\cos 2\pi + \pi \cos \pi + [\, \sin t \,]_{\pi}^{2\pi} \right]$$

$$- \frac{\pi}{2} \left[4\pi^2 - \pi^2 \right] - \pi \lambda (C_1 + C_2)\, [\, -\cos 2\pi + \cos \pi \,]$$

$$= \frac{7\pi^3}{3} + \lambda\, (C_1 + C_2) \{\, [-2\pi - \pi] + 0 \,\} - \frac{3\pi^3}{2} - \pi \lambda\, (C_1 + C_2)\, [-1 - 1]$$

$$= \frac{7\pi^3}{3} + \lambda\, (C_1 + C_2)\, (-3\pi) - \frac{3\pi^3}{2} + 2\lambda \pi\, (C_1 + C_2)$$

$$= \frac{5\pi^3}{6} + \lambda\, (C_1 + C_2)\, [\, -3\pi + 2\pi \,]$$

$$C_2 = \frac{5\pi^3}{6} - \lambda\pi\,(C_1 + C_2) \qquad \ldots\ldots(6)$$

Solving (5) and (6), we have

$$\left.\begin{aligned} C_1 &= \frac{\pi^3}{6} + \pi^4\lambda \\ C_2 &= \frac{5\pi^3}{6} - \pi^4\lambda \end{aligned}\right\} \qquad \ldots\ldots(7)$$

Using the values of C_1 and C_2 from (7) in (4), we have

$$u(x) = x + \lambda\left[\frac{\pi^3}{6} + \lambda\pi^4\right]\sin x + \lambda\sin x\left[\frac{5\pi^3}{6} - \lambda\pi^4\right]$$

$$= x + \frac{\lambda\pi^3}{6}\sin x + \frac{5\lambda\pi^3}{6}\sin x + \lambda^2\pi^4\sin x - \lambda^2\pi^4\sin x = x + \frac{6\lambda\pi^3}{6}\sin x$$

$$u(x) = x + \lambda\pi^3\sin x$$

is the required solution of the given integral equation.

EXAMPLE 16

Solve the following integral equation, using method of degenerate kernels

$$u(x) - \lambda\int_0^1 \cos(\log t^q)\, u(t)\, dt = 1$$

Solution : We have

$$u(x) = 1 + \lambda\int_0^1 \cos(\log t^q)\, u(t)\, dt \qquad \ldots\ldots(1)$$

Let $\quad C = \int_0^1 \cos(\log t^q)\, u(t)\, dt$

i.e., $\quad u(x) = 1 + \lambda C \Rightarrow u(t) = 1 + \lambda C$

Putting the value of u(t) in C, we get

$$C = \int_0^1 \cos(\log t^q)(1 + \lambda C)\, dt = (1 + \lambda C)\int_0^1 \cos(\log t^q)\, dt$$

Now, let $\log t^q = z \Rightarrow t^q = e^z \Rightarrow t = e^{z/q} \Rightarrow dt = \frac{e^{z/q}}{q}dz$

$$= (1 + \lambda C)\int_0^1 \frac{\cos z . e^{z/q}}{q} . dz = \frac{(1 + \lambda C)}{q}\int_0^1 e^{z/q} . \cos z\, dz$$

$$= \frac{(1 + \lambda C)}{q}\left[\frac{e^{z/q}}{(1/q)^2 + 1}\left[\frac{1}{q}\cos z + \sin z\right]_0^1\right]$$

$$= \frac{(1 + \lambda C)}{q}\left[\frac{q^2 t}{(1 + q^2)}\left[\frac{1}{q}\cos(\log t^q) + \sin(\log t^q)\right]_0^1\right]$$

$$= \frac{q(1 + \lambda C)}{(1 + q^2)}\left[\frac{1}{q}\cos 0 + \sin 0\right] - 0(\lambda)$$

$$\Rightarrow \qquad C = \frac{(1+\lambda C)}{(1+q^2)} \Rightarrow C[(1+q^2)-\lambda] = 1 \Rightarrow C = \frac{1}{(1+q^2)-\lambda}$$

i.e., $$u(x) = 1 + \lambda \frac{1}{(1+q^2)-\lambda} = \frac{-\lambda+1+q^2+\lambda}{(1+q^2-\lambda)} = \frac{1+q^2}{1+q^2-\lambda}$$

EXAMPLE 17

Using the method of degenerate kernels, solve the integral equation

$$u(x) - \lambda \int_0^1 \log(1/t)^p\, u(t)\, dt = 1$$

Solution : We have

$$u(x) = 1 + \lambda \int_0^1 \log(1/t)^p\, u(t)\, dt \qquad \text{......(1)}$$

Let $$C = \int_0^1 \log(1/t)^p\, u(t)\, dt \qquad \text{......(2)}$$

becomes equation (1), we get

$$u(x) = 1 + \lambda C$$

or $$u(t) = 1 + \lambda C \qquad \text{......(3)}$$

From equation (2) and (3), we get

$$C = \int_0^1 \log(1/t)^p\, (1+\lambda C)\, dt$$

Putting $\log(1/t) = z \Rightarrow t = e^{-z} \Rightarrow dt = -e^{-z}\, dz$

and when t = 1, then z = 0 and when t = 0, then z = ∞ .

$$\therefore \qquad C = (1+\lambda C) \int_0^\infty e^{-z} z^p . dz \qquad \left[\begin{array}{l} \because \overline{|p} = \int_0^\infty e^{-z} z^{p-1}\, dz \\ \text{and } \overline{|p+1} = \int_0^\infty e^{-z} z^p\, dz \end{array} \right]$$

$$\therefore \qquad C = (1+\lambda C) \overline{|p+1}$$

$$C\,(1 - \lambda \overline{|p+1}) = \overline{|p+1}$$

Hence, $$u(x) = 1 + \frac{\lambda . \overline{|p+1}}{1-\lambda \overline{|p+1}} = \frac{1 - \lambda\overline{|p+1} + \lambda\overline{|p+1}}{1-\lambda\overline{|p+1}} = \frac{1}{1-\lambda\overline{|p+1}}$$

EXERCISE - 2

(1) Solve the following integral equations, using method of degenerate kernels

(i) $u(x) = f(x) + \lambda \int_0^1 xt\, u(t)\, dt$.

(ii) $u(x) = \tan^{-1} x + \int_{-1}^1 e^{\sin^{-1} x}\, u(t)\, dt$

(iii) $u(x) = \sec x \tan x - \lambda \int_0^1 u(t)\, dt$

(iv) $u(x) = \lambda \int [illegible] u(t)\, dt$

(v) $u(x) = \sin x + \lambda \int_0^{\pi/2} \sin x \cos t\, u(t)\, dt$

(vi) $u(x) = 2x - \pi + 4\int_0^{\pi/2} \sin^2 x\, u(t)\, dt$ [GARHWAL-2002, 05]

(vii) $u(x) - \lambda \int_0^1 (4xt - x^2)\, u(t)\, dt = x$

(viii) $u(x) = 1 + x^2 + \int_0^1 (xt + x^2t^2)\, u(t)\, dt$

(2) Show that the characteristic values of λ for the equation $u(x) = \lambda\int_0^{2\pi} \sin(x + t)\, u(t)\, dt$ are $\lambda_1 = 1/\pi$ and $\lambda_2 = -1/\pi$ with corresponding characteristic functions of the form $u_1(x) = \sin x + \cos x$ and $u_2(x) = \sin x - \cos x$

(3) Solve the integral equation $u(x) = f(x) + \lambda\int_0^{2\pi} \cos(x + t)\, u(t)\, dt$ and find the condition that $f(x)$ must satisfy in order that this equation has a solution when λ is an eigen value. Find the general solution if $f(x) = \sin x$, considering all cases.

[Garhwal-2001]

(4) Solve the integral equation $u(x) = 1 + \lambda\int_{-\pi}^{\pi} e^{i\omega(x-t)}\, u(t)\, dt$ considering separately all exceptional cases.

(5) Find an approximate solution of the integral equation $u(x) = x^2 + \int_0^1 \sin(xt)\, u(t)\, dt$ by replacing $\sin(xt)$ by the first two terms of its power series expansion $\sin(xt) = (xt) - \frac{(xt)^3}{3!} + \dots$

Hints to the Selected Problems

1.(i) $u(x) = f(x) + \lambda x \int_0^1 t\, u(t)\, dt$

Let $C = \int_0^1 t\, u(t)\, dt$

$u(x) = f(x) + \lambda Cx \Rightarrow u(t) = f(t) + \lambda Ct$

Thus $C = \int_0^1 [f(t) + \lambda Ct]\, t\, dt = \int_0^1 t f(t)\, dt + \lambda C\int_0^1 t^2\, dt$. Then solve for C.

(ii) $u(x) = \tan^{-1} x + e^{\sin^{-1} x} \int_{-1}^{1} u(t)\, dt$

Let $C = \int_{-1}^{1} u(t)\, dt$

$\therefore$ $u(x) = \tan^{-1} x + e^{\sin^{-1} x} C \Rightarrow u(t) = \tan^{-1} t + e^{\sin^{-1} t} C$

Thus, $C = \int_{-1}^{1} [\tan^{-1} t + C.e^{\sin^{-1} t}]\, dt$. Now, simplify it.