PARTIAL ACTIONS, PARADOXICALITY AND TOPOLOGICAL FULL GROUPS

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PhD Thesis

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Abstract

We study how paradoxicality properties affect the way groups partially act on topological spaces and C^*-algebras. We also investigate the real rank zero and AF properties for certain classes of group C^*-algebras.

Specifically, in article A, we characterize supramenable groups in terms of existence of invariant probability measures for partial actions on compact Hausdorff spaces and existence of tracial states on partial crossed products. These characterizations show that, in general, one cannot decompose a partial crossed product of a C^*-algebra by a semidirect product of groups as two iterated partial crossed products. We give conditions which ensure that such decomposition is possible.

In Article B, we show that an action of a group on a set X is locally finite if and only if X is not equidecomposable with a proper subset of itself. As a consequence, a group is locally finite if and only if its uniform Roe algebra is finite.

In Article C, we analyze the C^*-algebra generated by the Koopman representation of a topological full group, showing, in particular, that it is not AF and has real rank zero. We also prove that if G is a finitely generated, elementary amenable group, and $C^*(G)$ has real rank zero, then G is finite.

Resumé

Vi studerer hvordan paradoksikalitet påvirker måden, som grupper partielt virker på topologiske rum og C^*-algebraer. Vi undersøger også de reel rang nul og AF egenskaber for visse klasser af gruppe C^*-algebraer.

I Artiklen A karakteriserer vi supramenable grupper i termer af eksitens af invariante sandsynlighedsmål for partielle virkninger på kompakte Hausdorff rum og eksitens af sportilstande på partielle krydsprodukter. Disse karakteriseringer viser at generelt kan man ikke nedbryde et partielt krydsprodukt af en C^*-algebra ved et semidirekte produkt af grupper, som to gentaget partielle krydsprodukter. Vi giver betingelser som sikre at en sådan nedbrydning er mulig.

I Artiklen B viser vi at en virkning af en gruppe på en mængde X er lokalt endelig hvis og kun hvis X ikke er ækvidekomposibel med en ægte delmængde af sig selv. Som en konsekvens er en gruppe lokalt endelig hvis og kun hvis dens uniform Roe algebra er endelig.

I Artiklen C analysere vi C^*-algebraen genereret af den Koopman repræsentation af en topologisk fuld gruppe. Specielt viser vi at den ikke er AF og har reel rang nul. Vi viser også at, hvis G er en endeligt frembragt, elementært amenabel gruppe, og $C^*(G)$ har reel rang nul, så er G endelig.

Contents

Part I

Thesis overview

Introduction

A group is amenable if and only if whenever it acts on a compact Hausdorff space, the space admits an invariant probability measure.

In [KMR13], Kellerhals, Monod and Rørdam characterized supramenability of a group in terms of existence of invariant measures for co-compact actions on locally compact Hausdorff space. Furthermore, they used non-supramenability of a group for constructing stable Kirchberg algebras out of actions on the locally compact, non-compact Cantor set.

In [Hop07], Hopenwasser exhibited a partial action of a metabelian group on the Cantor set, such that the associated partial crossed product is isomorphic to the Cuntz algebra $\mathcal{O}_2$. In particular, such partial action does not admit an invariant probability measure.

In [Sca16b], we showed that a group is supramenable if and only if whenever it partially acts on a compact Hausdorff space, then the space admits an invariant probability measure. Moreover, we proved the non-commutative version of this result in terms of invariant tracial states. These characterizations implied that, in general, one cannot decompose a partial crossed product of a C^*-algebra by a semidirect product of groups as two iterated partial crossed products. We gave conditions which ensure that such decomposition is possible.

In [KMR13], it was observed that, if a group is locally finite, then its uniform Roe algebra is finite, and asked whether the converse holds. In [Sca16a], we showed that a group is locally finite if and only if it is not equidecomposable to a proper subset, if and only if its uniform Roe algebra is finite.

Motivated by this result, we started to work on the problem of characterizing local finiteness of a group in terms of its group C^*-algebra. Clearly, if a countable group G is locally finite, then $C^*(G)$ is AF. Conversely, in [Kan93, Theorem 2], Kaniuth proved that if G is a nilpotent group and $C^*(G)$ has real rank zero, then G is locally finite.

In [Sca17], we showed that if G is a finitely generated, elementary amenable group, and $C^*(G)$ has real rank zero, then G is finite. A class of groups that is not covered by this result is that of commutators of topological full groups. Given scuch a group, we showed that its group C^*-algebra is not AF by analyzing its canonical Koopman representation.

Background

In this chapter, we collect some definitions and facts that are relevant for the thesis. More basic aspects about C^*-algebras can be consulted in [Mur90].

Crossed product

We refer the reader to [Exe15] (which treats the more general setting of partial actions and Fell bundles) for the material in this section.

Let θ be an action of a group G on a C^*-algebra A. Denote by $A \rtimes_{\mathrm{alg}} G$ the vector space of finitely supported functions from G into A.

For every $g \in G$ and $a_g \in A$, let $a_g\delta_g \in A \rtimes_{\mathrm{alg}} G$ be defined by

$$a_g\delta_g(h) := \begin{cases} a_g & \text{if } h = g \\ 0 & \text{if } g \neq h \end{cases}, \quad h \in G.$$

Notice that $\{a_g\delta_g : g \in G, a_g \in A\}$ spans $A \rtimes_{\mathrm{alg}} G$. Hence, we can define a product and an involution on $A \rtimes_{\mathrm{alg}} G$ by

$$(a_g\delta_g)(b_h\delta_h) := a_g\theta_g(b_h)\delta_{gh}$$
$$(a_g\delta_g)^* := \theta_{g^{-1}}(a_g^*)\delta_{g^{-1}}$$

for $g, h \in G$ and $a_g, b_h \in A$. These operations turn $A \rtimes_{\mathrm{alg}} G$ into a $*$-algebra.

Consider the linear map $\psi\colon A \rtimes_{\mathrm{alg}} G \to A$ such that

$$\psi(a\delta_g) = \begin{cases} a & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}, \quad g \in G, a \in I_g. \tag{1}$$

There is a C^*-norm $\|\cdot\|_{\mathrm{red}}$ on $A \rtimes_{\mathrm{alg}} G$ (that is, a norm satisfying the C^*-identity) which makes ψ contractive and such that the extension of ψ to the completion of $A \rtimes_{\mathrm{alg}} G$ endowed with $\|\cdot\|_{\mathrm{red}}$ is faithful (for C^*-algebras B and C, a linear map $\varphi\colon B \to C$ is said to be faithful if, for every $b \in B$, $\varphi(b^*b) = 0$ implies that $b = 0$). Furthermore, $\| \cdot \|_{\text{red}}$ is the unique C^*-norm on $A \rtimes_{\text{alg}} G$ with these properties.

The completion of $A \rtimes_{\text{alg}} G$ endowed with $\| \cdot \|_{\text{red}}$ is called the reduced crossed product, and denoted by $A \rtimes_{\text{red}} G$ (or $A \rtimes_{\text{r}} G$). See also [BO08] for the more usual construction of the reduced crossed product.

Define another C^*-norm on $A \rtimes_{\text{alg}} G$ by

$$\|x\|_{\max} := \sup\{p(x) : p \text{ is a C}^*\text{-seminorm on } A \rtimes_{\text{alg}} G\}.$$

The full crossed product, denoted by $A \rtimes G$, is the completion of $A \rtimes_{\text{alg}} G$ endowed with $\| \cdot \|_{\max}$.

There is a surjective $*$-homomorphism π from $A \rtimes G$ to $A \rtimes_{\text{red}} G$ defined in an obvious way. If G is amenable, then π is an isomorphism.

If $G = \mathbb{Z}$, there is a tool, the so called Pimsner-Voiculescu exact sequence, for computing the K-groups of $A \rtimes \mathbb{Z}$ (see, for example, [Cun81] for more details and a proof):

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\text{id}-K_0(\theta_{-1})} & K_0(A) & \longrightarrow & K_0(A \rtimes \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(A \rtimes \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[\text{id}-K_1(\theta_{-1})]{} & K_1(A).
\end{array}$$

If a group G acts on a locally compact Hausdorff X topologically freely (that is, for any $g \in G \setminus \{e\}$, the interior of $\{x \in X : gx = x\}$ is empty) and minimally (for any $x \in X$, it holds that Gx is dense), then the associated reduced crossed product $C_0(X) \rtimes_{\text{r}} G$ is simple.

Supramenable groups

Let G be a group. A non-empty subset A of G is said to be paradoxical if there exist disjoint subsets B and C of A, finite partitions $\{B_i\}_{i=1}^n$ and $\{C_j\}_{j=1}^m$ of B and C and elements $s_1, ..., s_n, t_1, ..., t_m \in G$ such that $A = \sqcup_{i=1}^n s_i B_i = \sqcup_{j=1}^m t_j C_j$ (disjoint union).

A group is supramenable if none of its subsets is paradoxical. The class of supramenable groups is closed under taking direct limits, and, if a group is virtually supramenable, then it is supramenable. See [Wag85, Chapter 12] for proofs of these and other permamence properties.

If a group has subexponential growth, then it is supramenable. It is unknown whether the converse holds, or if the direct product of two supramenable groups is supramenable.

Let G be a supramenable group. Clearly, the direct product of G with a locally finite group is supramenable. To our best knowledge, it is not known whether $G \times \mathbb{Z}$ is necessarily supramenable. If $G\times\mathbb{Z}$ is not supramenable, then, by [KMR13, Proposition 3.4] and [Sca16a, Proposition 2.5], for every non-locally finite group H it holds that $G\times H$ is not supramenable.

In [Mon17], a fixed-point property for groups is introduced, and shown to imply supramenability and be implied by subexponential growth. Furthermore, this property is shown to be closed with respect to taking direct product with any group of subexponential growth.

In [KMR13], it was shown that a group is supramenable if and only if whenever it acts co-compactly on a locally compact Hausdorff X, then X admits a non-zero, invariant, regular measure. In [MR15], it was shown that the hypothesis of co-compactness is indispensable.

Let A be a C^*-algebra. Recall that a non-zero projection $p\in A$ is said to be properly infinite if it has two orthogonal subprojections which are Murray-von Neumann equivalent to p.

Also in [KMR13], it was proven that a group G is supramenable if and only if $\ell^\infty(G)\rtimes_{\text{red}} G$ (the uniform Roe algebra) contains no properly infinite projection. We end this section by proving a slight strengthening of this result. It will not be necessary for the remaining of the thesis, so the reader is welcome to skip it.

Proposition. *A group G is supramenable if and only if whenever it acts on a locally compact Hausdorff space X, then $C_0(X)\rtimes_{\text{red}}G$ contains no properly infinite projection.*

Proof. Assume first that G is supramenable and that there is a properly infinite projection $p\in C_0(X)\rtimes_{\text{red}} G$.

Let $\psi\colon C_0(X)\rtimes_{\text{red}} G\to C_0(X)$ and $\psi_G\colon \ell^\infty(G)\rtimes_{\text{red}} G\to \ell^\infty(G)$ be the canonical conditional expectations. Since they are faithful, there is $x\in X$ such that $\psi(p)(x)\neq 0$.

Let $\varphi_x : C_0(X)\to\ell^\infty(G)$ be given by $\varphi_x(f)(g):=f(gx)$, for $f\in C_0(X)$ and $g\in G$. Then φ_x is a G-equivariant $*$-homomorphism. In particular, it can be naturally extended to a $*$-homomorphism $\Phi_x\colon C_0(X)\rtimes_{\text{red}} G\to \ell^\infty(G)\rtimes_{\text{red}} G$.

Notice that $\varphi_x\circ\psi=\psi_G\circ\Phi_x$. Hence, Φ_x maps p to a non-zero element, which clearly will be a properly infinite projection.

Since G was supramenable, this cotradicts [KMR13, Proposition 5.3]. Therefore, $C_0(X)\rtimes_{\text{red}} G$ contains no properly infinite projection.

Now assume G that is not supramenable, and take $A\subset G$ paradoxical. Then $1_A\in\ell^\infty(G)\rtimes_{\text{red}} G$ is a properly infinite projection, as proved in [RS12].

□

Abstract of the articles with perspectives for further research

Article A ([Sca16b])

We show that a group is supramenable if and only if whenever it partially acts on a compact Hausdorff space, the space admits an invariant probability measure.

We also show that a group is supramenable if and only if whenever it partially acts on a unital C^*-algebra which has a tracial state, the associated partial crossed product also has a tracial state.

Furthermore, we show that if G is a countable, amenable, non-supramenable group, then there exists a free, minimal partial action of G on the Cantor set such that the associated partial crossed product is a Kirchberg algebra.

In [Li16], Li showed that if G is an exact group which contains a non-abelian free semigroup (in particular, it is non-supramenable), and $\mathcal{E}$ is a countable graph, then there is a partial action of G on the boundary-path space of $\mathcal{E}$, such that the partial crossed product is isomorphic to the graph C^*-algebra of $\mathcal{E}$.

Given a (global) action of a semidirect product $G \rtimes H$ on a C^*-algebra A, one can decompose the crossed product $A \rtimes (G \rtimes H)$ as two iterated crossed products by G and H:

$$A \rtimes (G \rtimes H) \cong (A \rtimes G) \rtimes H. \tag{2}$$

The class of supramenable groups is not closed under taking semidirect products. Therefore, our results show that, in general, one cannot have a decomposition such as in (2) for partial actions.

We give conditions under which (2) holds for partial actions. These conditions involve assuming that A is stable or requiring that the domains of the partial isomorphisms satisfy a certain condition.

Article B ([Sca16a])

Given a group G acting on a set X, we show that the action is locally finite if and only no subset of X is equidecomposable to a proper subset of itself, if and only if $\ell^\infty(X) \rtimes_{\text{red}} G$ is a finite C^*-algebra. Consequently, a group is locally finite if and only if its uniform Roe algebra if finite.

After this paper was made available on arXiv, we became aware of [Wei11], in which Wei showed that, if a group is infinite and finitely generated, then its uniform Roe algebra is infinite.

In [LW17], Li and Willett investigate further properties of uniform Roe algebras, like stable rank one and real rank zero. They leave open, for example, the question of whether $\ell^\infty(\mathbb{Z}) \rtimes \mathbb{Z}$ has real rank zero.

Article C ([Sca17])

We show that if G is a finitely generated, elementary amenable group, and $C^*(G)$ has real rank zero, then G is finite.

Let (X, T, μ) be a Cantor minimal system and π the Koopman representation associated to the action of the topological full group $[[T]]$ on (X, μ).

We prove that $C^*([[T]]')$ is not AF. This is done by showing that $C^*_\pi([[T]]) = C^*_\pi([[T]]')$, and that the kernel of the character τ on $C^*_\pi([[T]])$ coming from weak containment of the trivial representation is a hereditary C^*-subalgebra of $C(X) \rtimes \mathbb{Z}$ (Theorem 3.7). Consequently, $\ker \tau$ is stably isomorphic to $C(X) \rtimes \mathbb{Z}$, and $C^*_\pi([[T]]')$ is not AF and has real rank zero.

The notion of a topological full group has been extended by Matui to essentially principal, étale groupoids (see [Mat16] for a survey).

We would like to see Theorem 3.7 from Article C generalized to this more general context. Specifically, let $\mathcal{G}$ be an essentially principal, minimal, étale groupoid, with unit space homeomorphic to the Cantor set. Let H be a subgroup of the topological full group of $\mathcal{G}$. Furthermore, assume that the canonical unitary representation π of H in $C^*(\mathcal{G})$ weakly contains the trivial representation (this is the case, for example, for a certain representation of Thompson's group F in $\mathcal{O}_2$, as proved in [HO17]). Under which conditions is the kernel of the associated character on $C^*_\pi(H)$ a hereditary C^*-subalgebra of $C^*(\mathcal{G})$?

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Part II

Articles

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Supramenable groups and partial actions

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Abstract. We characterize supramenable groups in terms of the existence of invariant probability measures for partial actions on compact Hausdorff spaces and the existence of tracial states on partial crossed products. These characterizations show that, in general, one cannot decompose a partial crossed product of a C*-algebra by a semidirect product of groups into two iterated partial crossed products. However, we give conditions which ensure that such decomposition is possible.

1. *Introduction*

Partial actions of groups on C*-algebras were introduced by Exel [**2**] and McClanahan [**8**] as a means of computing invariants and describing the structure of C*-algebras. Since then they have been studied in other categories as well: notably, sets, topological spaces and algebras (see [**3**] for details).

A group G is called *amenable* if it carries an invariant, finitely additive measure μ such that $\mu(G) = 1$. Following Rosenblatt [**12**], a group G is called *supramenable* if, for every non-empty $A \subset G$, there is an invariant, finitely additive measure μ on G such that $\mu(A) = 1$.

The class of supramenable groups is closed under taking subgroups, quotients and direct limits. Abelian groups and, more generally, groups of subexponential growth are supramenable (see [**13**, Ch. 12] for a proof of these facts). It is not known if the direct product of supramenable groups is supramenable and if every supramenable group has subexponential growth.

It is a well-known fact that a group is amenable if and only if, whenever it acts on a compact Hausdorff space, the space admits an invariant probability measure. There is also a non-commutative version of this result which says that a group is amenable if and only if, whenever it acts on a unital C*-algebra which has a tracial state, the associated crossed product also has a tracial state.

However, these results do not hold for partial actions; Hopenwasser showed in [**6**] that the Cuntz algebras can be realized as partial crossed products associated with partial actions of certain amenable groups on the Cantor set.

In [**7**], Kellerhals *et al* showed that a group is supramenable if and only if, whenever it acts cocompactly on a locally compact Hausdorff space, the space admits an invariant, non-zero and regular measure.

The main purpose of this paper is to show that the role of amenable groups is played by supramenable groups in the context of partial actions.

In §2, we prove that a group is supramenable if and only if, whenever it partially acts on a compact Hausdorff space, the space admits an invariant probability measure (Proposition 2.7).

In §3, we prove the non-commutative version of Proposition 2.7. Namely, we show that a group is supramenable if and only if, whenever it partially acts on a unital C*-algebra which has a tracial state, the associated partial crossed product also has a tracial state (Theorem 3.6).

In [**4**], Giordano and Sierakowski gave conditions which ensure that a partial crossed product is purely infinite. Our results, on the other hand, show that one cannot get a purely infinite C*-algebra out of a partial action of a supramenable group on a unital C*-algebra which has a tracial state.

Recall that a *Kirchberg algebra* is a simple, nuclear, separable and purely infinite C*-algebra. Non-amenability of a group was used by Rørdam and Sierakowski in [**10**] for constructing unital Kirchberg algebras out of actions on the Cantor set. Analogously, non-supramenability of a group was used by Kellerhals *et al* in [**7**] for constructing stable Kirchberg algebras out of actions on the locally compact, non-compact Cantor set.

Using a result of [**7**], we show that if G is a countable, amenable, non-supramenable group, then there exists a free, minimal, purely infinite and non-global partial action of G on the Cantor set K (Proposition 2.10). It is a consequence of results from [**3**, **4**], that the partial crossed product associated with any such partial action of G on K is a Kirchberg algebra.

Given a (global) action of a semidirect product $G \rtimes H$ on a C*-algebra A, one can decompose the crossed product $A \rtimes (G \rtimes H)$ into two iterated crossed products by G and H: that is

$$A \rtimes (G \rtimes H) \cong (A \rtimes G) \rtimes H. \tag{1}$$

The class of supramenable groups is not closed under taking semidirect products. For example, the *lamplighter group* $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ contains a free monoid on two generators, and hence cannot be supramenable. Therefore, our results show that, in general, one cannot have a decomposition such as in (1) for partial actions.

In §4, we give conditions under which (1) holds for partial actions. These conditions involve assuming that A is stable or requiring that the domains of the partial isomorphisms satisfy a certain condition.

2. *Supramenability and partial actions on compact Hausdorff spaces*

Throughout this article, we denote the identity element of a group by e and all groups are assumed to be discrete.

We begin by recalling the definition of partial actions on topological spaces. See [**3**] for details and historical notes.

Definition 2.1. A *partial action* θ of a group G on a topological space X is a pair $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ where $\{D_g\}_{g\in G}$ is a family of open subsets of X and each θ_g is a homeomorphism from $D_{g^{-1}}$ onto D_g such that:

(i) $\theta_e = \mathrm{Id}_X$; and

(ii) for every $g, h \in G$ and $x \in D_{g^{-1}}$, if $\theta_g(x) \in D_{h^{-1}}$, then $x \in D_{(hg)^{-1}}$ and $\theta_{hg}(x) = \theta_h \circ \theta_g(x)$.

A consequence of the above definition is that, for every $g, h \in G$,

$$\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}. \tag{2}$$

If $D_g = X$ for every $g \in G$, then the partial action is just a usual action of a group on a topological space. In this case, we might say that the partial action is *global*.

In the definition of a partial action, it is possible that some of the open subsets D_g are empty. For example, any group G partially acts trivially on a topological space X by letting $D_g := \emptyset$ for every $g \neq e$ and $\theta_e := \mathrm{Id}_X$.

Example 2.2. Let θ be an action of a group G on a topological space X. Given an open subset D of X, let $D_g := D \cap \theta_g(D)$ for every $g \in G$. Then

$$(\{D_g\}_{g\in G}, \{\theta_g|_{D_{g^{-1}}}\}_{g\in G})$$

is a partial action of G on D, called the *restriction* of θ to D.

Definition 2.3. Let $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of a group G on a topological space X. We say a measure ν on X is *invariant* if, for all $E \in \mathcal{B}(X)$ and $g \in G$,

$$\nu(\theta_g(E \cap D_{g^{-1}})) = \nu(E \cap D_{g^{-1}}).$$

Next, we recall the definitions of supramenable groups and paradoxical subsets of a group. See [**13**] for historical notes.

Definition 2.4. A group G is *supramenable* if, for every non-empty subset A of G, there is an invariant, finitely additive measure $\mu : \mathcal{P}(G) \to [0, +\infty]$ such that $\mu(A) = 1$.

Definition 2.5. Let G be a group. We say a non-empty subset A of G is *paradoxical* if there exist disjoint subsets B and C of A, finite partitions $\{B_i\}_{i=1}^n$ and $\{C_j\}_{j=1}^m$ of B and C and elements $s_1, \ldots, s_n, t_1, \ldots, t_m \in G$ such that $A = \bigsqcup_{i=1}^n s_i B_i = \bigsqcup_{j=1}^m t_j C_j$ (disjoint union).

By Tarski's theorem, a non-empty subset A of a group G is not paradoxical if and only if there exists an invariant, finitely additive measure $\mu : \mathcal{P}(G) \to [0, +\infty]$ such that $\mu(A) = 1$. Therefore, a group is supramenable if and only if it contains no paradoxical subsets.

In order to prove Proposition 2.7, we will need the following lemma, the proof of which can be found in [**7**, Proposition 2.1].

LEMMA 2.6. *Let G be a group and let μ be a finitely additive measure on G. Let V_μ be the subspace of $\ell^\infty(G)$ consisting of all $f \in \ell^\infty(G)$ such that $\mu(\mathrm{supp}(f)) < \infty$. It follows that there is a unique positive linear functional $I_\mu : V_\mu \to \mathbb{C}$ such that $I_\mu(1_E) = \mu(E)$ for all $E \subset G$ with $\mu(E) < \infty$. If μ is G-invariant, then so is I_μ.*

PROPOSITION 2.7. *A group is supramenable if and only if, whenever it partially acts on a compact Hausdorff space, the space admits an invariant probability measure.*

Proof. First, assume that G is a non-supramenable group. Then it has a paradoxical subset A. Let $j : G \to \beta G$ be the embedding of G into its Stone–Čech compactification. Consider the partial action obtained by restricting the canonical action of G on βG to $\overline{j(A)}$. By using [**7**, Lemma 2.4], one concludes that this partial action does not admit an invariant probability measure.

Conversely, let $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of a supramenable group G on a compact Hausdorff space X. Fix $x_0 \in X$. Given $f \in C(X)$, let $\hat{f} \in \ell^\infty(G)$ be defined by

$$\hat{f}(g) := \begin{cases} 0 & \text{if } x_0 \notin D_{g^{-1}}, \\ f(\theta_g(x_0)) & \text{if } x_0 \in D_{g^{-1}}. \end{cases}$$

By using (2), one can easily check that, for every $g \in G$ and $f \in C_0(D_{g^{-1}})$ (seen as an ideal of $C(X)$), it holds that

$$g \cdot \hat{f} = \widehat{f \circ \theta_{g^{-1}}}. \tag{3}$$

Notice that, formally speaking, $f \circ \theta_{g^{-1}}$ is only defined on D_g. We see it as being defined on X by extending it as zero outside of D_g.

Let $A := \{g \in G : x_0 \in D_{g^{-1}}\}$. Since G is supramenable, there exists an invariant, finitely additive measure μ on G such that $\mu(A) = 1$.

In the notation of Lemma 2.6, let

$$\begin{aligned} \varphi : C(X) &\to \mathbb{C} \\ f &\mapsto I_\mu(\hat{f}). \end{aligned}$$

Notice that φ is well defined because, for every $f \in C(X)$, we have that $\mathrm{supp}(\hat{f}) \subset A$.

Since $\widehat{1_{C(X)}} = 1_A$, we obtain that $\varphi(1_{C(X)}) = 1$. Therefore, φ is a state, since it is clearly positive. By (3) and the G-invariance of I_μ, we conclude that, for every $g \in G$ and $f \in C_0(D_{g^{-1}})$, it holds that

$$\varphi(f \circ \theta_{g^{-1}}) = \varphi(f). \tag{4}$$

Let ν be the regular probability measure on X associated with φ through the Riesz representation theorem. By (4) and the inner regularity of ν, we conclude that ν is invariant on open subsets. By outer regularity, it follows that ν is invariant for all Borel measurable sets. □

Next, we recall the definition and some facts about partial actions on C*-algebras and partial crossed products. See [**3**] for the details.

Definition 2.8. A *partial action* θ of a group G on a C*-algebra A is a pair $(\{I_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ where $\{I_g\}_{g\in G}$ is a family of closed two-sided ideals of A and each θ_g is a $*$-isomorphisms from $I_{g^{-1}}$ onto I_g such that:

(i) $\theta_e = \mathrm{Id}_A$; and
(ii) for every $g, h \in G$ and $x \in I_{g^{-1}}$, if $\theta_g(x) \in I_{h^{-1}}$, then $x \in I_{(hg)^{-1}}$ and $\theta_{hg}(x) = \theta_h \circ \theta_g(x)$.

The quadruple $(A, G, \{I_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is called a *partial dynamical system.*

Given a partial dynamical system

$$(A, G, \{I_g\}_{g\in G}, \{\theta_g\}_{g\in G}),$$

denote by $C_c(G, A)$ the vector space of finitely supported functions from G into A. Let

$$A \rtimes_{\theta,\mathrm{alg}} G := \{f \in C_c(G, A) : f(g) \in I_g \text{ for every } g \in G\}.$$

For every $g \in G$ and $a_g \in I_g$, let $a_g\delta_g \in A \rtimes_{\theta,\mathrm{alg}} G$ be defined by

$$a_g\delta_g(h) := \begin{cases} a_g & \text{if } h = g, \\ 0 & \text{if } g \neq h, \end{cases} \qquad h \in G.$$

Notice that $\{a_g\delta_g : g \in G, a_g \in I_g\}$ spans $A \rtimes_{\theta,\mathrm{alg}} G$. Hence we can define a product and an involution on $A \rtimes_{\theta,\mathrm{alg}} G$ by

$$(a_g\delta_g)(b_h\delta_h) := \theta_g(\theta_{g^{-1}}(a_g)b_h)\delta_{gh}$$
$$(a_g\delta_g)^* := \theta_{g^{-1}}(a_g^*)\delta_{g^{-1}}$$

for $g, h \in G$, $a_g \in I_g$ and $b_h \in I_h$. These operations turn $A \rtimes_{\theta,\mathrm{alg}} G$ into a $*$-algebra.

Define a seminorm on $A \rtimes_{\theta,\mathrm{alg}} G$ by

$$\|x\|_{\max} := \sup\{p(x) : p \text{ is a C}^*\text{-seminorm on } A \rtimes_{\theta,\mathrm{alg}} G\}.$$

The *partial crossed product* associated with the partial action θ, denoted by $A \rtimes_\theta G$ or $A \rtimes G$, is the enveloping C*-algebra of the $*$-algebra $A \rtimes_{\theta,\mathrm{alg}} G$ endowed with the seminorm $\|\cdot\|_{\max}$.

There is an embedding

$$i : A \to A \rtimes_\theta G$$
$$a \mapsto a\delta_e$$

and a conditional expectation $\psi : A \rtimes_\theta G \to A$ such that

$$\psi(a\delta_g) = \begin{cases} a & \text{if } g = e, \\ 0 & \text{if } g \neq e, \end{cases} \qquad g \in G, a \in I_g. \tag{5}$$

Notice that the construction of the partial crossed product is a generalization of the usual crossed product associated with an action of a group on a C*-algebra.

Given a partial action $\theta = (\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ of a group G on a locally compact Hausdorff space X, one can associate with it a partial action on $C_0(X)$, with domains $C_0(D_g)$ (seen as ideals of $C_0(X)$) and $*$-isomorphisms given by composition with the homeomorphisms θ_g.

Definition 2.9. Let $\theta = (\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of a group G on a Hausdorff space X. We say θ is *free* if, for every $g \in G$ and $x \in D_{g^{-1}}$,

$$\theta_g(x) = x \implies g = e.$$

The partial action is said to be *minimal* if, given a closed set $F \subset X$ such that for every $g \in G$, $\theta_g(F \cap D_{g^{-1}}) \subset F$, we necessarily have that $F = \emptyset$ or $F = X$.

That is, the partial action is minimal if X has no non-trivial invariant closed subsets.

If X is totally disconnected, then the partial action is said to be *purely infinite* if, for every compact-open subset K of X, there exist pairwise disjoint compact-open subsets $K_1, \ldots, K_{n+m}$ and elements $t_1, \ldots, t_{n+m} \in G$ such that $K_j \subset K \cap D_{t_j^{-1}}$ for all j and

$$K = \bigcup_{j=1}^{n} \theta_{t_j}(K_j) = \bigcup_{j=n+1}^{n+m} \theta_{t_j}(K_j).$$

This definition of purely infinite partial actions generalizes the one in [**7**, Definition 4.4]. Also, if the partial action is purely infinite, then every compact-open subset of X is (G, τ_X)-paradoxical, in the sense of [**4**, Definition 4.3].

PROPOSITION 2.10. *Let G be an amenable, non-supramenable, countable group. Then G admits a free, minimal, purely infinite and non-global partial action on the Cantor set K. The partial crossed product associated with any such partial action of G on K is a Kirchberg algebra.*

Proof. Let G be as in the statement. By [**7**, Theorem 1.2], there is a free, minimal, purely infinite global action θ of G on the non-compact, locally compact Cantor space K^*. The restriction of θ to a compact-open subset K (which is automatically homeomorphic to the Cantor set) is clearly free and purely infinite. It is not global because it clearly does not admit an invariant probability measure and G is amenable. Let us show that it is minimal.

Suppose that there exists a non-empty closed set $F \subsetneq K$ such that, for every $g \in G$,

$$\theta_g(F \cap \theta_{g^{-1}}(K)) \subset F. \tag{6}$$

By minimality of θ, the orbit of every point is dense. Hence, given $x \in F$, there is $g \in G$ such that $\theta_g(x)$ belongs to the non-empty open set $K \setminus F$, but, since $x \in F \cap \theta_{g^{-1}}(K)$, this contradicts (6).

The fact that the associated partial crossed product is simple follows from [**3**, Corollary 29.8]. The fact that it is nuclear is due to [**3**, Proposition 25.10]. It is purely infinite because of [**4**, Theorem 4.4]. □

3. *Tracial states on partial crossed products*

In this section, we prove the main result of this article (Theorem 3.6). We start by proving some lemmas about traces in C*-algebras.

LEMMA 3.1. *Let A be a C*-algebra, I an ideal of A and τ a positive linear functional on I. Then there is a unique positive linear functional τ' on A such that τ' extends τ and $\|\tau\| = \|\tau'\|$. Moreover, if τ is a trace, then τ' is also a trace.*

Proof. The existence and uniqueness of τ' is the content of [**9**, Theorem 3.3.9]. Assume that τ is a trace and we will show that τ' is also a trace.

By taking the unitization, we can assume A is unital.

Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for I. Notice that

$$\tau'(1) = \|\tau'\| = \|\tau\| = \lim_\lambda \tau(u_\lambda).$$

Now suppose that $(x_\lambda)_{\lambda\in\Lambda}$ is a bounded net in A such that $\lim_\lambda \tau'(x_\lambda)$ exists. Let us show that $\lim_\lambda \tau'(x_\lambda) = \lim_\lambda \tau(x_\lambda u_\lambda)$. Indeed,

$$\begin{aligned}|\tau'(x_\lambda) - \tau(x_\lambda u_\lambda)| = |\tau'(x_\lambda(1-u_\lambda))| &\overset{(*)}{\leq} \tau'(x_\lambda x_\lambda^*)^{1/2}\tau'((1-u_\lambda)^2)^{1/2}\\ &\overset{(**)}{\leq} \|\tau'\|^{1/2}\|x_\lambda\|\tau'(1-u_\lambda)^{1/2} \to 0.\end{aligned}$$

The inequality in (*) is due to the Cauchy–Schwarz inequality and the one in (**) is due to the fact that, for every $\lambda \in \Lambda$, it holds that $(1-u_\lambda)^2 \leq 1 - u_\lambda$.

Let $a, b \in A$ and let us show that $\tau'(ab) = \tau'(ba)$.

$$\begin{aligned}\tau'(ab) &= \lim_\lambda \tau(abu_\lambda) = \lim_\lambda \tau(u_\lambda^{1/2}abu_\lambda^{1/2}) = \lim_\lambda \tau(bu_\lambda a)\\ &= \lim_\lambda \tau(bu_\lambda a u_\lambda) = \lim_\lambda \tau(au_\lambda bu_\lambda) = \tau'(ba).\end{aligned}$$

□

Definition 3.2. Let A be a C*-algebra, G a group and $(A, G, \{I_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ a partial dynamical system. We say a linear functional τ on A is *G-invariant* if, for every $g \in G$ and $a \in I_g$,

$$\tau(a) = \tau(\theta_{g^{-1}}(a)).$$

LEMMA 3.3. *Let A be a C*-algebra, G a group and $(A, G, \{I_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ a partial dynamical system. Let ψ be the canonical conditional expectation from $A \rtimes G$ onto A as in (5). If φ is a tracial state on A, then $\varphi\circ\psi$ is a tracial state on $A \rtimes_\theta G$ if and only if φ is a G-invariant.*

Proof. Assume φ is G-invariant. Let us show that $\varphi\circ\psi$ is a tracial state. By the fact that ψ is positive and by (5), it follows that $\varphi\circ\psi$ is a state.

By linearity and continuity, in order to conclude that $\varphi\circ\psi$ is tracial, we only need to prove that, given $a_g\delta_g, b_h\delta_h \in A \rtimes_\theta G$,

$$\varphi\circ\psi((a_g\delta_g)(b_h\delta_h)) = \varphi\circ\psi((b_h\delta_h)(a_g\delta_g)).$$

This is true, since

$$\varphi\circ\psi(a_g\delta_g b_h\delta_h) = \varphi\circ\psi(\theta_g(\theta_{g^{-1}}(a_g)b_h)\delta_{gh}) = \begin{cases}0 & \text{if } h \neq g^{-1},\\ \varphi(a_g\theta_g(b_{g^{-1}})) & \text{if } h = g^{-1}.\end{cases}$$

By using the fact that φ is a G-invariant tracial state, one concludes that $\varphi\circ\psi(a_g\delta_g b_h\delta_h) = \varphi\circ\psi(b_h\delta_h a_g\delta_g)$. Hence, $\varphi\circ\psi$ is a tracial state.

Conversely, assume that $\varphi\circ\psi$ is a tracial state and let us show that φ is G-invariant.

Given $g \in G$, $a \in I_g$ and $(u_\lambda)_{\lambda\in\Lambda}$ an approximate unit for $I_{g^{-1}}$,

$$\varphi(a) = \varphi \circ \psi(a\delta_e) = \lim_\lambda \varphi \circ \psi(a\delta_g u_\lambda \delta_{g^{-1}}) = \lim_\lambda \varphi \circ \psi(u_\lambda \delta_{g^{-1}} a \delta_g)$$
$$= \varphi \circ \psi(\theta_{g^{-1}}(a)\delta_e) = \varphi(\theta_{g^{-1}}(a)).$$

Hence, φ is G-invariant. □

Just as in the case of global actions, the following proposition holds for partial actions.

PROPOSITION 3.4. *Let θ be a partial action of a group G on a compact Hausdorff space X. Then X admits an invariant probability measure if and only if $C(X) \rtimes_\theta G$ has a tracial state.*

Proof. Suppose X has an invariant probability measure μ. Consider the state on $C(X)$ given by integration with respect to μ. By invariance of μ, it follows that the state is also G-invariant. By Lemma 3.3, we conclude that $C(X) \rtimes_\theta G$ has a tracial state.

Now assume that $C(X) \rtimes_\theta G$ has a tracial state τ. Let $i : C(X) \to C(X) \rtimes_\theta G$ be the canonical embedding and let $\psi : C(X) \rtimes_\theta G \to C(X)$ be the canonical conditional expectation. We claim that $\tau \circ i$ is G-invariant. By Lemma 3.3, we only need to check that $\tau \circ i \circ \psi$ is a tracial state. Clearly, it is a state.

Given $a_g\delta_g, b_h\delta_h \in C(X) \rtimes G$, we have that $\psi(a_g\delta_g b_h\delta_h) = \psi(b_h\delta_h a_g\delta_g) = 0$ if $h \neq g^{-1}$. If $h = g^{-1}$, then $\tau \circ i \circ \psi(a_g\delta_g b_h\delta_h) = \tau(a_g\delta_g b_h\delta_h) = \tau(b_h\delta_h a_g\delta_g) = \tau \circ i \circ \psi(b_h\delta_h a_g\delta_g)$.

By linearity, we conclude that $\tau \circ i \circ \psi$ is a tracial state. Therefore, $\tau \circ i$ is a G-invariant state. Let μ be the regular probability measure on X associated with $\tau \circ i$. Since this state is G-invariant, we conclude, just as in the proof of Proposition 2.7, that μ is invariant. □

In the following, we denote by $T(A)$ the set of tracial states of a C*-algebra A.

LEMMA 3.5. *Let A be a unital C*-algebra such that $T(A) \neq \emptyset$. Then every extreme point τ of $T(A)$ satisfies the fact that, for every ideal I of A, $\|\tau|_I\|$ is either zero or one.*

Proof. Let τ be an extreme point of $T(A)$. Given an ideal $I \trianglelefteq A$, suppose that $t := \|\tau|_I\| \in (0, 1)$. By Lemma 3.1, there exists an extension τ_I of $\tau|_I$ to all of A such that $\|\tau_I\| = t$.

Let us show that $\tau_I \leq \tau$. Let $(u_\lambda)_{\lambda\in\Lambda}$ an approximate unit for I. Given $a \in A^+$, we know, by [**9**, Theorem 3.3.9], that $\tau_I(a) = \lim_\lambda \tau(u_\lambda a u_\lambda)$. Since τ is a tracial state, for any $\lambda \in \Lambda$,

$$\tau(u_\lambda a u_\lambda) = \tau(a^{1/2} u_\lambda u_\lambda a^{1/2}) \leq \|(u_\lambda)^2\| \tau(a) \leq \tau(a).$$

Therefore, $\tau_I(a) \leq \tau(a)$. Since a was arbitrary, it follows that $\tau_I \leq \tau$.

By evaluating on 1_A, we get that $\|\tau - \tau_I\| = 1 - t$. Hence,

$$\tau = t\left(\frac{\tau_I}{\|\tau_I\|}\right) + (1-t)\left(\frac{\tau - \tau_I}{\|\tau - \tau_I\|}\right).$$

Since τ is an extreme point, $\tau = \tau_I/\|\tau_I\|$. But $\tau|_I = \tau_I|_I \neq 0$. Thus we get a contradiction. □

The reader will notice that the idea of the proof of the next result is the same as that of Proposition 2.7.

THEOREM 3.6. *A group is supramenable if and only if, whenever it partially acts on a unital* C*-*algebra which has a tracial state, the associated partial crossed product also has a tracial state.*

Proof. Let G be a supramenable group, A a unital C*-algebra which has a tracial state and let $(A, G, \{I_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial dynamical system.

By Lemma 3.5, A has a tracial state τ such that, for every ideal I of A, $\|\tau|_I\|$ is either zero or one.

For each $g \in G$, let τ_g be the extension of $\tau \circ \theta_{g^{-1}}$ to A, as in Lemma 3.1. For each $a \in A$, let $\hat{a} \in \ell^\infty(G)$ be defined by $\hat{a}(g) := \tau_g(a)$, $g \in G$. Since each τ_g is a trace, we have that, for any $a, b \in A$, $\widehat{ab} = \widehat{ba}$.

Notice that for each $g \in G$, $\widehat{1_A}(g) = \|\tau_g\| = \|\tau|_{I_{g^{-1}}}\|$.

Since G is supramenable, there is an invariant, finitely additive measure μ on G such that $1 = \mu(\{g \in G : \|\tau|_{I_{g^{-1}}}\| = 1\})$.

In the notation of Lemma 2.6, define

$$\varphi : A \to \mathbb{C},$$
$$a \mapsto I_\mu(\hat{a}).$$

Notice that φ is a tracial state on A.

Let ψ be the canonical conditional expectation from $A \rtimes G$ onto A as in (5). Define $\rho := \varphi \circ \psi$. By Lemma 3.3, in order to show that ρ is a tracial state, it is sufficient to show that φ is G-invariant.

Given $x, y \in G$, let us prove that

$$\|\tau_x|_{I_y}\| = \|\tau_x|_{I_y \cap I_x}\|.$$

Let (u_λ^y) and (u_λ^x) be approximate units for I_y and I_x, respectively. Then

$$\|\tau_x|_{I_y}\| = \lim_{\lambda_1} \tau_x(u_{\lambda_1}^y) = \lim_{\lambda_1} \lim_{\lambda_2} \tau_x(u_{\lambda_1}^y u_{\lambda_2}^x).$$

Since, for each λ_1, λ_2, we have that $u_{\lambda_1}^y u_{\lambda_2}^x \in I_x \cap I_y$, it follows that $\|\tau_x|_{I_y}\| \leq \|\tau_x|_{I_y \cap I_x}\|$. The opposite inequality is evident.

For each $g, t \in G$, one can easily check that $(\tau_t \circ \theta_{g^{-1}})|_{I_g \cap I_{gt}} = \tau_{gt}|_{I_g \cap I_{gt}}$. Therefore,

$$\begin{aligned}\|\tau_t \circ \theta_{g^{-1}}\| &= \|\tau_t|_{I_{g^{-1}}}\| = \|\tau_t|_{I_{g^{-1}} \cap I_t}\| = \|\tau_t \circ \theta_{g^{-1}} \circ \theta_g|_{I_{g^{-1}} \cap I_t}\| \\ &= \|\tau_t \circ \theta_{g^{-1}}|_{I_g \cap I_{gt}}\| = \|\tau_{gt}|_{I_g \cap I_{gt}}\| = \|\tau_{gt}|_{I_g}\|.\end{aligned}$$

It follows, by Lemma 3.1, that $\tau_t \circ \theta_{g^{-1}} = \tau_{gt}|_{I_g}$. Hence, given $a \in I_g$,

$$g^{-1} \cdot \hat{a}(t) = \hat{a}(gt) = \widehat{\theta_{g^{-1}}(a)}(t).$$

Therefore,

$$g^{-1} \cdot \hat{a} = \widehat{\theta_{g^{-1}}(a)}.$$

Thus, for every $g \in G$ and $a \in I_g$,

$$\varphi(a) = I_\mu(\hat{a}) \overset{(*)}{=} I_\mu(g^{-1}\hat{a}) = I_\mu(\widehat{\theta_{g^{-1}}(a)}) = \varphi(\theta_{g^{-1}}(a)).$$

The equality (*) is due to the fact that μ is invariant. Hence, φ is G-invariant and ρ is a tracial state on $A \rtimes G$.

The converse follows from Propositions 3.4 and 2.7. □

4. *Decomposition of partial crossed products*

In this section, we investigate conditions under which one can decompose a partial crossed product into two iterated partial crossed products by simpler groups.

Let A be a C*-algebra, G a group and let $(A, G, \{I_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial dynamical system. Given a G-invariant ideal I of A (i.e. $\theta_g(I \cap I_{g^{-1}}) \subset I$ for every $g \in G$), one can consider the restricted partial dynamical system $(I, G, \{I \cap I_g\}_{g\in G}, \{\theta_g|_{I\cap I_{g^{-1}}}\}_{g\in G})$. It is a consequence of [**3**, Theorem 22.9] that the map

$$i : I \rtimes G \to A \rtimes G,$$
$$a\delta_g \mapsto a\delta_g$$

is an embedding onto an ideal of $A \rtimes G$. In the statement of the following lemma, we use this identification.

LEMMA 4.1. *Let $(A, G, \{I_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial dynamical system and let I and J be G-invariant ideals of A. Then $(I \rtimes G) \cap (J \rtimes G) = (I \cap J) \rtimes G$.*

Proof. Take $x \in (I \rtimes G) \cap (J \rtimes G)$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for J. Then $x = \lim_\lambda x(u_\lambda\delta_e)$. Since $x \in I \rtimes G$, it follows that x can be approximated by finite sums $\sum a_g\delta_g$ such that each $a_g \in I$. Due to G-invariance of I and J, if $a_g \in I$, then $a_g\delta_g u_\lambda\delta_e \in (I \cap J) \rtimes G$. Hence, for every $\lambda \in \Lambda$, $x(u_\lambda\delta_e) \in (I \cap J) \rtimes G$. From this, we conclude that $x \in (I \cap J) \rtimes G$.

The opposite inclusion is evident. □

PROPOSITION 4.2. *Let A be a C^*-algebra, $G \rtimes_\alpha H$ a semidirect product of groups and*

$$(A, G \rtimes_\alpha H, \{I_{(g,h)}\}_{(g,h)\in G\rtimes_\alpha H}, \{\theta_{(g,h)}\}_{(g,h)\in G\rtimes_\alpha H})$$

a partial dynamical system such that, for every $g \in G$ and $h \in H$,

$$I_{(g,h)} \subset I_{(g,e)} \cap I_{(e,h)}. \tag{7}$$

Consider the partial dynamical system

$$(A, G, \{I_{(g,e)}\}_{g\in G}, \{\theta_{(g,e)}\}_{g\in G})$$

obtained by restricting the partial action θ to the subgroup $G \times \{e\}$. There is a partial action of H on $A \rtimes G$ and a $*$-isomorphism

$$\varphi : A \rtimes (G \rtimes_\alpha H) \to (A \rtimes G) \rtimes H,$$
$$a_{(g,h)}\delta_{(g,h)} \mapsto (a_{(g,h)}\delta_g)\delta_h.$$

Proof. Notice that condition (7) and [**3**, Proposition 2.6] imply that, for each $h \in H$, $I_{(e,h)}$ is a G-invariant ideal for the partial dynamical system

$$(A, G, \{I_{(g,e)}\}_{g\in G}, \{\theta_{(g,e)}\}_{g\in G}).$$

Also, for each $g \in G$ and $h \in H$, it is true that $\theta_{(e,h)}(I_{(e,h^{-1})} \cap I_{(g,e)}) \subset I_{(\alpha_h(g),e)}$. For every $h \in H$, define

$$\beta_h : I_{(e,h^{-1})} \rtimes G \to I_{(e,h)} \rtimes G,$$
$$a_g\delta_g \mapsto \theta_{(e,h)}(a_g)\delta_{\alpha_h(g)}.$$

Let us check that β_h is a well-defined $*$-homomorphism. Given $g, l \in G$, $a_g \in I_{(e,h^{-1})} \cap I_{(g,e)}$ and $a_l \in I_{(e,h^{-1})} \cap I_{(l,e)}$,

$$\begin{aligned}
\beta_h((a_g\delta_g)(a_l\delta_l)) &= \beta_h(\theta_{(g,e)}(\theta_{(g^{-1},e)}(a_g)a_l)\delta_{gl}) \\
&= \theta_{(e,h)}(\theta_{(g,e)}(\theta_{(g^{-1},e)}(a_g)a_l))\delta_{\alpha_h(gl)} \\
&= \theta_{(\alpha_h(g),e)}(\theta_{(e,h)}(\theta_{(g^{-1},e)}(a_g)a_l))\delta_{\alpha_h(gl)} \\
&= \theta_{(\alpha_h(g),e)}(\theta_{(\alpha_h(g^{-1}),e)}(\theta_{(e,h)}(a_g))\theta_{(e,h)}(a_l))\delta_{\alpha_h(gl)} \\
&= (\theta_{(e,h)}(a_g)\delta_{\alpha_h(g)})(\theta_{(e,h)}(a_l)\delta_{\alpha_h(l)}) \\
&= \beta_h(a_g\delta_g)\beta_h(a_l\delta_l).
\end{aligned}$$

We leave it to the reader to check that, for every $g \in G$ and $a_g \in I_{(e,h^{-1})} \cap I_{(g,e)}$, it holds that $\beta_h((a_g\delta_g)^*) = \beta_h(a_g\delta_g)^*$. Therefore, β_h is a well-defined $*$-homomorphism. Also $\beta_{h^{-1}} = (\beta_h)^{-1}$ for every $h \in H$.

We claim that

$$(A \rtimes G, H, \{I_{(e,h)} \rtimes G\}_{h\in H}, \{\beta_h\}_{h\in H})$$

is a partial dynamical system.

Obviously, condition (i) of Definition 2.8 holds, so we just need to check that (ii) also holds.

Given $h_1, h_2 \in H$ and $x \in I_{(e,h_1^{-1})} \rtimes G$, suppose that $\beta_{h_1}(x) \in I_{(e,h_2^{-1})} \rtimes G$. Let us show that $x \in I_{(e,(h_2h_1)^{-1})} \rtimes G$.

Since the image of β_{h_1} is $I_{(e,h_1)} \rtimes G$,

$$\beta_{h_1}(x) \in (I_{(e,h_2^{-1})} \rtimes G) \cap (I_{(e,h_1)} \rtimes G) = (I_{(e,h_2^{-1})} \cap I_{(e,h_1)}) \rtimes G.$$

Since $\theta_{(e,h_1^{-1})}(I_{(e,h_2^{-1})} \cap I_{(e,h_1)}) \subset I_{(e,(h_2h_1)^{-1})}$, we conclude, by using the definition of $\beta_{h_1^{-1}}$, that

$$\beta_{h_1^{-1}}((I_{(e,h_2^{-1})} \cap I_{(e,h_1)}) \rtimes G) \subset I_{(e,(h_2h_1)^{-1})} \rtimes G.$$

Hence, $x \in I_{(e,(h_2h_1)^{-1})} \rtimes G$.

Now let us show that

$$\beta_{h_2h_1}(x) = \beta_{h_2} \circ \beta_{h_1}(x). \tag{8}$$

Since $x \in (I_{(e,(h_2h_1)^{-1})} \cap I_{(e,h_1^{-1})}) \rtimes G$, by using approximation arguments, we can assume that $x = a\delta_g$ for some $g \in G$ and $a \in I_{(e,(h_2h_1)^{-1})} \cap I_{(e,h_1^{-1})}$. Since $\theta_{(e,h_2h_1)}(a) = \theta_{(e,h_2)} \circ \theta_{(e,h_1)}(a)$, we conclude that (8) holds.

Hence, we have proven that

$$(A \rtimes G, H, \{I_{(e,h)} \rtimes G\}_{h\in H}, \{\beta_h\}_{h\in H})$$

is a partial dynamical system.

Define

$$\varphi : A \rtimes (G \rtimes_\alpha H) \to (A \rtimes G) \rtimes H,$$
$$a_{(g,h)}\delta_{(g,h)} \mapsto (a_{(g,h)}\delta_g)\delta_h.$$

For every $h \in H$, define

$$\psi_h : I_{(e,h)} \rtimes G \to A \rtimes (G \rtimes H),$$
$$a_g\delta_g \mapsto a_g\delta_{(g,e)}.$$

It is a straightforward computation to check that φ and, for every $h \in H$, ψ_h are well-defined $*$-homomorphisms.

For every $h \in H$, let $(u^h_\lambda)_{\lambda\in\Lambda_h}$ be an approximate unit for $I_{(e,h)}$. Given $g \in G$, $h \in H$ and $a \in I_{(e,h)} \cap I_{(g,e)}$, notice that

$$\lim_\lambda (a\delta_{(g,e)})(u^h_\lambda\delta_{(e,h)}) = \lim_\lambda \theta_{(g,e)}(\theta_{(g^{-1},e)}(a)u^h_\lambda)\delta_{(g,h)} = a\delta_{(g,h)}, \tag{9}$$

since $\theta_{(g^{-1},e)}(a) \in I_{(g^{-1},h)} \subset I_{(e,h)}$.

Define

$$\psi : (A \rtimes G) \rtimes H \to A \rtimes (G \rtimes H),$$
$$f_h\delta_h \mapsto \lim_\lambda (\psi_h(f_h)(u^h_\lambda\delta_{(e,h)})).$$

In order to see that the above limit is well defined, notice that, due to (9), convergence is ensured for $f_h = a_g\delta_g$. Then, one observes that $(\psi_h(f)(u^h_\lambda\delta_{(e,h)}))$ is a Cauchy net for any $f \in I_{(e,h)} \rtimes G$.

For the proof that ψ is a well-defined $*$-homomorphism, it is worthwhile remarking that, for every $h \in H$, $\psi|_{(I_{(e,h)} \rtimes G)\delta_h}$ is a contractive linear map.

Notice that, given $g \in G$, $h \in H$ and $a \in I_{(e,h)} \cap I_{(g,e)}$, it holds that $\psi((a\delta_g))\delta_h) = a\delta_{(g,h)}$. Hence, ψ and φ are inverses of each other. □

The purpose of the next proposition is to show that condition (7) is stronger than it seems at first sight. We give a direct proof of it, even though it can also be obtained as a corollary of the proof of the preceding proposition.

PROPOSITION 4.3. *Let A be a* C*-*algebra, $G \rtimes_\alpha H$ a semidirect product of groups and*

$$(A, G \rtimes_\alpha H, \{I_{(g,h)}\}_{(g,h)\in G\rtimes_\alpha H}, \{\theta_{(g,h)}\}_{(g,h)\in G\rtimes_\alpha H})$$

a partial dynamical system such that, for every $g \in G$ and $h \in H$, (7) is satisfied. Then, for every $g \in G$ and $h \in H$,

$$I_{(g,h)} = I_{(g,e)} \cap I_{(e,h)}.$$

Proof. Given $g \in G$ and $h \in H$,

$$\begin{aligned} I_{(g,e)} \cap I_{(e,h)} &= \theta_{(g,e)} \circ \theta_{(g^{-1},e)}(I_{(g,e)} \cap I_{(e,h)}) \\ &= \theta_{(g,e)}(I_{(g^{-1},e)} \cap I_{(g^{-1},h)}) \\ &\subset \theta_{(g,e)}(I_{(g^{-1},e)} \cap I_{(e,h)}) \\ &= I_{(g,e)} \cap I_{(g,h)} \\ &= I_{(g,h)}. \end{aligned}$$ □

Next, we give an example to which Proposition 4.2 can be applied.

Example 4.4. Let G and H be groups and $j : G \times H \to \beta(G \times H)$ be the embedding of $G \times H$ into its Stone–Čech compactification. Given $A \subset G$ and $B \subset H$ non-empty subsets, consider the partial action $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ of $G \times H$ on $\overline{j(A \times B)}$ obtained by restricting to $\overline{j(A \times B)}$ the canonical action of $G \times H$ on $\beta(G \times H)$, as in Example 2.2. By definition of the restriction, given $(g, h) \in G \times H$,

$$\begin{aligned} D_{(g,h)} = \overline{j((A \cap gA) \times (B \cap hB))} &= \overline{j(A \times (B \cap hB))} \cap \overline{j((A \cap gA) \times B)} \\ &= D_{(g,e)} \cap D_{(e,h)}. \end{aligned}$$

From this, it follows easily that the induced partial action on $C(\overline{j(A \times B)})$ satisfies (7).

Notice that, if G and H are supramenable groups, we can combine the previous example with Proposition 4.2 and Theorem 3.6 to show that $A \times B \subset G \times H$ is non-paradoxical for every $A \subset G$ and $B \subset H$ non-empty subsets. We remark, however, that this fact also follows easily from the results of [**11**].

As mentioned in the introduction, the class of supramenable groups is not closed under taking semidirect products. For example, let $(x_n)_{n\in\mathbb{Z}} \in \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ be such that $x_0 = 1$ and $x_n = 0$ for $n \neq 0$. Also, let $e \in \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ be the neutral element. Now consider the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} = (\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$. It is well known and easy to check that $((x_n)_{n\in\mathbb{Z}}, 1), (e, 1) \in G$ generate a free monoid SF_2. Since $SF_2 \subset \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is obviously paradoxical, it follows that $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is not supramenable.

Let $G \rtimes H$ be a non-supramenable group which is the semidirect product of supramenable groups. By Theorem 3.6, there is a partial action of $G \rtimes H$ on a unital C*-algebra with a tracial state such that the associated partial crossed product does not admit a tracial state. By applying Theorem 3.6 twice, we conclude that a decomposition such as in Proposition 4.2 cannot hold in this case.

Next, we are going to provide another condition which allows a decomposition as in (1). For the proof, we will use the concepts of Fell bundle, graded C*-algebra, cross sectional C*-algebra of a Fell bundle and reduced partial crossed product. We refer the reader to [**3**] for the appropriate definitions.

Given a partial action θ of a group G on a C*-algebra A and H a subgroup of G, one can consider the partial action of H on A obtained by restricting θ to H. We shall denote this restricted partial action also by θ.

LEMMA 4.5. *Let $(A, G, \{I_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial dynamical system and H a subgroup of G. Then $A \rtimes_\theta H$ embeds naturally into $A \rtimes_\theta G$ and there exists a conditional expectation $\psi_H : A \rtimes_\theta G \to A \rtimes_\theta H$ such that, for every $g \in G$ and $a \in I_g$,*

$$\psi_H(a\delta_g) = \begin{cases} a\delta_g & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases} \tag{10}$$

Proof. The fact that $A \rtimes_\theta H$ embeds naturally into $A \rtimes_\theta G$ was proven in [**1**, Corollary 6.3].

Let us now prove the existence of ψ_H. By [**1**, Proposition 6.1], the reduced partial crossed product $A \rtimes_{\theta,r} H$ embeds naturally into $A \rtimes_{\theta,r} G$ and there exists a conditional expectation $\varphi_H : A \rtimes_{\theta,r} G \to A \rtimes_{\theta,r} H$ satisfying the same condition as in (10).

Let $C^*_r(G)$ and $C^*_r(H)$ denote the reduced group C*-algebras of G and H, respectively, and $E_H : C^*_r(G) \to C^*_r(H)$ the conditional expectation satisfying the same condition as in (10).

Let

$$i_G : A \rtimes_\theta G \to (A \rtimes_{\theta,r} G) \bigotimes_{\max} C^*_r(G)$$

and

$$i_H : A \rtimes_\theta H \to (A \rtimes_{\theta,r} H) \bigotimes_{\max} C^*_r(H)$$

be the injective $*$-homomorphisms given by [**3**, Proposition 18.9] such that $i_G(a_g\delta_g) = a_g\delta_g \otimes \delta_g$ and $i_H(a_h\delta_h) = a_h\delta_h \otimes \delta_h$ for every $g \in G$, $a_g \in I_g$, $h \in H$ and $a_h \in I_h$.

Then $\psi_H := (i_H)^{-1} \circ (\varphi_H \otimes E_H) \circ i_G$ is the desired conditional expectation. □

THEOREM 4.6. *Let A be a separable and stable C*-algebra, G a countable group, N a normal subgroup of G and $(A, G, \{I_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ a partial dynamical system. Then there is a partial action β of G/N on $A \rtimes_{\tilde\theta} N$ such that*

$$A \rtimes_\theta G \cong (A \rtimes_\theta N) \rtimes_\beta \frac{G}{N}.$$

Proof. For each $gN \in G/N$, define

$$B_{gN} := \overline{\text{span}}\{b_h\delta_h : h \in gN, b_h \in I_h\}.$$

We want to show that $(A \rtimes_\theta G, \{B_{gN}\}_{gN\in G/N})$ is a G/N-graded C*-algebra.

Clearly, for every $g, g' \in G$, we have that $B_{gN}B_{g'N} \subset B_{gg'N}$ and $(B_{gN})^* = B_{g^{-1}N}$. Notice that, by Lemma 4.5, $B_{eN} \cong A \rtimes_\theta N$ and there exists a conditional expectation $\psi_N : A \rtimes_\theta G \to B_{eN}$ with the same property as in (10).

By [**3**, Theorem 19.1], it follows that

$$(A \rtimes_\theta G, \{B_{gN}\}_{gN\in G/N})$$

is a G/N-graded C*-algebra. Let $\mathcal{B}$ be the Fell bundle associated with this grading. There is a surjective $*$-homomorphism from the cross sectional C*-algebra $C^*(\mathcal{B})$ into $A \rtimes_\theta G$ which is the identity on each B_{gN}.

On the other hand, $C^*(\mathcal{B})$ can also be seen as a G-graded C^*-algebra, and the Fell bundle over G associated with this grading is isomorphic to the Fell bundle of $A \rtimes_\theta G$. From these considerations, we conclude that $A \rtimes_\theta G$ is isomorphic to $C^*(\mathcal{B})$.

Because of [**5**, Proposition 4.4], it holds that $B_e N$ is stable.

Due to [**3**, Theorem 27.11], it follows that there is a partial action β of G/N on $A \rtimes_\theta N$ such that

$$(A \rtimes_\theta N) \rtimes_\beta \frac{G}{N} \cong C^*(\mathcal{B}) \cong A \rtimes_\theta G. \qquad \square$$

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CHARACTERIZATIONS OF LOCALLY FINITE ACTIONS OF GROUPS ON SETS

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Abstract. We show that an action of a group on a set X is locally finite if and only if X is not equidecomposable with a proper subset of itself. As a consequence, a group is locally finite if and only if its uniform Roe algebra is finite.

1. Introduction

Given a group acting on a set X, a property that has been well-studied is the existence of an invariant mean on X, that is, amenability of the action (see [1] for historical remarks). By Tarski's Theorem ([6, Corollary 9.2]), this is equivalent to X not being equidecomposable with two disjoint subsets of itself.

In this note, we address the following question: given an action of a group G on a set X, when is X not equidecomposable with a proper subset of itself? We show that this holds if and only if the action is locally finite (Definition 2.2), if and only if $\ell^\infty(X) \rtimes_r G$ is a finite C*-algebra (Theorem 2.3). It follows from this that a group is locally finite if and only if its uniform Roe algebra ($\ell^\infty(G) \rtimes_{\mathrm{r}} G$) is finite (Proposition 2.5). In [3], it was shown that $\ell^\infty(G) \rtimes_{\mathrm{r}} G$ is finite if G is locally finite and asked if the converse holds.

It was already known that amenability of a group G is equivalent to $\ell^\infty(G) \rtimes_{\mathrm{r}} G$ not being properly infinite, and supramenability is equivalent to $\ell^\infty(G) \rtimes_{\mathrm{r}} G$ not containing any properly infinite projections ([3, Proposition 5.3]). Therefore, Proposition 2.5 completes the dictionary between equidecomposition properties of groups and the type of projections in the uniform Roe algebra.

2. Characterizations of locally finite actions of groups on sets

We start by recalling some definitions:

Definition 2.1. Let be G be a group acting on a set X. Two subsets A and B of X are said to be *equidecomposable* if there are finite partitions $\{A_i\}_{i=i}^n$ and $\{B_i\}_{i=i}^n$ of A and B, respectively, and elements $s_1, \dots, s_n \in G$ such that $B_i = s_i A_i$ for $1 \leq i \leq n$. When we say that two subsets of G are equidecomposable, it is with respect to the left action of G on itself.

The next definition has already been introduced in [5] for actions on semilattices.

Definition 2.2. An action of a group G on a set X is said to be *locally finite* if, for every finitely generated subgroup H of G and every $x \in X$, the H-orbit of x is finite.

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The left action of a group on itself is locally finite if and only if the group is locally finite.

The following result shows that the notion of locally finite action is a natural strengthening of the notion of amenable action on a set.

Theorem 2.3. *Let G be a group acting on a set X. The following conditions are equivalent:*

(1) *The action is locally finite;*
(2) *$\ell^\infty(X) \rtimes_r G$ is finite;*
(3) *X is not equidecomposable with a proper subset of itself;*
(4) *No subset of X is equidecomposable with a proper subset of itself.*

Proof. (1) ⇒ (2). Since the inductive limit of finite unital C*-algebras with unital conneting maps is finite, it suffices to show that $\ell^\infty(X) \rtimes_{\mathrm{r}} H$ is finite for every finitely generated subgroup H of G. Let H be such a subgroup and $X = \sqcup_{i\in I} X_i$ be the partition of X into its H-orbits.

For every $i \in I$, the restriction map $\ell^\infty(X) \to \ell^\infty(X_i)$ is H-equivariant. Therefore, there is a homomorphism $\psi\colon \ell^\infty(X) \rtimes_{\mathrm{r}} H \to \prod(\ell^\infty(X_i) \rtimes_{\mathrm{r}} H)$. We claim that ψ is injective.

Let $\varphi\colon \ell^\infty(X) \rtimes_{\mathrm{r}} H \to \ell^\infty(X)$ and, for every $i \in I$, $\varphi_i\colon \ell^\infty(X_i) \rtimes_{\mathrm{r}} H \to \ell^\infty(X_i)$ be the canonical conditional expectations. Also let $\varphi_I\colon \prod(\ell^\infty(X_i) \rtimes_{\mathrm{r}} H) \to \prod \ell^\infty(X_i)$ be the product of the maps φ_i, and $T\colon \ell^\infty(X) \to \prod \ell^\infty(X_i)$ be the isomorphism which arises from the product of the restriction maps. The following diagram commutes:

$$\begin{array}{ccc}
\ell^\infty(X) \rtimes_{\mathrm{r}} H & \xrightarrow{\ \psi\ } & \prod(\ell^\infty(X_i) \rtimes_{\mathrm{r}} H) \\
{\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_I} \\
\ell^\infty(X) & \xrightarrow[\ T\ ]{} & \prod \ell^\infty(X_i).
\end{array}$$

Since φ is faithful, we conclude that ψ is injective. Since the product of finite unital C*-algebras is finite, it suffices to show that $\ell^\infty(X_i) \rtimes_{\mathrm{r}} H$ is finite for every $i \in I$ in order to conclude that $\ell^\infty(X) \rtimes_{\mathrm{r}} H$ is finite.

Given $i \in I$, let τ_i be the tracial state on $\ell^\infty(X_i)$ which arises from the uniform probability measure on the finite set X_i. Since τ_i is H-invariant and faithful, it follows that the map $\tau_i \circ \varphi_i\colon \ell^\infty(X_i) \rtimes_{\mathrm{r}} H \to \mathbb{C}$ is a faithful tracial state. Therefore, $\ell^\infty(X_i) \rtimes_{\mathrm{r}} H$ is finite.

(2) ⇒ (3). This follows from the fact that, if A and B are equidecomposable subsets of X, then the projections 1_A and 1_B are equivalent in $\ell^\infty(X) \rtimes_{\mathrm{r}} G$.

(3) ⇒ (4). If $A \subset X$ is equidecomposable with $B \subsetneq A$, then $X = A \sqcup A^c$ is equidecomposable with $B \sqcup A^c \subsetneq X$.

(4) ⇒ (1). Suppose that there is $H < G$ generated by a finite and symmetric set S and $x \in X$ such that Hx is infinite. Then there exists a sequence $(s_n)_{n\in\mathbb{N}} \subset S$ such that

$$\forall n, m \in \mathbb{N}\colon n \neq m \Rightarrow s_n \cdots s_1 x \neq s_m \cdots s_1 x.$$

The sequence $(s_n \cdots s_1 x)_{n\in\mathbb{N}}$ can be seen as an infinite simple path in the graph whose vertex set is Hx and whose edges come from S.

We claim that $\gamma := \{s_n \cdots s_1 x\colon n \in \mathbb{N}\}$ is equidecomposable with $\gamma \setminus \{s_1 x\}$.

Given $s \in S$, let $A_s := \{s_n \cdots s_1 x \colon s_{n+1} = s\}$. It is easy to check that $\{A_s\}_{s\in S}$ partitions γ and $\{sA_s\}_{s\in S}$ partitions $\gamma \setminus \{s_1 x\}$. Hence, γ is equidecomposable with its proper subset $\gamma \setminus \{s_1 x\}$.

□

We now proceed to give a characterization of locally finite groups which can be seen as an analogy to parts of [3, Theorem 1.1].

The next definition is from [4].

Definition 2.4. Let H and G be groups. A map $f\colon H \to G$ is said to be a *uniform embedding* if, for every finite set $S \subset H$, there is a finite set $T \subset G$ such that:

$$\forall x, y \in H\colon xy^{-1} \in S \implies f(x)f(y)^{-1} \in T$$

and, for every finite set $T \subset G$, there is $S \subset H$ finite such that

$$\forall x, y \in H\colon f(x)f(y)^{-1} \in T \implies xy^{-1} \in S.$$

The implication (1) ⇒ (2) in the next result had already been observed in [3, Remark 5.4], and (5) ⇒ (1) is an immediate consequence of [8, Lemma 1].

Proposition 2.5. *Let G be a group. The following conditions are equivalent:*

(1) *G is locally finite;*
(2) *The uniform Roe algebra $\ell^\infty(G) \rtimes_{\mathrm{r}} G$ is finite;*
(3) *G is not equidecomposable with a proper subset of itself;*
(4) *No subset $A \subset G$ is equidecomposable with a proper subset of itself;*
(5) *There is no injective uniform embedding from $\mathbb{Z}$ into G.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) (and (4) ⇒ (1)) are a consequence of Theorem 2.3.

(4) ⇒ (5). This follows from the fact that $\mathbb{N} \subset \mathbb{Z}$ is equidecomposable with a proper subset of itself and [3, Lemma 3.2].

(5) ⇒ (1). This is a consequence of [8, Lemma 1]. □

Remark 2.6. After this note was made available on arXiv, we became aware of [7], where it is shown that if a group is infinite and finitely generated, then its uniform Roe algebra is infinite.

Any locally finite group acts on itself in a transitive, faithful and locally finite way. If a finitely generated group admits a faithful, transitive, locally finite action on a set, then the group is finite. This is in stark contrast to the fact that there are finitely generated, non-amenable groups which admit faithful, transitive, amenable actions on sets (see [1] for various examples).

A finitely generated group admits a faithful, locally finite action if and only if it is residually finite.

Proposition 2.7. *If a group admits a faithful, locally finite action, then it embeds into a group which admits a faithful, locally finite and transitive action.*

Proof. Let G be a group which acts on a set X in a faithful and locally finite way.

Take a set $Y \subset X$ of representatives of all G-orbits, and let S_Y be the group of finitely supported permutations of Y. Consider the natural action of S_Y on X and the associated action of $H := G * S_Y$ on X. This action is transitive and locally finite. By taking the quotient of H by the kernel of this action, we get a faithful, transitive, locally finite action on X by a group which contains G.

□

In analogy to what is known for amenable actions ([2, Lemma 3.2]), the following holds for locally finite actions:

Proposition 2.8. *Let G be a group acting on a set X in a locally finite way. If, for every $x \in X$, the stabilizer subgroup G_x is locally finite, then G is locally finite.*

Proof. Take $H < G$ finitely generated and $x \in X$. Since the action is locally finite, it follows that Hx is finite. Hence, there is H_0 a subgroup of finite index in H such that $H_0 < G_x$. In particular, H_0 is locally finite. Therefore, H is also locally finite. Since H is finitely generated, we conclude that it is finite. □

Remark 2.9. One can define in a natural way an action of a group on a set X to be supramenable if no subset of X is equidecomposable with two disjoint proper subsets of itself. It is not true that if the action of a group G is supramenable, and all the stabilizer subgroups are supramenable, then G is supramenable.

Indeed, it is well-known that the class of supramenable groups is not closed by taking extensions (the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is such an example). Let then G be a non-supramenable group which contains a supramenable normal subgroup N such that $\frac{G}{N}$ is also supramenable.

Consider the left action of G on $\frac{G}{N}$. Since $\frac{G}{N}$ is supramenable, it follows easily that this action is supramenable. The stabilizer subgroups of the action are all equal to N, hence are supramenable.

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ON THE C^*-ALGEBRA GENERATED BY THE KOOPMAN REPRESENTATION OF A TOPOLOGICAL FULL GROUP

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ABSTRACT. Let (X, T, μ) be a Cantor minimal sytem and $[[T]]$ the associated topological full group. We analyze $C^*_\pi([[T]])$, where π is the Koopman representation attached to the action of $[[T]]$ on (X, μ).

Specifically, we show that $C^*_\pi([[T]]) = C^*_\pi([[T]]')$ and that the kernel of the character τ on $C^*_\pi([[T]])$ coming from weak containment of the trivial representation is a hereditary C^*-subalgebra of $C(X) \rtimes \mathbb{Z}$. Consequently, $\ker \tau$ is stably isomorphic to $C(X) \rtimes \mathbb{Z}$, and $C^*_\pi([[T]]')$ is not AF.

We also prove that if G is a finitely generated, elementary amenable group and $C^*(G)$ has real rank zero, then G is finite.

1. Introduction

In this work, we study the real rank zero and AF properties for certain classes of group C^*-algebras. The motivations are the classical equivalence between amenability of a group and nuclearity of its C^*-algebra, and the equivalence between local finiteness of a group and finiteness of its uniform Roe algebra worked out in [18], [10] and [17].

For a compact metric space X, both $C(X)$ being AF and having real rank zero are equivalent to total disconnectedness of X.

If a group G is countable and locally finite, then $C^*(G)$ is clearly AF. Conversely, in [9, Theorem 2], Kaniuth proved that if G is a nilpotent group and $C^*(G)$ has real rank zero, then G is locally finite.

In section 2, we show that if G is a finitely generated, elementary amenable group, and $C^*(G)$ has real rank zero, then G is finite. Our proof relies on the fact (Lemma 2.2) that infinite, finitely generated, elementary amenable groups virtually map onto $\mathbb{Z}$.

Let (X, T, μ) be a Cantor minimal system and π the Koopman representation associated to the action of the topological full group $[[T]]$ on (X, μ).

Notice that $C^*([[T]])$ does not have real rank zero, since $[[T]]$ maps onto $\mathbb{Z}$ (by [16, Theorem 1.1(i)], or [5, Proposition 5.5]). On the other hand, by results of Matui, the commutator subgroup $[[T]]'$ is simple ([12]) and non-locally finite (this follows from much sharper results from [13]). Hence, commutators of topological full groups form a class which is not covered by Theorem 2.4.

Futhermore, it was proven by Juschenko and Monod ([7]) that $[[T]]$ is amenable. In Section 3, we prove that $C^*([[T]]')$ is not AF. This is done by showing that $C^*_\pi([[T]]) = C^*_\pi([[T]]')$, and that the kernel of the character τ on $C^*_\pi([[T]])$ coming from weak containment of the trivial representation is a hereditary C^*-subalgebra

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of $C(X) \rtimes \mathbb{Z}$. Consequently, $\ker \tau$ is stably isomorphic to $C(X) \rtimes \mathbb{Z}$, and $C^*_\pi([[T]]')$ is not AF and has real rank zero.

In Section 4, we discuss examples coming from odometers.

2. Elementary amenable groups and real rank zero

Recall that the class of elementary amenable groups is the smallest class of groups containing all abelian and all finite groups, and closed under taking subgroups, quotients, extensions and inductive limits.

Let B denote the class of groups consisting of finite groups and $\mathbb{Z}$. The next lemma is immediate from the proof of [14, Corollary 2.1]. For the conveninence of the reader, we give a proof of it by recalling Osin's argument and notation.

Lemma 2.1. *The class of elementary amenable groups is the smallest class of groups containing B and closed under taking direct limits with injective connecting maps and extensions by groups from B.*

Proof. Let $\mathcal{E}(B)$ be the smallest class of groups containing B and closed under taking subgroups, quotients, extensions and inductive limits. Clearly, $\mathcal{E}(B)$ is equal to the class of elementary amenable groups.

Let $\mathcal{E}_0(B)$ be the class consisting only of the trivial group. Supposing that $\alpha > 0$ is an ordinal and we have already defined the classes $\mathcal{E}_\beta(B)$ for all ordinals $\beta < \alpha$, put

$$\mathcal{E}_\alpha(B) = \bigcup_{\beta<\alpha} \mathcal{E}_\beta(B)$$

if α is a limit ordinal. If α is a successor ordinal, define $\mathcal{E}_\alpha(B)$ to be the class of groups that can be obtained from groups of the class $\mathcal{E}_{\alpha-1}(B)$ by taking direct limits or by taking an extension of a given group by a group from B.

By [14, Lemma 3.1], each $\mathcal{E}_\alpha$ is closed under taking quotients. Hence, in the definition of $\mathcal{E}_\alpha$ for α a successor ordinal, we could have considered only direct limits with injective connecting maps.

Therefore, it follows from [14, Theorem 2.1] that $\mathcal{E}(B)$ is the smallest class of groups containing B and closed under taking direct limits with injective connecting maps and extensions by groups from B. □

Lemma 2.2. *If G is an infinite, finitely generated, elementary amenable group, then there is a subgroup of finite index of G which admits a homomorphism onto $\mathbb{Z}$.*

Proof. Let $\mathcal{A}$ be the class of all finite groups, all non-finitely generated groups, and all groups containing a finite index subgroup which maps onto $\mathbb{Z}$.

We claim that $\mathcal{A}$ contains the class of elementary amenable groups. Obviously, $\mathcal{A}$ contains all finite groups, it contains $\mathbb{Z}$, and it is closed under taking inductive limits with injective connecting maps, and extensions by $\mathbb{Z}$.

Let us check that $\mathcal{A}$ is also closed under taking extensions by finite groups. Let $H \in \mathcal{A}$, F be a finite group, and G a group which fits into the short exact sequence

$$1 \to H \to G \to F \to 1.$$

If G is infinite and finitely generated, then also H is infinite and finitely generated. Hence, H contains a finite index subgroup H' which maps onto $\mathbb{Z}$. Since F is finite, also H' has finite index in G. Therefore, $G \in \mathcal{A}$.

By Lemma 2.1, it follows that $\mathcal{A}$ contains the elementary amenable groups. □

A C^*-algebra A is said to have real rank zero if every hereditary C^*-subalgebra of A has an approximate unit of projections (not necessarily increasing). We refer the reader to, for example, [3, Section V.7] for this and other equivalent definitions of real rank zero.

Lemma 2.3. *If A is an infinite-dimensional, real rank zero C^*-algebra, then it contains a sequence of non-zero, orthogonal projections.*

Proof. By [8, Exercise 4.6.13], and since A is infinite-dimensional, there is a sequence $(a_n)_{n\in\mathbb{N}} \subset A$ of non-zero, positive elements such that $a_j a_k = 0$ when $j \neq k$. For each $n \in \mathbb{N}$, take a non-zero projection p_n in the hereditary (hence real rank zero) C^*-subalgebra $\overline{a_n A a_n}$. By construction, $p_j p_k = 0$ when $j \neq k$. □

Theorem 2.4. *If G is a finitely generated, elementary amenable group and $C^*(G)$ has real rank zero, then G is finite.*

Proof. Suppose G is infinite. By Lemma 2.2, there is a subgroup H of G with finite index n, and $\Phi\colon H \to \mathbb{Z}$ a surjective homomorphism. Let $\varphi\colon C^*(H) \to C^*(\mathbb{Z})$ be the $*$-homomorphism induced by Φ, and $\varphi_n\colon M_n(C^*(H)) \to M_n(C^*(\mathbb{Z}))$ the inflation of φ.

There is an injective $*$-homomorphism $\psi\colon C^*(G) \to M_n(C^*(H))$ such that the image of $\varphi_n \circ \psi$ is infinite-dimensional. For the convenience of the reader, we sketch the construction of ψ, which is standard.

Let $x_1, \ldots, x_n \in G$ be such that $x_1 = e$ and $G = \sqcup_{i=1}^n x_i H$. Consider the following unitary defined on canonical basis vectors:

$$U\colon \bigoplus_{i=1}^{n} \ell^2(H) \to \ell^2(G)$$
$$\delta_{i,h} \mapsto \delta_{x_i h}.$$

Let $S\colon B(\ell^2(G)) \to M_n(B(\ell^2(H))$ be the isomorphism induced by U.

By using the left regular representations λ_G and λ_H, we see $C^*(G)$ as contained in $B(\ell^2(G))$ and analogously for $C^*(H)$.

It is easy to check that $S(\lambda_G(g)) \in M_n(C^*(H))$ for every $g \in G$. Hence, $S(C^*(G)) \subset M_n(C^*(H))$. Furthermore, for $h \in H$, we have that $S(\lambda_G(h))_{1,1} = \lambda_H(h)$. Let $\psi := S|_{C^*(G)}$. Then $\varphi_n(\psi(C^*(G)))$ is infinite-dimensional.

Hence, by Lemma 2.3, $M_n(C^*(\mathbb{Z})) \simeq M_n(C(\mathbb{T}))$ contains a sequence of non-zero, orthogonal projections. Since $\mathbb{T}$ is connected, we get a contradiction. Hence, G is finite. □

Remark 2.5. Recall that a C^*-algebra A is said to have property (SP) if every non-zero hereditary C^*-subalgebra of A contains a non-zero projection. Furthermore, A is said to have residual property (SP) if every quotient of A has property (SP) (see [15, Section 7] for more details about these properties).

In the proof of Theorem 2.4, the only aspects of real rank zero that were used are that it implies property (SP) and that having real rank zero is closed under taking quotients. In particular, Theorem 2.4 remains true if one replaces "real rank zero" by "residual property (SP)".

3. Koopman representation of a topological full group

Given a unitary representation π of a group G, we denote by $C^*_\pi(G)$ the C^*-algebra generated by the image of π.

We will denote the Cantor set by X.

Let α be an action of a group G on X by homeomorphisms. The topological full group associated to α, denoted by $[[\alpha]]$, is the group of all homeomorphisms γ on X for which there exists a finite partition of X into clopen sets $\{A_i\}_{i=1}^n$ and $g_1, \ldots, g_n \in G$ such that $\gamma|A_i = \alpha_{g_i}|_{A_i}$ for $1 \leq i \leq n$. That is, $[[\alpha]]$ consists of the homeomorphisms on X which are locally given by the action α.

Fix T a minimal homeomorphim on X. We denote by $[[T]]$ the topological full group associated to the $\mathbb{Z}$-action induced by T.

Let μ be a T-invariant probability measure on X. Note that μ is also invariant under the action of $[[T]]$ on X. Let $\pi\colon [[T]] \to B(L^2(X,\mu))$ be given by $\pi(g)(f) := f \circ g^{-1}$, for $g \in [[T]]$ and $f \in L^2(X,\mu)$. This π is the so called Koopman representation associated to the action of $[[T]]$ on (X,μ).

We will use the faithful representation of $C(X) \rtimes \mathbb{Z}$ in $B(L^2(X,\mu))$, with $C(X)$ acting by multiplication operators, and, for n$\in \mathbb{Z}$, $\delta_n := \pi(T^n)$, so that $C(X) \rtimes \mathbb{Z} := \overline{\text{span}}\{f\delta_n : f \in C(X), n \in \mathbb{Z}\}$.

Given $g \in [[T]]$ and $\{A_i\}_{i=1}^n$ a partition of X into clopen sets such that $g|_{A_i} = T^{n_i}|A_i$ for $1 \leq i \leq n$, notice that $\pi(g) = \sum 1_{T^{n_i}(A_i)}\delta_{n_i}$. In particular, $C^*_\pi([[T]]) \subset C(X) \rtimes \mathbb{Z}$.

Definition 3.1. Given $n \in \mathbb{N}$, we say that a subset $A \subset X$ is n-disjoint if

$$A, T(A), \ldots, T^{n-1}(A)$$

are pairwise disjoint.

Suppose $A \subset X$ is a clopen and n-disjoint set. Consider the symmetric group S_n acting on $\{0, \ldots, n-1\}$. For $\sigma \in S_n$, let $\sigma_A \in [[T]]$ be given by

$$\sigma_A(x) = \begin{cases} T^{\sigma(i)-i}(x), & \text{if } 0 \leq i < n \text{ and } x \in T^i(A) \\ x, & \text{if } x \notin \sqcup_{i=0}^{n-1} T^i(A), \end{cases} \qquad x \in X. \tag{1}$$

Lemma 3.2. *Let $n \geq 4$ and $A \subset X$ be a clopen and n-disjoint set. For every $\sigma \in S_n$, it holds that $\pi(\sigma_A) \in C^*_\pi([[T]]')$.*

Proof. Notice first that $\{1_{T^i(A)}\delta_{i-j}\}_{0 \leq i,j<n}$ forms a system of matrix units in $C(X) \rtimes \mathbb{Z}$ of type $M_n(\mathbb{C})$ (we see $M_n(\mathbb{C})$ as matrices indexed by the set $\{0, \ldots, n-1\}$).

Let $B := (\sqcup_{i=0}^{n-1} T^i(A))^c$ and $\varphi\colon \mathbb{C} \oplus M_n(\mathbb{C}) \to C(X) \rtimes \mathbb{Z}$ be the $*$-homomorphism given by $\varphi(\alpha, e_{ij}) := \alpha 1_B + 1_{T^i(A)}\delta_{i-j}$, for $\alpha \in \mathbb{C}$ and $0 \leq i,j \leq n-1$.

Let $\rho\colon S_n \to \mathbb{C} \oplus M_n(\mathbb{C})$ be the direct sum of the trivial representation and the permutation representation. Then, for $\sigma \in S_n$, it holds that $\varphi(\rho(\sigma)) = \pi(\sigma_A)$.

Furthermore, since $n \geq 4$, we have that $C^*_\rho((S_n)') = C^*_\rho(S_n)$.

Therefore, $\sigma_A \in C^*_\pi([[T]]')$ for any $\sigma \in S_n$.

□

Given $A \subset X$ clopen, consider the continuous function

$$t_A\colon A \to \mathbb{N}$$
$$x \mapsto \min\{k \in \mathbb{N} : T^k(x) \in A\}.$$

This is the so called function of first return to A.

Notice that, for $j \in \mathbb{Z}$, it holds that

$$t_{T^j(A)} \circ T^j|_A = t_A. \tag{2}$$

Let $T_A \in [[T]]$ be defined by

$$T_A(x) = \begin{cases} T^{t_A(x)}(x), & \text{if } x \in A \\ x, & \text{otherwise.} \end{cases}, \quad x \in X. \tag{3}$$

If $B \subset X$ is a clopen set disjoint from A, then T_A and T_B commute.

In order to prove Lemma 3.4, we will have to analyze the spectrum of C^*-algebras generated by certain commuting unitaries, and the next lemma will be useful for it.

We consider the circle $\mathbb{T}$ as a pointed space with basepoint 1.

Lemma 3.3. *The universal C^*-algebra generated by commuting unitaries $z_1, \ldots, z_n$ subject to the relations $\{(z_i - 1)(z_j - 1) = 0 : 1 \leq i \neq j \leq n\}$ is $C(\bigvee_{k=1}^n \mathbb{T})$, with z_k being given by*

$$z_k \colon \bigvee_{i=1}^{n} \mathbb{T} \to \mathbb{C}$$
$$(x, i) \mapsto \begin{cases} x, & \text{if } i = k \\ 1, & \text{if } i \neq k. \end{cases}$$

Proof. Consider the embedding $F\colon \bigvee_{i=1}^n \mathbb{T} \to \mathbb{T}^n$ which takes x in the i-th copy of $\mathbb{T}$ and sends it into $(F(x)_i)_{1 \leq i \leq n} \in \mathbb{T}^n$ such that $F(x)_i := x$ and $F(x)_j := 1$ if $j \neq i$. Also let $F' \colon C(\mathbb{T}^n) \to C(\bigvee_{i=1}^n \mathbb{T})$ be given by $F'(f) := f \circ F$, for $f \in C(\mathbb{T}^n)$.

For $1 \leq i \leq n$, let $w_i \in C(\mathbb{T}^n)$ be given by $w_i(y) := y_i$, for $y \in \mathbb{T}^n$. Then $F'(w_i) = z_i$.

Assume $n > 1$. Let $A := C^*(\{(w_i - 1)^k (w_j - 1)^l : i \neq j \text{ and } k, l \in \mathbb{N}\})$. We claim that $\ker F' = A$. Clearly, $A \subset \ker F'$.

Let $Y := \mathbb{T}^n \setminus \mathrm{Im}(F)$. Notice that $\ker F' = \{f \in C(\mathbb{T}^n) : f|_{\mathrm{Im}(F)} = 0\} \simeq C_0(Y)$. By the Stone-Weierstrass Theorem, in order to show that $A = C_0(Y)$, it is sufficient to show that, for every $y \in Y$, there is $f \in A$ such that $f(y) \neq 0$, and that A separates the points of Y. The proof of the former condition is trivial, so we only show that A separates the points of Y.

Take $(x_1, \ldots, x_n), (y_1, \ldots y_n) \in Y$ distinct points. There is i such that $x_i \neq y_i$. Without loss of generality, assume $x_i \neq 1$. Take $j \neq i$ such that $x_j \neq 1$. Then, by choosing $k \in \mathbb{N}$ appropriately, we get $(x_i - 1)^k (x_j - 1) \neq (y_i - 1)^k (y_j - 1)$.

Since $C(\mathbb{T}^n)$ is the universal C^*-algebra generated by n commuting unitaries and $C(\bigvee_{i=1}^n \mathbb{T})$ is generated by $\{z_1, \ldots, z_n\}$, the result follows. $\square$

Lemma 3.4. *Let $A \subset X$ be a clopen and 3-disjoint set. Then $\pi(T_A) \in C^*([[T]]')$.*

Proof. Given $\sigma \in S_3$, $x \in A$ and $0 \leq i, j < 3$, we have that $\sigma_A T_{T^i(A)} \sigma_A^{-1}(T^j(x)) = T^j(x)$ if $j \neq \sigma(i)$ and

$$\begin{aligned}\sigma_A T_{T^i(A)}\sigma_A^{-1}(T^{\sigma(i)}(x)) &= \sigma_A T_{T^i(A)}(T^i(x))\\ &= T^{\sigma(i)-i}T^{t_{T^i(A)}(T^i(x))}(T^i(x))\\ &\overset{(*)}{=} T^{\sigma(i)-i}T^{t_{T^{\sigma(i)}(A)}(T^{\sigma(i)}(x))}(T^i(x))\\ &= T^{t_{T^{\sigma(i)}(A)}(T^{\sigma(i)}(x))}(T^{\sigma(i)}(x))\\ &= T_{T^{\sigma(i)}(A)}(T^{\sigma(i)}(x)),\end{aligned}$$

where the equality in (*) is due to (2). Hence, $\sigma_A T_{T^i(A)}\sigma_A^{-1} = T_{T^{\sigma(i)}(A)}$.

In particular, for $0 \leq i, j < 3$, we have that $T_{T^i(A)}(T_{T^j(A)})^{-1} \in [[T]]'$.

If $0 \leq i \neq j < 3$, it is easy to check that $(\pi(T_{T^i(A)})-1)(\pi(T_{T^j(A)})-1) = 0$. Then, by Lemma 3.3, there is a $*$-homomorphism from $C(\bigvee_{i=1}^3 \mathbb{T})$ into $C^*(\{\pi(T_{T^i(A)}) : 0 \leq i < 3\}$ mapping z_i into $\pi(T_{T^{i-1}(A)})$ for $1 \leq i \leq 3$. Furthermore, by the Stone-Weierstrass theorem, $C(\bigvee_{i=1}^3 \mathbb{T})$ is generated by $\{z_i z_j^* : 1 \leq i, j \leq 3\}$. Hence, $\pi(T_A) \in C^*_\pi([[T]]')$.

□

Theorem 3.5. *Let (X, T, μ) be a Cantor minimal system and π the Koopman representation associated to the action of $[[T]]$ on (X, μ). Then $C^*_\pi([[T]]) = C^*_\pi([[T]]')$.*

Proof. By [6, Theorem 4.7], given $m \in \mathbb{N}$, [[T]] is generated by

$$\bigcup_{n \geq m} \{T_A, \sigma_A : \sigma \in S_n, A \subset X \text{ is clopen and } n\text{-disjoint}\}.$$

By Lemmas 3.2 and 3.4, the result follows.

□

Notice that $1_X \in L^2(X, \mu)$ is invariant under $\pi([[T]])$. Therefore, π weakly contains the trivial representation. Denote by τ the associated character on $C^*_\pi([[T]])$.

Lemma 3.6. *Let τ be the character on $C^*_\pi([[T]])$ coming from weak containment of the trivial representation. Then $\ker\tau = \overline{\operatorname{span}}\{1 - \pi(g) : g \in [[T]]\}$.*

Proof. Given $d \in \ker\tau$ and $\epsilon > 0$, take $d' \in \operatorname{span}\pi([[T]])$ such that $\|d - d'\| < \frac{\epsilon}{2}$. Then $\|d - (d' - \tau(d'))\| = \|(d - d') + \tau(d' - d)\| < \epsilon$. Furthermore, $d' - \tau(d') \in \ker\tau \cap \operatorname{span}\pi([[T]])$.

Since $\ker\tau \cap \operatorname{span}\pi([[T]]) = \operatorname{span}\{1 - \pi(g) : g \in [[T]]\}$, the result follows.

□

Theorem 3.7. *Let τ be the character on $C^*_\pi([[T]])$ coming from weak containment of the trivial representation. Then $\ker\tau$ is a hereditary C^*-subalgebra of $C(X) \rtimes \mathbb{Z}$.*

Proof. We are going to show that, for $a \in C(X) \rtimes \mathbb{Z}$ and $b, c \in \ker\tau$, it holds that $bac \in \ker\tau$.

Given $A \subset X$ clopen and 2-disjoint, notice that $(\delta_0 - \delta_1)1_A(\delta_0 - \delta_{-1}) = \delta_0 - (1_{(A\cup T(A))^c}\delta_0 + 1_{T(A)}\delta_1 + 1_A\delta_{-1}) \in C^*_\pi([[T]])$.

By using telescoping sums, it follows that, for $n, m \in \mathbb{Z}$ and $A \subset X$ 2-disjoint and clopen, $(\delta_0 - \delta_n)1_A(\delta_0 - \delta_m) \in C^*_\pi([[T]])$.

Given $g, h \in [[T]]$, take a basis $\mathcal{B}$ of 2-disjoint, clopen sets for the topology of X. Moreover, assume that, for each $A \in \mathcal{B}$, there is $n(A), m(A) \in \mathbb{Z}$ such that $g|_A = T^{n(A)}|_A$ and $h|_{h^{-1}(A)} = T^{m(A)}|_{h^{-1}(A)}$.

Then

$$(\delta_0 - \pi(g))1_A(\delta_0 - \pi(h)) = 1_A - \delta_{n(A)}1_A - 1_A\delta_{m(A)} + \delta_{n(A)}1_A\delta_{m(A)}$$
$$= (\delta_0 - \delta_{n(A)})1_A(\delta_0 - \delta_{m(A)}) \in C^*_\pi([[T]]).$$

Since $C(X) = \overline{\mathrm{span}}\{1_A : A \in \mathcal{B}\}$, we conclude that, for $g, h \in [[T]]$ and $f \in C(X)$, $(\delta_0 - \pi(g))f(\delta_0 - \pi(h)) \in C^*_\pi([[T]])$.

By Lemma 3.6 and the fact that $C(X) \rtimes \mathbb{Z} = \overline{\mathrm{span}}\{f\delta_n : f \in C(X), n \in \mathbb{Z}\}$, we conclude that, for $b, c \in \ker\tau$ and $a \in C(X) \rtimes \mathbb{Z}$, $bac \in C^*_\pi([[T]])$.

Since τ is a character, the result follows. □

Corollary 3.8. *Let τ be the character on $C^*_\pi([[T]])$ coming from weak containment of the trivial representation. Then $\ker\tau$ is stably isomorphic to $C(X) \rtimes \mathbb{Z}$. In particular, $C^*_\pi([[T]]')$ has real rank zero and $C^*([[T]]')$ is not AF.*

Proof. By Theorem 3.7 and the fact that $C(X) \rtimes \mathbb{Z}$ is simple, it follows that $\ker\tau$ is a full, hereditary C^*-subalgebra of $C(X) \rtimes \mathbb{Z}$. Therefore, [2, Theorem 2.8] implies that $\ker\tau$ is stably isomorphic to $C(X) \rtimes \mathbb{Z}$.

Since $C(X) \rtimes \mathbb{Z}$ has real rank zero, and $K_1(C(X) \rtimes \mathbb{Z}) \simeq \mathbb{Z}$, and $K_1(A) = 0$ for any AF-algebra A, the conclusion follows. □

4. Odometers

We start this section by giving a description of $C^*_\pi([[T]])$ when T is an odometer map.

Given $m \in \mathbb{N}$, let $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$.

Example 4.1. Let (n_k) be a strictly increasing sequence of natural numbers such that, for every k, $n_k | n_{k+1}$. Let $\rho_k \colon \mathbb{Z}_{n_{k+1}} \to \mathbb{Z}_{n_k}$ be the surjective homomorphism such that $\rho_k(1) = 1$, and define

$$X := \{(x_k) \in \prod_{k\in\mathbb{N}} \mathbb{Z}_{n_k} : \rho_k(x_{k+1}) = x_k, \forall k \in \mathbb{N}\}.$$

Consider

$$T \colon X \to X$$
$$(x_k) \mapsto (x_k + 1).$$

Then (X, T) is a Cantor minimal system, the so called odometer of type (n_k).

For $k \in \mathbb{N}$ and $l \in \mathbb{Z}_{n_k}$, let $U(k, l) := \{(x_m) \in X : x_k = l\}$.

Using the notation from (1) and (3), let, for $k \in \mathbb{N}$, $\Gamma_k := \langle\{T_{U(k,l)}, \sigma_{U(k,0)} \in [[T]] : l \in \mathbb{Z}_{n_k}, \sigma \in S_{n_k}\}\rangle$. As proven by Matui in [13, Proposition 2.1], $\Gamma_k \subset \Gamma_{k+1}$, $\Gamma_k \simeq \mathbb{Z}^{n_k} \rtimes S_{n_k}$, and $\bigcup_k \Gamma_k = [[T]]$.

For $k \in \mathbb{N}$, let $A_k := \overline{\mathrm{span}}\{1_{U(k,l)}\delta_m : l \in \mathbb{Z}_{n_k}, m \in \mathbb{Z}\}$. Then $A_k \subset A_{k+1}$, and $C(X) \rtimes \mathbb{Z} = \overline{\bigcup_k A_k}$.

Fix $k \in \mathbb{N}$ and consider the isomorphism $\varphi_k \colon A_k \to C(\mathbb{T}, M_{\mathbb{Z}_{n_k}}(\mathbb{C}))$, such that $\varphi_k(1_{U(k,l)}) = e_{l,l}$, for $l \in \mathbb{Z}_{n_k}$, and, for $z \in \mathbb{T}$,

$$(\varphi_k(\delta_1)(z))_{i,j} := \begin{cases} 1, & \text{if } 0 < i \leq n_k - 1 \text{ and } j = i - 1 \\ z, & \text{if } i = 0 \text{ and } j = n_k - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\pi\colon [[T]] \to U(C(X) \rtimes \mathbb{Z})$ be the homomorphism coming from the Koopman representation and $B_k := \{b \in M_{\mathbb{Z}_{n_k}}(\mathbb{C}) : \forall i, j \in \mathbb{Z}_{n_k}, \sum_r b_{i,r} = \sum_s b_{s,j}\}$.

Then, for $\sigma \in S_{n_k}$, we have that $\varphi_k(\pi(\sigma_{U(k,0)})) = \sum e_{\sigma(i),i}$ and

$$C^*(\{\varphi_k(\pi(\sigma_{U(k,0)})) : \sigma \in S_{n_k}\}) \simeq B_k.$$

Furthermore, $\varphi_k(C^*(\pi(\{T_{U(k,l)} : l \in \mathbb{Z}_{n_k}\}))) \simeq C(\bigvee_{l\in\mathbb{Z}_{n_k}} \mathbb{T})$ and $\varphi_k(C^*_\pi(\Gamma_k)) = \{f \in C(\mathbb{T}, M_{\mathbb{Z}_{n_k}}(\mathbb{C})) : f(1) \in B_k\}$.

In [4], Dykema and Rørdam gave examples of non-locally finite groups G such that $C^*_{\mathrm{red}}(G)$ has real rank zero. As far as we are aware, there is no known example of non-locally finite group G such that $C^*(G)$ has real rank zero.

Question 4.2. *Let (X, T) be an odometer as in Example 4.1. Does $C^*([[T]]')$ have real rank zero?*

Example 4.3. Let (X, T) be an odometer of type (n_k) as in Example 4.1. Consider

$$\begin{aligned} J\colon X &\to X \\ (x_k) &\mapsto (-x_k). \end{aligned}$$

Then J is an involutive homeomorphism on X such that $JTJ = T^{-1}$. Hence, T and J induce an action α of the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$ on X. We will use Matui's technique ([13, Proposition 2.1]) in order to compute $[[\alpha]]$.

For every $\gamma \in (\mathbb{Z} \rtimes \mathbb{Z}_2) \setminus \{e\}$, it holds that $\{x \in X : \alpha_\gamma(x) = x\}$ has empty interior (it consists of at most two elements). Hence, given $g \in [[\alpha]]$, there exists a unique continuous function $c_g\colon X \to \mathbb{Z} \rtimes \mathbb{Z}_2$ such that, for $x \in X$, $g(x) = \alpha_{c(g)}(x)$.

For $k \in \mathbb{N}$ and $l \in \mathbb{Z}_{n_k}$, let $U(k, l)$ be as in Example 4.1 and

$$\Gamma_k := \{g \in [[\alpha]] : c_g \text{ is constant on } U(k, l) \text{ for } l \in \mathbb{Z}_{n_k}\}.$$

Define $J_{k,l} \in [[\alpha]]$ by

$$J_{k,l}(x) = \begin{cases} T^{2l}J(x), & \text{if } x \in U(k, l) \\ x, & \text{otherwise,} \end{cases} \qquad x \in X.$$

Then $\Gamma_k = \langle \{T_{U(k,l)}, J_{k,l}, \sigma_{U(k,0)} : l \in \mathbb{Z}_{n_k}, \sigma \in S_{n_k}\}\rangle$ and

$$\Gamma_k \simeq (\mathbb{Z} \rtimes \mathbb{Z}_2)^{n_k} \rtimes S_{n_k},\ \Gamma_k \subset \Gamma_{k+1},\ \text{and } \bigcup_k \Gamma_k = [[\alpha]]. \tag{4}$$

Notice that the constant sequence $(0) \in X$ is a fixed point for J. Hence, [1, Theorem 3.5] implies that $C(X) \rtimes (\mathbb{Z} \rtimes \mathbb{Z}_2)$ is AF (see also [11]). Moreover, it follows from (4) that the abelianization of $[[\alpha]]$ is locally finite.

Therefore, the two obstructions that were used for ruling out the possibility of $C^*([[T]])$ and $C^*([[T]]')$ being AF do not hold for $C^*([[\alpha]])$.

Question 4.4. *Let α be as in Example 4.3. Is $C^*([[\alpha]])$ AF?.*

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