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Player Aggregation in Noncooperative Games, II*

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Conditions are given under which subsets of the players of a noncooperative game can be combined into "aggregate players" without changing the set of equilibrium-point solutions of the game. These conditions are shown to be the weakest possible ones with a certain specified kind of generality. "Approximate" versions of the results are also formulated and proven.

Key words: Aggregation; equilibrium; game theory; mathematical economics; noncooperative games; total orders.

1. Introduction

Like its predecessor [1],[1] this paper is motivated by the following observation: in applied game-theoretic modeling, it may prove useful to reduce model-complexity and data-needs by combining, into a single "aggregate player," some subset of the original players whose interests are sufficiently "parallel" for this purpose. It is natural, therefore, to investigate from a mathematical viewpoint the conditions under which such an aggregation is "valid" in the sense of leaving the game's set of "solutions" undisturbed. Results of this type were needed, for example, in connection with a class of models [2, 3, 4] involving an inspection agency with insufficient resources to visit all of a number of sites at which "cheating" might or might not occur; the issue was whether the individual site-managers could be aggregated into a single "inspectee" entity so that the situation could be analyzed as a two-person game.

Our treatment here is limited to noncooperative games and to the equilibrium-point notion of "solution." For completeness, these concepts are defined in section 2 below, where the process of aggregation is also formalized. A natural special case of aggregation ("group equilibrium") is discussed in [5], but without considering the question of solution-set preservation.

In [1], a simple condition was given under which aggregation does not change a game's set of solutions. That condition stated that each individual player's payoff (in the original game) is independent of the strategy choices by the other individuals comprising the same aggregate player. Though applicable to the inspector-inspectee games mentioned above, this condition is clearly rather restrictive, and captures the "independence" or "indifference" of the aggregated players' interests rather than (as desired) the "parallelism" of those interests. In section 3 of the present paper, we give less restrictive sufficient conditions for aggregation (a) to avoid introducing extraneous solutions, and (b) to preserve all solutions of the original game. Further theorems and examples show that these conditions, though not necessary as well as sufficient, are actually the weakest possible sufficient conditions with a certain specified kind of generality. Additional examples show that the family of player-subsets which are "aggregable," in the sense of obeying the minimal conditions mentioned above, need not have certain properties to be expected if aggregability fully corresponded to some natural notion of "parallel interest;" for example, a subset of an aggregable set need not be aggregable.

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[1] Numbers in brackets indicate literature references at the end of the paper.

Consistent with [1], section 4 extends the preceding material to "approximate solutions" in cases in which the conditions described above are satisfied only approximately. This topic reflects an expectation that in applied contexts, many mathematical relationships will not (or cannot be known to) hold exactly.

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2. Games, equilibria, aggregation

Let $n \geqslant 1$ be an integer, and $N = \{1,2,\ldots,n\}$. An n-person noncooperative game $G = (X, f)$ consists of an n-tuple $(X_1, \ldots, X_n)$ of nonempty sets X_i with Cartesian product X, and an n-tuple $f = (f_1, \ldots, f_n)$ of functions f_i: $X \to R_i$ where R_i is equipped with a strict linear order ϱ_i. Here X_i is interpreted as the set of strategies or actions open to the i-th player, f_i as that player's "payoff function," R_i as the set of possible payoffs or outcomes experienced by that player, and ϱ_i as the relation of (strict) preference by that player among outcomes. The fact that the domain of f_i is X, rather than X_i, expresses the idea that each player's payoff depends not only on what strategy that player chooses, but also on the choices made by other players.

For any $x \varepsilon X$, any $i \varepsilon N$, and any $x_i \varepsilon X_i$, we denote by (x, i, x_i) the member of X obtained from x by changing its i-th coordinate to x_i. With this notation, a "solution" concept can be defined: $x^\circ \varepsilon X$ is called an *equilibrium point* (EP) for game G if, for every $i \varepsilon N$ and every $x_i \varepsilon X_i$, the relation

$$f_i(x^\circ, i, x_i)\, \varrho_i f_i(x^\circ) \tag{1}$$

is *false*. That is, if one thinks of the coordinates of x° as the players' "current" choices of strategies, then no player has an incentive to deviate unilaterally from his or her current choice. Since the game is regarded as "noncooperative," only unilateral shifts come into consideration, and so the falsity of all relations (1) is sufficient to describe the "stability" of x°. If $n = 1$, an EP is simply a strategy that yields a preference-maximal outcome for the (sole) player.

Next we describe an "aggregation" of game G. Let m be an integer with $1 \leqslant m \leqslant n$, and let $M = \{1, 2, \ldots, m\}$. An m-player aggregation $G[B,F]$ of G is specified by the following structure. $B = \{B_1, \ldots, B_m\}$ is a partition of N into nonempty sets; note that the relation $i \varepsilon B_{j(i)}$ defines a function j: $N \to M$. Let S_j be the Cartesian product of the sets $\{R_i : i \varepsilon B_j\}$; also let $F = (F_1, \ldots, F_m)$ be an m-tuple of functions $F_j : S_j \to T_j$ where each set T_j is equipped with a strict linear order τ_j, and function F_j is strictly monotone in each of its arguments. This last condition means that for any $s_j \varepsilon S_j$, for any $i \varepsilon B_j$ with r_i the i-th coordinate of s_j, and for any $r_i' \varepsilon R_i$,

$$r_i' \varrho_i r_i \text{ implies } F_j(s_j, i, r_i')\, \tau_j\, F_j(s_j). \tag{2}$$

Such a structure defines an m-person game as follows. The "players" are $\{B_j : j \varepsilon M\}$. The set of strategies of B_j is Y_j, the Cartesian product of $\{X_i : i \varepsilon B_j\}$. Note that the Cartesian product of the player's strategy-sets, i.e. of $\{Y_j : j \varepsilon M\}$, is the same set X as for the original game; this permits the symbols "x" and "y" to be used interchangeably, and corresponds to the idea that we are dealing with aggregation of players and payoffs, but not of strategies. (The same observation justifies the later use of notation like (x, j, y_j), as an extension of the previous symbol (x, i, x_i).) In the aggregated game, the payoff function for player B_j is g_j : $X \to T_j$, defined by[2]

$$g_j(x) = F_j[\{f_i(x) : i \varepsilon B_j\}]. \tag{3}$$

The definition of an EP for game $G[B,F]$ is directly analogous to that for G.

[2] In the following notation, the argument of F_j is the member $s_j \varepsilon S_j$ whose i-th coordinate, for $i \varepsilon B_j$, is $f_i(x)$.

3. Preservation of the solution-set

For any subset S of N, let X_S denote the Cartesian product of the strategy-sets $\{X_i : i\varepsilon S\}$. We will call $x\varepsilon X$ *S-stable* if there is no $x_S\varepsilon X_S$ such that (with an obvious extension of previous notation)

$$f_i\,(x,S,x_S)\varrho_i f_i(x) \text{ or } f_i\,(x,S,x_S) = f_i(x)$$

holds for each $i\varepsilon S$, with the first relation holding for at least one $i\varepsilon S$. Intuitively, this means that even if the players in S could act together, they could effect no deviation from the *status quo* represented by x which would be "advantageous to S" in the sense of being preferable to some members of S and a matter of indifference to the others. Thus x° is an *EP* if and only if it is $\{i\}$-stable for all $i\varepsilon N$.

Let X^S denote the set of S-stable members of X. We call S *nonfrustrating* (*NF*) if

$$X^S \subseteq \cap\,\{X^{\{i\}} : i\varepsilon S\}. \tag{4}$$

The reason for this terminology is best seen by taking complements in (4): for each $x\varepsilon X$, if some player $i\varepsilon S$ can find a unilateral deviation preferable (for i) to the *status quo* represented by x, then there is a joint deviation $x_S\varepsilon X_S$ which is "advantageous to S" in the sense described above and which could therefore be adopted by S without internal dissension were it the only available alternative to continuation of x. That is, there is no opposition of interests within S that would preclude deviating from a *status quo* which some single player in S could profitably deviate from. Thus no player, by becoming part of an aggregate player S, would risk the frustration of being "stuck with" an undesirable *status quo* against which he or she had a unilateral counteraction as an individual player.

THEOREM 1. *If each* B_j *is* NF *and* x° *is an* EP *for* G [B,F], *then* x° *is also an* EP *for* G.

PROOF. Suppose, to the contrary, that (1) holds for some $i\varepsilon N$. Then x° is not in $X^{\{i\}}$; since $B_{j(i)}$ is NF, x° cannot be $B_{j(i)}$-stable. Thus there exists $y_{j(i)}\varepsilon Y_{j(i)}$ such that changing from x° to $(x^\circ, j(i), y_{j(i)})$ is "advantageous to $B_{j(i)}$" in the sense defined above. Consider changing the arguments of $F_{j(i)}\,[\{f_k\,(x^\circ) : k\varepsilon B_{j(i)}\}]$ one at a time to those of $F_{j(i)}[\{f_k\,(x^\circ, j(i), y_{j(i)}) : k\varepsilon B_{j(i)}\}]$; by the monotonicity of $F_{j(i)}$ and the transitivity of $\tau_{j(i)}$ it follows—cf. (3)—that $g_{j(i)}\,(x^\circ, j(i), y_{j(i)})\tau_{j(i)}\, g_{j(i)}(x^\circ)$, contradicting the hypothesis that x° is an *EP* for $G[B,F]$.

We will call a subset S of N *unilateral-deviation unanimous* (*UDU*) if, for each $x\varepsilon X$ and $i\varepsilon S$, any $x_i\varepsilon X_i$ for which $f_i(x,i,x_i)\varrho_i f_i(x)$ also has the property that

$$f_k(x,i,x_i)\varrho_k f_k(x) \text{ or } f_k(x,i,x_i) = f_k(x)$$

holds for all $k\varepsilon S - \{i\}$. Intuitively, this says that whenever a single player in S has incentive for some particular unilateral deviation from a *status quo*, then no other player in S would regret seeing that single deviation effected. This condition, introduced only for the sake of the following Corollary, may be easier to recognize than NF.

COROLLARY. *If each* B_j *is* UDU, *and* x° *is an* EP *for* G[B,F], *then* x° *is also an* EP *for* G.

PROOF. In view of Theorem 1, it suffices to observe that UDU implies NF; the proof of that implication is straightforward.

As preparation for the next result, observe that condition NF when applied to subset S of N is "local to S," in that it does not refer to the attributes $\{R_i, \varrho_i, f_i\}$ of the players $i\varepsilon N\text{-}S$. This is important for ease of checking the condition. Furthermore, *NF* is "general" in that it does not refer to the attributes (T_S, τ_S, F_S) of S as an aggregate player in any particular aggregated version of G. The next theorem shows that condition *NF* is the *weakest* one, with these two properties, which would suffice for an analog of Theorem 1.

THEOREM 2. *Let all the information needed to define* G *and* G[B,F] *be specified except for* (T_1, τ_1, F_1) *and* $\{f_i : i \varepsilon N\text{-}B_1\}$. *If* B_1 *is not* NF, *then the missing information can be specified in such a way that* G[B,F] *has an* EP *which is not an* EP *for* G.

PROOF. Since B_1 is not NF, there exists a B_1-stable $x \varepsilon X$, an $i \varepsilon B_1$, and an $x_i \varepsilon X_i$, such that

$$f_i(x,i,x_i) \varrho_i f_i(x). \tag{5}$$

It follows from (5) that x cannot be an EP for G. We shall specify the missing information in such a way that x is an EP for $G[B,F]$.

Choose each of $\{f_t : t \varepsilon N\text{-}B_1\}$ to be constant, and write "$B(1)$" instead of "B_1" for subscripting purposes. For x to be an EP for $G[B,F]$, it suffices to choose (T_1, τ_1, F_1) in such a way that for no $x_{B(1)} \varepsilon X_{B(1)}$ is

$$F_1[\{f_p(x, B_1, x_{B(1)}) : p \varepsilon B_1\}] \tau_1 F_1[\{f_p(x) : p \varepsilon B_1\}]. \tag{6}$$

Choose $T_1 = S_1$, and F_1 to be the identity map of S_1. Define τ_1^* as the direct product of $\{\varrho_p : p \varepsilon B_1\}$; that is, given s_1 and s_1' in S_1, with respective coordinates $\{r_p : p \varepsilon B_1\}$ and $\{r_p' : p \varepsilon B_1\}$, then $s_1' \tau_1^* s_1$ holds if and only if

$$r_p' \varrho_p r_p \text{ or } r_p' = r_p$$

holds for all $p \varepsilon B_1$, with the former relation holding for at least one $p \varepsilon B_1$. This τ_1^* is a strict partial order on T_1, but not in general a linear order. To make F_1 strictly monotone in each of its arguments, as required, we must (and it suffices to) choose the linear order τ_1 to be an extension of τ_1^*.

The partial order τ_1^* can be viewed as a collection of ordered pairs of distinct members of T_1, where $(t,t') \varepsilon T_1 \times T_1$ is in the collection if and only if $t \tau_1^* t'$. Now let t° be the member of T_1 with coordinates $\{f_p(x): p \varepsilon B_1\}$, and for each $x_{B(1)} \varepsilon X_{B(1)}$ let $h(x_{B(1)})$ be the member of T_1 with coordinates $\{f_p(x, B_1, x_{B(1)}) : p \varepsilon B_1\}$. Define a second collection of ordered pairs by

$$T^\circ = [h(X_{B(1)}) - \{t^\circ\}] \times \{t^\circ\}.$$

Then the fact that x is B_1-stable can be written as $T^\circ \cap \tau_1^* = \phi$, and the desired condition that (6) hold for no $x_{B(1)} \varepsilon X_{B(1)}$ can be written $T^\circ \cap \tau_1 = \phi$.

What remains to be proved, then, is the existence of a strict linear order τ_1 on T_1 which is an extension of τ_1^* and which satisfies $T^\circ \cap \tau_1 = \phi$. Since in general $T^\circ \neq \phi$, we cannot simply appeal to the general theorem [6] that every partial order can be extended to a linear order. Let $\boldsymbol{\tau}$ be the family of all strict partial orders on T_1 which contain τ_1^* and are disjoint from T°. $\boldsymbol{\tau}$ is not empty, since it includes τ_1^*. Considered as subsets of $T_1 \times T_1$, the members of $\boldsymbol{\tau}$ are partially ordered by set-inclusion, and the union of any linearly-ordered subfamily of $\boldsymbol{\tau}$ is easily shown to be again in $\boldsymbol{\tau}$. By Zorn's Lemma, $\boldsymbol{\tau}$ has a maximal member τ_1. The proof will be completed by showing that τ_1 is a linear order.

Suppose, to the contrary, that there exist distinct elements t,t' of T_1 such that neither (t,t') nor (t',t) lies in τ_1. (It is in the balance of this paragraph that a more delicate argument than that given in [6] for the case $T^\circ = \phi$ is required.) At most one of t,t' can coincide with t°; if one of them does, choose the notation for t and t' so that $t = t^\circ$ and $t' \neq t^\circ$. Observe that at least one of (t,t') and (t',t) must fail to be a (t'',t''') with the property that $(t'',t^\circ) \varepsilon \tau^\circ$ and $(t''',t^\circ) \varepsilon \tau_1$ (this is true because $T^\circ \cap \tau_1 = \phi$); if either (t,t') or (t',t) is a (t'',t''') with the property just mentioned, choose the notation for t and t' so that (t',t) rather than (t,t') is such a (t'',t'''). (This precaution will only be needed if neither t nor t' coincides with t°.) Note for future use that in this case, there cannot be an $s \varepsilon T_1$ such that $(s,t) \varepsilon \tau_1$, $(t',t^\circ) \varepsilon \tau_1$ and $(s,t^\circ) \varepsilon T^\circ$. Observe further that at least one of (t,t') and (t',t) must fail to be a $(\bar{t}'',\bar{t}''')$ with the property that for some $s \varepsilon T_1$, the relations $(s,\bar{t}'') \varepsilon T_1$, $(\bar{t}''',t^\circ) \varepsilon \tau_1$ and $(s,t^\circ) \varepsilon T^\circ$ all hold. In this case, choose the notation for t and t' so that (t,t') is not a $(\bar{t}'',\bar{t}''')$ with the last-mentioned property. (This precaution will only be needed if neither t nor t' coincides with t°, and if in addition neither (t,t') nor (t',t) is a (t'',t''') of the type described above.)

Now set

$$\bar{\tau}_1 = \tau_1 \cup \{(t,t')\} \cup \{(s,t'): (s,t)\varepsilon\tau_1\} \cup \{(t,s'): (t',s')\varepsilon\tau_1\}$$

$$\cup \{(s,s'): (s,t), (t',s')\varepsilon\tau_1\}.$$

Then $\bar{\tau}_1$ is a binary relation on T_1 which is a proper extension of τ_1. A straightforward enumeration of cases, ignoring the care with which the notation for t and t' was chosen above, verifies that $\bar{\tau}_1$ is a strict partial order on T_1. With that care taken into account, another enumeration of cases verifies that $T^\circ \cap \bar{\tau}_1 = \phi$, yielding a contradiction to the maximality of τ_1 in $\mathbf{T}$. So τ_1 is a linear order, and Theorem 2 is proved.

Having considered what conditions rule out the introduction through aggregation of "extraneous" solutions (*EP*'s), we now turn to ruling out the *loss* of solutions under aggregation. A subset S of N will be called *unilateral-deviation strong* (*UDS*) if for each $x\varepsilon X$, in case $f_k(x,S,x_S)\varrho_k f_k(x)$ holds for any $k\varepsilon S$ and $x_S\varepsilon X_S$, it follows that $f_i(x,i,x_i)\varrho_i f_i(x)$ for some $i\varepsilon S$ and $x_i\varepsilon X_i$. This (somewhat unsatisfactory) terminology is intended to reflect the idea that the unilateral deviations desirable to the individual members of S (as deviators) are, taken together, sufficiently "strong" to reject any *status quo* x from which S as a collective could possibly deviate to the advantage of *any* of its members.

THEOREM 3. *If each* B_j *is* UDS, *and* x° *is an* EP *for* G, *then* x° *is also an* EP *for* G[B,F].

PROOF. Suppose, to the contrary, that $g_j(x^\circ,j,y_j)\tau_j g_j(x^\circ)$ for some $j\varepsilon M$ and some $y_j\varepsilon Y_j$. Because each of $\{\varrho_k : k\varepsilon B_j\}$ is a linear order and F_j is monotone, it follows (proof by contradiction) that

$$f_k(x^\circ,j,y_j)\varrho_k f_k(x^\circ)$$

must hold for at least one $k\varepsilon B_j$. From this, and the fact that B_j is *UDS*, there follows immediately a contradiction to the hypothesis that x° is an *EP* of *G*.

Note that *UDS*, applied to subset S of N, has the same "local to S" and "generality" properties described before Theorem 2. The next theorem shows that *UDS* is the *weakest* condition, with these two properties, which would suffice for an analog of Theorem 3.

THEOREM 4. *Let all the information needed to define* G *and* G[B,F] *be specified except for* (T_1,τ_1,F_1) *and* $\{f_i : i\varepsilon N - B_1\}$. *If* B_1 *is not* UDS, *then the missing information can be specified in such a way that* G *has an* EP *which is not an* EP *for* G[B,F].

PROOF. Write "$B(1)$" instead of "B_1" for subscripting purposes. Since B_1 is not *UDS*, there exists an $x\varepsilon X$, an $x_{B(1)}\varepsilon X_{B(1)}$, and a $k\varepsilon B_1$ such that

$$f_k(x,B_1,x_{B(1)})\varrho_k f_k(x), \tag{7}$$

$$f_i(x,i,x_i)\varrho_i f_i(x) \text{ for no } i\varepsilon B_1 \text{ and } x_i\varepsilon X_i. \tag{8}$$

Choose each of $\{f_i : i\varepsilon N - B_1\}$ to be constant; then (8) implies that x is an *EP* for *G*. Choose $T_1 = S_1$, and F_1 to be the identity map of S_1. Let $|B_1| = b$, and let $\{k(1),k(2),\ldots,k(b)\}$ be any enumeration of B_1 in which $k(1) = k$. Take τ_1 to be the lexicographic product of the sequence $\{\varrho_k = \varrho_{k(1)},\varrho_{k(2)},\ldots,\varrho_{k(b)}\}$; then τ_1 is a linear order and F_1 is monotone. It follows from (7) that $g_1(x,B_1,x_{B(1)})\tau_1 g_1(x)$, and so x cannot be an *EP* for *G*[*B*,*F*], completing the proof.

Combining the previous results, we see that if each B_j is both *NF* and *UDS*, then *G*[*B*,*F*] has the same set of *EP*'s as *G*. Furthermore, the conjunction "*NF* and *UDS*" is the *weakest* condition, with the properties of being "local" and "general" as defined earlier, which suffices for such a conclusion; one might well refer to this condition as "aggregability." The question of how this condition might be systematically and efficiently checked-for is a natural one, but will not be addressed here.

To relate the preceding material to that of [1], we define a subset S of N to be *limited-dependent* (LD) if, for each $i \varepsilon S$, $f_i(x)$ does not depend on the coordinates of x associated with the players in $S - \{i\}$. Theorems 1 and 2 of [1] are then equivalent to the following assertion.

THEOREM 5. *If each* B_j *is* LD, *then* G[B,F] *has the same set of* EP's *as* G.

PROOF. By Theorems 1 and 3, it suffices to show, if S is an LD subset of the players, that S is both UDU (hence, NF) and UDS. First, suppose some $x \varepsilon X$ is not $\{i\}$-stable for some $i \varepsilon S$, i.e. $f_i(x,i,x_i) \varrho_i f_i(x)$ for some $x_i \varepsilon X_i$. Since S is LD, $f_k(x,i,x_i) = f_k(x)$ for all $k \varepsilon S - \{i\}$. This shows that S is UDU. Next, consider any $x \varepsilon X$ such that $f_k(x,S,x_S) \varrho_k f_k(x)$ for some $k \varepsilon S$ and $x_S \varepsilon X_S$. Let x_k be the coordinate of x_S corresponding to player k; since S is LD, $f_k(x,S,x_S) = f_k(x,k,x_k)$ and so $f_k(x,k,x_k) \varrho_k f_k(x)$. This shows that S is UDS, completing the proof of the theorem.

To show that Theorem 5's sufficient condition for preservation of the solution-set was not also a necessary condition, the following example was given in [1]. It had $n = 2$, $m = 1$, and $B_1 = \{1,2\}$. Each $\varrho_i (i = 1,2)$ was the numerical ">" relation, $X_1 = \{A,B\}$, $X_2 = \{a,b\}$, and the payoff functions f_1 and f_2 were identical ($f_1 = f_2 = \bar{f}$) with

$$\bar{f}(A,a) = 2, \bar{f}(A,b) = \bar{f}(B,a) = 1, \bar{f}(B,b) = 0.$$

Here B_1 is not LD, but both G and $G[B,F]$—for any choice of (T_1,τ_1,F_1)—have (A,a) as the unique equilibrium point. It is easily verified that B_1 is both NF and UDS, so that this example is "explained" by Theorems 1 and 3 of the present paper.

To illustrate that the hypotheses of Theorems 1 and 3, though sufficient conditions for the Theorems' conclusions to hold, are not also necessary conditions, we will give an example in which NF and UDS both fail, but G and $G[B,F]$ have the same set of equilibrium points. We would like the set of G's equilibrium points to be a *proper* subset of X (so that aggregation has a fair chance to introduce one or more extraneous EP's), and to be nonempty (so that aggregation has a fair chance to lose one or more EP's). Also, to keep the example simple, we would like to have $n = 2$, $m = 1$, and $B_1 = \{1,2\} = N$. For an example meeting all these stipulations, take $(X_1,X_2,\varrho_1,\varrho_2)$ as in the last paragraph, but set

$$f_1(A,a) = 1, f_1(B,a) = 2, f_1(A,b) = f_1(B,b) = 0$$

and $f_2 = -f_1$. Then (in the notation introduced above eq (4)) $X^{\{1\}} = X - \{(A,a)\}$ and $X^{\{2\}} = \{(A,b), (B,b)\}$. Because G is zero-sum, $X^{\{1,2\}}$ is all of X, and comparison with (4) shows that NF fails. That UDS fails can be seen from the statement of this condition by taking $k = 1$, $x = (A,b)$, $x_S = (B,a)$. The set of equilibrium points of G is given by $X^{\{1\}} \cap X^{\{2\}} = X^{\{2\}}$, a nonempty proper subset of X. Now take $T_1 = S_1$, F_1 to be the identity map of T_1, and τ_1 to be the lexicographic product of the ordered pair (ϱ_2,ϱ_1); with this choice, $G[B,F]$ has the same equilibrium-point set as G.

In the remainder of this Section, we investigate how well the conditions appearing above—NF, UDU, UDS, "NF and UDS," and LD—conform to some intuitively plausible requirements for representing the notion of "parallel interests." Let $\mathcal{G}$ be a family of nonempty subsets of N, which includes all singletons. We will say that $\mathcal{G}$ is *equivalence-derived* if there is an equivalence relation on N such that subset S of N belongs to $\mathcal{G}$ if and only if S lies in a single equivalence class. Also, we will call $\mathcal{G}$ *hereditary* if $\emptyset \neq T \subset S$ and $S \varepsilon \mathcal{G}$ imply $T \varepsilon \mathcal{G}$, and will call $\mathcal{G}$ *connected* if $S \varepsilon \mathcal{G}$, $T \varepsilon \mathcal{G}$ and $S \cap T \neq \emptyset$ imply $S \cup T \varepsilon \mathcal{G}$. All three of these properties are plausible requirements if $\mathcal{G}$ is to represent the family of all subsets whose players "have parallel interests." It is easy to show that a family $\mathcal{G}$ is equivalence-derived if and only if it is both hereditary and connected, so that we will deal with these last two properties.

None of the conditions appearing in the previous theorems necessarily yields a connected family of subsets of N. To show this, it suffices to give an example in which subsets S and T of the players are both LD (hence NF, UDU and UDS), satisfy $S \cap T \neq \emptyset$, but have $S \cup T$ neither NF nor UDS. Such an example can be constructed from the preceding one by adjoining a third player as "dummy." Specifically, take $(X_1,X_2,\varrho_1,\varrho_2)$ as before; (X_3,ϱ_3) need not be specified, for what follows. Adjoin a dummy argument $x_3 \varepsilon X_3$ to the previous definitions of f_1 and f_2, and take f_3 to be constant. Choosing $S = \{1,3\}$ and $T = \{2,3\}$ then yields an example with the desired characteristics.

It is easily shown that the *LD* subsets of *N* form a hereditary family, as do the *UDU* subsets. To dispose of the "hereditary question" for the remaining properties, it suffices to give an example in which subsets *S* and *T* of the players satisfy $\emptyset \neq T \subset S$, *S* is both *NF* and *UDS*, but *T* is neither *NF* nor *UDS*. This example, too, will be constructed from a preceding one by suitably adjoining a third player. Take $(X_1, X_2, \varrho_1, \varrho_2)$ as before, and take $X_3 = \{\alpha, \beta\}$; ϱ_3 is the numerical ">" relation. Following the model of the previous example, let

$$f_1(A,a,\alpha) = 1, f_1(B,a,\alpha) = 2, f_1(A,b,\alpha) = f_1(B,b,\alpha) = 0$$

and $f_2(x_1,x_2,\alpha) = -f_1(x_1,x_2,\alpha)$ for all $(x_1,x_2) \varepsilon X_1 \times X_2$. This is already enough to assure that $T = \{1,2\}$ is neither *NF* nor *UDS*. To make $S = \{1,2,3\}$ both *NF* and *UDS*, set

$$f_1(x_1,x_2,\beta) = 3 \text{ for all } (x_1,x_2) \varepsilon X_1 \times X_2,$$

$$f_2(x_1,a,\beta) = 3, f_2(x_1,b,\beta) = 4 \text{ for all } x_1 \varepsilon X_1,$$

$$f_3(x_1,x_2,\alpha) = 0 \text{ for all } (x_1,x_2) \varepsilon X_1 \times X_2,$$

$$f_3(x_1,a,\beta) = 0, f_3(x_1,b,\beta) = 1 \text{ for all } x_1 \varepsilon X_1.$$

An initial impression from these findings might be, that despite the motivating observation with which the present paper began, aggregability of a subset of the players does not really have much to do with the parallelism of those players' interests. I presently prefer an alternative interpretation, namely that the findings are corrective to an implicit assumption that parallelism of players' interests must (like parallelism of sets of lines in the Euclidean plane) be viewed as equivalence-derived from a binary relation of "parallel interests" between individual *pairs* of players. The formulation of a different concept of parallel interests for a subset of the players, and the study of the relationship between that concept and aggregability, are planned for a subsequent paper.

4. Approximations

Since the topics of this section deal with quantitative rather than qualitative relationships, we now take all sets R_i and T_j to be the set of real numbers, and take all the relations ϱ_i and τ_j to be the ordinary numerical "greater than" relation. Theorems 1 and 3 involve the notions of an equilibrium point, of an aggregation of a given game *G*, and of conditions *NF* and *UDS*. The definitions of these concepts involve elements of an essentially order-theoretic nature (the linear-order properties of ϱ_i and τ_j; the monotonicity of F_j). To obtain "approximate" versions of the theorems, it will be necessary to replace these "ordinal" elements by suitable "cardinal" ones. The particular replacements introduced below appear reasonable, but other plausible alternatives may be more appropriate in particular contexts.

For each $i \varepsilon N$ and each $x \varepsilon X$, the quantity

$$M_i(x) = \sup\{f_i(x,i,x_i) : x_i \varepsilon X_i\} - f_i(x)$$

is nonnegative. If $\delta = (\delta_1, \ldots, \delta_n)$ is an *n*-tuple of positive real numbers, and if $x^\circ \varepsilon X$ satisfies

$$M_i(x^\circ) \leqslant \delta_i \text{ (all } i \varepsilon N), \tag{9}$$

then x° will be called a δ-*EP* of game *G*. Approximate *EP*'s of $G[B,F]$ are defined similarly.

We will say that F_j is (k_j^+,k_j^-)-*bounded monotone* in each argument, where k_j^+ and k_j^- are positive constants, if for each $s_j \varepsilon S_j$, and for any $i \varepsilon B_j$ with r_i the i-th coordinate of s_j, and for any $r_i' \varepsilon R_i$,

$$r_i' > r_i \text{ implies } 0 < k_j^- \leqslant [F_j(s_j,i,r_i') - F_j(s_j)]/(r_i' - r_i) \leqslant k_j^+. \tag{10}$$

This hypothesis expresses the plausible idea that an aggregate player's payoff should respond "in a bounded manner" to changes in the payoffs of the individual players comprising the aggregate. Note that (10) holds with $k_j^+ = k_j^- = 1$ if $F_j(s_j)$ is the sum (as in [5]) of the coordinates of s_j.

For a given $c > 0$, a subset S of N will be called $NF(c)$ if for all $x \varepsilon X$, if there is an $i \varepsilon S$ and an $x_i \varepsilon X_i$ such that $f_i(x,i,x_i) > f_i(x)$, then there is an $x_S \varepsilon X_S$ such that $f_k(x,S,x_S) \geqslant f_k(x)$ for all $k \varepsilon S$ and

$$f_k(x,S,x_S) - f_k(x) \geqslant c\,[f_i(x,i,x_i) - f_i(x)] \tag{11}$$

holds for at least one $k \varepsilon S$. (The intuitive interpretation of this condition follows readily from that given earlier for NF.)

THEOREM 6. *Assume* F_j *is* (k_j^+,k_j^-)*-bounded monotone and* B_j *is* $NF(c_j)$ *for all* $j \varepsilon M$. *Let* δ *be a positive* n*-tuple and* δ' *a positive* m*-tuple such that* $\delta_i \geqslant \delta'_{j(i)}/k_{j(i)}^- c_{j(i)}$ *for all* $i \varepsilon N$. *If* x° *is a* δ'-EP *for* G[B,F], *then* x° *is also a* δ-EP *for* G.

PROOF. Suppose, to the contrary, that $f_i(x^\circ,i,x_i) - f_i(x^\circ) > \delta_i$ for some $i \varepsilon S$ and some $x_i \varepsilon X_i$. Let $j = j(i)$. Since B_j is $NF(c_j)$, there is a $y_j \varepsilon Y_j$ such that $f_k(x^\circ,j,y_j) \geqslant f_k(x^\circ)$ for all $k \varepsilon B_j$, with

$$f_k(x^\circ,j,y_j) - f_k(x^\circ) \geqslant c_j\,[f_i(x^\circ,i,x_i) - f_i(x^\circ)] > c_j \delta_i \tag{12}$$

holding for at least one $k \varepsilon B_j$. Consider changing the arguments of $F_j[\{f_k(x^\circ) : k \varepsilon B_j\}]$ one at a time to those of $F_j[\{f_k(x^\circ,j,y_j) : k \varepsilon B_j\}]$, beginning with a $k \varepsilon B_j$ for which (12) holds. It follows from (10) and the last display that

$$g_j(x^\circ,j,y_j) - g_j(x^\circ) > k_j^- c_j \delta_i,$$

yielding a contradiction to the assumption that x° is a δ'-EP for G[B,F].

Theorem 6 is an "approximate" version of Theorem 1. To obtain an "approximate" version of Theorem 3, we first define subset S of N to be $UDS(c)$, where $c > 0$, if for all $x \varepsilon X$, if $f_k(x,S,x_S) > f_k(x)$ holds for some $k \varepsilon S$ and $x_S \varepsilon X_S$, then

$$f_i(x,i,x_i) - f_i(x) \geqslant c\,[f_k(x,S,x_S) - f_k(x)]$$

holds for some $i \varepsilon S$ and $x_i \varepsilon X_i$. (The intuitive interpretation follows from that given earlier for UDS.)

THEOREM 7. *Assume* F_j *is* (k_j^+,k_j^-)*-bounded monotone and* B_j *is* $UDS(c_j)$ *for all* $j \varepsilon M$. *Let* δ *be a positive* n*-tuple and* δ' *a positive* m*-tuple such that* $\delta_i \leqslant c_{j(i)}\delta'_{j(i)}/k_{j(i)}^+|B_{j(i)}|$ *for all* $i \varepsilon N$. *If* x° *is a* δ-EP *for* G, *then* x° *is also a* δ'-EP *for* G[B,F].

PROOF. Suppose, to the contrary, that $g_j(x^\circ,j,y_j) - g_j(x^\circ) > \delta_j'$ for some $j \varepsilon M$ and $y_j \varepsilon Y_j$. Consider changing the arguments of $F_j[\{f_k(x^\circ) : k \varepsilon B_j\}]$ one at a time to those of $F_j[\{f_k(x^\circ,j,y_j) : k \varepsilon B_j\}]$; it follows from (10) that

$$g_j(x^\circ,j,y_j) - g_j(x^\circ) \leqslant k_j^+ \,\Sigma\{\max[0, f_k(x^\circ,j,y_j) - f_k(x^\circ)]: k \varepsilon B_j\}$$

$$\leqslant k_j^+ |B_j| \max\{f_k(x^\circ,j,y_j) - f_k(x^\circ) : k \varepsilon B_j\}.$$

Thus there must be a $k \varepsilon B_j$ for which

$$f_k(x^\circ,j,y_j) - f_k(x^\circ) > \delta'_j / k_j^+ |B_j|.$$

Since B_j is $UDS(c_j)$, it follows that there must be an $i \varepsilon B_j$ and an $x_i \varepsilon X_i$ such that

$$f_i(x^\circ,i,x_i) - f_i(x^\circ) > c_j \delta'_j / k_j^+ |B_j|,$$

yielding a contradiction to the assumption that x° is a δ-EP for G.

Note that the presence of k_j^+ in (10) is required for Theorem 7 but not for Theorem 6, and *vice versa* for k_j^- (so long as F_j is assumed monotone). The two theorems are not intended to apply simultaneously to the same pair (δ,δ'). Theorem 6 directly generalizes Theorem 3 of [1], an extension of which is obtained by taking all $c_j = 1$, all $k_j^+ = k_j^- = 1$, and each $F_j(s_j)$ to be the sum of the coordinates of s_j. Making the same choices in Theorem 7 yields a result closely related to Theorem 4 of [1].

5. References

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