Bilinear Fractal Interpolation and Box Dimension

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Abstract

In the context of general iterated function systems (IFSs), we introduce bilinear fractal interpolants as the fixed points of certain Read-Bajraktarević operators. By exhibiting a generalized "taxi-cab" metric, we show that the graph of a bilinear fractal interpolant is the attractor of an underlying contractive bilinear IFS. We present an explicit formula for the box-counting dimension of the graph of a bilinear fractal interpolant in the case of equally spaced data points.

Keywords: Iterated function system (IFS), attractor, fractal interpolation, Read-Bajraktarević operator, bilinear mapping, bilinear IFS, box counting dimension
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1. Introduction

Bilinear filtering or bilinear interpolation is used in computer graphics to compute intermediate values for a two-dimensional regular grid. One of the main objectives is the smoothening of textures when they are enlarged or reduced in size. In mathematical terms, the interpolation technique is based on finding a function $f(x, y)$ of the form $f(x, y) = a + bx + cy + dxy$, where $a, b, c, d \in \mathbb{R}$, that passes through prescribed data points.

As textures reveal, in general, a non-smooth or even fractal characteristic, a description in terms of fractal geometric methods seems reasonable. To this end, the classical bilinear approximation method is replaced by a bilinear

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fractal interpolation procedure. The latter allows for additional parameters, such as the box dimension, that are related to the regularity and appearance of an underlying texture pattern.

We introduce a class of fractal interpolants that are based on bilinear functions of the above form. We do this by considering a more general class of iterated function systems (IFSs) and by using a more general definition of attractor of an IFS. These more comprehensive concepts are primarily based on topological considerations. In this context, we extend and correct some known results from [15] concerning fractal interpolation functions that are fixed points of so-called Read-Bajraktarević operators. Theorem 4 relates the fixed point in Theorem 3 to the attractor of an IFS and generalizes known results to the case where the IFS is not contractive.

As a special example of the preceding theory we introduce bilinear fractal interpolants and show that their graphs are the attractors of an underlying contractive bilinear IFS. Such bilinear IFSs have been investigated in [5] in connection with fractal homeomorphisms and address structures underlying an IFS. Finally, we present an explicit formula for the box dimension of the graph of a bilinear fractal interpolant in the case where the data points are equally spaced.

2. General iterated function systems

The terminology here for iterated function system, attractor, and contractive iterated function system is from [4]. Throughout this paper, $(\mathbb{X}, d)$ denotes a complete metric space with metric $d = d_{\mathbb{X}}$.

Definition 1. *Let $N \in \mathbb{N} := \{1, 2, 3, \ldots\}$. If $f_n : \mathbb{X} \to \mathbb{X}$, $n = 1, 2, \ldots, N$, are continuous mappings, then $\mathcal{F} = (\mathbb{X}; f_1, f_2, ..., f_N)$ is called an **iterated function system** (IFS).*

By slight abuse of terminology we use the same symbol $\mathcal{F}$ for the IFS, the set of functions in the IFS, and for the following mappings. We define $\mathcal{F} : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ by

$$\mathcal{F}(B) := \bigcup_{f \in \mathcal{F}} f(B)$$

for all $B \in 2^{\mathbb{X}}$, the set of subsets of $\mathbb{X}$. Let $\mathbb{H} = \mathbb{H}(\mathbb{X})$ be the set of nonempty compact subsets of $\mathbb{X}$. Since $\mathcal{F}(\mathbb{H}) \subset \mathbb{H}$, we can also treat $\mathcal{F}$ as a mapping $\mathcal{F} : \mathbb{H} \to \mathbb{H}$. When $U \subset \mathbb{X}$ is nonempty, we may write $\mathbb{H}(U) = \mathbb{H}(\mathbb{X}) \cap 2^U$. We denote by $|\mathcal{F}|$ the number of distinct mappings in $\mathcal{F}$.

Let $d_{\mathbb{H}}$ denote the Hausdorff metric on $\mathbb{H}$, defined in terms of $d_{\mathbb{X}}$. A convenient definition (see for example [7, p.66]) is

$$d_{\mathbb{H}}(B, C) := \inf\{r > 0 : B \subset C + r, C \subset B + r\},$$

for all $B, C \in \mathbb{H}$. For $S \subset \mathbb{X}$ and $r > 0$, $S + r$ denotes the set $\{y \in \mathbb{X} : \exists x \in S$ so that $d_{\mathbb{X}}(x, y) < r\}$.

We say that a metric space $\mathbb{X}$ is *locally compact* to mean that if $C \subset \mathbb{X}$ is compact and r is a positive real number then $\overline{C + r}$ is compact. Here, $\overline{S}$ denotes the closure of a set S. (For an equivalent definition of local compactness, see for instance [8, 3.3].)

The following information is foundational.

Theorem 1. *(i) The metric space* $(\mathbb{H}, d_{\mathbb{H}})$ *is complete.*

(ii) If $(\mathbb{X}, d_{\mathbb{X}})$ *is compact then* $(\mathbb{H}, d_{\mathbb{H}})$ *is compact.*

(iii) If $(\mathbb{X}, d_{\mathbb{X}})$ *is locally compact then* $(\mathbb{H}, d_{\mathbb{H}})$ *is locally compact.*

(iv) If $\mathbb{X}$ *is locally compact, or if each* $f \in \mathcal{F}$ *is uniformly continuous, then* $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ *is continuous.*

(v) If $f : \mathbb{X} \to \mathbb{X}$ *is a contraction mapping for each* $f \in \mathcal{F}$*, then* $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ *is a contraction mapping.*

Proof. (i) This is well-known. A short proof can be found in [7, p.67, Theorem 2.4.4].

(ii) This is well-known; see for example [10]. Here is a short proof. Let $\varepsilon > 0$ be given. Since $\mathbb{X}$ is compact we can find a finite set of points $\mathbb{X}_\varepsilon \subset \mathbb{X}$ such that $\mathbb{X} = \cup_{x \in \mathbb{X}_\varepsilon} \mathcal{B}(x, \varepsilon)$ where $\mathcal{B}(x, \varepsilon) \subset \mathbb{X}$ denote the open ball with center at x and radius ε. Let $\mathbb{H}_\varepsilon := 2^{\mathbb{X}_\varepsilon}$, a finite set of points in $\mathbb{H}$. It is readily verified that $\mathbb{H} = \cup_{C \in \mathbb{H}_\varepsilon} \mathcal{B}(C, \varepsilon)$ where now $\mathcal{B}(C, \varepsilon) \subset \mathbb{H}$ denotes the open ball with center at $C \in \mathbb{H}$ and radius ε, measured using the Hausdorff metric. It follows that $\mathbb{H}$ is totally bounded. It follows that $\mathbb{H}$ is compact.

(iii) Let $C \in \mathbb{H}$. Consider the set $\overline{C + r}$. It belongs to $\mathbb{H}$ since $\mathbb{X}$ is locally compact. Let $\varepsilon > 0$ be given. Since $\overline{C + r}$ is a compact subset of $\mathbb{X}$ we can find a finite set of points $C_\varepsilon \subset \overline{C + r}$ such that $\overline{C + r} \subset \cup_{c \in C_\varepsilon} \mathcal{B}(c, \varepsilon)$. Let $\mathbb{C}_\varepsilon := 2^{C_\varepsilon}$, a finite set of points in $\mathbb{H}$. It is readily verified that $\overline{C + r} \subset \cup_{c \in \mathbb{C}_c} \mathcal{B}(C, \varepsilon)$ where now $\mathcal{B}(c, \varepsilon) \subset \mathbb{H}$ denotes the open ball with center at $c \in \mathbb{H}$ and radius ε, measured using the Haudorff metric. It follows that $\overline{C + r}$ is totally bounded. It follows that $\overline{C + r}$ is compact.

(iv) Let $B \in \mathbb{H}$. We show that $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ is continuous at B. We restrict attention to the action of $\mathcal{F}$ on $B+1$. If $\mathbb{X}$ is locally compact, it follows that $\overline{B+1}$ is compact. It follows that each $f \in \mathcal{F}$ is uniformly continuous on $\overline{B+1}$. It follows that if $\mathbb{X}$ is locally compact, or if each $f \in \mathcal{F}$ is uniformly continuous, we can find $\delta_\varepsilon > 0$ such that $d_{\mathbb{X}}(f(x), f(y)) < \varepsilon$ whenever $d_{\mathbb{X}}(x,y) < \delta_\varepsilon$, for all $x, y \in \overline{B+1}$, for all $f \in \mathcal{F}$. Let $C \in \mathbb{H}$ with $d_{\mathbb{H}}(B,C) < \delta_\varepsilon$ and let $f \in \mathcal{F}$. We can suppose that $\delta_\varepsilon < 1$.

Let $b' \in f(B)$. Then there is $b \in B$ such that $f(b) = b'$. Since $d_{\mathbb{H}}(B,C) < \delta_\varepsilon$ there is $c \in C$ such that $d(b,c) < \delta_\varepsilon$. Since $\delta_\varepsilon < 1$ we have $c \in B+1$. It follows that $d(f(b), f(c)) < \varepsilon$. It follows that $f(B) \subset f(C) + \varepsilon$. By a similar argument $f(C) \subset f(B) + \varepsilon$. Hence $d_{\mathbb{H}}(f(B), f(C)) < \varepsilon$.

(v) This is Hutchinson's theorem, [11, p. 731], proved as follows. Verify that if $\lambda \geq 0$ is a uniform Lipschitz constant for all $f \in \mathcal{F}$, namely $d_{\mathbb{X}}(f(x), f(y)) \leq \lambda d_{\mathbb{X}}(x,y)$ for all $f \in \mathcal{F}$, for all $x, y \in \mathbb{X}$, then λ is also a Lipschitz constant for $\mathcal{F}$, namely $d_{\mathbb{H}}(\mathcal{F}(B), \mathcal{F}(C)) \leq \lambda d_{\mathbb{H}}(B,C)$ for all $B, C \in \mathbb{H}$. If $f : \mathbb{X} \to \mathbb{X}$ is a contraction mapping for each $f \in \mathcal{F}$, then we can choose $\lambda < 1$. It follows that $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ is a contraction mapping. $\square$

For $B \subset \mathbb{X}$, let $\mathcal{F}^k(B)$ denote the k-fold composition of $\mathcal{F}$, i.e., the union of $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(B)$ over all finite words $i_1 i_2 \cdots i_k$ of length k over the alphabet $\{1, \ldots, N\}$. Define $\mathcal{F}^0(B) := B$.

Definition 2. *A nonempty compact set $A \subset \mathbb{X}$ is said to be an* ***attractor*** *of the IFS $\mathcal{F}$ if*

(i) $\mathcal{F}(A) = A$ *and*

(ii) there exists an open set $U \subset \mathbb{X}$ such that $A \subset U$ and $\lim_{k\to\infty} \mathcal{F}^k(B) = A$, for all $B \in \mathbb{H}(U)$, where the limit is with respect to the Hausdorff metric.

The largest open set U such that (ii) *is true is called the* ***basin of attraction*** *(for the attractor A of the IFS $\mathcal{F}$).*

Note that if U_1 and U_2 satisfy condition (ii) in Definition 2 for the same attractor A then so does $U_1 \cup U_2$. We also remark that the invariance condition (i) is not needed; it follows from (ii) for $B := A$.

Example 1. *An IFS $\mathcal{F} = (\mathbb{X}; f_1, f_2, \ldots, f_N)$ is called* ***contractive*** *if each $f \in \mathcal{F}$ is a contraction (with respect to the metric d), i.e., if there is a* *constant $s \in [0,1)$ such that $d(f(x_1), f(x_2)) \leq s\, d(x_1, x_2)$, for all $x_1, x_2 \in \mathbb{X}$. By item* (v) *in Theorem 1, the mapping $\mathcal{F} : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ is then also contractive on the complete metric space $(\mathbb{H}(\mathbb{X}); d_{\mathbb{H}})$ and thus possesses a unique attractor A. In this case, the basin of attraction for A is $\mathbb{X}$.*

We will use the following observation [13, Proposition 3 (vii)], [7, Proposition 2.5.6].

Lemma 1. *Let $\{B_k\}_{k=1}^{\infty}$ be a sequence of nonempty compact sets such that $B_{k+1} \subset B_k$, for all k. Then $\cap_{k\geq 1} B_k = \lim_{k\to\infty} B_k$ where convergence is with respect to the Haudorff metric.*

Theorem 2. *Let $\mathcal{F}$ be an IFS with attractor A and basin of attraction U. If $\mathcal{F} : \mathbb{H}(U) \to \mathbb{H}(U)$ is continuous then*

$$A = \bigcap_{K\geq 1} \overline{\bigcup_{k\geq K} \mathcal{F}^k(B)} \quad \text{for all } B \subset U \text{ such that } \overline{B} \in \mathbb{H}(U).$$

The quantity on the right-hand side here is sometimes called the *topological upper limit* of the sequence $\left\{\mathcal{F}^k(B)\right\}_{k=1}^{\infty}$.

Proof. Note that the proof can be carried out under the assumption that $B \in \mathbb{H}(U)$. (It then follows from [12, Proposition 3(i)] that Theorem 2 is true for all $B \subset U$ such that $\overline{B} \in \mathbb{H}(U)$.) Under this assumption and with the fact that $\overline{\bigcup_k \mathcal{F}^k(B)} = \overline{\bigcup_k \overline{\mathcal{F}^k(B)}}$ (see, for instance, [8]) the statement follows from Theorem 3.82 in [1]. □

We will also need the following observation.

Lemma 2. *Let $\mathbb{X}$ be locally compact. Let $\mathcal{F} = (\mathbb{X}; f_1, f_2, ..., f_N)$ be an IFS with attractor A and basin of attraction U. For any given $\varepsilon > 0$ there is an integer L such that for each $x \in A + \varepsilon$ there is an integer $l \leq L$ such that*

$$d_{\mathbb{H}}(A, \mathcal{F}^l(\{x\})) < \varepsilon.$$

Proof. For each $x \in \overline{A + \varepsilon}$ there is an integer $l(x, \varepsilon)$ so that $d_H(A, \mathcal{F}^{l(x,\varepsilon)}(\{x\})) < \varepsilon/2$.

Since $\mathbb{X}$ is locally compact it follows that $\mathcal{F}^{l(x,\varepsilon)} : \mathbb{H} \to \mathbb{H}$ is continuous. Since $\mathcal{F}^{l(x,\varepsilon)} : \mathbb{H} \to \mathbb{H}$ is continuous there is an open neighborhood $N(\{x\})$ (in $\mathbb{H}$) of $\{x\}$ such that $d_{\mathbb{H}}(A, \mathcal{F}^{l(x,\varepsilon)}(Y)) < \varepsilon$ for all $Y \in N(\{x\})$. It follows, in particular, that there is an open neighborhood $N(x)$ (in $\mathbb{X}$) of x such that $d_{\mathbb{H}}(A, \mathcal{F}^{l(x,\varepsilon)}(\{y\})) < \varepsilon$ for all $y \in N(x)$. Also since $\mathbb{X}$ is locally compact, there is a finite set of points $\{x_1, x_2, ..., x_q\}$ such that $\overline{A + \varepsilon} \subset \cup_{i=1}^{q} N(x_i)$. Choose $L := \max_i l(x_i, \varepsilon)$. □

3. Fractal interpolants as fixed points of operators

Let $\{(X_j, Y_j) : j = 0, 1, ..., N\}$ denote the cartesian coordinates of a finite set of points in the Euclidean plane, with

$$X_0 < X_1 < ... < X_N.$$

Let I denote the closed interval $[X_0, X_N]$. For $n = 1, 2, ..., N$, let $l_n : I \to [X_{n-1}, X_n]$ be a continuous bijection. Let $L : I \to I$ be such that

$$L(x) := l_n^{-1}(x) \text{ for } x \in [X_{n-1}, X_n)$$

for $n = 1, 2, ..., N$ (with the tacit understanding that for $n = N$ the interval is $[X_{N-1}, X_N]$). Let $S : I \to \mathbb{R}$ be bounded and piecewise continuous, where the only possible discontinuities are finite jumps occuring at the points in $\{X_1, X_2, ..., X_{N-1}\}$. Let

$$s := \max\{|S(x)| : x \in I\}.$$

Denote by $C = C(I)$ the set of continuous functions $f : I \to \mathbb{R}$. It is well-known that (C, d_∞) is a complete metric space, where

$$d_\infty(f, g) = \max\{|f(x) - g(x)| : x \in I\}.$$

Let

$$C^* := \{f \in C : f(X_0) = Y_0, f(X_N) = Y_N\},$$
$$C^{**} := \{f \in C : f(X_j) = Y_j \text{ for } j = 0, 1, ..., N\}.$$

Note that C^* and C^{**} are closed subspaces of C with $C^{**} \subset C^* \subset C$. We say that each of the functions in C^{**} *interpolates the data* $\{(X_j, Y_j) : j = 0, 1, \ldots, N\}$.

Let $b \in C^*$ and $h \in C^{**}$. Define $T : C^* \to C^{**}$ by

$$Tg := h + S \cdot (g \circ L - b \circ L). \tag{3.1}$$

T is a form of Read-Bajraktarević operator as defined in [15]. The following result is a corrected version of [15, Theorem 5.1, p. 136]. See also [11, Theorem 3, p. 731].

Theorem 3. *The mapping $T : C^* \to C^{**}$ obeys*

$$d_\infty(Tg_1, Tg_2) \leq s\, d_\infty(g_1, g_2)$$

for all $g_1, g_2 \in C^*$. In particular, if $s < 1$ then T is a contraction and it possesses a unique fixed point $f \in C^{**}$.

Proof. The operator T is well-defined. Indeed, for $i = 1, \ldots, N-1$,

$$Tg(X_i-) = h(X_i) = Tg(X_i+).$$

To prove contractivity in the Chebyshev norm $\|\cdot\|_\infty$, observe that

$$\begin{aligned} d_\infty(Tg_1, Tg_2) &= \max\{|S(x)(g_1(L(x)) - g_2(L(x)))| : x \in I\} \\ &\leq s \max\{\left|(g_1(l_n^{-1}(x)) - g_2(l_n^{-1}(x)))\right| : x \in [X_{n-1}, X_n], n = 1, 2, ..., N\} \\ &= s\, d_\infty(g_1, g_2). \end{aligned}$$

The existence of a unique fixed point $f \in C$ (when $s < 1$) follows from the contraction mapping theorem. Since $f(C^*) \subset C^{**}$ and (C^{**}, d_∞) is closed, hence complete, it follows that $f \in C^{**}$. □

Note that $Tg = H + S \cdot g \circ L$ where $H = h - S \cdot b \circ L$. This tells us that a fractal interpolation function f is uniquely defined by three functions H, S, and L, of the special forms defined above.

The fixed point f of T interpolates the data $\{(X_j, Y_j) : j = 0, 1, 2, ..., N\}$ and is an example of a fractal interpolation function [2]. One way to evaluate f is to use

$$f = \lim_{k\to\infty} T^k(f_0),$$

where $f_0 \in C^*$. The proof of the contraction mapping theorem gives also an estimate for the rate of convergence (cf. [18]], Theorem 5.2.3.):

$$\left\|f - T^k(f_0)\right\|_\infty \leq \frac{s^k}{1-s}\, \|f_1 - f_0\|_\infty\,. \tag{3.2}$$

In addition, an estimate for the operator T can also be derived (cf. [16]):

$$\|T\|_\infty \leq \frac{1+s}{1-s}.$$

4. The metric space $(I \times \mathbb{R}, d_q)$

The following metric generalizes the "taxi-cab" metric. We will need it in the proof of Theorem 4.

Proposition 1. *Let $\alpha, \beta > 0$ and $q : I \to \mathbb{R}$. Let $d_q : (I \times \mathbb{R}) \times (I \times \mathbb{R}) \to [0, \infty)$ be defined by*

$$d_q((x_1, y_1), (x_2, y_2)) := \alpha |x_1 - x_2| + \beta |(y_1 - q(x_1)) - (y_2 - q(x_2))|,$$

for all (x_1, y_1), $(x_2, y_2) \in I \times \mathbb{R}$. Then d_q is a metric on $I \times \mathbb{R}$. If q is continuous then $(I \times \mathbb{R}, d_q)$ is a complete metric space.

Proof. Clearly $d_q((x_2, y_2), (x_1, y_1)) = d_q((x_1, y_1), (x_2, y_2)) \geq 0$. Suppose that $d_q((x_1, y_1), (x_2, y_2)) = 0$. Then $\alpha |x_1 - x_2| + \beta |(y_1 - q(x_1)) - (y_2 - q(x_2))| = 0$ which implies $x_1 = x_2$. Hence $|(y_1 - q(x_1)) - (y_2 - q(x_1))| = 0$ which implies $y_1 = y_2$.

Demonstration that d obeys the triangle inequality. Let $(x_i, y_i) \in I \times \mathbb{R}$, for $i = 1, 2, 3$. Write $q_i = q(x_i)$ for $i = 1, 2, 3$. We have

$$\begin{aligned}
&d_q((x_1, y_1), (x_2, y_2)) + d_q((x_2, y_2), (x_3, y_3)) \\
&= \alpha |x_1 - x_2| + \beta |(y_1 - q_1) - (y_2 - q_2)| + \alpha |x_2 - x_3| + \beta |(y_2 - q_2) - (y_3 - q_3)| \\
&= \alpha(|x_1 - x_2| + |x_2 - x_3|) + \beta(|(y_1 - q_1) - (y_2 - q_2)| + |(y_2 - q_2) - (y_3 - q_3)|) \\
&\geq \alpha(|x_1 - x_3|) + \beta(|(y_1 - q_1) - (y_2 - q_2)| + |(y_2 - q_2) - (y_3 - q_3)|) \\
&\geq \alpha(|x_1 - x_3|) + \beta(|(y_1 - q_1) - (y_3 - q_3)|) = d_q((x_1, y_1), (x_3, y_3)).
\end{aligned}$$

To prove completeness in the case that q is continuous, let $\{(x_k, y_k)\}_{k=1}^{\infty}$ denote a Cauchy sequence with respect to the metric d_q. Given $\varepsilon > 0$ we can find an integer $N(\varepsilon)$ so that

$$\alpha |x_k - x_l| + \beta |(y_k - q(x_k)) - (y_l - q(x_l))| < \varepsilon$$

whenever $k, l > N(\varepsilon)$. It follows that $\{x_k\}$ is a Cauchy sequence with respect to the Euclidean norm, and so it converges, with limit $x^* \in I$. Since q is continuous, it now follows that $\{q(x_k)\}$ converges to some limit $q^* \in \mathbb{R}$. In turn, it follows that $\{y_k\}$ converges to some $y^* \in \mathbb{R}$. Hence $\{(x_k, y_k)\}_{k=1}^{\infty}$ converges to $(x^*, y^*) \in I \times \mathbb{R}$. It follows that $(I \times \mathbb{R}, d_q)$ is complete. ☐

5. Fractal interpolants as attractors of iterated function systems

Here we characterize the graph of the fixed point f of T as an attractor of an IFS. Define $w_n : I \times \mathbb{R} \to I \times \mathbb{R}$ by

$$w_n(x,y) := (l_n(x), h(l_n(x)) + S(l_n(x))(y - b(x))).$$

Define an IFS by

$$\mathcal{W} := (I \times \mathbb{R}; w_1, w_2, ..., w_N).$$

Here we make use of the metric d_q of Proposition 1 with $q = f$, the fixed point of T. Let $\eta > 0$ and let

$$\mathbb{X} := \{(x,y) \in I \times \mathbb{R} : |y - f(x)| \leq \eta\}.$$

It is readily verified that, when Theorem 3 holds, namely when $s < 1, \mathcal{W}(\mathbb{X}) \subset \mathbb{X}$. The following theorem gives conditions under which (i) the IFS $(\mathbb{X}; w_1, w_2, ..., w_N)$ is contractive with respect to d_f and (ii) $\mathcal{W}$ has a unique attractor. This result is a substantial generalization of [15, Theorem 5.3, p. 140] which would require, in the present setting, that h is uniformly Lipschitz. Here, we avoid this restriction by using the metric d_q with $q = f$.

Theorem 4. *Let $s < 1$ and let $f \in C^{**}$ be the fixed point of T, as in Theorem 3. Let $l_n : I \to I$ have uniform Lipschitz constant $\lambda_l < 1$, such that $|l_n(x_1) - l_n(x_2)| \leq \lambda_l |x_1 - x_2|$ for all $x_1, x_2 \in I$, for all n. Let $S : I \to [-s, s]$ have Lipschitz constant λ_S, so that $|S(x_1) - S(x_2)| \leq \lambda_S |x_1 - x_2|$ for all $x_1, x_2 \in I$. Then the IFS $(\mathbb{X}; w_1, w_2, ..., w_N)$ is contractive with respect to the metric d_f with $\alpha = 1$ and $0 < \beta < (1 - \lambda_l)/\lambda_l \lambda_S \eta$. In particular, under these conditions, the IFS $\mathcal{W}$ has a unique attractor $A = \Gamma(f)$, the graph of f, with basin of attraction $I \times \mathbb{R}$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{X}$. We have

$$\begin{aligned}
&d_f(w_n(x_1,y_1), w_n(x_2,y_2)) - \alpha |l_n(x_1) - l_n(x_2)| \\
&= \beta |h(l_n(x_1)) + S(l_n(x_1))(y_1 - b(x_1)) - f(l_n(x_1)) \\
&- (h(l_n(x_2)) + S(l_n(x_2))(y_2 - b(x_2)) - f(l_n(x_2)))| \\
&= \beta | (S(l_n(x_1))(y_1 - f(x_1))) - (S(l_n(x_2))(y_2 - f(x_2))) | \\
&\leq \beta |S(l_n(x_1))| \cdot |(y_1 - f(x_1)) - (y_2 - f(x_2))| \\
&+ |S(l_n(x_1)) - S(l_n(x_2))| \cdot |(y_2 - f(x_2))| \\
&\leq \beta s |(y_1 - f(x_1)) - (y_2 - f(x_2))| + \beta \lambda_l \lambda_S \eta |x_1 - x_2|.
\end{aligned}$$

To obtain the second equality, we used the fact that f is the fixed point of (3.1).

Hence

$$\begin{aligned}
&d_f(w_n(x_1,y_1), w_n(x_2,y_2)) \\
&\leq (\alpha\lambda_l + \beta\lambda_S\lambda_l\eta)|x_1 - x_2| + \beta s|(y_1 - f(x_1)) - (y_2 - f(x_2))| \\
&\leq (\alpha + \beta\lambda_S\eta)\lambda_l|x_1 - x_2| + \beta s|(y_1 - f(x_1)) - (y_2 - f(x_2))| \\
&\leq c \cdot (\alpha|x_1 - x_2| + \beta|(y_1 - f(x_1)) - (y_2 - f(x_2))|)
\end{aligned}$$

where $c := \max\{s, \lambda_l + \beta\lambda_l\lambda_S\eta/\alpha\}$. Since $\lambda_l < 1$ we can choose $\alpha, \beta > 0$ so that $c < 1$. For example, we can choose $\alpha = 1$ and $0 < \beta < (1 - \lambda_l)/\lambda_l\lambda_S\eta$.

It follows that the IFS $\widetilde{\mathcal{W}} := (\mathbb{X}; w_1, w_2, ..., w_N)$ is contractive, and hence it has a unique attractor. This attractor must be $\Gamma(f)$ because a contractive IFS has a unique nonempty compact invariant set and it is readily verified that $\widetilde{\mathcal{W}}(\Gamma(f)) = \Gamma(f)$. Since we can choose the constant η arbitrarily large, it now follows that $\mathcal{W}$ has a unique attractor, namely $\Gamma(f)$. Note that we have *not* provided a metric with respect to which $\mathcal{W}$ is contractive! □

We remark that

$$\Gamma(Tg) = \mathcal{W}(\Gamma(g)), \text{ for all } g \in C^*.$$

When, for example, S is Lipschitz continuous with Lipschitz constant $s < 1$, and the functions l_n are contractive, the graph of the fractal interpolant f can be approximated by the "chaos game" algorithm. (See [3] and [6] for new topological viewpoints of the "chaos game.")

6. Bilinear fractal interpolation

We consider a specific example of the preceding theory. Let $l_n : I \to [X_{n-1}, X_n]$ be given by

$$l_n(x) := X_{n-1} + \left(\frac{X_n - X_{n-1}}{X_N - X_0}\right)(x - X_0) \tag{6.1}$$

and $S : I \to \mathbb{R}$ by

$$S := S_n \circ l_n^{-1},$$

for $x \in [X_{n-1}, X_n]$, $n = 1, \ldots, N$, where $S_n : I \to \mathbb{R}$,

$$S_n(x) := s_{n-1} + \left(\frac{s_n - s_{n-1}}{X_N - X_0}\right)(x - X_0),$$

with $\{s_j : j = 0, 1, 2, ..., N\} \subset (-1, 1)$. Note that the s_j, $j = 0, 1, \ldots, N$, need not be ordered.

Then S is continuous and

$$\begin{aligned}|S(x)| &\leq \max\{|S_n(l_n^{-1}(x))| : x \in [X_{n-1}, X_n], n \in \{1, 2, ..., N\}\} \\ &= \max\{|s_j| : j = 0, 1, ..., N\} =: s < 1.\end{aligned}$$

Furthermore, let $b : I \to \mathbb{R}$ be given by

$$b(x) := Y_0 + \left(\frac{Y_N - Y_0}{X_N - X_0}\right)(x - X_0) \tag{6.2}$$

and $h : I \to \mathbb{R}$ by

$$h(x) := \sum_{n=1}^{N} \left[Y_{n-1} + \left(\frac{Y_n - Y_{n-1}}{X_n - X_{n-1}}\right)(x - X_{n-1})\right] \chi_{[X_{n-1}, X_n]}(x), \tag{6.3}$$

where χ_M denotes the characteristic function of a set M.

Note that $b \in C^*$ and $h \in C^{**}$. Theorem 3 implies that T has a unique fixed point f. Specifically, f is the unique solution of the set of functional equations of the form

$$f(l_n(x)) - h(l_n(x)) = S_n(l_n(x))[f(x) - b(x)], \quad n = 1, \ldots, N; \; x \in I. \tag{6.4}$$

We refer to f as a **bilinear fractal interpolant**. The reason for this name is that in this case the functions w_n of the IFS $\mathcal{W}$ take the form

$$w_n(x, y) := (l_n(x), a + bx + cy + dxy),$$

where a, b, c, d are real constants. Functions of the form $B : (x, y) \mapsto a + bx + cy + dxy$ are called *bilinear* in the computer graphics literature. We will adhere to this terminology but like to point out that B is for fixed x or fixed y affine in the other variable. More precisely,

$$\begin{aligned}B((1-t)x_1 + tx_2, y) &= (1-t)B(x_1, y) + tB(x_2, y) \\ B(x, (1-t)y_1 + ty_2) &= (1-t)B(x, y_1) + tB(x, y_2),\end{aligned}$$

for all $x_1, x_2, y_1, y_2, t \in \mathbb{R}$.

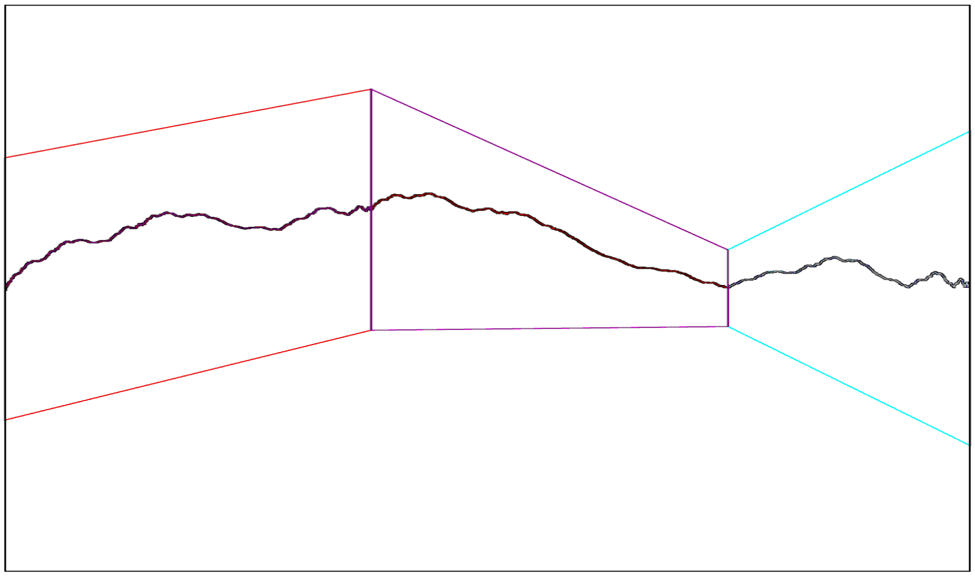

Figure 1: A fractal interpolation function defined by three bilinear transformations. See text.

Using the expressions for l_n, S_n, and h above, we can write the functions w_n in the form

$$w_n(x, y) = \left(X_{n-1} + \left(\frac{X_n - X_{n-1}}{X_N - X_0} \right) (x - X_0), Y_{n-1} + \left(\frac{Y_n - Y_{n-1}}{X_N - X_0} \right) (x - X_0) \right. \tag{6.5}$$
$$\left. + \left[s_{n-1} + \left(\frac{s_n - s_{n-1}}{X_N - X_0} \right) (x - X_0) \right] \left[y - Y_0 - \left(\frac{Y_N - Y_0}{X_N - X_0} \right) (x - X_0) \right] \right).$$

In particular note that

$$w_n(X_N, y) = (X_n, Y_n + s_n(y - Y_N)) \quad \text{and} \quad w_{n+1}(X_0, y) = (X_n, Y_n + s_n(y - Y_0)).$$

It follows that the images of any (possibly degenerate) parallelogram with vertices at $(X_0, Y_0 \pm H)$ and $(X_N, Y_N \pm H)$, for $H \in \mathbb{R}$ under the IFS fit together neatly, as illustrated in Figure 1.

7. Box dimension of bilinear interpolants

In this section, we derive a formula for the box dimension of the graphs of a class of bilinear interpolants. To this end, let $1 < N \in \mathbb{N}$, let $I := [0, 1] \subset \mathbb{R}$

be the unit interval, and let $\blacksquare := I\times I$ denote the filled unit square. Suppose that $\{0 =: X_0 < X_1 < \cdots < X_N := 1\}$ is a set of knots in I. Furthermore, suppose that $\{\underline{Y}_j \in I : j = 0,1,\ldots,N\}$ and $\{\overline{Y}_j \in I : j = 0,1,\ldots,N\}$ are two sets of points with the property that $0 \le \underline{Y}_j \le \overline{Y}_j < 1$, $\forall j = 0,1,\ldots,N$.

Denote by Q_n the trapezoid with vertices $A_n := (X_{n-1},\underline{Y}_{n-1})$, $B_n := (X_n,\underline{Y}_n)$, $C_n := (X_n,\overline{Y}_n)$, and $D_n := (X_{n-1},\overline{Y}_{n-1})$, $n = 1,\ldots,N$. For each $n = 1,\ldots,N$, let $l_n : I \to [X_{n-1},X_n]$ be a family of affine mappings and $B_n : I\times\mathbb{R}\to\mathbb{R}, (x,y)\mapsto a_n x + b_n y + c_n xy + d_n = (d_n + a_n x) + (b_n + c_n x)y$, $a_n, b_n, c_n, d_n \in \mathbb{R}$, a family of bilinear mappings.

Define mappings $w_n := (l_n, B_n) : \blacksquare \to Q_n$ by requiring that

$$(0,0) \overset{w_n}{\longmapsto} A_n, \quad (1,0) \overset{w_n}{\longmapsto} B_n, \quad (1,1) \overset{w_n}{\longmapsto} C_n, \quad (0,1) \overset{w_n}{\longmapsto} D_n. \tag{7.1}$$

It follows readily from (7.1) that the affine mappings l_n are given by (6.1) and the bilinear mappings B_n by

$$B_n(x,y) = a_n x + [s_{n-1} + (s_n - s_{n-1})x]\, y + \underline{Y}_{i-1}, \tag{7.2}$$

where we set $a_n = \underline{Y}_n - \underline{Y}_{n-1}$, $n = 1,\ldots,N$, and $s_j := \overline{Y}_j - \underline{Y}_j$, $j = 0,1,\ldots,N$. Note that $0 \le s_j < 1$, for all $j = 0,1,\ldots,N$.

Definition 3. *The IFS $\mathcal{F} := (\blacksquare; w_1,\ldots,w_N)$ where $w_n := (l_n, B_n)$ with l_n and B_n, $n = 1,\ldots,N$, given by* (6.1) *and* (7.2)*, respectively, is called* ***bilinear****.*

In [5] such bilinear IFSs are investigated in more generality and in connection with fractal homeomorphisms. The approach undertaken in [5] makes substantial use of the geometric properties that functions in an bilinear IFS possess, namely that they take horizontal and vertical lines to lines and that they preserve proportions along horizontal and vertical lines. For further details and results, we refer the interested reader to [5].

Recall the definition of the metric d_q given in Proposition 1. For our current purposes, we set $q \equiv 1$. As in Theorem 4 we denote the Lipschitz constant of the l_n by λ_n.

Theorem 5. *The bilinear IFS $\mathcal{F} = (\blacksquare; w_1,\ldots,w_N)$ is contractive in the metric d_1 with $\alpha := 1$ and $0 < \beta < \frac{1-\lambda_n}{2}$.*

Proof. It suffices to show that each $w_n \in \mathcal{F}$ is contractive with respect to the metric d_1. To this end, let $(x,y),(x',y') \in \blacksquare$ and set $\Delta s_n := s_n - s_{n-1}$,

$n = 1, \ldots, N$. Then

$$
\begin{aligned}
d_1(w_n(x,y), w_n(x',y')) &= |l_n(x) - l_n(x')| + \beta |B_n(x,y) - B_n(x',y')| \\
&\le \lambda_n |x - x'| + \beta(|a_n||x - x'| + |[s_{n-1} + \Delta s_n x]y - [s_{n-1} + \Delta s_n x']y'|) \\
&= \lambda_n |x - x'| + \beta(|a_n||x - x'| + |[s_{n-1} + \Delta s_n x]y \\
&\qquad - |[s_{n-1} + \Delta s_n x']y + |[s_{n-1} + \Delta s_n x']y - |[s_{n-1} + \Delta s_n x']y') \\
&\le \lambda_n |x - x'| + \beta(|a_n||x - x'| + |\Delta s_n y||x - x'| + |s_{n-1} + \Delta s_n x'||y - y|) \\
&\le (\lambda_n + \beta(|a_n| + |\Delta s_n|)\, |x - x'| + \beta |s_n||y - y'| \quad (\text{since } |x'|, |y| \le 1) \\
&\le \max\{\lambda_n + 2\beta, s_n\}\; d_1((x,y),(y',y')) \quad (\text{since } |a_n|, |\Delta s_n| \le 1).
\end{aligned}
$$

By choosing $0 < \beta < \frac{1-\lambda_n}{2}$ the maximum can be made strictly smaller than 1. □

Adapting (6.1), (6.2), and (6.3) to the current setting using $\{\underline{Y}_j : j = 0, 1, \ldots, N\}$ instead of $\{Y_j : j = 0, 1, \ldots, N\}$, we see by Theorem 3 that the associated operator $T : C^* \to C^{**}$ defined by

$$Tg := h + [s_{n-1} + \Delta s_n\,(\bullet)] \cdot (g - b) \circ L \tag{7.3}$$

is contractive and its unique fixed point f is an element of C^{**}. Moreover, f satisfies the functional equations set forth in (6.4).

Next, we derive a formula for the box dimension of the graphs of bilinear fractal interpolants arising from the above bilinear IFS $\mathcal{F} = (\blacksquare; w_1, \ldots, w_N)$. For this purpose, we may assume, without loss of generality, that $\underline{Y}_0 = \underline{Y}_N = 0$. This special case can always be achieved by means of an affine transformation (which does not change the box dimension).

To this end, we recall the definition of box-counting or box dimension of a bounded set $M \subset \mathbb{R}^n$:

$$\dim_B M := \lim_{\varepsilon \to 0+} \frac{\log \mathcal{N}_\varepsilon(M)}{\log \varepsilon^{-1}}, \tag{7.4}$$

where $\mathcal{N}_\varepsilon(M)$ is the minimum number of square boxes with sides parallel to the axes, whose union contains M. By the statement "$\dim_B M = D$" we mean that the limit in equation (7.4) exists and equals D.

In the case where M is the graph $\Gamma(f)$ of a function f, knowledge of the box dimension of $\Gamma(f)$ provides information about the smoothness of f since $\dim_B \Gamma(f)$ is related to Hölder exponents associated with f. (See, for example, [19, Section 12.5].)

The following result gives an explicit formula for the box dimension of the graph of a bilinear fractal interpolant defined via the operator (7.3). The proof is based on arguments first applied in [9].

Theorem 6. *Let $\mathcal{F}$ denote the bilinear IFS defined above and let $\Gamma(f)$ denote its attractor. Suppose that the knots $\{X_j : j = 0, 1, \ldots, N\}$ are uniformally spaced on I, i.e., $X_j = j/N$, $\forall j = 0, 1, \ldots, N$, and suppose that $s_0 = s_N$. If $\gamma := \sum_{n=1}^{N} \frac{s_{n-1}+s_n}{2} > 1$ and $\Gamma(f)$ is not a straight line segment then*

$$\dim_B \Gamma(f) = 1 + \frac{\log \gamma}{\log N};$$

otherwise $\dim_B \Gamma(f) = 1$.

Proof. Note that in the computation of the box dimension of $\Gamma(f)$ it suffices to consider covers of $\Gamma(f)$ whose elements are squares of side N^{-r}, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Denote by $\mathcal{C}_0(r)$ a cover of $\Gamma(f)$ consisting of a finite number of squares of side N^{-r}, $r \in \mathbb{N}_0$. Now consider a specific cover $\mathcal{C}(r)$ of $\Gamma(f)$ of the form

$$\mathcal{C}(r) := \left\{ \left[\frac{k-1}{N^r}, \frac{k}{N^r}\right] \times \left[a, a + \frac{1}{N^r}\right] : r \in \mathbb{N}_0;\ k = 1, \ldots, N^r;\ a \in \mathbb{R} \right\}. \tag{7.5}$$

By the compactness of $\Gamma(f)$, there exists a minimal cover $\mathcal{C}_0^*(r)$ of $\Gamma(f)$ and also a minimal cover $\mathcal{C}^*(r)$ of $\Gamma(f)$ of the form (7.5). Denote by $\mathcal{N}_0(r)$, respectively, $\mathcal{N}(r)$ the cardinality of these minimal covers. Since covers of the form (7.5) are more restrictive, we have $\mathcal{N}_0(r) \le \mathcal{N}(r)$. On the other hand, every $(N^{-r} \times N^{-r})$-square in $\mathcal{C}_0^*(r)$ can be covered by at most two $(N^{-r} \times N^{-r})$-squares from a cover of the form (7.5). Thus, $\mathcal{N}(r) \le 2\mathcal{N}_0(r)$. Hence, when computing the box dimension of $\Gamma(f)$ it suffices to consider covers of the form (7.5).

To this end, let $r \in \mathbb{N}_0$ be fixed. Let $\mathcal{C}(r)$ be a minimal cover of $\Gamma(f)$ of cardinality $\mathcal{N}(r)$ consisting of squares of side N^{-r} whose interiors are disjoint. Let $\mathcal{C}(r,k)$ be the collection of all squares in $\mathcal{C}(r)$ that lie between $x = \frac{k-1}{N^r}$ and $x = \frac{k}{N^r}$, $k = 1, \ldots, N^r$. Denote by $\mathcal{N}(r,k)$ the cardinality of $\mathcal{C}(r,k)$, and let

$$\mathcal{R}(r,k) := \bigcup_{C \in \mathcal{C}(r,k)} C.$$

As $\mathcal{C}(r)$ is a cover of $\Gamma(f)$ of minimal cardinality, every square in $\mathcal{C}(r)$ must meet $\Gamma(f)$, and since f is continuous on I, the set $\mathcal{R}(r,k)$ must be a rectangle of width N^{-r} and height $N^{-r}\mathcal{N}(r,k)$. Note that $\mathcal{N}(r) = \sum_{k=1}^{N^r} \mathcal{N}(r,k)$.

Now apply the mappings w_n, $n = 1, \ldots, N$, defined in (7.1) to the rectangle $\mathcal{R}(r,k)$. The image of $\mathcal{R}(r,k)$ under w_n is a trapezoid contained in the strip $\left[\frac{l(k,n)-1}{N^{r+1}}, \frac{l(k,n)}{N^{r+1}}\right] \times \mathbb{R}$, with $l(k,n) := k + (n-1)N^r$. Observe that

$$\mathcal{N}(r+1) = \sum_{n=1}^{N}\sum_{k=1}^{N^r} \mathcal{N}(r+1, l(k,n)).$$

The fixed point equation for $\Gamma(f)$, namely, $\Gamma(f) = \bigcup_{n=1}^{N} w_i(\Gamma(f))$, implies that

$$\Gamma(f) \subseteq \bigcup_{n=1}^{N} w_n \left(\bigcup_{k=1}^{N^r} \mathcal{R}(r,k)\right).$$

Depending on the sign of Δs_n, there are ten possible geometric shapes for the trapezoid $w_n(\mathcal{R}(r,k))$. In Figure 2 one of these trapezoids is depicted and the relevant geometric quantities identified.

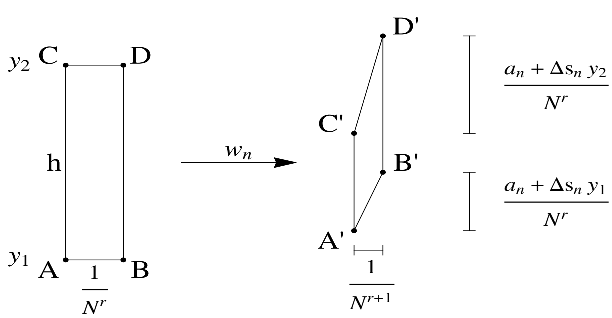

Figure 2: An image of a rectangle under the map w_n.

Employing the notation in Figure 2, we write $A' < B'$ if the y–coordinate of the point A' is less than the y–coordinate of the point B'. Similarly, we define $A' \leq B'$.

Case I: $\Delta s_n \geq 0$**.** Note that in this scenario, distance(A', C') $\leq$ distance(B', D'). The five possible shapes are given by the location of the vertices A', B', C', and D'. They are: $B' < D' \leq A' < C'$, $B' < A' \leq D' \leq C'$, $B' \leq A' < C' \leq D'$, $A' \leq B' \leq C' < D'$, and $A' < C' \leq B' < D'$. Each one of these trapezoids is contained in a rectangle of width $N^{-(r+1)}$ and height at most

$$\left(s_{n-1} + \Delta s_n \cdot \frac{k}{N^r}\right)(N^{-r}\mathcal{N}(r,k)) + \frac{2(|a_n| + |\Delta s_n|)}{N^r}, \tag{7.6}$$

and meets a rectangle of width $N^{-(r+1)}$ and height at least

$$\left(s_{n-1} + \Delta s_n \cdot \frac{k-1}{N^r}\right)(N^{-r}\mathcal{N}(r,k)) - \frac{2(|a_n| + |\Delta s_n|)}{N^r}. \tag{7.7}$$

Hence,

$$\begin{aligned}\mathcal{N}(r+1, l(k,n)) &\leq \left[\left(s_{n-1} + \Delta s_n \cdot \frac{k}{N^r}\right)(N^{-r}\mathcal{N}(r,k)) + \frac{2(|a_n| + |\Delta s_n|)}{N^r}\right] N^{r+1} + 2 \\ &= N\left(s_{n-1} + \Delta s_n \cdot \frac{k}{N^r}\right)\mathcal{N}(r,k) + 2N(|a_n| + |\Delta s_n|) + 2,\end{aligned} \tag{7.8}$$

and, similarly,

$$\mathcal{N}(r+1, l(k,n)) \geq N\left(s_{n-1} + \Delta s_n \cdot \frac{k-1}{N^r}\right)\mathcal{N}(r,k) - 2N(|a_n| + |\Delta s_n|) - 2. \tag{7.9}$$

Case II: $\Delta s_n \leq 0$**.** Here, distance$(A', C') \geq$ distance(B', D') and the five possible shapes are as above. Each one of these trapezoids is contained in a rectangle of width $N^{-(r+1)}$ and height at most

$$\left(s_{n-1} + \Delta s_n \cdot \frac{k-1}{N^r}\right)(N^{-r}\mathcal{N}(r,k)) + \frac{2(|a_n| + |\Delta s_n|)}{N^r}, \tag{7.10}$$

and meets a rectangle of width $N^{-(r+1)}$ and height at least

$$\left(s_{n-1} + \Delta s_n \cdot \frac{k}{N^r}\right)(N^{-r}\mathcal{N}(r,k)) - \frac{2(|a_n| + |\Delta s_n|)}{N^r}. \tag{7.11}$$

Thus, similar to Case I, we obtain an upper, respectively lower, bound for $\mathcal{N}(r+1, l(k,n))$ of the form

$$\mathcal{N}(r+1, l(k,n)) \leq N\left(s_{n-1} + \Delta s_n \cdot \frac{k-1}{N^r}\right)\mathcal{N}(r,k) + 2N(|a_n| + |\Delta s_n|) + 2, \tag{7.12}$$

and

$$\mathcal{N}(r+1, l(k,n)) \geq N\left(s_{n-1} + \Delta s_n \cdot \frac{k}{N^r}\right)\mathcal{N}(r,k) - 2N(|a_n| + |\Delta s_n|) - 2. \tag{7.13}$$

Denote by $N_\pm$ the set of all indices $n \in \{1, \ldots, N\}$ for which $\Delta s_n \geq 0$, respectively, $\Delta s_n \leq 0$. Then, using Equations (7.6) and (7.10), summation over n yields

$$\begin{aligned}\sum_{n=1}^{N} \mathcal{N}(r+1, l(k,n)) &= \sum_{n\in N_+} \mathcal{N}(r+1, l(k,n)) + \sum_{n\in N_-} \mathcal{N}(r+1, l(k,n)) \\ &\leq N \sum_{n\in N_+} \left(s_{n-1} + \Delta s_n \cdot \frac{k}{N^r}\right)\mathcal{N}(r,k) \\ &+ N \sum_{n\in N_-} \left(s_{n-1} + \Delta s_n \cdot \frac{k-1}{N^r}\right)\mathcal{N}(r,k) \\ &+ \sum_{n=1}^{N} \left[2N(|a_n| + |\Delta s_n|) + 2\right]\end{aligned}$$

Now,

$$s_{n-1} + \Delta s_n \cdot \frac{k}{N^r} = \frac{s_{n-1} + s_n}{2} + \Delta s_n \left(\frac{k}{N^r} - \frac{1}{2}\right), \quad n \in N_+,$$

and

$$\begin{aligned}s_{n-1} + \Delta s_n \cdot \frac{k-1}{N^r} &= \frac{s_{n-1} + s_n}{2} + \Delta s_n \left(\frac{k-1}{N^r} - \frac{1}{2}\right) \\ &= \frac{s_{n-1} + s_n}{2} + \Delta s_n \left(\frac{k}{N^r} - \frac{1}{2}\right) + \frac{-\Delta s_n}{N^r}, \quad n \in N_-.\end{aligned}$$

Substitution into the expression for $\sum_{n=1}^{N} \mathcal{N}(r+1, l(k,n))$ gives

$$\begin{aligned}\sum_{n=1}^{N} \mathcal{N}(r+1, l(k,n)) \leq N \sum_{n=1}^{N} \left(\frac{s_{n-1} + s_n}{2}\right)\mathcal{N}(r,k) + N\left(\sum_{n=1}^{N} \Delta s_n\right)\left(\frac{k}{N^r} - \frac{1}{2}\right)\mathcal{N}(r,k) \\ + \sum_{n\in N_-} \frac{-\Delta s_n}{N^r} + \sum_{n=1}^{N} \left[2N(|a_n| + |\Delta s_n|) + 2\right].\end{aligned}$$

As $\sum_{n=1}^{N} \Delta s_n = s_N - s_0 = 0$ by assumption, we obtain

$$\sum_{n=1}^{N} \mathcal{N}(r+1, l(k,n)) \leq (N\gamma)\,\mathcal{N}(r,k) + \sum_{n\in N_-} \frac{-\Delta s_n}{N^r} + \sum_{n=1}^{N} [2N(|a_n| + |\Delta s_n|) + 2]$$
$$\leq (N\gamma)\,\mathcal{N}(r,k) + c_1, \tag{7.14}$$

where we set $c_1 := \sum_{n=1}^{N} \left[2N(|a_n| + |\Delta s_n|) + \frac{|\Delta s_n|}{N} + 2\right]$.

Summing Equation 7.14 over k produces an upper bound for $\mathcal{N}(r+1)$ in terms of $\mathcal{N}(r)$:

$$\mathcal{N}(r+1) \leq (N\gamma)\,\mathcal{N}(r) + c_1 N^r.$$

Induction on r yields

$$\mathcal{N}(r) \leq (N\gamma)^r\,\mathcal{N}(0) + c_1 N^{r-1} \sum_{\varrho=0}^{r-1} \gamma^{\varrho}.$$

Depending on the value of γ, two cases need to be considered.

Case A: $\gamma \leq 1$**.** This implies that $\mathcal{N}(r) \leq N^r(\mathcal{N}(0) + c_1 r)$. Hence,

$$\dim_B \Gamma(f) \leq \lim_{r\to\infty} \frac{\log N^r(\mathcal{N}(0) + c_1 r)}{\log N^r} = 1.$$

Case B: $\gamma > 1$**.** Observing that in this situation

$$\sum_{\varrho=0}^{r-1} \gamma^{\varrho} \leq \frac{\gamma^r}{\gamma - 1},$$

we obtain

$$\mathcal{N}(r) \leq (\gamma N)^r\,\mathcal{N}(0) + \frac{c_1 (N\gamma)^r}{\gamma - 1} =: c_2 (N\gamma)^r.$$

Thus,

$$\dim_B \Gamma(f) \leq \lim_{r\to\infty} \frac{\log c_2\,(\gamma N)^r}{\log N^r} = 1 + \frac{\log \gamma}{\log N}.$$

Note that since f is a continuous function, $\dim_B \Gamma(f) \geq 1$. If $\Gamma(f)$ is a line segment, i.e., if the set of data $\mathcal{J} := \{(j/N, \underline{Y}_j) \,:\, j = 0, 1, \ldots, N)\}$ is collinear, then $\Gamma(f) = [0,1]$ implying that $\dim_B \Gamma(f) = 1$.

To obtain a nontrivial lower bound for $\Gamma(f)$, the following lemma is required.

Lemma 3. *If* $\gamma := \sum_{i=1}^{N} \frac{s_{i-1}+s_i}{2} > 1$, $s_0 = s_N$, *and* $\Gamma(f)$ *is not a line segment then*

$$\lim_{r\to\infty} \frac{\mathcal{N}(r)}{N^r} = \infty.$$

Proof. The assumption that $\Gamma(f)$ is not a line segment implies the existence of at least one index $n_0 \in \{1, \ldots, N-1\}$ so that

$$\delta := \underline{Y}_{n_0} > 0.$$

Since f is continuous on I, we have that $\mathcal{N}(r) \geq \delta N^r$. Note that I is mapped to the line segments $\overline{(X_{n-1}, \underline{Y}_{n-1}), (X_n, \underline{Y}_n)}$, implying that for $r \geq 1$

$$\begin{aligned}
\mathcal{N}(r) &\geq \sum_{n=1}^{N} \left[s_{n-1} + \Delta s_n \frac{n_0}{N} \right] \delta N^r \\
&= \left[\sum_{n=1}^{N} \left(\frac{s_{n-1} + s_n}{2} \right) + \sum_{n=1}^{N} \left(\frac{n_0}{N} - \frac{1}{2} \right) \Delta s_n \right] \delta N^r \\
&= \sum_{n=1}^{N} \left(\frac{s_{n-1} + s_n}{2} \right) (\delta N^r). \qquad \text{(As the sum over } \Delta s_n \text{ equals zero.)}
\end{aligned}$$

Proceeding inductively, we arrive at

$$\mathcal{N}(r) \geq \sum_{n_1, \cdots, n_k = 1}^{N} \prod_{\ell=1}^{k} \left[\frac{s_{n_\ell - 1} + s_{n_\ell}}{2} \right] (\delta N^r), \qquad r \geq k.$$

Therefore,

$$\mathcal{N}(r) \geq [\gamma^r \delta - 1] N^r,$$

which, since $\gamma > 1$, finishes the proof of the lemma. □

Suppose then that $\gamma > 1$ and that $\Gamma(f)$ is not a line segment, i.e., $\mathcal{J}$ is not collinear. Since each $C \in \mathcal{C}(r, k)$ meets $\Gamma(f)$, the image of C under the maps w_n, $n = 1, \ldots, N$, must also meet $\Gamma(f)$.

Thus, using Equations (7.9) and (7.13), we obtain

$$
\begin{aligned}
\sum_{n=1}^{N} \mathcal{N}(r+1, l(k,n)) &= \sum_{n\in N_+} \mathcal{N}(r+1, l(k,n)) + \sum_{n\in N_-} \mathcal{N}(r+1, l(k,n)) \\
&\geq N \sum_{n\in N_+} \left(s_{n-1} + \Delta s_n \cdot \frac{k-1}{N^r} \right) \mathcal{N}(r,k) \\
&+ N \sum_{n\in N_-} \left(s_{n-1} + \Delta s_n \cdot \frac{k}{N^r} \right) \mathcal{N}(r,k) \\
&- \sum_{n=1}^{N} \left[2N(|a_n| + |\Delta s_n|) + 2 \right]
\end{aligned}
$$

Algebra similar to that applied in the estimate for the upper bound, yields

$$
\begin{aligned}
\sum_{n=1}^{N} \mathcal{N}(r+1, l(k,n)) \geq N \sum_{n=1}^{N} \left(\frac{s_{n-1} + s_n}{2} \right) \mathcal{N}(r,k) \\
- \sum_{n\in N_+} \frac{\Delta s_n}{N^r} - \sum_{n=1}^{N} \left[2N(|a_n| + |\Delta s_n|) + 2 \right].
\end{aligned}
$$

Summation over k gives

$$
\mathcal{N}(r+1) \geq (N\gamma)\, \mathcal{N}(r) - c_1 N^r.
$$

Hence,

$$
\begin{aligned}
\mathcal{N}(r) &\geq (N\gamma)^{r-m} \mathcal{N}(m) - c_1 N^{r-1} \sum_{\varrho=0}^{r-m-1} \gamma^{\varrho} \\
&\geq (N\gamma)^{r-m} \left[\mathcal{N}(m) - \frac{c_1 N^{m-1}}{1-\gamma^{-1}} \right],
\end{aligned}
$$

for all $m \in \mathbb{N}$ with $1 \leq m \leq r$.

Lemma 3 implies that we can choose r and m large enough so that

$$
\mathcal{N}(m) - \frac{c_1 N^{m-1}}{1-\gamma^{-1}} > 0.
$$

Therefore, $\mathcal{N}(r) \geq c_2\, (\gamma\, N)^r$, for a constant $c_2 > 0$ and for large enough r. Hence, $\dim_B \Gamma(f) \geq 1 + \dfrac{\log \gamma}{\log N}$. □

Remark 1. *Recall that the code space associated with an IFS is given by* $\Omega := \{1, \ldots, N\}^\infty$*. The elements of* Ω *are called codes. The set of all finite codes is defined as* $\Omega' := \bigcup_{k=0}^\infty \{1, \ldots, N\}^k$*, where the empty set represents a code of length zero.*

The proof of Theorem 6 shows in particular that for a given $\sigma = \sigma_1 \ldots \sigma_{|\sigma|} \in \Omega'$ *of finite length* $|\sigma|$*, there exist constants* $0 < \underline{c} \le \overline{c}$ *such that*

$$\underline{c}\,(\gamma\,N)^{|\sigma|} \le \mathcal{N}(|\sigma|) \le \overline{c}\,(\gamma\,N)^{|\sigma|}.$$

Moreover, if $w_{\sigma_1\cdots\sigma_r}(\Gamma(f))$ *denotes the image of* $\Gamma(f)$ *under the maps* $w_{\sigma_1\cdots\sigma_r} := w_{\sigma_1} \circ \cdots \circ w_{\sigma_r}$ *over the subinterval* $l_{\sigma_1\cdots\sigma_r}(I)$*, then there also exist constants* $0 < \underline{c}^* \le \overline{c}^*$ *such that*

$$\underline{c}^*\,\gamma_{\sigma_1}\cdots\gamma_{\sigma_r}\,N^{|\sigma|} \le \mathcal{N}_{\sigma_1\cdots\sigma_r}(|\sigma|) \le \overline{c}^*\,\gamma_{\sigma_1}\cdots\gamma_{\sigma_r}\,N^{|\sigma|}, \tag{7.15}$$

where $\mathcal{N}_{\sigma_1\cdots\sigma_r}(|\sigma|)$ *denotes the minimum number of* $N^{-|\sigma|} \times N^{-|\sigma|}$*-squares from a cover of the form (7.5) needed to cover* $w_{\sigma_1\cdots\sigma_r}(\Gamma(f))$ *and* $\gamma_n := \frac{s_{n-1}+s_n}{2}$*,* $n = 1, \ldots, N$*.*

Estimates of this type are important for box dimension calculations in the context of V*-variable fractals and superfractals. We refer the interested reader to [17] where such computations were made for affine fractal interpolants.*

Remark 2. *Bilinear interpolants may be used to model or describe planar data sets that exhibit highly irregular behavior for which classical interpolation and approximation schemes such as polynomials and splines do not succeed. As in the case of affine interpolants, the determination of the free parameters, namely the scaling factors* s_0*,* $s_1, \ldots s_N$*, is essential for an accurate approximation of data sets using the error estimate* (3.2)*, or for modeling data with a pre-described or numerically computed box dimension. However, the particular nature of the problem dictates what type of optimization needs to be employed. For instance, an* L^2*-optimization may be applied to a functional setting as in [12], or bounding volumes may be used for parameter identification as in [14]. These and related questions will be investigated elsewhere.*

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