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Critical and Anticritical Edges with respect to Perfectness

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Abstract

We call an edge e of a perfect graph G critical if $G - e$ is imperfect and call e anticritical if $G + e$ is imperfect. The present paper surveys several questions in this context.

We ask in which perfect graphs critical and anticritical edges occur and how to detect such edges. The main result by Hougardy & Wagler [32] shows that a graph does not admit any critical edge if and only if it is Meyniel.

The goal is to order the edges resp. non-edges of certain perfect graphs s.t. deleting resp. adding all edges in this order yields a sequence of perfect graphs only. Results of Hayward [15] and Spinrad & Sritharan [27] show the existence of such edge orders for weakly triangulated graphs; the line-perfect graphs are precisely these graphs where all edge orders are perfect [33].

Such edge orders cannot exist for every subclass of perfect graphs that contains critically resp. anticritically perfect graphs where deleting resp. adding an arbitrary edge yields an imperfect graph. We present several examples and properties of such graphs, discuss constructions and characterizations from [31, 32].

An application of the concept of critically and anticritically perfect graphs is a result due to Hougardy & Wagler [23] showing that perfectness is an elusive graph property.

Keywords: Perfect graphs, critical edges, perfect edge orders, critically perfect graphs.

1 Introduction

1.1 Perfect graphs

Berge proposed to call a graph $G = (V, E)$ *perfect* if, for each of its induced subgraphs $G' \subseteq G$, the chromatic number $\chi(G')$ equals the clique number $\omega(G')$ (i.e., as many stable sets cover all nodes of G' as a maximum clique of G' has nodes).

Berge [2] conjectured two characterizations of perfect graphs. His first conjecture was that a graph G is perfect iff the clique covering number $\overline{\chi}(G')$ equals the stability number $\alpha(G')$ for all $G' \subseteq G$ (i.e., as many cliques cover all nodes of G' as a maximum stable set of G' has nodes). Since complementation transforms stable sets into cliques and colorings into clique coverings, this is equivalent to saying that a graph G is perfect if and only if its complement $\overline{G}$ is perfect. This was proved by Lovász [24] and is nowadays known as the *Perfect Graph Theorem*.

The second conjecture, called *Strong Perfect Graph Conjecture*, concerns a characterization via forbidden subgraphs. Berge observed that chordless odd cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, and their complements $\overline{C}_{2k+1}$, called *odd antiholes*, are imperfect. Clearly, each graph containing an odd hole or an odd antihole as an induced subgraph is imperfect as well. Berge conjectured in [2] that a graph is perfect if and only if it contains neither odd holes nor odd antiholes as induced subgraphs; such graphs are called *Berge* by a suggestion of Chvátal and Sbihi.
Imperfect graphs with the property that removing any of its nodes yields a perfect graph are called *minimally imperfect*. Using this term, the Strong Perfect Graph Conjecture reads that odd holes and odd antiholes are the only minimally imperfect graphs. Considerable effort has been spent over the years to verify or falsify the Strong Perfect Graph Conjecture. For example, many structural properties of minimally imperfect graphs have been discovered and the conjecture has been proved for many classes of Berge graphs.

A common way to show the perfectness of a class $\mathcal{C}$ uses structural results proving that all graphs in $\mathcal{C}$ have a basic form or there are certain structural faults which cannot occur in minimally imperfect graphs. For example, *triangulated graphs* (having no hole C_k for $k \geq 4$) are perfect due to Berge [1] since every subgraph is either a clique or contains a clique cutset. *Meyniel graphs* where every odd cycle of length ≥ 5 has at least two chords belong to a basic class or else contain a so-called amalgam by Burlet & Fonlupt [7]. BIP* is a hereditary class constructed from *bipartite graphs* (having chromatic number ≤ 2) with the help of star-cutsets in the graph and in the complement due to Chvátal [8]. C_4-free Berge graphs are either bipartite or line graphs of bipartite graphs or else have star-cutsets or 2-joins by Conforti, Cornuéjols & Vuŝković [11]. The *line graph* $L(F)$ is obtained by taking the edges of a graph F as nodes of the line graph and by joining two nodes of $L(F)$ if the corresponding edges of F are incident.

In May 2002, Chudnovsky, Robertson, Seymour & Thomas announced a proof of the Strong Perfect Graph Conjecture with the help of the following decomposition theorem [10]: Every Berge graph G either belongs to five basic classes (G or $\overline{G}$ is bipartite or the line graph of a bipartite graph, or G is a double split graph) or has one of four structural faults (G or $\overline{G}$ has a 2-join, G has an M-join, or G has a balanced skew partition). Recently, Chudnovsky showed in her thesis [9] that M-joins can be dropped. Since odd holes are the only minimally imperfect graphs containing 2-joins by [12] and a minimum counterexample to the Strong Perfect Graph Conjecture has no balanced skew partition due to [10], this decomposition theorem implies the *Strong Perfect Graph Theorem*: every Berge graph is perfect.

1.2 A graph evolution process

The motivation to investigate critical and anticritical edges w.r.t. perfectness comes from the following graph evolution process. Imagine you have an arbitrary perfect graph and you consecutively delete one edge after the other until the empty graph is reached, or you consecutively add edges until the graph is complete. So you create a sequence of graphs starting and ending up with a perfect graph. But, if you choose the edges to be deleted or added randomly, most graphs of your sequence will be imperfect. The aim is to avoid the occurrence of imperfect graphs in our sequence.

First, one could try to delete or add one edge keeping perfectness in each step:

Problem 1.1 *Given a certain perfect graph, is there a rule how to choose an edge to be deleted or added in order to preserve perfectness?*

Let G be a perfect graph. We call an edge e *critical* if $G - e$ is imperfect and call e *anticritical* if $G + e$ is imperfect. Note that a critical edge of G is anticritical in the complementary graph $\overline{G}$: due to the Perfect Graph Theorem [24]. Section 2 presents some rules to detect whether an edge is critical or anticritical. Problem 1.1 could be easier to solve if the graph in question belongs to a certain *subclass* of perfect graphs. In particular, certain perfect graphs may not possess any critical or anticritical edge, in this case we could choose an arbitrary edge keeping perfectness.

Problem 1.2 *In which perfect graphs do critical (anticritical) edges occur at all?*

The answer leads to a new characterization of Meyniel graphs and their complements. Removing any edge from a Meyniel graph preserves perfectness by Hertz [17]. Hougardy & Wagler showed that a graph is Meyniel if and only if it does not admit any critical edge (see [32] for the proof). This implies that every perfect graph G has at least one critical (resp. anticritical) edge if and only if G (resp. $\overline{G}$) is not Meyniel.

In Section 2 we list how to specify critical and anticritical edges for certain classes of perfect graphs, e.g., for line graphs of bipartite graphs, for *strongly perfect graphs* whose subgraphs admit a stable set meeting all maximal cliques, and for *strict quasi parity graphs* whose non-complete subgraphs have an even pair x, y (all induced paths between x and y have even length).

In Section 3, we treat the following problem:

Problem 1.3 *For any given perfect graph, is there an order of all the edges to be deleted (added) so that we get a sequence of perfect graphs ending up with a stable set (clique)?*

It turns out that it does not suffice to identify non-critical or non-anticritical edges: we can certainly remove an arbitrary edge from a Meyniel graph keeping perfectness but, at present, we do not know anything about critical edges of the resulting graph. Thus we have to look for edges the deletion or addition of which preserves membership within the corresponding *subclass* of perfect graphs. This is easy for bipartite and triangulated graphs. Results of Hayward [15] and Spinrad & Sritharan [27] show the existence of such edge orders for *weakly triangulated graphs* that have neither holes C_k nor antiholes $\overline{C}_k$ with $k \geq 5$.
It is clear that no such edge orders exist for classes of perfect graphs containing a graph with only critical or anticritical edges. Perfect graphs s.t. deleting (adding) and arbitrary edge yields an imperfect graph have been termed *critically (anticritically) perfect*. The question is:

Problem 1.4 *Are there critically or anticritically perfect graphs?*

Section 4.1 shows that such graphs exist indeed and provides some properties. In particular, the critically and anticritically perfect *line* graphs have been characterized in [31, 32]. With the help of this characterization, it is easy to obtain critically perfect line graphs belonging to the classes of, e.g., *quasi parity graphs* where G' or $\overline{G}'$ owns an even pair for each subgraph G' and *locally perfect graphs* whose subgraphs can be colored using $\omega(N(x))$ in the neighborhood $N(x)$ of every node x. Furthermore, anticritically perfect line graphs can be found in the classes of, e.g., strongly perfect, strict quasi parity, or planar Berge graphs. (See Section 4.3 for more examples.)
Section 4.2 presents further techniques of constructing critically and anticritically perfect graphs by, e.g., substitution, clique identification, and composition from not necessarily critical or anticritical components.

A typical question to ask is:

Problem 1.5 *Characterize the class of all critically or anticritically perfect graphs!*

A study of this problem revealed that the classes of critically and anticritically perfect graphs can neither be characterized by means of forbidden subgraphs nor by composition techniques that construct all such graphs from some basic classes. (Note that all critically or anticritically perfect graphs known so far are either line graphs of certain bipartite graphs, complements of line graphs of certain bipartite graphs, or can be constructed from such graphs.) Moreover, the two classes are incomparable to almost all subclasses of perfect graphs.

Section 5 is devoted to an application of the concept of critically and anticritically perfect graphs: a result due to Hougardy & Wagler [23] shows that perfectness is an elusive graph property. A graph property is called *elusive* (or evasive) if, for every number of nodes, the best possible algorithm for testing this property has to read, in the worst case, all entries of the adjacency matrix of the given graph. This is equivalent to the following task: start with an empty graph and build a graph by consecutively adding edges (in random order) s.t. no imperfect induced subgraph appears before the last node pair has been probed but that the last step can create both a perfect or an imperfect graph. This is obviously possible if the constructed graph is *bicritically perfect*, i.e., if it is critically *and* anticritically perfect. Hougardy & Wagler constructed bicritically perfect line graphs for all large numbers of nodes and applied a slightly different concept and a parity argument to solve the small cases (where no bicritically perfect graphs exist).

2 Critical and Anticritical Edges

In this section we present few observations which will be frequently used later to detect critical or anticritical edges (Problem 1.1). The question which perfect graphs contain critical edges at all (Problem 1.2) will be answered by characterizing the perfect graphs without any critical edge.

For every critical (anticritical) edge e of a perfect graph G there is necessarily at least one subgraph $G_e \subseteq G$ such that $G_e - e$ ($G_e + e$) is *minimally* imperfect, i.e., an odd hole or odd antihole by the Strong Perfect Graph Theorem.

If $G_e - e \subseteq G - e$ is an odd hole, then G_e is obviously isomorphic to an odd cycle of length ≥ 5 which admits precisely one short chord, namely e (the case $G_e - e = C_7$ is depicted in Figure 1(a) with $e = xy$). In the complementary graph $\overline{G}$, e is clearly an anticritical edge and $\overline{G}_e + e$ an odd antihole (see Figure 1(b)).

If $G_e - e \subseteq G - e$ is an odd antihole, then G_e is the complement $\overline{P}_{2k+1}$ of an induced path with $k \geq 2$ (the case $G_e - e = \overline{C}_7$ is depicted in Figure 1(c) with $e = xy$). In the complement $\overline{G}$, e is an anticritical edge, $\overline{G}_e + e$ an odd hole, and $\overline{G}_e$ an induced path P_{2k+1} (see Figure 1(d)).

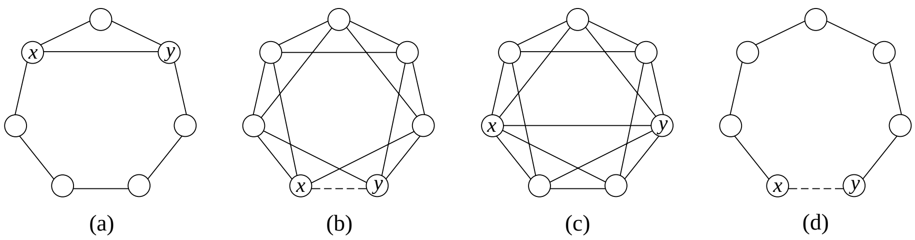


Figure 1: The cases $|G_e| = 7$ with critical or anticritical edge $e = xy$.

In order to decide whether an edge is critical or anticritical in G, several structural properties of minimally imperfect graphs turned out to be useful. It is, e.g., well-known that no minimally imperfect graph has a *comparable pair* (two nodes x, y s.t. all neighbors of x except possibly y belong to the neighborhood of y). Furthermore, minimally imperfect graphs do not contain *twins* (*antitwins*) (two nodes x, y s.t. all remaining nodes of the graph are adjacent to both or to none of x and y (to either x or y)) due to Lovász [24] (Olariu [25]). A result of Berge [2] shows that, for every minimally imperfect graph, there is no *simplicial node* having a clique as neighborhood.

The above observations on critical edges together with these results imply:

Lemma 2.1 [32] *If* $e = xy$ *is a critical edge of a perfect graph* G*, then*

(i) e *belongs to a triangle and an even hole,*

(ii) x *and* y *do not form twins, antitwins, or a comparable pair,*

(iii) *neither* x *nor* y *is simplicial.*

If two nodes x and y form twins, antitwins, or a comparable pair in G, they do so in the complementary graph $\overline{G}$, too. In addition, two non-adjacent nodes x and y must not form a 2-pair (all induced paths connecting x and y have length two) if xy is supposed to be an anticritical edge.

Lemma 2.2 [32] *If* $e = xy$ *is an anticritical edge of a perfect graph* G*, then*

(i) x *and* y *belong to a stable set of size 3 and to an even antihole,*

(ii) x *and* y *do not form twins, antitwins, a comparable pair, or a 2-pair,*

(iii) *neither* x *nor* y *is simplicial in* $\overline{G}$.

We can use these lemmas in order to investigate Problem 1.2: In which perfect graphs do critical or anticritical edges occur at all? More precisely, are there classes of perfect graphs so that no graph in this class has a critical or anticritical edge? E.g., Lemma 2.1(i) says that every critical edge is contained in a triangle and in an even hole. Hence neither bipartite nor triangulated graphs contain any critical edge. Moreover, the existence of an odd cycle of length ≥ 5 in the union of the triangle and the even hole implies that *line-perfect graphs* cannot admit critical edges, too (since they contain no odd cycles of length ≥ 5). The main result is a characterization of perfect graphs without any critical edge by Hougardy & Wagler (see [32] for the proof).

Theorem 2.3 *A perfect graph does not admit any critical edge if and only if it is Meyniel.*

This theorem was originally proved with the help of a result due to Hertz [17] which implies that removing any edge from a Meyniel graph yields a perfect graph. Due to the Strong Perfect Graph Theorem, this is now obvious since every subgraph $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect is isomorphic either to an odd cycle of length ≥ 5 with precisely one chord e or to $\overline{P}_{2k+1}$, $k \geq 2$ which contains $\overline{P}_5$.

We immediately obtain two corollaries from the above theorem:

Corollary 2.4 *A perfect graph* G *does not admit any anticritical edge if and only if* $\overline{G}$ *is Meyniel.*

Corollary 2.5 *A perfect graph G does neither admit critical nor anticritical edges if and only if G and $\overline{G}$ are Meyniel.*

It is worth noting that Corollary 2.5 can be used to prove that the $C_5, P_5, \overline{P}_5$-free graphs are precisely the graphs G where both G and $\overline{G}$ are Meyniel [18].

The above results show further that critical resp. anticritical edges exist in all perfect graphs G such that G resp. $\overline{G}$ is not Meyniel. The next lemma presents examples of perfect graphs where deleting a critical resp. adding an anticritical edge creates only a certain type of minimally imperfect subgraphs. These facts are easily obtained by combining the knowledge on forbidden subgraphs in the respective graph classes with Lemma 2.1 and Lemma 2.2 (see [32] for more examples).

Lemma 2.6 *Let G be a perfect graph, $e \in E(G)$ a critical edge of G, and $e' \notin E(G)$ an anticritical edge of G.*

(i) *If G is diamond-free or K_4-free, then $G - e$ has odd holes as only minimally imperfect subgraphs.*

(ii) *If G is strict quasi parity or strongly perfect, then $G + e'$ has odd holes as only minimally imperfect subgraphs.*

(iii) *If G is weakly triangulated or bull-free, then $G - e$ has odd antiholes and $G + e'$ odd holes as only minimally imperfect subgraphs.*

(iv) *If G is murky (i.e., $C_5, P_6, \overline{P}_6$-free), then $G - e$ and $G + e'$ have the C_5 as only minimally imperfect subgraph.*

Note that a diamond is a graph obtained from a K_4 by deleting one edge and a bull is a graph with five nodes a, b, c, d, e and edges ab, bc, bd, cd, de.

An even stronger result holds for perfect line graphs $L(F)$. It provides a constructive way to characterize critical and anticritical edges e in $L(F)$ and the minimally imperfect subgraphs in the resulting graphs $L(F) - e$ and $L(F) + e$. For that, we define two structures in the underlying graph F. Note that $L(F)$is perfect if and only if F is line-perfect by Trotter [29].

We say that two incident edges x and y form an *H-pair* in F if there is an edge z incident to the common node of x and y and if there is a (not necessarily induced) even cycle C containing x and y but only one endnode of z (see Figure 2(a)). $L(C)$ is an even hole and the node in $L(F)$ corresponding to z has precisely two neighbors on $L(C)$, namely x and y (see Figure 2(b)). Two non-incident edges x and y are called an *A-pair* if they are the endedges of a (not necessarily induced) odd path P with length at least five (see Figure 2(c)). $L(P)$ is an even, chordless path of length at least four with endnodes x and y (see Figure 2(d)).

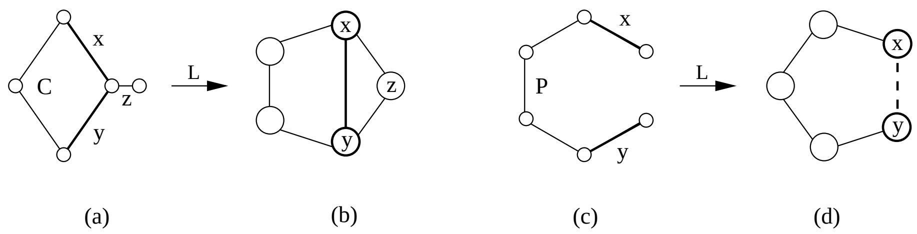


Figure 2: Definition of H-pairs and A-pairs.

It is straightforward that deleting resp. adding the edge xy in $L(C \cup z)$ resp. $L(P)$ yields an odd hole. In [31, 32] it is established that xy is a critical (anticritical) edge in $L(F)$ *only if* x and y form an H-pair (A-pair) in F.

Theorem 2.7 [31, 32] *Consider the line graph $G = (V, E)$ of a line-perfect graph F.*

(i) *An edge $xy \in E$ is critical iff x and y form an H-pair in F.*

(ii) *An edge $xy \notin E$ is anticritical iff x and y form an A-pair in F.*

It is easy to see that $L(F) - xy$ and $L(F) + xy$ do not contain odd antiholes of length ≥ 7. The difficulty in the original proof of Theorem 2.7 was to exclude the occurrence of minimally imperfect Berge graphs in $L(F) - xy$ and $L(F) + xy$ which is now obvious due to the Strong Perfect Graph Theorem. A consequence of Theorem 2.7 is:

Corollary 2.8 *If G is a perfect line graph, then $G - e$ and $G + e'$ have odd holes as only minimally imperfect subgraphs for all edges e and non-edges e' of G.*

3 Perfect and Co-Perfect Edge Orders

In this section, we turn to Problem 1.3: is it possible, for graphs in a certain class $\mathcal{C}$ of perfect graphs, to successively delete or add edges keeping perfectness until a stable set or a clique is reached. The existence of such edge orders for all graphs in $\mathcal{C}$ would provide a constructive method to generate the graphs in $\mathcal{C}$ by consecutively deleting or adding edges.

Let $G = (V, E)$ be a perfect graph. We call a numbering $(e_1, \ldots, e_m)$ of its edge set E a *perfect edge order* if, for $G = G_0$, all graphs $G_i := G_{i-1} - e_i$ are perfect for $1 \leq i \leq m$. Clearly, e_i has to be a non-critical edge of G_{i-1} for $1 \leq i \leq m$, and G_m is a stable set. Analogously, we say that a perfect graph G admits a *co-perfect edge order* iff its complement $\overline{G}$ has a perfect edge order. Here we simply use the numbering of the edges of $\overline{G}$ for the non-edges of G and get finally a clique.

Note that it does not suffice to identify non-critical or non-anticritical edges in the perfect graphs in question. E.g., we can certainly delete an arbitrary edge of a Meyniel graph keeping perfectness by [17], but we may obtain a non-Meyniel graph and do not know anything about its non-critical edges. We cannot provide perfect edge orders for Meyniel graphs, although the graphs in this class are even characterized to contain no critical edge due to Theorem 2.3. Thus we mainly have to look for edges such that their deletion or addition preserves the membership to the corresponding *subclass* of perfect graphs.

In general, we have to look for *critical (anticritical) graphs* with respect to the *subclass* $\mathcal{C}$ of perfect graphs under consideration: that are graphs which lose the studied property by deleting (adding) an arbitrary edge. If we can ensure that no (anti)critical graph with respect to $\mathcal{C}$ exists, then we know there is a (co-)perfect edge order for all graphs in $\mathcal{C}$: every graph G in $\mathcal{C}$ admits at least one edge e such that $G - e$ $(G + e)$ still belongs to $\mathcal{C}$. It is "only" left to find that edge e.

Looking for graph classes without critical graphs, we first observe that there are no critical graphs with respect to every *monotone* class (preserved under deleting edges). The simplest example is the class of bipartite graphs. Obviously, deleting an *arbitrary* edge of a bipartite graph yields a bipartite graph again. Hence, bipartite graphs obviously admit perfect edge orders and, in particular, *every* edge order is perfect.

A superclass of bipartite graphs consists of all line-perfect graphs which clearly admit perfect edge orders, too. (Recall that line-perfect graphs do not admit odd cycles of length ≥ 5.) Moreover, it is easy to see that they are *precisely* the perfect graphs such that *every* edge order is perfect.

Theorem 3.1 [32, 33] *A graph is line-perfect iff all edge orders are perfect.*

Note that bipartite and line-perfect graphs are Meyniel graphs. A further class of Meyniel graphs for which we know a perfect edge order consists of all triangulated graphs. For those graphs, we have a well-known structural result due to Dirac [13]: a graph is triangulated iff every subgraph has a simplicial node x. Deleting an arbitrary edge incident to x yields a graph which is obviously still triangulated. As a consequence, we obtain a perfect edge order for triangulated graphs:

Theorem 3.2 [32, 33] *Every triangulated graph G admits a perfect edge order $(e_1, \ldots, e_m)$ with $G = G_0$, $G_i = G_{i-1} - e_i$, and e_i incident to a simplicial node of G_{i-1} for $1 \leq i \leq m$.*

Clearly, the complements of bipartite, line-perfect, and triangulated graphs admit the corresponding co-perfect edge orders.

For one class of perfect (but non-Meyniel) graphs, both a perfect and a co-perfect edge order are known, namely, for weakly triangulated graphs. Every non-complete subgraph of a weakly triangulated graph G has a 2-pair x, y due to Hayward, Hoàng & Maffray [16]. The graph $G + xy$ is not only perfect by Lemma 2.2 but still weakly triangulated by a result of Spinrad & Sritharan [27]. Consequently, we obtain a co-perfect edge order for weakly triangulated graphs by consecutively adding edges between 2-pairs (called 2-pair non-edge order in [27]).

Theorem 3.3 [27] *Every weakly triangulated graph G admits a co-perfect edge order $(e_1, \ldots, e_{\overline{m}})$ with $G = G_0$, $G_i = G_{i-1} + e_i$ such that the endnodes of e_i form a 2-pair in G_{i-1} for $1 \leq i \leq \overline{m} = |E(\overline{G})|$.*

The class of weakly triangulated graphs is closed under complementation. This yields particularly a perfect edge order for every weakly triangulated graph: Hayward proved in [15] that the following perfect edge order of a weakly triangulated graph G corresponds to the 2-pair non-edge order for $\overline{G}$ given by Spinrad & Sritharan [27].

Theorem 3.4 [15] *Weakly triangulated graphs G admit perfect edge orders $(e_1, \ldots, e_m)$ with $G = G_0$, $G_i = G_{i-1} - e_i$ such that e_i is not the middle edge of any P_4 in G_i for $1 \leq i \leq m = |E(G)|$.*

Weakly triangulated graphs are defined as a common generalization of triangulated graphs and their complements. Hence, the above theorems provide also a co-perfect edge order for triangulated graphs and a perfect edge order for their complements. In addition, the 2-pair non-edge order for weakly triangulated graphs enables us to establish a co-perfect edge order for bipartite graphs G: either there are non-adjacent nodes a, b in different color classes of G and $G + ab$ is still bipartite, or G is as complete bipartite graph weakly triangulated.

At present, perfect or co-perfect edge orders are not known to the author for other subclasses of perfect graphs. However, there are some classes $\mathcal{C}$ of perfect graphs that cannot have such edge orders. If $\mathcal{C}$ contains a critically (anticritically) *perfect* graph G then G is, in particular, critical (anticritical) with respect to $\mathcal{C}$ and there is no perfect (co-perfect) edge order for the graphs in $\mathcal{C}$. In the next section, we present examples of critically and anticritically perfect graphs showing, e.g., that there are neither perfect nor co-perfect edge orders for line graphs of bipartite graphs, planar Berge graphs, and K_4-free Berge graphs (see Section 4.3 for more examples).

4 Critically and Anticritically Perfect Graphs

Section 4.1 affirmatively answers the question whether there exist critically and anticritically perfect graphs at all (Problem 1.4). We present properties and some examples. Section 4.2 provides different ways to construct such graphs. The main result is the characterization of critically and anticritically perfect line graphs. Furthermore, some techniques of constructing critically and anticritically perfect graphs from not necessarily critical or anticritical components are presented. We discuss two consequences for treating Problem 1.5: The classes of critically and anticritically perfect graphs can neither be characterized by forbidden subgraphs nor in a constructive way, e.g., by taking critically perfect line graphs or complements of line graphs as basic components. Moreover, the two classes are incomparable to almost all subclasses of perfect graphs. Section 4.3 therefore discusses the intersection of critically and anticritically perfect graphs with some well-known subclasses of perfect graphs.

4.1 Existence and properties

This subsection solves the problem whether there exist critically and anticritically perfect graphs at all: the answer is in the affirmative. In order to figure out whether a graph is critical or anticritical, the following immediate consequences of Lemma 2.1 and Lemma 2.2 are helpful.

Lemma 4.1 *If G is a critically perfect graph, then*

(i) *every edge belongs to a triangle and an even hole,*

(ii) *two adjacent nodes never form twins, antitwins, or a comparable pair,*

(iii) *no node is simplicial.*

Lemma 4.2 *If G is an anticritically perfect graph, then*

(i) *every non-edge belongs to a stable set of size 3 and to an even antihole,*

(ii) *two non-adjacent nodes never form twins, antitwins, a comparable pair, or a 2-pair,*

(iii) *no node is simplicial in $\overline{G}$.*

Further properties of critically and anticritically perfect graphs have been found in [32], e.g., bounds on the following graph parameters. Additionally, we have a restriction on the connectivity of anticritically perfect graphs.

Lemma 4.3 [32] *Let G be critically perfect and $n = |G|$.*

(i) *G has minimum degree $4 \leq \delta(G)$ and maximum degree $\Delta(G) \leq n - 3$.*

(ii) *G has clique number $3 \leq \omega(G) \leq n - 5$ and stability number $3 \leq \alpha(G) \leq n - 6$.*

(iii) *$\overline{G}$ is 2-connected.*

With the help of some easily detectable properties from the previous lemmas, Hougardy [19] provided the complete answer to Problem 1.4 for graphs on ≤ 11 nodes by enumeration.

Theorem 4.4 [19] *None of the perfect graphs on fewer than 9 nodes is critical. There are 3 critically perfect graphs on 9 nodes, one of them is bicritical. There are 10 resp. 52 critically perfect graphs on 10 resp. 11 nodes, none of them is bicritical.*

Clearly, Theorem 4.4 remains true if "critically perfect" is replaced by "anticritically perfect". Figure 3 shows the three critically perfect graphs on nine nodes. The first graph is self-complementary and, therefore, also anticritical. The other two graphs are not anticritical, but their complements are. Note that these three graphs already show that the bounds in Lemma 4.3 are sharp. None of the critically perfect graphs with 10 and 11 nodes is anticritical (see next subsection for an explanation).

4.2 Constructions and consequences for possible characterizations

This subsection provides further examples of critically and anticritically perfect graphs. On the one hand, we characterize critically and anticritically

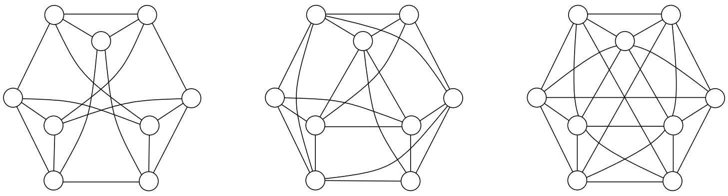


Figure 3: The three smallest critically perfect graphs.

perfect line graphs with the help of Theorem 2.7 and discuss some consequences. On the other hand, we present some operations preserving critical and anticritical perfectness from [32]. Both techniques enable us to easily construct critically and anticritically perfect graphs. Finally, we discuss two consequences for possible characterizations of critically and anticritically perfect graphs.

Due to Theorem 2.7, critical (anticritical) edges of perfect line graphs $L(F)$ correspond to H-pairs (A-pairs) in the underlying graphs F. Consequently, if $L(F)$ is supposed to be critically (anticritically) perfect, *every* pair of incident (non-incident) edges in F must form an H-pair (A-pair). We define a graph with at least two incident (non-incident) edges to be an *H-graph* (*A-graph*) if each pair of incident (non-incident) edges forms an H-pair (A-pair). Theorem 2.7 implies that F has to be a line-perfect H-graph (A-graph) if $L(F)$ is critically (anticritically) perfect. In [31, 32] it was shown that F must be *bipartite* in both cases.

Theorem 4.5 [31, 32] *Let G be the line graph of some graph F.*

(i) *G is critically perfect if and only if F is a bipartite H-graph.*

(ii) *G is anticritically perfect if and only if F is a bipartite A-graph.*

Finding examples of critically (anticritically) perfect line graphs means, therefore, to look for bipartite H-graphs (A-graphs). The three smallest critically perfect graphs are the complements of the line graphs of the three bipartite A-graphs presented in Figure 4. A_1 is also an H-graph, hence $L(A_1)$ is bicritical (it is in fact self-complementary).

All anticritically perfect graphs on 10 and 11 nodes are line graphs of bipartite A-graphs (which arise from A_1, A_2, and A_3 by duplicating existing or adding new edges and are no H-graphs). Note that duplicating edges preserves the property of being an A-graph (since no new pair of non-incident

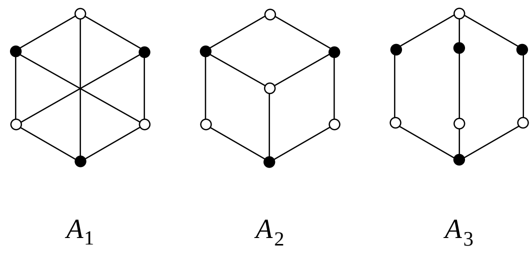


Figure 4: The three smallest bipartite A-graphs.

edges occurs) while it does not preserve the property of being an H-graph since parallel edges are incident but never form an H-pair. This follows also from Lemma 2.1 since the corresponding nodes in the line graphs form twins.

A_1 is the only bipartite H-graph with 3 nodes in each color class. If there are 4 nodes in one color class, then an H-graph has at least 12 edges since it has minimum degree 3 by definition. Hence, the second smallest H-graph admits 12 edges.

The following sufficient condition for a bipartite graph F to be an H-graph *and* an A-graph is established in [32].

Lemma 4.6 [32] *Every simple, 3-connected, bipartite graph is an H-graph as well as an A-graph.*

This provides an easy tool to construct bicritically perfect graphs. For example, every complete bipartite graph $K_{a,b}$ with $a, b \geq 3$ is 3-connected, hence $L(K_{a,b})$ is bicritically perfect. Further examples of bipartite H-graphs and A-graphs can be found in [32].

Once examples of critically and anticritically perfect graphs are known the next natural step is to wonder whether further examples can be constructed using perfectness preserving graph transformations.

It is obvious that the disjoint union preserves critical perfectness while the complete join preserves anticritical perfectness. Further transformations where studied in [32], e.g., clique identification, substitution, composition (also called 1-join), resp. amalgamation which are known to preserve perfectness by Berge [1], Lovász [24], Bixby [4], resp. Burlet & Fonlupt [7].

Let $Q_1 \subseteq G_1$ and $Q_2 \subseteq G_2$ be cliques with $|Q_1| = |Q_2|$. Then *clique identification* means to choose a bijection $\phi : Q_1 \to Q_2$ and to identify every $v \in Q_1$ with $\phi(v) \in Q_2$. The *substitution* of a node v of G_1 by a graph G_2 joins every neighbor of v in G_1 with every node in G_2 and removes v. Let v_1 be a node of G_1 and v_2 of G_2. Then *composition* of G_1 and G_2 w.r.t. v_1

and v_2 means to add an edge between every neighbor of v_1 in G_1 and of v_2 in G_2 and to delete v_1, v_2. The amalgamation is a common generalization of clique identification and composition [7].

The results from [32] show that clique identification preserves critical perfectness but not anticritical perfectness (since anticritically perfect graphs have to be 2-connected by Lemma 4.3(iii)). Substituting nodes of critically (anticritically) perfect graphs by stable sets (cliques) or critically (anticritically) perfect graphs yields again critically (anticritically) perfect graphs. The composition preserves critical perfectness while the more general amalgamation cannot be applied to critically perfect graphs. Both transformations do not preserve anticritical perfectness (see [32] for more details).

It is worth noting that all the *known* examples of critically and anticritically perfect graphs are either line graphs of bipartite graphs, complements of line graphs of bipartite graphs, or can be obtained with the help of the transformations studied in [32]. However, the hope that all critically and anticritically perfect graphs can be obtained by composing some basic *line* graphs turned out to be wrong (although the studied transformations yield graphs with special skew partitions). The reason is that the results in [32] even establish the possibility to construct critically perfect graphs with the help of *non*-critical components by, e.g., clique identification.

Lemma 4.7 [32] *Let G arise by identifying perfect graphs G_1 and G_2 in a clique Q. An edge xy of G is critical if and only if one of the following conditions is satisfied:*

(i) *xy is a critical edge of G_1 or G_2,*

(ii) *$x, y \in Q$, there is an even hole through x and y in G_1, and $G_2 - Q$ contains a common neighbor of x and y (or vice versa).*

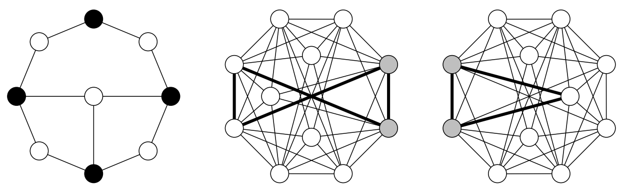
Then G inherits all critical edges from G_1 and G_2 by (i). This provides the opportunity to construct, for each perfect graph F, a critically perfect graph G with $F \subseteq G$ as follows. Cover all non-critical edges of F by cliques $Q_1, \ldots, Q_k$. Choose critically perfect graphs $G_1, \ldots, G_k$ containing cliques Q_i of suitable size and identify F with all graphs G_i in Q_i. The arising graph G is perfect, contains F as subgraph, and has only critical edges by Lemma 4.7(i). Analogously, we obtain, for each perfect graph F, an anticritically perfect graph G with $F \subseteq G$. Thus we have:

Theorem 4.8 [32] *Every perfect graph may occur as a subgraph of a critically or an anticritically perfect graph.*

This theorem has a strong consequence. Namely, if we want to characterize critically or anticritically perfect graphs by means of forbidden subgraphs, we have to exclude all imperfect subgraphs (as in the case of general perfect graphs). To distinguish between general perfect and critically or anticritically perfect graphs further subgraphs need to be excluded. But this is not possible by Theorem 4.8. We obtain as immediate consequence:

Corollary 4.9 *Critically and anticritically perfect graphs cannot be characterized by means of forbidden subgraphs.*

The above construction shows already that there is no way to compose critically or anticritically perfect graphs from some basic (line) graphs, since *every* perfect graph may be involved. Even more, Lemma 4.7(ii) provides the opportunity to create *new* critical edges via clique identification: $xy \in Q$ may be a critical edge of G but neither of G_1 nor G_2. I.e., it is possible to create critically perfect graphs from two perfect but non-critical components.



(a) (b)

Figure 5

Figure 5(a) shows a bipartite graph F and Figure 5(b) two copies of the complement of its line graph. $\overline{L(F)}$ admits precisely one non-critical edge, joining the grey nodes in the picture. Identifying the two copies of this graph in the grey nodes yields a critically perfect graph due to Lemma 4.7(ii) (the hole and the triangle are emphasized by bold lines in the picture). This implies:

Corollary 4.10 *Critically and anticritically perfect graphs cannot be characterized by decomposition, i.e., by a procedure that decomposes such graphs along certain structural faults into basic components.*

4.3 Intersection with subclasses of perfect graphs

This subsection asks in which subclasses of perfect graphs critically and anticritically perfect graphs occur at all. As an immediate consequence of Theorem 4.8, the class of all (anti)critically perfect graphs is not contained in any subclass of perfect graphs for which one perfect forbidden subgraph F exists. Conversely, if F is a minimal forbidden subgraph of a class, then every proper subgraph $F' \subset F$ belongs to that class. If F' is sufficiently small (e.g., if F' has fewer than nine nodes), it cannot be (anti)critically perfect due to Theorem 4.4. We conclude that no subclass of perfect graphs with a perfect forbidden subgraph is contained in the class of all (anti)critically perfect graphs. This implies:

Corollary 4.11 *There is no inclusion relation between the class of critically (anticritically) perfect graphs and any subclass of perfect graphs with a perfect forbidden subgraph.*

For most classes of perfect graphs there exists a perfect graph F which is minimal forbidden for all graphs in this class. We study, therefore, the *intersection* of critically and anticritically perfect graphs with other subclasses of perfect graphs (see [21] for definitions and inclusion relations).

It is obvious that no Meyniel graph is critically perfect graph by Theorem 2.3. The existence of a (co-)perfect edge order clearly implies that there is no (anti)critically perfect graph in such a class. Hence the results in [15] and [27] show that weakly triangulated graphs are neither critically nor anticritically perfect. The same is true for so-called clique separable graphs by [32]. No similar result has been established for the graphs in BIP*, bull-free Berge graphs, strict quasi parity graphs, and strongly perfect graphs but in none of these classes are critically perfect graphs known. Note that absorbantly perfect graphs are characterized in [14] to possess either a strong stable set (that meets all maximal cliques of the graph) or a comparable pair of adjacent nodes. Hence the class of absorbantly perfect graphs cannot contain any critically perfect graph that is not strongly perfect due to Lemma 2.2. No anticritically perfect graphs are known belonging to BIP* and the class of bull-free Berge graphs. The bicritically perfect graph $L(A_1)$ admits an embedding in the plane, is alternately colorable, locally perfect, and belongs to the classes of F-free Berge graphs where F is a claw, a diamond, or a K_4. The anticritical graph $L(A_3)$ is strongly perfect and perfectly contractile. $L(A_3)$ and its critically perfect complement $\overline{L(A_3)}$ are preperfect and quasi parity. (See [32] for more results.)

5 Perfectness is an Elusive Graph Property

A graph property P is called *elusive* (or also evasive) if every algorithm for testing this property has to read in the worst case all $\binom{n}{2}$ entries of the adjacency matrix of a given graph. This is equivalent to considering the following two-players game. Player **A** wants to know whether an unknown simple graph on a given node set has the graph property P in question by asking Player **B** one by one, whether a certain pair of nodes is an edge. At each stage Player **A** makes full use of the information of edges and non-edges he has up to that point in order to decide whether the graph has property P or not. Player **A** wants to minimize the number of his queries, Player **B** wants to force him to ask as many queries as possible. The number of questions needed for the decision if both players play optimally from their point of view is the recognition complexity $c(P)$ of the studied graph property. The property is said to be elusive for graphs on n nodes if there is a strategy enabling Player **B** to force Player **A** to test *every* node pair before coming to a decision, i.e., if $c(P) = \binom{n}{2}$. The property is said to be elusive if such a strategy exists for all non-trivial cases (all n such that there are graphs on n nodes with and without the studied property P).

Several graph properties are known to be elusive, e.g., having a clique of a certain size or a coloring with a certain number of color classes (Bollobás [5]) and being planar for graphs on ≥ 5 nodes (Best et al. [3]), see [6, 28] for more examples. An application of the concept of critically and anticritically perfect graphs is the following result by Hougardy & Wagler [23]:

Theorem 5.1 [23] *Perfectness is an elusive graph property.*

From the Strong Perfect Graph Theorem we know that identifying one induced odd hole or odd antihole would enable Player **A** to make the final decision: the graph in question is not perfect. Consequently, Player **B** has to answer in such a way that no induced odd hole or odd antihole appears until Player **A** asks the last question but that the last answer can create such an induced subgraph.

The odd hole of length five is the smallest imperfect graph. Hence, the cases with $n \leq 4$ nodes are trivial: Player **A** knows without asking any question that the studied graph is perfect. In order to prove Theorem 5.1 a strategy for all non-trivial cases $n \geq 5$ is required.

The odd hole C_5 is the *only* imperfect graph on five nodes (note: the C_5 is self-complementary, hence also the odd antihole on five nodes). Thus, one cannot reach another imperfect graph from the C_5 by deleting or adding one edge. This provides a simple strategy for Player **B** in the smallest non-trivial case: answer all queries but the last as in the C_5.

Proposition 5.2 [23] *Perfectness is elusive for graphs on five nodes.*

The case $n = 5$ is, however, the only case where precisely one imperfect graph exists. But the main idea for proving Theorem 5.1 is to try precisely the opposite way: Find perfect graphs such that you cannot reach another perfect graph by deleting or adding one edge. Hence, we look for bicritically perfect graphs which are, by definition, both critically *and* anticritically perfect: the deletion and addition of an *arbitrary* edge yields an imperfect graph. If there exists a bicritically perfect graph G_n on n nodes, then Player **B** has only to answer all but the last query "$ij \in E$?" of Player **A** as in G_n. I.e., Player **B** has only to apply the following strategy for graphs on n nodes.

Strategy 1 Let G_n be a bicritically perfect graph on n nodes.
For queries 1 to $\binom{n}{2} - 1$:
  Answer "$ij \in E$?" with YES if $ij \in E(G_n)$, NO otherwise.

Then no induced imperfect subgraph appears during the first $\binom{n}{2} - 1$ questions, and the answer to the last query "$ij \in E$?" yields the decision. In order to prove Theorem 5.1, the task is, therefore: Find, for as many n as possible, a bicritically perfect graph G_n on n nodes.
Hougardy & Wagler constructed in [23] with the help of Theorem 4.5 and Lemma 4.6 bicritically perfect line graph G_n for each $n \geq 12$ as follows. Consider the graphs $F_{3k} = (A \cup B, E_1 \cup E_2)$ with

$$\begin{aligned} A &= \{1, 3, \ldots, 2k-1\} \\ B &= \{2, 4, \ldots, 2k\} \\ E_1 &= \{ii+1 : 1 \leq i \leq 2k\} \\ E_2 &= \{ii+3 : i \in A\} \end{aligned}$$

for each $k \geq 3$ (all indices are taken modulo $2k$). The three smallest examples of graphs F_{3k} for $k \in \{3, 4, 5\}$ are shown in Figure 6 (note $A_1 = F_9$). F_{3k} is an even cycle $(A \cup B, E_1)$ on its $2k$ nodes with k chords E_2 outgoing from a node in A with odd index and ending in a node in B with even index. Thus, the graphs F_{3k} are bipartite and simple by construction. In [23] it is shown that they are 3-connected. Hence, Lemma 4.6 ensures that the line graphs of F_{3k} are bicritically perfect for $k \geq 3$.

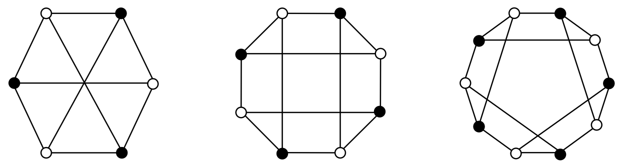


Figure 6: The graphs F_{3k} with $k = 3, 4, 5$.

The gaps with $n = 3k + 1, 3k + 2$ can be closed using the following immediate consequence of Lemma 4.6: If $F = (A \cup B, E)$ is a simple, 3-connected, bipartite graph and $ab \notin E$ with $a \in A, b \in B$, then $F + ab$ is a simple bipartite A- and H-graph. This yields the studied bipartite A- and H-graphs F_n for $n = 3k+1$ and $n = 3k+2$ if $k \geq 4$ (but not for the complete bipartite graph F_9). Hence, Strategy 1 can be applied in almost all cases:

Lemma 5.3 [23] *Perfectness is elusive for graphs on $n = 9$ and $n \geq 12$ nodes.*

However, Theorem 4.4 shows that there are no bicritically perfect graphs G_n with $n \leq 8$ and $n = 10, 11$ nodes. The cases $n = 10, 11$ are treated in [23] with the help of a slightly different concept. Let us call a graph G *almost bicritically perfect* if G is anticritically perfect and all edges but one are critical. We modify Strategy 1 for almost bicritically perfect graphs as follows:

Strategy 2 Let G_n be an almost bicritically perfect graph on n nodes and let uv be its only non-critical edge.
Query 1:
  Answer "$ij \in E$?" with YES. Number the nodes of G_n s.t. $i = u$, $j = v$.
For queries 2 to $\binom{n}{2} - 1$:
  Answer "$ij \in E$?" with YES if $ij \in E(G_n)$, NO otherwise.

Then no imperfect subgraph appears during the first $\binom{n}{2} - 1$ queries, and the answer to the last question yields the decision again.

Almost bicritically perfect graphs G_{10} and G_{11} are constructed in [23] as the line graphs of the bipartite A-graphs F_{10} and F_{11} depicted in Figure 7 which both admit precisely one non-H-pair. This implies:

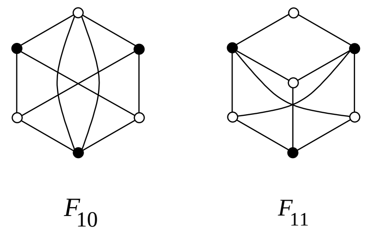


Figure 7: The graphs F_{10} and F_{11}.

Lemma 5.4 [23] *Perfectness is elusive for graphs on* $n = 10, 11$ *nodes.*

The remaining cases $6 \leq n \leq 8$ are solved in [23] with the help of a parity argument due to Rivest and Vuillemin [26]: if a graph property P is *not* elusive for graphs on n nodes then the number $G(P, n, \text{even})$ of labeled graphs on n nodes with property P having an even number of edges *equals* the number $G(P, n, \text{odd})$ of labeled graphs on n nodes with property P that have an odd number of edges. In particular, $G(P, n, \text{even}) \neq G(P, n, \text{odd})$ implies that P *is* elusive for graphs on n nodes.
In [23] is shown that $G(P, n, \text{even})$ and $G(P, n, \text{odd})$ differ for perfect graphs on eight nodes. An extension of this argument has been used for $n = 6, 7$: If perfectness is not elusive for graphs on n nodes then it is also not elusive for the graphs containing one fixed edge.

Lemma 5.5 [23] *Perfectness is elusive for graphs on* $6 \leq n \leq 8$ *nodes.*

Proposition 5.2, Lemma 5.3, Lemma 5.4, and Lemma 5.5 finally imply Theorem 5.1: Perfectness is an elusive graph property.
Note that the Strong Perfect Graph Theorem has neither been used to prove Theorem 5.1 nor would this deep result be useful to simplify the proof.

6 Concluding Remarks

The previous sections provide an overview on the investigation of critical and anticritical edges w.r.t. perfectness. We summarize the main results and conclude with some remarks and open problems.

Section 2 lists several rules from [32] how to detect critical and anticritical edges in perfect graphs (Problem 1.1) and presents a new characterization of Meyniel graphs (Theorem 2.3) in order to answer which perfect graphs admit critical or anticritical edges at all (Problem 1.2).

Section 3 studies perfect and co-perfect edge orders (Problem 1.3). We pointed out that one has to look for edges whose deletion or addition preserves the membership in the corresponding subclass of perfect graphs in order to find such edge orders for all graphs in this subclass. That way, perfect edge orders are obtained for bipartite, line-perfect, and triangulated graphs [32, 33]. The existence of perfect and co-perfect edge orders for weakly triangulated graphs was established by Hayward [15] and Spinrad & Sritharan [27].

We observed further that no perfect edge order is known for Meyniel graphs although the graphs in this class do not admit any critical edge. The reason is that deleting an edge from a Meyniel graph does not necessarily yield a Meyniel graph again. In order to prove or disprove the existence of perfect edge orders for Meyniel graphs, one has to solve:

Problem 6.1 *Show that every Meyniel graph G has an edge e s.t. $G - e$ is still Meyniel or find a critical Meyniel graph.*

Problem 6.1 may be solved with the help of a stronger result:

Problem 6.2 *Show that there is an edge e for all graphs in* BIP^* *s.t.* $G - e$ *still belongs to* BIP^*.

The decomposition of graphs G in BIP^* along star-cutsets in G and $\overline{G}$ yields finally bipartite components. Removing any edge e from such a bipartite component creates again a graph in BIP^*, provided one can ensure that e is not contained in any of the star-cutsets. An answer to Problem 6.2 would imply perfect edge orders for all graphs in BIP^*, hence also for all Meyniel, perfectly orderable, and alternately orientable graphs.

Clearly, the existence of perfect resp. co-perfect edge orders for a subclass $\mathcal{C}$ of perfect graphs excludes the existence of critically and anticritically perfect graphs in $\mathcal{C}$, and vice versa. Hence, neither critical nor anticritical perfect graphs are weakly triangulated due to Hayward [15] and Spinrad & Sritharan [27]. Section 4 answers affirmatively whether critically and anticritically perfect graphs exist at all (Problem 1.4) by presenting examples from [32]. In particular, certain bicritically perfect graphs show that there are neither perfect nor co-perfect edge orders for, e.g., line graphs of bipartite graphs (hence for alternately colorable, claw-free, and diamond-free Berge graphs), locally perfect, quasi parity, preperfect, and planar Berge graphs.

Moreover, there are classes of perfect graphs for which it is neither known to admit (co-)perfect edge orders nor to contain (anti)critically perfect graphs, e.g., for BIP* and bull-free Berge graphs. Neither perfect edge orders nor critically perfect graphs are known in the classes of strict quasi parity and strongly perfect graphs. Further investigations are needed here.

Section 4.2 shows that two popular ways to characterize subclasses of perfect graphs fail for critically and anticritically perfect graphs: the characterization via forbidden subgraphs and the decomposition with the help of certain structural faults into graphs of a basic form. No other characterization has been found so far, hence Problem 1.5 is still unsolved.

It is, moreover, worth to mention that anticritically resp. bicritically perfect graphs play a role for constructing minimally even pair-free and minimally non-preperfect graphs.

Even pairs play an important role in conjunction with perfectness and many classical families of perfect graphs were proven to be strict quasi parity during the last 20 years. However, there are perfect graphs with no even pair, e.g., all the line graphs of 3-connected bipartite graphs [20]. Hougardy conjectured in [20] that the minimally even pair-free graphs (i.e., the minimally non-strict quasi parity graphs) are either odd holes, antiholes of length at least seven, or line graphs of bipartite graphs.

Hougardy proved his conjecture for all graphs on less than 17 nodes by enumeration. Besides odd holes and antiholes of length at least seven, his list contains only line graphs $L(F)$ where F is either the $K_{2,3}$, a bipartite A-graph, or a graph obtained from another bipartite graph F' with minimally even pair-free line graph by a simple operation (insert a C_4 on an edge of F').

The latter operation creates bipartite graphs F such that $\overline{L(F)}$ admits an even pair. The odd holes, odd antiholes, and anticritically perfect graphs among the above minimally even pair-free graphs are, therefore, precisely the minimally non-quasi parity graphs on less than 17 nodes. It is interesting to prove Hougardy's conjecture and to investigate the relation of anticritically perfect and minimally even pair-free graphs in general.

Hammer & Maffray introduced in [14] the class of preperfect graphs (by relaxing the concept of comparable pairs). They presented an infinite sequence of perfect but minimally non-preperfect graphs and asked for further examples. A different sequence of such graphs was found (but not published) by Hougardy, Maffray & Sebö [22]. All these graphs are line graphs of special bipartite graphs. Motivated by this observation, minimally non-preperfect graphs with small maximum degree have been investigated by Tuza & Wagler [30]. In fact, a graph of maximum degree 4 is minimally non-preperfect if and only if it is either an odd hole, the $\overline{C}_7$, or the line graph of a 3-regular, 3-connected bipartite graph [30]. Note that no other perfect but minimally non-preperfect graphs are known yet, hence all of them are bicritically perfect.

Finally, Chudnovsky introduces in her thesis [9] a concept related to critical and anticritical edges. She defines a *trigraph* $T = (V, E \cup N \cup S)$ to have node set V where the set of node pairs is partitioned into three disjoint sets: the set E of strong edges, the set N of strong non-edges, and the set S of switchable pairs which may become edges or non-edges. In this notation, T is a graph if $S = \emptyset$. A *realization* of a trigraph $T = (V, E \cup N \cup S)$ is any graph $G = (V, E \cup S')$ for some $S' \subseteq S$. A trigraph is *perfect* (resp. *imperfect*) if every of its realizations is perfect (resp. imperfect). That means: one has to ensure for a perfect trigraph that the node pairs in S cannot form critical or anticritical edges.
Note that trigraphs describe the intermediate stages in the two-players game defining elusiveness: we have node pairs known to be edges or non-edges and node pairs which are undecided yet. Providing a strategy for Player **B** means, therefore, to avoid the occurrence of perfect and imperfect trigraphs with $S \neq \emptyset$ otherwise Player **A** would be able to decide "is perfect" or "is imperfect" without probing the remaining node pairs in S.
Recall that Hougardy & Wagler [23] proved elusiveness for perfect graphs on 6,7, and 8 nodes with the help of a parity argument due to Rivest and Vuillemin [26] but not by giving an *explicit* strategy for these cases. In fact, such an explicit strategy is known for graphs on 6 nodes but not on 7 and 8 nodes. The knowledge of all the perfect and imperfect trigraphs on 7 and 8 nodes with S maximal may help to find such strategies for the cases on 7 and 8 nodes.

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