PENSION MATHEMATICS
with Numerical Illustrations

Second Edition

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Published by

Pension Research Council
Wharton School of the University of Pennsylvania

and

University of Pennsylvania Press
Philadelphia

Library of Congress Cataloging-in-Publication Data
Winklevoss, Howard E.
Pension mathematics with numerical illustrations / Howard E. Winklevoss. –2nd ed.
p. cm.
Includes bibliographical references and index.
ISBN 0-8122-3196-1
1. Pensions–Mathematics. 2. Pensions–Costs–Mathematics. 3. Pension trusts–Accounting. I. Title.
HD7105.W55 1993
331.25'2–dc20 92-44652
CIP

Printed in the United States of America

Chapter 9

Multiple Retirement Ages

The mathematics presented thus far have assumed that retirement occurs at a single retirement age r. Most plans, however, provide for early and late retirement, sometimes on an *actuarially equivalent* basis. If the benefit provided at early or late retirement has the same present value as the benefit payable at normal retirement, then the retirement benefit is deemed to be actuarially equivalent to the normal retirement benefit. One might surmise that such a benefit would be cost neutral to the plan sponsor; however, as we will see, the cost effect is dependent on the actuarial cost method in use.

The first section of this chapter defines the concept of actuarial equivalence, while the second section defines pension costs for alternative actuarial cost methods under the assumption of multiple retirement ages. Although the aggregate cost methods are not considered explicitly, the previously discussed analogies of weighting the numerator and denominator of the individual cost methods to obtain the results under the aggregate methods still hold.

ACTUARIAL EQUIVALENCE

The symbol ${}^{*}g_k^{(r)}$ represents a retirement grading function which, when multiplied by a participant's natural (or formula) accrued benefit, B_x, produces actuarially equivalent benefits.[1] The principle underlying an actuarially equivalent benefit is one

[1] The asterisk is used to denote the special case when the value of the grading function is equal to the *actuarially equivalent* grading function.

that seeks to provide the participant with an adjusted benefit such that its present value evaluated at the retirement age in question is equal to the present value of the *unadjusted* benefit payable at the normal retirement age of the plan. This relationship for retirement at age k relative to normal retirement at age r is given by

$$^{*}g_k^{(r)}\ B_k\ \ddot{a}_k = B_k\ {}_{r-k}p_k^{(m)}\ v^{r-k}\ \ddot{a}_r. \tag{9.1}$$

This equation assumes that $r > k$. If $r < k$, then the reciprocal of the survival probability is used. The remaining discussion assumes that $r > k$; however, the same principles apply to the opposite case.

The left side of (9.1) is the present value at age k of the reduced early retirement benefit payable immediately, while the right side represents the present value (also evaluated at age k) of the participant's accrued benefit assumed to be payable at age r. An expression for the actuarially equivalent grading function is obtained by solving (9.1):

$$^{*}g_k^{(r)} = \frac{{}_{r-k}p_k^{(m)}\ v^{r-k}\ \ddot{a}_r}{\ddot{a}_k} = \frac{{}_{r-k|}\ddot{a}_x}{\ddot{a}_k}. \tag{9.2a}$$

The actuarially equivalent grading function, consisting of the ratio of an $(r-k)$-year deferred life annuity at age k to a nondeferred life annuity at age k, is clearly less than unity for all $k < r$ (and greater than unity for all $k > r$). The effect of alternative interest and mortality rates can be seen more clearly by rewriting (9.2a) in the following manner:

$$^{*}g_k^{(r)} = \frac{{}_{r-k}p_k^{(m)}\ v^{r-k}\ \ddot{a}_r}{\ddot{a}_{k:\overline{r-k|}} + {}_{r-k}p_k^{(m)}\ v^{r-k}\ \ddot{a}_r}. \tag{9.2b}$$

The denominator of (9.2b) is less sensitive to changes in interest and mortality because of the temporary annuity factor. Thus, the actuarial equivalent grading function is inversely related to interest and mortality: the higher the interest and/or mortality rates the smaller will be the actuarially equivalent early retirement benefit. Males, for example, have higher mortality than females; hence, their actuarially equivalent reduction for early retirement would exceed the reduction for females if such reductions were based on gender as opposed to using a unisex mortality assumption. Similarly, actuarially equivalent reductions would be greater in a high interest rate environment compared to a low interest rate environment. As a practical matter, the plan document must define the actuarially equivalent methodology, which typically does not change with changes in the valuation interest rate or the market rate of interest. Thus, early retirement adjustments may be greater or less than the plan-based or market-based actuarial adjustment.

Table 9–1 gives the value of the actuarial equivalent grading function for ages 55 through 70, with age 65 representing the plan's normal retirement age. The effect of alternative interest and mortality rates are also shown. Interestingly, the actuarially equivalent grading function under the standard assumptions (8 percent interest and GAM-71 mortality assumptions) decreases by nearly 10 percent for each successively younger early retirement age. Thus, the grading function is approximately equal to $.9^{k-r}$ for $k < r$. An employee retiring early at age 61, for example, would receive about two-thirds of the benefit earned to date, while an employee retiring at age 55 would receive about one-third. The data in Table 9–1 show that rather wide variations in

TABLE 9–1

Actuarial Equivalent Grading Function

		Actuarial Equivalent Grading Function Under Alternative Assumptions				
		Interest Rate Sensitivity		*Mortality Rate Sensitivity*		
Age	${}^{\bullet}g_k^{(r)}$	$i = .06$	$i = .10$	$.5\,q_k^{'(m)}$	$1.5\,q_k^{'(m)}$	$\frac{1}{{}^{\bullet}g_k^{(r)}}$
55	0.33	0.39	0.28	0.38	0.29	3.02
56	0.37	0.42	0.32	0.41	0.33	2.73
57	0.41	0.46	0.36	0.46	0.37	2.46
58	0.45	0.50	0.40	0.50	0.41	2.22
59	0.50	0.55	0.46	0.55	0.46	1.99
60	0.56	0.60	0.52	0.61	0.52	1.79
61	0.62	0.66	0.59	0.67	0.59	1.60
62	0.70	0.73	0.67	0.74	0.67	1.43
63	0.79	0.81	0.76	0.81	0.76	1.27
64	0.89	0.90	0.87	0.90	0.87	1.13
65	1.00	1.00	1.00	1.00	1.00	1.00
66	1.13	1.12	1.15	1.11	1.15	0.88
67	1.29	1.25	1.33	1.24	1.34	0.78
68	1.47	1.41	1.54	1.38	1.56	0.68
69	1.68	1.59	1.79	1.54	1.82	0.59
70	1.94	1.80	2.09	1.73	2.15	0.52

the interest and mortality rate assumptions have a relatively modest impact on the actuarially equivalent grading function at most ages. Below age 60, however, the impact becomes more significant.

If the plan were to provide accrued benefits instead of actuarially equivalent benefits at early or late retirement, then the added cost would be represented by the reciprocal of the grading function, shown in the last column of Table 9–1. While this factor represents the true economic cost, the nominal cost depends on the actuarial cost method in use.

PRESENT VALUE OF FUTURE BENEFITS

The present value of future benefits under multiple retirement ages is presented as an introduction to the notation and because of its close relationship to the various actuarial cost methods. Equation (9.3) defines this function for a participant currently age x who is eligible to retire at any age from r' to r''.

$$^{r'}(PVFB)_x = \sum_{k=x}^{r''} g_k^{(r)} \ B_k \ {}_{k-x}^{\ r}p_x^{(T)} \ q_k^{(r)} \ v^{k-x} \ \ddot{a}_k \qquad (9.3)$$

where

r' = first age for which the employee qualifies for early retirement

r'' = age at which all employees are assumed to have retired

$g_k^{(r)}$ = proportion of the participant's accrued benefit payable if retirement occurs at the beginning of age k

${}_{k-x}^{\ r}p_x^{(T)}$ = probability of surviving in employment $(k - x)$ years, where retirement decrements are included with mortality, termination, and disability decrements

$q_k^{(r)}$ = probability of retiring at the *beginning* of age k.

The prescript r' to the $(PVFB)_x$ function is used to signify that costs are determined according to multiple retirement ages, whereas the r prescript has been used to denote this function based on a single retirement age, usually the normal retirement age but not restricted to that age. This formulation assumes, for simplicity, that all retirements occur at the beginning of the age in question, an assumption made throughout this book. The grading function gives the portion of the participant's accrued benefit available for retirement at age k. If full benefits are payable at all retirement ages, this function takes on the value of unity. It is more likely, however, that the benefit payable at some early retirement ages will be less than the attained age accrued benefit, and in some cases will be actuarially reduced. Beyond the normal retirement age, the grading function might exceed unity if the plan provides for actuarial or formula increases.

The present value of future benefits function evaluated with multiple retirement ages, and assuming actuarially equivalent early retirement benefits, is denoted by ${}^{*r'}(PVFB)_x$ and found by substituting (9.2a) for the grading function in (9.3): [2]

$$
{}^{*r'}(PVFB)_x = \sum_{k=x}^{r} \left[\frac{{}_{r-k}p_k^{(m)}\, v^{r-k}\, \ddot{a}_r}{\ddot{a}_k} \right] \cdot B_k\, {}_{k-x}^{\;r}p_x^{(T)}\, q_k^{(r)}\, v^{k-x}\, \ddot{a}_k. \qquad (9.4a)
$$

By expressing the survival function in terms of individual survival probabilities and noting that there are no termination rates beyond the first early retirement eligibility age, equation (9.4a) can be written as follows:

$$
{}^{*r'}(PVFB)_x = \left[\sum_{k=x}^{r} B_k\, {}_{k-x}p_x^{(d)}\, {}_{k-x}p_x^{(r)}\, q_k^{(r)} \right] \cdot {}_{r-x}p_r^{(m)}\, {}_{r-x}p_r^{(t)}\, v^{r-x}\, \ddot{a}_r. \qquad (9.4b)
$$

An intuitively appealing approximation to the ${}^{*r'}(PVFB)_x$ function evaluated for an actuarially equivalent grading function is found by assuming no disability rates beyond r'. This allows the disability survival function to be extracted from the summation as follows:

$$
{}^{*r'}(PVFB)_x \approx \left[\sum_{k=x}^{r} B_k\, {}_{k-x}p_x^{(r)}\, q_k^{(r)} \right] {}_{r-x}p_r^{(T)}\, v^{r-x}\, \ddot{a}_r. \qquad (9.4c)
$$

[2]For simplicity, this discussion assumes that the maximum retirement age is the normal retirement age r (i.e., $r'' = r$). The same general conclusions hold for retirements beyond age r, however, the reciprocal of the survival probability in (9.4a) would have to be used for these ages.

Note the use of the composite survival function in place of the individual survival rates for termination, disability, and mortality. The quantity in brackets represents the *expected* early retirement benefit, denoted by $E(B)$. Thus, an approximation to the actuarially equivalent early retirement function is

$$^{*r'}(PVFB)_x \approx E(B)\ {}_{r-x}p_x^{(T)}\ v^{r-x}\ \ddot{a}_r = \frac{E(B)}{B_r}\ {}^{r}(PVFB)_x. \quad (9.4d)$$

Equation (9.4c) shows that, if early retirement is permitted on an actuarially equivalent basis, the $^{r}(PVFB)_x$ function evaluated at $E(B)$ instead of B_r provides an approximation to $^{*r'}(PVFB)_x$. The approximation under estimates the true value of $^{*r'}(PVFB)_x$, but the error is relatively small for the typical set of disability rates. If B_x increases with age then $E(B) < B_r$; hence, the present value of future benefits is reduced for plans providing actuarially equivalent early retirement benefits.

ACCRUED BENEFIT METHOD

The normal cost under the accrued benefit method with multiple retirement ages is given by equation (9.5), with the benefit function equal to the natural (or formula) accruals:

$$^{AB\ r'}(NC)_x = b_x \sum_{k=x}^{r''-1} g_k^{(r)}\ {}_{k-x}^{\ r}p_x^{(T)}\ q_k^{(r)}\ v^{k-x}\ \ddot{a}_k. \quad (9.5)$$

The actuarial liability is found by substituting B_x for b_x in (9.5).

Evaluation (9.5) with actuarially equivalent early retirement benefits (again, assuming for simplicity, that r is the last retirement age) produces the following equation:

$$^{AB\ *r'}(NC)_x = b_x \left[\sum_{k=x}^{r-1} {}_{k-x}p_x^{(d)}\ {}_{k-x}p_x^{(r)}\ q_k^{(r)} \right] \cdot {}_{r-x}p_x^{(m)}\ {}_{r-x}p_x^{(t)}\ v^{r-x}\ \ddot{a}_r. \quad (9.6)$$

If the disability survival probability from r' to r is reasonably constant (i.e., low or nonexistent disability rates), it too can be extracted from the summation sign without introducing a substantial error. The remaining terms inside the bracket sum to unity; hence, an approximation to (9.6) is simply the accrued benefit method evaluated for retirement at age r. The same approximation holds for this method's actuarial liability. Thus, unlike ${}^{r'}(PVFB)_x$, the normal cost and actuarial liability under the accrued benefit method are practically unaffected if early retirement occurs with actuarially reduced benefits. Although this is the result that one would expect, it turns out to be the only actuarial cost method where this it true, as shown subsequently.

BENEFIT PRORATE METHODS

The normal cost for the constant dollar version of the benefit prorate method is given by

$$^{BD}\,{}^{r'}(NC)_x = \sum_{k=x}^{r''-1} \frac{B_k}{(k-y)}\, g_k^{(r)}\ {}_{k-x}^{\;\;r}p_x^{(T)}\ q_k^{(r)}\ v^{k-x}\ \ddot{a}_k. \tag{9.7}$$

The actuarial liability can be expressed as

$$^{BD}\,{}^{r'}(AL)_x = \sum_{k=x}^{r''} \frac{B_k}{(k-y)}\,(x-y)\ g_k^{(r)}\ {}_{k-x}^{\;\;r}p_x^{(T)}\ q_k^{(r)}\ v^{k-x}\ \ddot{a}_k. \tag{9.8}$$

In each case, the benefit is projected to each successive retirement age and prorated from that age back to the employee's entry age. Clearly, if there is only one retirement age, r, then (9.7) simplifies to (6.8) and (9.8) simplifies to (5.6a).

The corresponding equations for the constant percent method are as follows: [3]

$$^{BP}\,{}^{r'}(NC)_x = \sum_{k=x}^{r''-1} \frac{B_k}{S_k}\, s_x\, g_k^{(r)}\ {}_{k-x}^{\;\;r}p_x^{(T)}\ q_k^{(r)}\ v^{k-x}\ \ddot{a}_k; \tag{9.9}$$

$$^{BP}\,{}^{r'}(AL)_x = \sum_{k=x}^{r''} \frac{B_k}{S_k}\, S_x\, g_k^{(r)}\ {}_{k-x}^{\;\;r}p_x^{(T)}\ q_k^{(r)}\ v^{k-x}\ \ddot{a}_k. \tag{9.10}$$

Evaluating the normal cost given in (9.7) with actuarially equivalent early retirement benefits (assuming $k < r$) gives

$$^{BD}\,{}^{*r'}(NC)_x = \left[\sum_{k=x}^{r-1} \frac{B_k}{(k-y)}\ {}_{k-x}p_x^{(d)}\ {}_{k-x}p_x^{(r)}\ q_k^{(r)}\right] \cdot {}_{r-x}p_x^{(m)}\ {}_{r-x}p_x^{(t)}\ v^{r-x}\ \ddot{a}_r. \tag{9.11}$$

---

[3]The normal cost under these methods can be defined by deriving a benefit allocation factor that is, indeed, a constant value (as a dollar amount or as a percent of salary) which can then be substituted for the benefit factor in (9.5). This methodology was presented in the first edition of this book. The equations given above, however, have become the accepted approaches.

Again, if the disability survival function is reasonably constant over the various early retirement ages, the bracketed term represents the *average* benefit allocation factor. Since this term will be less than the benefit factor evaluated at normal retirement [i.e., $B_r \div (r-y)$ from equation (6.8)], actuarially equivalent early retirement under the constant dollar benefit prorate method will be less expensive than normal retirement. The corresponding actuarial liability, of course, will be less also.

The same conclusion can be reached for the constant percent benefit prorate method. In this case, the analogous term to the bracketed term in (9.11) involves the salary function.

COST PRORATE METHODS

The normal cost under the constant dollar cost prorate method with multiple retirement ages is equal to the following ratio:

$$^{CD\,r'}(NC)_x = \frac{^{r'}(PVFB)_y}{^{r'}\ddot{a}^T_{y:\overline{r-y}|}} \tag{9.12}$$

where $^{r'}\ddot{a}^T_{y:\overline{r-y}|}$ represents the present value of a temporary employment-based annuity including early retirement decrements from age r' to r''. The corresponding equation for the normal cost under the constant percent version is given by

$$^{CP\,r'}(NC)_x = \frac{^{r'}(PVFB)_y}{s_y \; ^{r's}\ddot{a}^T_{y:\overline{r-y}|}} \, s_x \tag{9.13}$$

where the r' prescript to the temporary annuity symbol again denotes the assumption of early retirement decrements.

The actuarial liability under the constant dollar version, evaluated prospectively, may be expressed as

$$^{CD\,r'}(AL)_x = {}^{r'}(PVFB)_x - \frac{^{r'}(PVFB)_y}{^{r'}\ddot{a}^T_{y:\overline{r-y}|}} \, {}^{r'}\ddot{a}^T_{x:\overline{r-x}|}, \tag{9.14}$$

while that of the constant percent version is found by replacing the annuities in (9.14) with salary-based annuities multiplied by the appropriate salary function. Equations (5.8b) and (5.9b), which express the actuarial liability as a ratio of annuities times the present value of future benefits function, hold for multiple retirement ages but only for $x \le r'$.

The normal cost evaluated with actuarially equivalent early retirement benefits (again, assuming age r is the last retirement age) is found by the appropriate substitution of the grading function into the numerators of (9.12) and (9.13). An approximation, made by assuming the disability survival probability is constant from r' to r, is as follows:

$$^{CD\,*r'}(NC)_x \approx \frac{E(B)\ {}_{r-y}p_y^{(T)}\ v^{r-y}\ \ddot{a}_r}{{}^{r'}\ddot{a}^{T}_{y:\overline{r-y}|}} \tag{9.15}$$

where $E(B)$ was defined in conjunction with (9.4c). In words, (9.15) equals the expected early retirement benefit evaluated at normal retirement age, divided by an $(r-y)$-year employment based life annuity, where the latter includes the retirement decrements from r' to r. The decrease in $E(B)$ relative to B_r is generally greater than the decrease in the employment based life annuity for early retirement relative to its value at normal retirement. Thus, it can be concluded that actuarially reduced early retirement generally produces a *lower* cost than the cost for normal retirement. The same conclusion holds for the constant percent version, the analogous equation to (9.15) being as follows:

$$^{CP\,*r'}(NC)_x \approx \frac{E(B)\ {}_{r-y}p_y^{(T)}\ v^{r-y}\ \ddot{a}_r}{s_y\ {}^{r's}\ddot{a}^{T}_{y:\overline{r-y}|}}\, s_x. \tag{9.16}$$

The actuarial liability associated with actuarially equivalent early retirement benefits is given by (19.17a), which can be expressed more compactly for $x \le r'$ by (19.17b):

$$^{CD\,*r'}(AL)_x \approx E(B)\ {}_{r-x}p_x^{(T)}\ v^{r-x}\ \ddot{a}_r - \frac{E(B)\ {}_{r-y}p_y^{(T)}\ v^{r-y}\ \ddot{a}_r}{{}^{r'}\ddot{a}^{T}_{y:\overline{r-y}|}}\ {}^{r'}\ddot{a}^{T}_{x:\overline{r-x}|} \tag{9.17a}$$

$$\approx E(B)\ {}_{r-x}p_x^{(T)}\ v^{r-x}\ \ddot{a}_r\ \frac{{}^{r'}\ddot{a}^{T}_{y:\overline{x-y}|}}{{}^{r'}\ddot{a}^{T}_{y:\overline{r-y}|}}.\quad (x \le r') \tag{9.17b}$$

In both cases the disability survival probability is assumed to be constant from r' to r. The corresponding actuarial liability equations for the constant percent version can be obtained by substituting the salary-based temporary annuities for the temporary annuities in (9.17a) and (9.17b). The annuities in (9.17a) must be multiplied by the appropriate salary function s_x in the numerator and s_y in the denominator, whereas the salary functions cancel out in (9.17b), since s_y is required for both the numerator and the denominator.

It turns out that, like the normal cost of the cost prorate method under actuarially equivalent early retirement benefits, the actuarial liability given by the above equations is *less* than the corresponding liability based on normal retirement.

RELATIVE COST OF EARLY RETIREMENT

The normal cost and actuarial liability *early retirement cost ratio* (ERCR), equal to the ratio of the cost of an early retirement benefit to the cost of a normal retirement benefit, is given by (9.18a) when the early retirement grading function is not specified and by (9.18b) when actuarially equivalent benefits are provided, both based on attained age x $(x < k)$:

$$_k(ERCR)_x = g_k^{(r)} \frac{B_k}{B_r} \frac{1}{_{r-k}p_k^{(T)}} \frac{1}{v^{r-k}} \frac{\ddot{a}_k}{\ddot{a}_r} C_k; \qquad (9.18a)$$

$$_k^*(ERCR)_x = \frac{B_k}{B_r} \frac{_{r-k}p_k^{(m)}}{_{r-k}p_k^{(T)}} C_k \qquad (9.18b)$$

where the coefficient C_k has the following values:

$= \dfrac{B_r}{B_k}$ for accrued benefit method

$= \dfrac{S_r}{S_k}$ for benefit prorate, constant percent method

$= \dfrac{r-y}{k-y}$ for benefit prorate, constant dollar method

$= \dfrac{^s\ddot{a}^T_{y:\overline{r-y}|}}{^s\ddot{a}^T_{y:\overline{k-y}|}}$ for cost prorate, constant percent method

$= \dfrac{\ddot{a}^T_{y:\overline{r-y}|}}{\ddot{a}^T_{y:\overline{k-y}|}}$ for cost prorate, constant dollar method

$= 1$ for the $(PVFB)x$ function.

All of the definitions of C_k exceed unity, implying that the relative cost of early retirement under each of the actuarial cost methods is *greater* than the relative cost of early retirement under the $(PVFB)_x$ function. Table 9–2 gives the $_k(ERCR)_x$ based on equation (9.18a) for each of the actuarial cost methods under the model assumptions and for the $(PVFB)_x$ function. The data show that the relative cost of early retirement, based on non-reduced accrued benefits, is significantly higher under the accrued benefit and benefit prorate methods than it is under the cost prorate methods.

TABLE 9–2

Relative Cost of Early Retirement With Full Benefit Accruals for Various Normal Costs

Age	*Accrued Benefit Method*	*Benefit Prorate Methods* Constant Percent	*Benefit Prorate Methods* Constant Dollar	*Cost Prorate Methods* Constant Percent	*Cost Prorate Methods* Constant Dollar	*PVFB*
65	1.00	1.00	1.00	1.00	1.00	1.00
64	1.16	1.15	1.10	1.08	1.07	1.07
63	1.34	1.30	1.20	1.16	1.14	1.13
62	1.52	1.46	1.30	1.22	1.19	1.19
61	1.73	1.63	1.39	1.29	1.25	1.24
60	1.95	1.82	1.49	1.35	1.29	1.28
59	2.19	2.01	1.58	1.40	1.33	1.31
58	2.46	2.21	1.67	1.45	1.36	1.34
57	2.74	2.43	1.76	1.49	1.39	1.36
56	3.06	2.66	1.84	1.53	1.41	1.37
55	3.40	2.90	1.93	1.56	1.42	1.38
54	3.91	3.27	2.09	1.65	1.48	1.43
53	4.50	3.68	2.25	1.74	1.54	1.48
52	5.17	4.14	2.43	1.82	1.60	1.53
51	5.93	4.65	2.61	1.91	1.65	1.57
50	6.79	5.21	2.80	2.00	1.70	1.60

Table 9–3 gives the $_k^*(ERCR)_x$ based on equation (9.18b), i.e., where actuarially equivalent benefits are provided. The relative cost of actuarially reduced benefits under the cost prorate methods is similar to the relative cost of the $(PVFB)_x$ function, all of which are considerably less than unity and decrease as the age is lowered. The accrued benefit method, at the other extreme, exceeds unity and represents an increasing function of lower attained ages.

The analysis up to this point has been in terms of the cost of early retirement at age k relative to the cost at age r. The cost of early retirement at a given age with full benefits versus actuari-

TABLE 9–3

Relative Cost of Early Retirement With Actuarially Reduced Benefit Accruals for Various Normal Costs

	Accrued	*Benefit Prorate Methods*		*Cost Prorate Methods*		
	Benefit	*Constant*	*Constant*	*Constant*	*Constant*	
Age	*Method*	*Percent*	*Dollar*	*Percent*	*Dollar*	*PVFB*
65	1.00	1.00	1.00	1.00	1.00	1.00
64	1.03	1.01	0.98	0.96	0.95	0.95
63	1.05	1.02	0.94	0.91	0.89	0.89
62	1.07	1.02	0.91	0.86	0.84	0.83
61	1.08	1.02	0.87	0.80	0.78	0.77
60	1.09	1.02	0.83	0.75	0.72	0.71
59	1.10	1.01	0.79	0.70	0.67	0.66
58	1.11	1.00	0.75	0.65	0.61	0.60
57	1.11	0.99	0.71	0.60	0.56	0.55
56	1.12	0.97	0.68	0.56	0.51	0.50
55	1.13	0.96	0.64	0.52	0.47	0.46
54	1.17	0.98	0.63	0.49	0.44	0.43
53	1.22	1.00	0.61	0.47	0.42	0.40
52	1.27	1.02	0.60	0.45	0.39	0.38
51	1.33	1.04	0.58	0.43	0.37	0.35
50	1.38	1.06	0.57	0.41	0.35	0.33

ally reduced benefits is equal to the reciprocal of ${}^{*}g_k^{(r)}$ and is valid for all actuarial cost methods. Table 9–1 provided this statistic, indicating a substantial cost increase for full benefits.

Figure 9–1 shows, for the model pension plan, the cost of normal retirement, early retirement with full benefits, and early retirement with actuarially reduced benefits based on the retirement rates presented in Table 2.9 of Chapter 2.

FIGURE 9–1a

Normal and Early Retirement Under Alternative Normal Costs

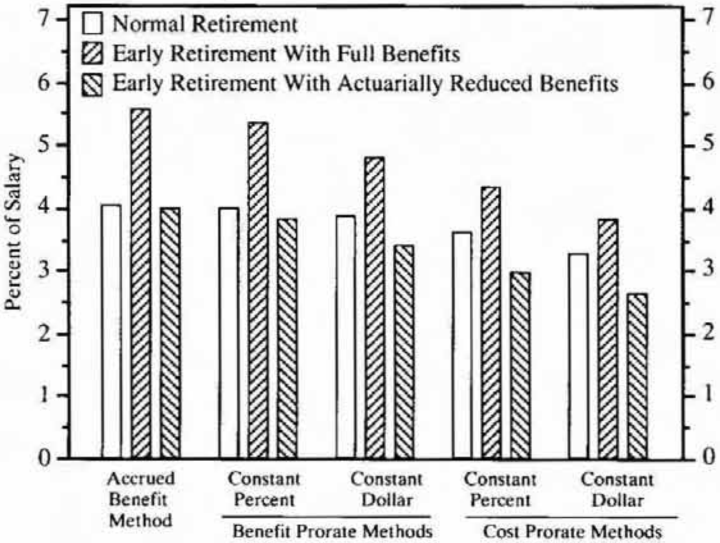

FIGURE 9–1b

Normal and Early Retirement Under Alternative Actuarial Liabilities