Resolution of Kleinian Singularities

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ABSTRACT

Firstly, the classification of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$, a result of Felix Klein in 1884, is presented. The polynomial invariant subrings of these groups are then found. The generators of these subrings satisfy a polynomial relation in three variables, which can be realised as a hypersurface in $\mathbb{C}^3$. Each of these surfaces have a singularity at the origin; these are the *Kleinian singularities*. These singularities are blown-up, and their resolution graphs are shown to be precisely the Coxeter-Dynkin diagrams ADE. The target readership of this project is intended to be undergraduates with a foundational knowledge of group theory, topology and algebraic geometry.

1 Classifying the Finite Subgroups of SL$(2, \mathbb{C})$

1.1 Important Subgroups of the General Linear Group

Recall: The *general linear group* of a vector space V over a field $\mathbb{F}$ is given by

$$\mathrm{GL}(V) = \{f : V \longrightarrow V \mid f \text{ is linear and invertible}\}.$$

In particular, we denote $\mathrm{GL}(\mathbb{F}^n)$ by $\mathrm{GL}(n, \mathbb{F})$. Since we can view linear maps as matrices, $\mathrm{GL}(n, \mathbb{F})$ can also be viewed as the set of invertible $n \times n$ matrices with entries in $\mathbb{F}$.

The next few definitions include important subgroups of $\mathrm{GL}(n, \mathbb{F})$.

Definition 1.1. The *special linear group* over $\mathbb{F}$ is given by

$$\mathrm{SL}(n, \mathbb{F}) = \{A \in \mathrm{GL}(n, \mathbb{F}) \mid \det A = 1\}.$$

Definition 1.2. The *orthogonal group* over $\mathbb{F}$ is given by

$$\mathrm{O}(n, \mathbb{F}) = \{A \in \mathrm{GL}(n, \mathbb{F}) \mid AA^T = I\}$$

where A^T denotes the transpose of A, and I denotes the $n \times n$ identity matrix.

Note that if $\mathbb{F} = \mathbb{R}$ we simply refer to $\mathrm{O}(n, \mathbb{R})$ as the *orthogonal group* and denote it by $\mathrm{O}(n)$. Additionally, if $\mathbb{F} = \mathbb{C}$ we have the *unitary group*, given by

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid AA^* = I\}$$

where A^* denotes the conjugate transpose of A.

Definition 1.3. The *special orthogonal group* over $\mathbb{F}$ is given by

$$\mathrm{SO}(n, \mathbb{F}) = \{A \in \mathrm{O}(n, \mathbb{F}) \mid \det A = 1\}.$$

Moreover, if $\mathbb{F} = \mathbb{R}$ we refer to $\mathrm{SO}(n, \mathbb{R})$ as the *special orthogonal group*, denoted $\mathrm{SO}(n)$, and if $\mathbb{F} = \mathbb{C}$ we have the *special unitary group*, given by

$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det A = 1\}.$$

Remark 1.4. Routine calculations prove that the sets defined in Definitions 1.1-1.3 are indeed subgroups of the general linear group under matrix multiplication.

In this project we are especially interested in $\mathrm{SL}(2, \mathbb{C})$, the set of complex 2×2 matrices with determinant equal to 1.

1.2 Platonic Solids and the Finite Subgroups of SO(3)

To present the classification of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$, we will make use of the classification of finite subgroups of $\mathrm{SO}(3)$. The latter arises as a consequence of there existing exactly five platonic solids. We will construct these now.

Definition 1.5. A *half-space* is either of the two parts into which a plane divides $\mathbb{R}^3$. A *convex polyhedron* in $\mathbb{R}^3$ is the intersection of finitely many half-spaces. Loosely speaking then, a polyhedron is a solid in three dimensions such that its faces are flat and its edges are straight. By *convex* we mean that for any two points inside (including on the boundary of) the polyhedron, all of the points on the line joining them are also contained inside the polyhedron.

Definition 1.6. Let P be a polyhedron. A *flag* of P is a triple (v, e, F) consisting of a vertex v, an edge e and a face F such that v is one of the endpoints of e, and e is one of the sides of F. We now say that P is *regular* if for any two flags of P there is a symmetry (rotation or reflection) of P mapping one to the other.

Theorem 1.7. (The Platonic Solids) Let P be a regular convex polyhedron. By the regularity of P, each face must be a regular polygon and an equal number of them must meet at each vertex. Suppose that the faces of P are p-gons and that q faces meet at each vertex. The pair $\{p, q\}$ is then called the *Schläfli symbol* of P. Let v be a vertex of P. We know that at v, the total of the angles between each pair of edges connecting to v must be less than 2π. Since q faces meet at v, there are q such angles and since each face is a regular p-gon, these angles are all of size $\pi - 2\pi/p$. Thus we have the condition

$$q(\pi - 2\pi/p) < 2\pi$$

which simplifies to

$$(p-2)(q-2) < 4.$$

Together with the natural condition that $p, q \geqslant 3$, we have that the only integer solutions are {3,3}, {4,3}, {3,4}, {5,3}, {3,5}. These pairs identify the five platonic solids and are illustrated below. The table below that gives the number of vertices, edges and faces of each.

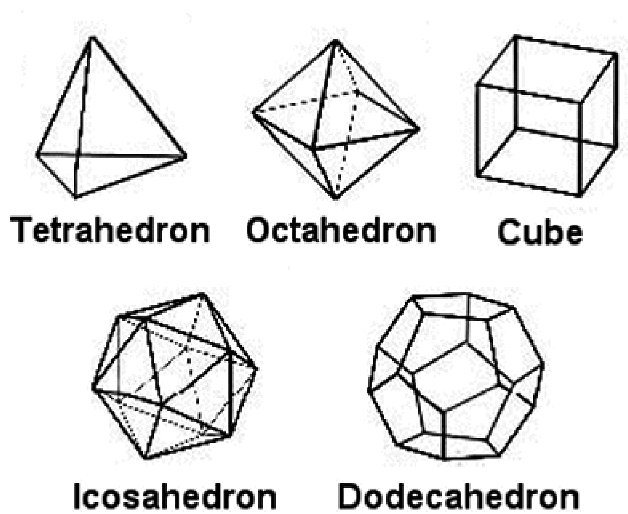


Figure 1: The Platonic Solids [1]

Polyhedron	Vertices	Edges	Faces	Schläfli symbol
Tetrahedron	4	6	4	{3,3}
Cube	8	12	6	{4,3}
Octahedron	6	12	8	{3,4}
Dodecahedron	20	30	12	{5,3}
Icosahedron	12	30	20	{3,5}

□

The classification of finite subgroups of SO(3) is now given, although the proof is not. A full proof given from first principles and perfect for undergraduate reading can be found in [2, pp. 10-15].

We say that a pair of solids are *dual* if one can be constructed from the other by connecting vertices placed at the centres of the faces of its dual. An example of this can be seen using the cube and octahedron below.

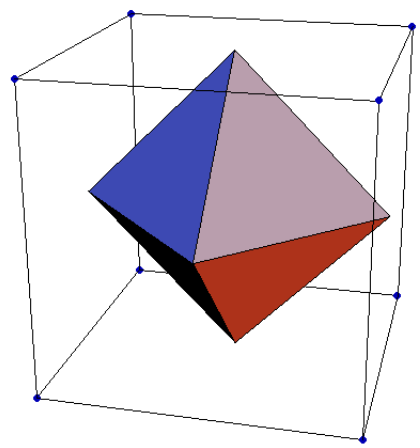


Figure 2: Illustration of the duality of the cube and octahedron [3]

Essentially then, dual solids are solids with their faces and vertices interchanged. Due to this, the symmetry groups of dual solids are the same.

Below is a table of the symmetry groups of the platonic solids. Note that the tetrahedron is self-dual.

Platonic Solid	Isomorphic to:	Order
Tetrahedron	A_4	12
Cube and Octahedron	S_4	24
Dodecahedron and Icosahedron	A_5	60

Recall that transformations in $\mathbb{R}^3$ that preserve orientation and distance from the origin are precisely rotations about the origin; these are the matrices comprising SO(3).

The classification: All finite subgroups of SO(3) are isomorphic to either:

- a cyclic group $\mathbb{Z}_n$, order n. We can view $\mathbb{Z}_n$ as a cyclic group of rotations around a particular axis. It is generated by a rotation a satisfying $a^n = 1$.
- a dihedral group D_n, order $2n$. We can view D_n as the rotations of a prism based on a regular n-gon. It is generated by two rotations a and b that satisfy the relations $a^n = 1$, b^2=1, and $bab^{-1} = a^{-1}$.

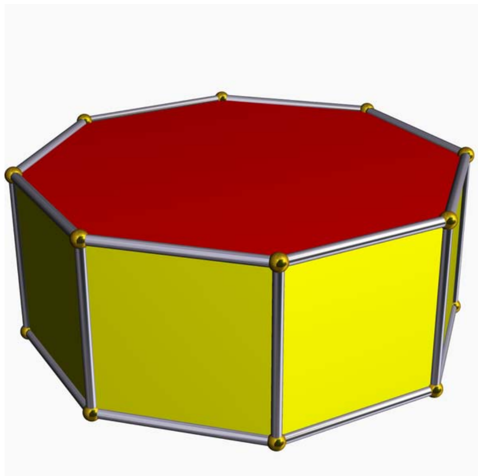


Figure 3: Octagonal Prism. Its rotational symmetry group is D_n, $n = 8$. [4]

- the rotational symmetry group of a platonic solid, either:
  - the tetrahedron $T \cong A_4$, order 12.
  - the octahedron $O \cong S_4$, order 24.
  - the icosahedron $I \cong A_5$, order 60.

1.3 From SL$(2,\mathbb{C})$ to SO(3)

Definition 1.8. Let G be a group. We say that two subgroups H_1, H_2 of G are *conjugate subgroups* if $\exists g \in G$ such that $gH_1g^{-1} = H_2$.

Lemma 1.9. Every finite subgroup of SL$(n,\mathbb{C})$ is conjugate to a subgroup of SU(n).

Proof. Let G be a finite subgroup of SL$(n,\mathbb{C})$. Denote the usual inner product on $\mathbb{C}^n$ by $\langle\ ,\ \rangle$ (so $\langle u, v\rangle = u \cdot v = \sum_{j=1}^{n} u_j \overline{v_j}$). We will need a new inner product $(\ ,\ )$ on $\mathbb{C}^n$ that is unitary with respect to G, i.e. $(Au, Av) = (u, v)\ \forall A \in G$ and $\forall u, v \in \mathbb{C}^n$. The inner product

$$(u, v) := \frac{1}{|G|} \sum_{A \in G} \langle Au, Av\rangle$$

will do. It is easy to check that this is an inner product on $\mathbb{C}^n$ and the fact that $AG = G\ \forall A \in G$ implies that it is unitary.

Now, since $\mathbb{C}^n$ with $(\ , \ )$ is a finite dimensional inner product space, there exists an orthonormal basis $\mathcal{B}$ for $(\ , \ )$ by the Gram-Schmidt process. Let $\rho : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the change of basis operator taking $\mathcal{B}$ to the standard basis. Then $\rho \in \mathrm{GL}(n, \mathbb{C})$ and $\rho G \rho^{-1}$ is a subset of $\mathrm{SU}(n)$ as

$$\langle \rho A \rho^{-1} u, \rho A \rho^{-1} v \rangle = (A\rho^{-1}u, A\rho^{-1}v) = (\rho^{-1}u, \rho^{-1}v) = \langle u, v \rangle$$

$\forall A \in G$ and $\forall u, v \in \mathbb{C}^n$. Moreover, $\rho G \rho^{-1}$ is a subgroup of $\mathrm{SU}(n)$ as it is the image of the conjugation map $A \longmapsto \rho A \rho^{-1}$ which is a homomorphism. By construction, G is conjugate to the subgroup $\rho G \rho^{-1}$. $\square$

Remark 1.10. Lemma 1.9 tells us that in order to classify all finite subgroups of $\mathrm{SL}(2, \mathbb{C})$, it is enough to classify all finite subgroups of $\mathrm{SU}(2)$ (up to conjugacy).

Lemma 1.11. There exists a natural surjective group homomorphism $\pi : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ with kernel $\{\pm I\}$.

Proof. For this proof we make use of the fact that the quaternions, denoted by $\mathbb{H}$, of norm equal to 1 can be used to describe rotations in $\mathbb{R}^3$, as can matrices in $\mathrm{SO}(3)$ (see page 4).

Recall that quaternions are of the form $a+bi+cj+dk$, where $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1$ and $ij = k$. Notice that we can write a quaternion $q \in \mathbb{H}$ as $q = z_1 + z_2 j$, where $z_1 = a + bi$ and $z_2 = c + di$ are complex numbers. Hence we have

$$\begin{aligned} |q|^2 &= a^2 + b^2 + c^2 + d^2 = |z_1|^2 + |z_2|^2, \\ \bar{q} &= a - bi - cj - dk = \bar{z_1} - z_2 j. \end{aligned}$$

A quaternion q is invertible if and only if $|q| \neq 0$, in which case $q^{-1} = \frac{1}{|q|}\bar{q}$. Remember that multiplication of quaternions is not commutative!

Recall from the definition that $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det A = 1\}$. In the case $n = 2$, we have

$$\mathrm{SU}(2) = \{\begin{pmatrix} z_1 & z_2 \\ -\bar{z_2} & \bar{z_1} \end{pmatrix} \ : \ z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\}.$$

Thus we have a group isomorphism (a routine calculation to check)

$$\Phi : \mathrm{SU}(2) \longrightarrow \mathbb{H}_1, \begin{pmatrix} z_1 & z_2 \\ -\bar{z_2} & \bar{z_1} \end{pmatrix} \mapsto z_1 + z_2 j$$

where $\mathbb{H}_1$ is the group of quaternions of norm 1 ($\mathbb{H}_1$ is a subgroup of $\mathbb{H}^* = \mathbb{H}\backslash\{0\}$. It is closed under multiplication, has identity 1 and inverses as described above).

Now let us identify $\mathbb{R}^3$ with the space of "pure quaternions", i.e. quaternions of the form $bi+cj+dk$. Recall that complex numbers of norm 1 can be written in the form $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Similarly we can write quaternions of norm 1 in the form $q = e^{q_1\theta} = \cos(\theta) + q_1 \sin(\theta)$, where q_1 is a pure quaternion of norm 1 (for a thorough explanation of general quaternions in polar form, see [5]). Akin to using $e^{i\theta}$ to represent a rotation in the plane by θ about the origin, $e^{q_1\theta}$ can be used to represent a rotation by θ around the axis given by the unit vector q_1 in Euclidean space.

For any pure quaternion q_0 (vector in $\mathbb{R}^3$) and any $q \in \mathbb{H}_1$ written in the form $q = e^{q_1\theta}$, the expression qq_0q^{-1} gives the resulting vector of rotating q_0 by 2θ around the axis q_1. A sketch proof of this is given in [6, pp. 20-23]. Therefore we can define the map $\pi : \mathbb{H}_1 \longrightarrow \mathrm{SO}(3)$, $q \mapsto qq_0q^{-1}$, which rotates each fixed vector q_0 by 2θ around the axis q_1 as explained above. This map is clearly a homomorphism (it is essentially just conjugation by q) and is surjective as any rotation in $\mathbb{R}^3$ can be expressed by a unit quaternion.

To work out the kernel, suppose $\pi(q) = qq_0q^{-1} = q_0$, i.e. $qq_0 = q_0q$. Then we are looking for quaternions of norm 1 that commute with every pure quaternion. The only ones are $\{1, -1\}$. In terms of SU(2), this is $\{I, -I\}$.

Since $\mathrm{SU}(2) \cong \mathbb{H}_1$, we have that $\pi : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$, via the isomorphism Φ above, is the surjective homomorphism required. $\square$

Lemma 1.12. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the only element of SU(2) of degree 2.

Proof. Any element of SU(2) with degree 2 satisfies

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives us the system of equations

$$\alpha^2 - |\beta|^2 = 1 \tag{1}$$
$$\alpha\beta + \beta\bar{\alpha} = 0 \tag{2}$$
$$-\bar{\beta}\alpha - \bar{\beta}\bar{\alpha} = 0 \tag{3}$$
$$\bar{\alpha}^2 - |\beta|^2 = 1 \tag{4}$$

Note that (1) implies $\alpha \neq 0$ since otherwise $-|\beta|^2 = 1$. Now (1) and (4) imply that $\alpha^2 = \bar{\alpha}^2$ so $\alpha = \pm\bar{\alpha}$. Thus $\alpha = x$ or $\alpha = ix$ for some $x \in \mathbb{R}\backslash\{0\}$. If $\alpha = ix$ then (1) implies $(ix)^2 - |\beta|^2 = 1$ so $-x^2 - |\beta|^2 = 1$, however $-x^2 - |\beta|^2 < 0$ so we cannot have $\alpha = ix$. Hence $\alpha = x$ and (2) implies that $2\alpha\beta = 0$ so $\beta = 0$ and finally (1) gives $\alpha = \pm 1$. Of course $\alpha = 1$ gives the identity matrix, and $\alpha = -1$ gives the degree 2 matrix that we seek. $\square$

Lemma 1.13. Let G be a finite subgroup of SU(2) and let π be the map constructed in Lemma 1.11. Then either G is cyclic of odd order, or $|G|$ is even and $G = \pi^{-1}(\pi(G))$ is the preimage of a finite subgroup of SO(3).

Proof. First suppose $|G|$ is odd. Then there are no elements of order 2 in G so $\ker(\pi) \cap G = \{I\}$. By the First Isomorphism Theorem, the restriction of π to G is isomorphic to its image, so by the classification of finite subgroups of SO(3), this can only be a cyclic group of odd order.

Now suppose $|G|$ is even. Then by Cauchy's Theorem, G must have an element of order 2, which is $-I$ by Lemma 1.12. Hence $\ker(\pi) = \{\pm I\} \subseteq G$, so $G = \pi^{-1}(\pi(G))$ is the preimage of a finite subgroup of SO(3). $\square$

Remark 1.14. When $|G|$ is even, $\ker(\pi)$ is of order 2 in G so π is a two-to-one surjection. A proof of this is as follows: suppose y is an element of the codomain. By surjectivity, there exists an x in the domain such that $\pi(x) = y$. Let e, I be the identity elements of G and SO(3) respectively. Since π is a homomorphism and $-e \in \ker(\pi)$, then $\pi(-x) = \pi(-e)\pi(x) = I\pi(x) = y$, so at least two elements in the domain map to y. If there was a third, say $\pi(a) = y$, then we have $\pi(x) = \pi(a) \Rightarrow \pi(x)\pi(a)^{-1} = I \Rightarrow \pi(xa^{-1}) = I \Rightarrow xa^{-1} \in \ker(\pi)$. We have just produced a third element in the kernel, a contradiction.

Therefore if $|\pi(G)| = n$, then $|G| = 2n$. We call G a *binary polyhedral group*; it corresponds to a finite subgroup of SO(3) but has order twice that of its image. This classifies finite subgroups of SU(2), and hence $\mathrm{SL}(2, \mathbb{C})$ up to conjugation.

Theorem 1.15. (Classification of the Finite Subgroups of SL$(2, \mathbb{C})$) The classification of the non-trivial finite subgroups of $\mathrm{SL}(2, \mathbb{C})$, up to conjugation, are precisely the binary polyhedral groups, which are given below. Hereafter we set $\varepsilon_k = \exp(\frac{2\pi i}{k})$.

A_n: For $n \geq 1$, the cyclic group $G \cong \mathbb{Z}_m$, where $m = n + 1$, order m, generated by

$$\begin{pmatrix} \varepsilon_m & 0 \\ 0 & \varepsilon_m^{-1} \end{pmatrix}.$$

D_n: For $n \geq 4$, the *binary dihedral group* $\mathbb{D}_m$, where $m = n - 2$, order $4m$, generated by A, B where

$$A = \begin{pmatrix} \varepsilon_{2m} & 0 \\ 0 & \varepsilon_{2m}^{-1} \end{pmatrix}, \; B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

E_6: The *binary tetrahedral group* $\mathbb{T}$, order 24, generated by σ, τ, μ where

$$\sigma = \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^{-1} \end{pmatrix}, \; \tau = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; \mu = \frac{1}{1-i} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

E_7: The *binary octahedral group* $\mathbb{O}$, order 48, generated by κ, τ, μ where

$$\kappa = \begin{pmatrix} \varepsilon_8 & 0 \\ 0 & \varepsilon_8^{-1} \end{pmatrix}, \; \tau = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; \mu = \frac{1}{1-i} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

E_8: The *binary icosahedral group* $\mathbb{I}$, order 120, generated by γ, τ, Ω where

$$\gamma = \begin{pmatrix} \varepsilon_{10} & 0 \\ 0 & \varepsilon_{10}^{-1} \end{pmatrix}, \; \tau = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; \Omega = \frac{1}{\sqrt{5}} \begin{pmatrix} \varepsilon_5 - \varepsilon_5^4 & \varepsilon_5^2 - \varepsilon_5^3 \\ \varepsilon_5^2 - \varepsilon_5^3 & -\varepsilon_5 + \varepsilon_5^4 \end{pmatrix}.$$

Remark 1.16. i) The generators of each group can be found via the map π, but these calculations are skipped and only the results given here. It is worth noting however that there are other ways of expressing the above groups using different generators.
ii) This theorem is an example of **ADE Classification**, which is a specific type of classification into two infinite sets of objects indexed by the natural numbers, namely A_n for $n \geq 1$ and D_n for $n \geq 4$ (for $n = 1, 2, 3$ we have $A_n \cong D_n$), with three sporadic cases denoted E_6, E_7, E_8. Why we use these names will become apparent later.

2 G-invariant Subrings

Now that we have classified all of the finite subgroups G of $\mathrm{SL}(2,\mathbb{C})$, we seek all of the G-invariant polynomials in two variables over $\mathbb{C}$.

To be explicit, let $\mathbb{C}[u,v]$ be the ring of polynomials in two variables with coefficients in $\mathbb{C}$, let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ (one of A_n, D_n, E_6, E_7, E_8 as in Theorem 1.15) and let G act on $\mathbb{C}[u,v]$. In this chapter we want to find the set of polynomials *invariant under the action of* G, i.e. the set

$$\mathbb{C}[u,v]^G := \{f \in \mathbb{C}[u,v] \mid gf = f \ \forall g \in G\}.$$

So what is the action we are interested in? We simply take the column vector $\begin{pmatrix} u & v \end{pmatrix}^T$ and left-multiply it by a generator of G (we need only consider the generators - see below) to see how it acts on $u, v \in \mathbb{C}[u,v]$. Observe however that u, v generate the algebra $\mathbb{C}[u,v]$, so by understanding how G acts on u, v we can understand how G acts on any $f \in \mathbb{C}[u,v]$.

This is perhaps best understood through an example. Consider the A_n case, so $G \cong \mathbb{Z}_m$, where $m = n+1$. Let $g = \begin{pmatrix} \varepsilon_m & 0 \\ 0 & \varepsilon_m^{-1} \end{pmatrix}$ be the generator of G. Then

$$\begin{pmatrix} \varepsilon_m & 0 \\ 0 & \varepsilon_m^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \varepsilon_m u \\ \varepsilon_m^{-1} v \end{pmatrix}$$

so $gu = \varepsilon_m u$ and $gv = \varepsilon_m^{-1} v$. For example then, if $f = u + v^2 + u^2 v^3 + 1$, we have $gf = \varepsilon_m u + \varepsilon_m^{-2} v^2 + \varepsilon_m^{-1} u^2 v^3 + 1$. An example of an invariant polynomial is $f = uv$, because $gf = \varepsilon_m u \varepsilon_m^{-1} v = uv = f$.

Observe that we only need to check the first power of each generator. If f is invariant under a generator g, i.e. $gf = f$, then it is invariant under all powers of g because $g^2 f = g(gf) = gf = f$, and an induction argument can be used for greater powers. A similar proof shows that we needn't investigate products of distinct generators either.

Lemma 2.1. Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ with action on $\mathbb{C}[u,v]$ as explained above. Then $\mathbb{C}[u,v]^G$ is a subring of $\mathbb{C}[u,v]$.

Proof. Clearly $1 \in \mathbb{C}[u,v]^G$. Suppose $g \in G$ and let $p, q \in \mathbb{C}[u,v]$ be monomials, i.e. $p = au^{r_1}v^{t_1}, q = bu^{r_2}v^{t_2}$ where $r_i, t_i \in \mathbb{Z}$ are non-negative. Then $gp = a(gu^{r_1})(gv^{t_1})$ and $gq = b(gu^{r_2})(gv^{t_2})$.

Hence $g(p+q) = a(gu^{r_1})(gv^{t_1}) + b(gu^{r_2})(gv^{t_2}) = gp + gq$ and $g(pq) = a(gu^{r_1})(gv^{t_1})b(gu^{r_2})(gv^{t_2}) = ab(gu^{r_1+r_2})(gv^{t_1+t_2}) = gpgq$, so g preserves addition and multiplication of monomials in $\mathbb{C}[u,v]$. By extension, g preserves addition and multiplication of every polynomial in $\mathbb{C}[u,v]$ because these are just sums of finitely many monomials. Therefore if f, h are invariant under g we have

- $g(-f) = -gf = -f$
- $g(f+h) = gf + gh = f + h$
- $g(fh) = gfgh = fh$

and so $\mathbb{C}[u,v]^G$ is a subring of $\mathbb{C}[u,v]$. □

Theorem 2.2. Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. Then the G-invariant subrings $\mathbb{C}[u,v]^G$ are generated by the following invariant polynomials:

A_n: $f_1 = u^m$ $(m = n+1)$
$f_2 = v^m$
$f_3 = uv$
We have the relation $f_3^m = f_1 f_2$.

D_n: $f_1 = uv(u^{2m} - 2u^m v^m + v^{2m}),\ (m = n-2)$
For n odd, $f_2 = u^{2m} - v^{2m},\ f_3 = u^2v^2$.
We have the relation $\hat{f_1}^2 = f_3 f_2^2 + 4f_3^{m+1}$, where $\hat{f_1} = f_1 + 2f_3^{\frac{m+1}{2}}$.
For n even, $f_2 = u^{2m} - 2u^m v^m + v^{2m},\ f_3 = uv(u^{2m} - v^{2m})$.

E_6: $f_1 = 2(u^{12} - 33u^8v^4 - 33u^4v^8 + v^{12})$
$f_2 = u^8 + 14u^4v^4 + v^8$
$f_3 = uv(u^4 - v^4)$
We have the relation $f_1^2 = f_3^4 + 4f_2^3$.

E_7: $f_1 = uv(u^8 - v^8)(u^8 + v^8 - 34u^4v^4)$
$f_2 = u^8 + 14u^4v^4 + v^8$
$f_3 = (u^5v - uv^5)^2$
We have the relation $f_1^2 = f_3 f_2^3 - 108 f_3^3$.

E_8: $f_1 = u^{30} + v^{30} + 522(u^{25}v^5 - u^5v^{25}) - 10005(u^{20}v^{10} + u^{10}v^{20})$
$f_2 = -(u^{20} + v^{20}) + 228(u^{15}v^5 - u^5v^{15}) - 494u^{10}v^{10}$
$f_3 = uv(u^{10} + 11u^5v^5 - v^{10})$
We have the relation $f_1^2 + f_2^3 = 1728 f_3^5$.

Proof. We only prove the A_n case here. Unfortunately it is beyond the scope of this project to reproduce the entire proof. For this, see [7, pp. 6-13].

So let $G \cong \mathbb{Z}_m$. As explained in the example on page 9, our lone generator $g = \begin{pmatrix} \varepsilon_m & 0 \\ 0 & \varepsilon_m^{-1} \end{pmatrix}$ acts on u, v as follows: $gu = \varepsilon_m u$ and $gv = \varepsilon_m^{-1} v$. Since the action of g in this case simply multiplies u and v by a constant, a polynomial will be invariant if and only if each of its terms are invariant. Hence, in this case, $\mathbb{C}[u,v]^G$ will be generated by monomials so we only need to consider them.

Suppose a monomial $f = u^a v^b$ is invariant. Then $gf = g$ iff $\varepsilon_m^a u^a \varepsilon_m^{-b} v^b = u^a v^b$ iff $\varepsilon_m^{a-b} = 1$ iff m divides $a - b$. Assume without loss of generality that $a \geq b$. Then since $u^a v^b = (uv)^b u^{a-b}$ we have that $u^a v^b$ must be a product of a power of uv and u^m. Similarly if $b \geq a$ then we have that $u^a v^b$ must be a product of a power of uv and v^m. This shows that u^m, v^m, uv generate the ring of invariants, and so $\mathbb{C}[u,v]^G = \mathbb{C}[u^m, v^m, uv]$. □

Remark 2.3. i) In the D_n case the relation between the invariant polynomials for n even is omitted. This is because the ring of invariants is isomorphic to the n odd case (proof omitted) so will not be needed for Theorem 2.4.
ii) Observe that in every case $\mathbb{C}[u,v]^G$ is generated by 3 homogeneous (no constant term) polynomials, and that there exists a relation between them. The relations F are homogeneous polynomials in three "variables"; using the A_n example again, we have $F = xy - z^m$, where $x = f_1 = u^m,\ y = f_2 = v^m,\ z = f_3 = uv$.

Theorem 2.4. Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ and let f_1, f_2, f_3 generate the ring of invariants, i.e. $\mathbb{C}[u,v]^G = \mathbb{C}[f_1, f_2, f_3]$. Let $F(x,y,z) = 0$ be the homogeneous polynomial relation between f_1, f_2, f_3 as given in Theorem 2.2. Then $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/\langle F(x,y,z)\rangle$.

Proof. First of all note that when writing down $F(x,y,z)$, we can scale the relation between f_1, f_2, f_3 so that all of the coefficients are equal to 1 simply by multiplying the invariants by appropriate constants (they will still be invariant and generate $\mathbb{C}[u,v]^G$). Unfortunately, only half of a proof of the A_n case can be given; the rest is beyond the scope of this project.

We have $G \cong \mathbb{Z}_m$, $\mathbb{C}[u,v]^G = \mathbb{C}[u^m, v^m, uv]$, and $F(x,y,z) = xy - z^m$. Let $\phi : \mathbb{C}[x,y,z] \longrightarrow \mathbb{C}[u^m, v^m, uv]$ be the map taking x, y, z to u^m, v^m, uv respectively. Clearly ϕ is a surjective ring homomorphism. It's also clear that the ideal generated by F, i.e. $\langle xy - z^m\rangle$ is contained in the kernel of ϕ. What's true but not so trivial is that $\ker(\phi) \subseteq \langle xy - z^m\rangle$. Armed with this fact, we have $\ker(\phi) = \langle xy - z^m\rangle$ and so by the First Isomorphism Theorem, $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/\langle xy - z^m\rangle$. $\square$

The results of Theorem 2.4 are as follows:

A_n: $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/\langle xy - z^{n+1}\rangle$, $n \geq 1$

D_n: $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/\langle x^2 + zy^2 + z^{n-1}\rangle$, $n \geq 4$

E_6: $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/\langle x^2 + y^3 + z^4\rangle$

E_7: $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/\langle x^2 + y^3 + yz^3\rangle$

E_8: $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/\langle x^2 + y^3 + z^5\rangle$

The Final Goal: The last step of this journey is to take the generators of the ideals in each of the above quotient rings above and realise them as hypersurfaces in $\mathbb{C}^3$. We will see that these hypersurfaces have exactly one singularity at the origin. Our aim is to blow-up these singularities and show that their *resolution graphs* (explained through example later) match those given by Figure 4. These singularities define a special class of surface singularities called the *Kleinian singularities*, named after Felix Klein (1849-1925) who first determined the classification of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ in 1884. These singularities also go by the names of *du Val singularities* or *simple surface* singularities.

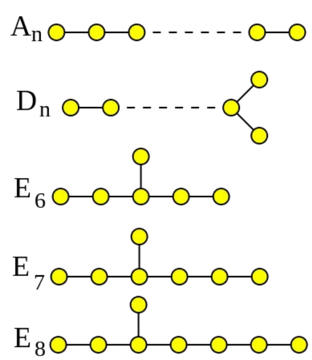


Figure 4: We want to show that the resolution graphs of the surfaces described above match the Coxeter-Dynkin diagrams of type ADE [8]

3 Blow-up

3.1 Definitions and Properties

First let us recall some definitions. Let $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$ denote n-dimensional affine and projective space respectively. The field in question will always be $\mathbb{C}$, unless otherwise specified.

Definition 3.1. Suppose $S \subseteq \mathbb{C}[x_1, \dots, x_n]$ is a set of polynomials in n variables. The *algebraic variety*, or just *variety*, defined by S is the locus given by

$$\mathbb{V}(S) = \{p \in \mathbb{A}^n \mid f(p) = 0 \ \forall f \in S\}.$$

Definition 3.2. The *Zariski topology* on affine or projective space is the topology such that algebraic varieties are precisely the closed sets. We will always assume this topology on $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$.

Definition 3.3. Let $X \in \mathbb{A}^n, Y \in \mathbb{A}^m$ be algebraic varieties. A polynomial map $\phi : X \to Y$ is an *isomorphism* if there exists another polynomial map $\psi : Y \to X$ satisfying $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_Y$.

Definition 3.4. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be irreducible and nonconstant. A point $p \in \mathbb{V}(f)$ is *singular* if $\dfrac{\partial f}{\partial x_i}(p) = 0 \ \forall i = 1, \dots, n$. If a variety has no singular points then we say that it is *smooth*.

So what exactly is blow-up? Intuitively, it's a process whereby we "pull apart" a variety at a singular point according to the different directions of lines through that point. An example of this is given by Example 3.8. If the resulting variety is smooth, then we are said to have achieved a *resolution of singularities*, i.e. we have smoothed the singular variety. Sometimes this process requires several iterations of blowing-up, as we shall see later.

To begin with, we will define the blow-up of the origin $O = (0, \dots, 0) \in \mathbb{A}^n$. Consider the Cartesian product $\mathbb{A}^n \times \mathbb{P}^{n-1}$. Denote the coordinates of $\mathbb{A}^n$ by $(x_1, \dots, x_n)$ and the homogeneous coordinates of $\mathbb{P}^{n-1}$ by $[y_1 : \dots : y_n]$. Note that the closed subsets (i.e. varieties) of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ are defined by polynomials in x_i, y_j which are also homogeneous with respect to the y_j.

Definition 3.5. The *blow-up* of $\mathbb{A}^n$ at the point O is the closed subset

$$X = \mathbb{V}(x_i y_j - x_j y_i \mid i, j = 1, \dots, n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$$

There is an important natural morphism $\varphi : X \to \mathbb{A}^n$ obtained by simply restricting the inclusion map from X into $\mathbb{A}^n \times \mathbb{P}^{n-1}$ to the affine part. Hereafter X and φ will always refer to this variety and map.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\varphi"] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d] \\ & \mathbb{A}^n \end{tikzcd}$$

Lemma 3.6. (Properties of X and φ)
i) If $O \neq p \in \mathbb{A}^n$, then $\varphi^{-1}(p)$ consists of a single point.
Proof. We show that φ gives an isomorphism between $X \backslash \varphi^{-1}(O)$ and $\mathbb{A}^n \backslash \{O\}$, i.e. we need to find an inverse morphism of φ. So let $p = (x_1, \ldots, x_n) \in \mathbb{A}^n$ with some $x_i \neq 0$. If $p \times [y_1 : \ldots : y_n] \in \varphi^{-1}(p)$, then by definition of X we must have $y_j = \frac{x_j}{x_i} y_i \ \forall j = 1, \ldots, n$. But this uniquely determines $[y_1 : \ldots : y_n] \in \mathbb{P}^{n-1}$ (up to a scalar multiple at least, so without loss of generality we can take $y_i = x_i \ \forall i = 1, \ldots, n$). Hence $\varphi^{-1}(p)$ consists of a single point and defining $\psi(p) = (x_1, \ldots, x_n) \times [x_1 : \ldots : x_n]$ gives an inverse morphism of φ, thus $X \backslash \varphi^{-1}(O) \cong \mathbb{A}^n \backslash \{O\}$.

ii) $\varphi^{-1}(O) \cong \mathbb{P}^{n-1}$.
Proof. $\varphi^{-1}(O)$ consists of all points of the form $O \times q$, where $q \in \mathbb{P}^{n-1}$ is subject to no constraints.

iii) The points of $\varphi^{-1}(O)$ are in one-to-one correspondence with the set of lines through O in $\mathbb{A}^n$.
Proof. Let L be a line through O in $\mathbb{A}^n$, given parametrically by $x_i = a_i t, i = 1, \ldots, n, t \in \mathbb{A}^1, a_i \in \mathbb{C}$ not all 0. Let L' be the line $\varphi^{-1}(L \backslash \{O\})$ in $X \backslash \varphi^{-1}(O)$. This line is given by the equations $x_i = a_i t, y_i = a_i t, t \in \mathbb{A}^1 \backslash \{O\}$. But since the y_i are homogeneous coordinates in $\mathbb{P}^{n-1}$, we can describe L' by $x_i = a_i t, y_i = a_i$, and these equations now make sense for all $t \in \mathbb{A}^1$; this gives the closure of L' in X. Now $\overline{L'}$ meets $\varphi^{-1}(O)$ at the point $O \times [a_1 : \ldots : a_n]$ where $[a_1 : \ldots : a_n] \in \mathbb{P}^{n-1}$. Hence the map sending L to $[a_1 : \ldots : a_n]$ gives a one-to-one correspondence between lines through O in $\mathbb{A}^n$ and points of $\varphi^{-1}(O)$.

iv) X is irreducible.
Proof. We have $X = (X \backslash \varphi^{-1}(O)) \cup (\varphi^{-1}(O))$. By i), the first piece is isomorphic to $\mathbb{A}^n \backslash \{O\}$ and thus irreducible. As for the second piece, we know from iii) that every point of $\varphi^{-1}(O)$ is in the closure of some subset (L' as constructed in iii)) of $X \backslash \varphi^{-1}(O)$. Hence $X \backslash \varphi^{-1}(O)$ is dense in X and thus $\overline{(X \backslash \varphi^{-1}(O))} = \overline{X} = X$ (where the latter equality comes from the fact that X is a variety). But the closure of an irreducible set is irreducible, therefore X is irreducible. □

Definition 3.7. If Y is an algebraic variety in $\mathbb{A}^n$ passing through the origin O, the *blow-up of Y at the point O* is $\widetilde{Y} = \overline{(\varphi^{-1}(Y \backslash \{O\}))} \subset X \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$. To blow up any other point $P \in Y$, we make a linear change of coordinates sending P to O.

Example 3.8. Let $Y = \mathbb{V}(y^2 - x^2(x+1)) \subset \mathbb{A}^2$. This variety has one singularity at the origin, which we will blow up. Let the projective coordinates of $\mathbb{A}^2 \times \mathbb{P}^1$ be given by $[t : u]$. Then the blow-up of $(0,0) \in Y$ is given by $\widetilde{Y} = Y \cap X = \mathbb{V}(xu - yt, y^2 - x^2(x+1)) \subset \mathbb{A}^2 \times \mathbb{P}^1$.

Observe that the result looks like our curve in $\mathbb{A}^2$ except that the origin has been replaced by a $\mathbb{P}^1$, see Figure 5. We call this $\mathbb{P}^1$ the *exceptional divisor* and denote it by E.

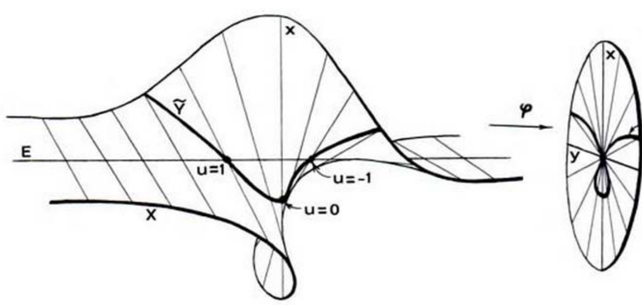


Figure 5: Blow-up of $(0,0) \in Y = \mathbb{V}(y^2 - x^2(x+1))$ [9]

Recall that $\mathbb{A}^2 \times \mathbb{P}^1$ is covered by two affine open sets given by $t \neq 0$ and $u \neq 0$, denoted U_t and U_u respectively. We will look at $\widetilde{Y}$ in the affine chart U_t. Denote this by $\widetilde{Y}_t$, i.e. $\widetilde{Y}_t = U_t \cap \widetilde{Y}$. Then we can divide our projective coordinates by t and use $\frac{u}{t}$ as an affine coordinate, and so study $\widetilde{Y}_t$ as if it lived in $\mathbb{A}^3$ with coordinates $(x, y, \frac{u}{t})$. Therefore we have $\widetilde{Y}_t = \mathbb{V}(y - x\frac{u}{t}, y^2 - x^2(x+1)) \subset U_t \cong \mathbb{A}^3$. Plugging the first equation into the second yields $x^2(\frac{u}{t})^2 - x^2(x+1) = 0$ which factors as $x^2((\frac{u}{t})^2 - x - 1) = 0$.

We must have either $x^2 = 0$ or $(\frac{u}{t})^2 = x + 1$. The first condition forces $y = 0$ and leaves $\frac{u}{t}$ arbitrary; this of course corresponds to E (or at least the $t \neq 0$ affine part of it). The second condition is $(\frac{u}{t})^2 = x + 1$. This is $\widetilde{Y}_t$. Observe that at $\widetilde{Y}_t \cap E$ we must have $(x, y) = (0, 0)$ in this equation. This leaves $(\frac{u}{t})^2 = \pm 1$. These two points, $(x, y, \frac{u}{t}) = (0, 0, \pm 1)$ correspond to the slopes of the two branches of Y at the origin. Note that with respect to Figure 5, t has been set equal to 1, as opposed to dividing through by it in the above calculations. $\lozenge$

The surfaces we want to blow-up are defined by the polynomials generating the ideals on page 11. Here they are again for convenience:

A_n: $xy - z^{n+1} = 0$, $n \geq 1$

D_n: $x^2 + zy^2 + z^{n-1} = 0$, $n \geq 4$

E_6: $x^2 + y^3 + z^4 = 0$

E_7: $x^2 + y^3 + yz^3 = 0$

E_8: $x^2 + y^3 + z^5 = 0$

Lemma 3.9. Each of these surfaces has exactly one singularity, which can be found at the origin.

Proof. The A_n, E_6 and E_8 cases are easy as taking the partial derivatives with respect to each variable and setting them equal to 0 forces each variable in turn to be equal to zero, thus giving a singularity at $(0, 0, 0)$ as required.

For E_7, let f be the polynomial given. Firstly $\frac{\partial f}{\partial x} = 0$ forces $x = 0$. Now $\frac{\partial f}{\partial z} = 0$ implies $3yz^2 = 0$ and $\frac{\partial f}{\partial y} = 0$ implies $3y^2 + z^3 = 0$. The only simultaneous solution to both of these equations is $y = 0 = z$, giving the desired singularity $(0, 0, 0)$. The D_n case is similar. $\square$

We are now ready to calculate the blow-ups of each of these surfaces at the origin and find their resolution graphs. These calculations will take us to the end of the project. Note that in Example 3.8 we were working in $\mathbb{R}^2$, and hence able to get nice pictures. Such pictures are impossible for blow-ups of surfaces in $\mathbb{C}^3$, although guiding diagrams have been attempted.

Notation: Denote the coordinates of $\mathbb{A}^3 \times \mathbb{P}^2$ by $(x, y, z; a : b : c)$, i.e. we use x, y, z for the affine part and a, b, c for the projective part. X is given by $\mathbb{V}(xb - ya, xc - za, yc - zb)$. In each case, denote the surface in question by the variety $Y = \mathbb{V}(f)$, where f is the defining polynomial. Let the blow-up of the surface at the origin be given by $\widetilde{Y} = \mathbb{V}(xb - ya, xc - za, yc - zb, f) \subset X$. Let $U_a \subset \mathbb{A}^3 \times \mathbb{P}^2$ be the open subset where the the a-coordinate is non-zero. Observe $U_a \cong \mathbb{A}^5$ and that the coordinates in this space are $(x, y, z, \frac{b}{a}, \frac{c}{a})$. We define U_b, U_c similarly. Denote the affine charts of $\widetilde{Y}$, for example $\widetilde{Y} \cap U_a$, by $\widetilde{Y}_a$. Denote the exceptional divisor of each blow-up by E. For cases requiring several blow-ups, we will alternate between the coordinates above and $(A, B, C; \alpha : \beta : \gamma) \in \mathbb{A}^3 \times \mathbb{P}^2$.

3.2 The A_n Case

The strategy will be to show that A_1 and A_2 have the desired resolution graphs and then complete an induction argument on the general A_n case.

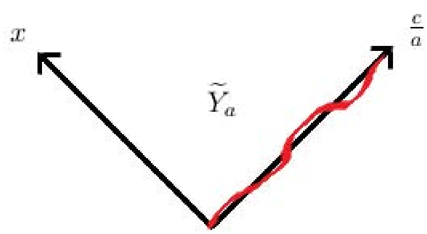
3.2.1 A_1

We have $f = xy - z^2$ and $\widetilde{Y} = \mathbb{V}(xb - ya, xc - za, yc - zb, xy - z^2)$.

First consider $\widetilde{Y}_a = \mathbb{V}(x\frac{b}{a} - y, x\frac{c}{a} - z, y\frac{c}{a} - z\frac{b}{a}, xy - z^2) \subset U_a \cong \mathbb{A}^5$. Substituting the first two equations into the last one yields $x^2\frac{b}{a} - x^2(\frac{c}{a})^2 = 0$, i.e. $x^2(\frac{b}{a} - (\frac{c}{a})^2) = 0$.

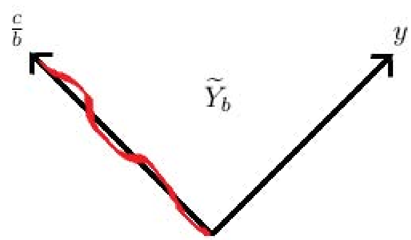
This last equation has two irreducible parts, namely x^2 and $\frac{b}{a} - (\frac{c}{a})^2$. If we set $x^2 = 0$, the other equations in $\widetilde{Y}_a$ force $y = 0 = z$. This part is the exceptional divisor E. It is the set that ϕ maps to the singularity $(0,0,0) \in Y$. Observe that in X, E is the set with coordinates $\{(0,0,0; a : b : c)\} \cong \mathbb{P}^2$. This is in accordance with Lemma 3.6.ii, which is reassuring.

The other part, $\frac{b}{a} - (\frac{c}{a})^2$, tells us more about $\widetilde{Y}_a$. Using $\frac{b}{a} = (\frac{c}{a})^2$ and the equations $y = x\frac{b}{a}$, $z = x\frac{c}{a}$, we can construct a polynomial isomorphism $(x, y, z, \frac{b}{a}, \frac{c}{a}) = (x, x(\frac{c}{a})^2, x(\frac{c}{a}), (\frac{c}{a})^2, \frac{c}{a}) \mapsto (x, \frac{c}{a}) \in \mathbb{A}^2$, i.e. we achieve $\widetilde{Y}_a \cong \mathbb{A}^2$. Since $\mathbb{A}^2$ is smooth, we know that $\widetilde{Y}_a$ must be smooth. The following diagram represents $\widetilde{Y}_a$ as $\mathbb{A}^2$ with coordinate axes x, $\frac{c}{a}$.



The red squiggle along the $\frac{c}{a}$-axis corresponds to where $\widetilde{Y}_a$ meets E, as $x = 0$ forces $y = 0 = z$.

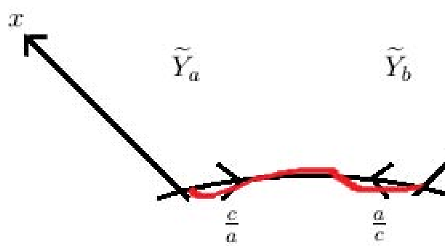
The calculations for $\widetilde{Y}_b$ are almost identical to the ones above. The important thing is that we get another isomorphism $(x, y, z, \frac{a}{b}, \frac{c}{b}) = (y(\frac{c}{b})^2, y, y(\frac{c}{b}), (\frac{c}{b})^2, \frac{c}{b}) \mapsto (y, \frac{c}{b}) \in \mathbb{A}^2$, thus showing that $\widetilde{Y}_b$ is also smooth. Below we have another similar diagram. The red squiggle again corresponds to where $\widetilde{Y}_b$ meets E.



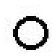
Finally, in the same manner again, considering $\widetilde{Y}_c$ yields the equation $z^2(\frac{a}{c}\frac{b}{c} - 1) = 0$. The interesting irreducible part of this is $\frac{a}{c}\frac{b}{c} = 1$. Observe that this implies both $a \neq 0$ and $b \neq 0$, but these two conditions are precisely those considered by looking in the charts U_a and U_b. Hence we have $\widetilde{Y}_c \subset \widetilde{Y}_a \cup \widetilde{Y}_b$, so there is nothing new to be found in this chart. It is superfluous.

The next step is to "glue" our two charts (or diagrams) together. Remember that affine charts collectively cover the same space and overlap each other almost entirely. In fact $\widetilde{Y}_a \backslash \{\frac{c}{a} = 0\} \cong \widetilde{Y}_b \backslash \{\frac{c}{b} = 0\}$ via the map $(x, \frac{c}{a}) \mapsto (x(\frac{c}{a})^2, (\frac{c}{a})^{-1}) = (y, \frac{c}{b})$, which has inverse $(y, \frac{c}{b}) \mapsto (y(\frac{c}{b})^2, (\frac{c}{b})^{-1}) = (x, \frac{c}{a})$.

Importantly, note that in this case we have $\frac{c}{b} = \frac{c/a}{b/a} = \frac{c/a}{(c/a)^2} = (\frac{c}{a})^{-1} = \frac{a}{c}$, so that the original axes $\frac{c}{a}$ and $\frac{c}{b}$ coincide almost entirely (we can see this by joining the diagrams together). The $\frac{c}{a}$ axis doesn't quite reach the y-axis in $\widetilde{Y}_b$ as this would correspond to $\frac{a}{c} = 0$ i.e. $a = 0$ (this is why when gluing $\widetilde{Y}_a$ to $\widetilde{Y}_b$ we must omit the line $\{\frac{c}{a} = 0\}$ in the domain). However by treating this as the "point at infinity", we actually see that this unified axis is a projective line $\mathbb{P}^1$. We thus have the following diagram, where the red squiggle is a $\mathbb{P}^1$.



So what is the *resolution graph*? Whenever we have a $\mathbb{P}^1$ we draw a node, and we connect two nodes with a line if the two projective lines they represent intersect. Since here we only have one $\mathbb{P}^1$, our graph is just a single node, which is the desired result (see Figure 4 on page 11).



$\Diamond$

3.2.2 A_2

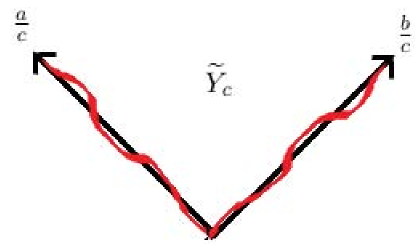
Now we have $f = xy - z^3$ and $\widetilde{Y} = \mathbb{V}(xb - ya, xc - za, yc - zb, xy - z^3)$.

As before, we consider $\widetilde{Y}$ in each of the three affine charts. $\widetilde{Y}_a$ and $\widetilde{Y}_b$ will be very similar, but this time $\widetilde{Y}_c$ will have a much more important role.

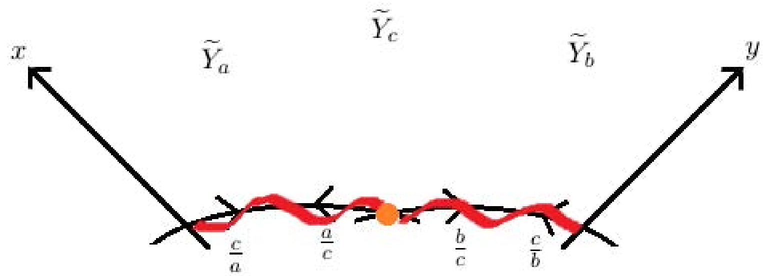
Now in $\widetilde{Y}_a$, we have $y = x\frac{b}{a}$, $z = x\frac{c}{a}$ again and substituting these into $xy - z^3 = 0$ yields $x^2\frac{b}{a} - x^3(\frac{c}{a})^3 = 0$, i.e. $x^2(\frac{b}{a} - x(\frac{c}{a})^3) = 0$. Again the x^2 part will give us E and the other part gives us $\frac{b}{a} = x(\frac{c}{a})^3$. Just like before, we have an isomorphism $(x, y, z, \frac{b}{a}, \frac{c}{a}) = (x, x^2(\frac{c}{a})^3, x(\frac{c}{a}), x(\frac{c}{a})^3, \frac{c}{a}) \mapsto (x, \frac{c}{a}) \in \mathbb{A}^2$. Hence this chart is smooth and we get a diagram identical to the first of those given in the A_1 case.

Similarly, considering $\widetilde{Y}_b$ yields the isomorphism $(x, y, z, \frac{a}{b}, \frac{c}{b}) = (y^2(\frac{c}{b})^3, y, y(\frac{c}{b}), y(\frac{c}{b})^3, \frac{c}{b}) \mapsto (y, \frac{c}{b}) \in \mathbb{A}^2$. We thus have smoothness and a diagram identical to the second of those given in the A_1 case.

In $\widetilde{Y}_c$, we have $x = z\frac{a}{c}$ and $y = z\frac{b}{c}$. Plugging these into $xy - z^3 = 0$ yields $z^2(\frac{a}{c}\frac{b}{c} - z) = 0$, from which the interesting irreducible part gives $z = \frac{a}{c}\frac{a}{b}$. This time our isomorphism $\widetilde{Y}_c$ to $\mathbb{A}^2$ will make all three affine coordinates redundant: $(x, y, z, \frac{a}{c}, \frac{b}{c}) = ((\frac{a}{c})^2\frac{b}{c}, \frac{a}{c}(\frac{b}{c})^2, \frac{a}{c}\frac{b}{c}, \frac{a}{c}, \frac{b}{c}) \mapsto (\frac{a}{c}, \frac{b}{c}) \in \mathbb{A}^2$. We still get a smooth result but with a different diagram. Here both coordinate axes have preimage in E.



Now we're ready to look at the big picture. We glue our three affine charts together and achieve the digram below. Note that we need to bend the coordinate axes of $\widetilde{Y}_c$ to do this on paper - the angle between the $\frac{a}{c}$,$\frac{b}{c}$ axes is still a right angle.



The glue between these charts are the following maps.

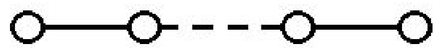
- $\widetilde{Y}_a \backslash \{\frac{c}{a} = 0\} \cong \widetilde{Y}_c \backslash \{\frac{a}{c} = 0\}$ via $(x, \frac{c}{a}) \mapsto ((\frac{c}{a})^{-1}, x(\frac{c}{a})^2) = (\frac{a}{c}, \frac{b}{c})$. The latter coordinate in the image comes from the fact that in $\widetilde{Y}_a$, $\frac{b}{a} = x(\frac{c}{a})^3$, so $\frac{b}{c} = \frac{b/a}{c/a} = \frac{x(c/a)^3}{c/a} = x(\frac{c}{a})^2$.

- $\widetilde{Y}_c \backslash \{\frac{b}{c} = 0\} \cong \widetilde{Y}_b \backslash \{\frac{c}{b} = 0\}$ via $(\frac{a}{c}, \frac{b}{c}) \mapsto ((\frac{b}{c})^{-1}, \frac{a}{c}(\frac{b}{c})^2) = (\frac{c}{b}, y)$.

So what is the resolution graph? We have two distinct projective lines intersecting in a single point (the orange dot), thus we have the following Coxeter-Dynkin diagram, representing A_2! We are done. $\Diamond$

3.2.3 A_n

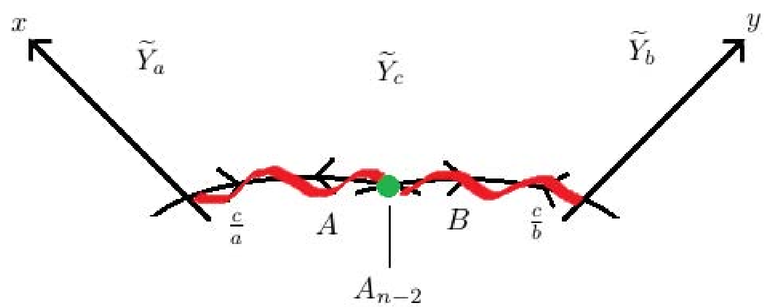
The A_1 and A_2 cases will be our base cases for an induction argument. Consider then the A_n case, i.e. $f = xy - z^{n+1}$ and $\widetilde{Y} = \mathbb{V}(xb - ya, xc - za, yc - zb, xy - z^{n+1})$, and suppose that for all $k < n$ we have that the resolution graph of A_k is k nodes in a line as shown below.



The results of looking in the U_a and U_b charts are exactly the same as before; we get that $\widetilde{Y}_a \cong \mathbb{A}^2$ and $\widetilde{Y}_b \cong \mathbb{A}^2$ via the isomorphisms listed below, and so the diagrams we get are the same as the first two given in the A_1 case.

- $\widetilde{Y}_a$. $(x, y, z, \frac{b}{a}, \frac{c}{a}) = (x, x^n(\frac{c}{a})^{n+1}, x\frac{c}{a}, x^{n-1}(\frac{c}{a})^{n+1}, \frac{c}{a}) \mapsto (x, \frac{c}{a})$
- $\widetilde{Y}_b$. $(x, y, z, \frac{a}{b}, \frac{c}{b}) = (y^n(\frac{c}{b})^{n+1}, y, y\frac{c}{b}, y^{n-1}(\frac{c}{b})^{n+1}, \frac{c}{b}) \mapsto (y, \frac{c}{b})$

For $\widetilde{Y}_c$ we will get the equation $z^2(\frac{a}{c}\frac{b}{c} - z^{n-1}) = 0$. Here we cannot make the z variable redundant and get a polynomial isomorphism $\widetilde{Y}_c \cong \mathbb{A}^2$. In fact, the equation $\frac{a}{c}\frac{b}{c} - z^{n-1} = 0$ does not even define a smooth variety. By renaming the variables $\frac{a}{c} \mapsto A$, $\frac{b}{c} \mapsto B$, $z \mapsto C$, it defines the singular variety $\mathbb{V}(AB - C^{n-1}) \subset \mathbb{A}^3$. Hence $\widetilde{Y}_c \cong \mathbb{V}(AB - C^{n-1}) \subset \mathbb{A}^3$. But look! This variety describes the A_{n-2} singularity (the green dot on the diagram below), so by induction it has resolution graph $n-2$ nodes in a line.



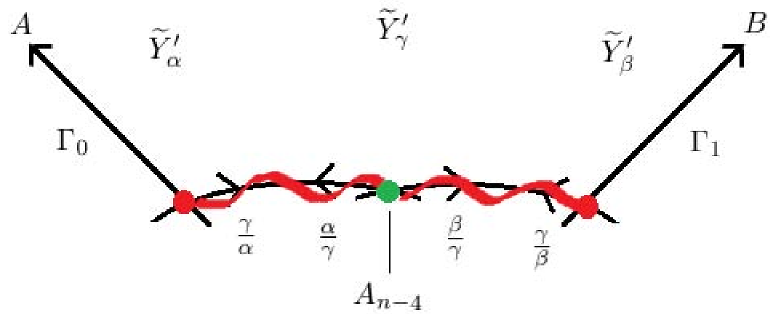
The glue between the charts is:

- $\widetilde{Y}_c \backslash \{A = 0\} \cong \widetilde{Y}_a \backslash \{\frac{c}{a} = 0\}$ via $(A, B, C) \mapsto (AC, A^{-1})$ with inverse $(x, \frac{c}{a}) \mapsto ((\frac{c}{a})^{-1}, x^{n-1}(\frac{c}{a})^n, x\frac{c}{a})$
- $\widetilde{Y}_c \backslash \{B = 0\} \cong \widetilde{Y}_b \backslash \{\frac{c}{b} = 0\}$ via $(A, B, C) \mapsto (BC, B^{-1})$ with inverse $(y, \frac{c}{b}) \mapsto (y^{n-1}(\frac{c}{b})^n, (\frac{c}{b})^{-1}, y\frac{c}{b})$

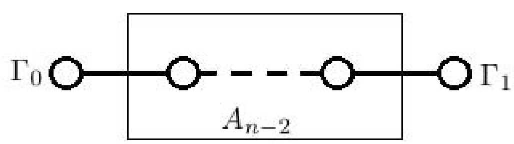
Hence the coincidence of the $\frac{c}{a}$-axis and the A-axis in $\widetilde{Y}_c$ forms a $\mathbb{P}^1$, call it Γ_0, and similarly the $\frac{c}{b}$ and B axes form another, call it Γ_1. Thus Γ_0, Γ_1 will contribute two extra nodes to the resolution graph representing the A_{n-2} singularity in $\widetilde{Y}_c$; we just need to make sure that these two nodes join the rest in the correct place. Using the new coordinates (A, B, C) with projective coordinates $[\alpha : \beta : \gamma]$ we blow-up $Y' := \mathbb{V}(AB - C^{n-1})$ and, like we've already seen, we will get smoothness in the $\widetilde{Y}'_\alpha$ and $\widetilde{Y}'_\beta$ charts via the following isomorphisms to $\mathbb{A}^2$:

- $\widetilde{Y}'_\alpha$. $(A,B,C,\frac{\beta}{\alpha},\frac{\gamma}{\alpha}) = (A, A^{n-2}(\frac{\gamma}{\alpha})^{n-1}, A\frac{\gamma}{\alpha}, A^{n-3}(\frac{\gamma}{\alpha})^{n-1}, \frac{\gamma}{\alpha}) \mapsto (A, \frac{\gamma}{\alpha})$
- $\widetilde{Y}'_\beta$. $(A,B,C,\frac{\alpha}{\beta},\frac{\gamma}{\beta}) = (B^{n-2}(\frac{\gamma}{\beta})^{n-1}, B, B\frac{\gamma}{\beta}, B^{n-3}(\frac{\gamma}{\beta})^{n-1}, \frac{\gamma}{\beta}) \mapsto (B, \frac{\gamma}{\beta})$

For $\widetilde{Y}'_\gamma$, after throwing away the irreducible part with preimage in E we will get the equation $\frac{\alpha}{\gamma}\frac{\beta}{\gamma} - C^{n-3} = 0$, a singularity of A_{n-4} type. Like above, the coincidence of the $\frac{\gamma}{\alpha}$,$\frac{\alpha}{\gamma}$ axes and the $\frac{\gamma}{\beta}$,$\frac{\beta}{\gamma}$ axes will create another two $\mathbb{P}^1$s that intersect in this A_{n-4} singularity. Meanwhile, Γ_0, Γ_1 will intersect these $\mathbb{P}^1$s also. See the diagram below.



This whole process of gaining two extra $\mathbb{P}^1$s repeats with each blow-up, overall yielding the desired n nodes in a straight line for A_n.



$\Diamond$

3.3 The D_n Case

Like A_n, we first study D_4 and D_5 and then complete an induction argument on the general case D_n. From hereafter the calculations for checking whether a chart may be smooth or not are skipped (these are easily done by taking partial derivatives).

3.3.1 D_4

We have $f = x^2 + zy^2 + z^3$ and $\widetilde{Y} = \mathbb{V}(xb - ya, xc - za, yc - zb, x^2 + zy^2 + z^3)$. Substituting the blow-up equations into f and looking in each affine chart, we have:

- $\widetilde{Y}_a$: $x^2(1 + x(\frac{b}{a})^2\frac{c}{a} + x^3(\frac{c}{a})^3) = 0$.
- $\widetilde{Y}_b$: $y^2((\frac{a}{b})^2 + y\frac{c}{b} + y(\frac{c}{b})^3) = 0$.
- $\widetilde{Y}_c$: $z^2((\frac{a}{c})^2 + z(\frac{b}{c})^2 + z) = 0$.

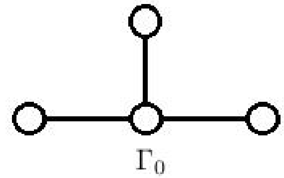
The irreducible parts x^2, y^2, z^2 of the respective charts above each have preimage entirely contained in E. In fact, we will get these factors every time we blow-up Kleinian singularities. Since we are not interested in these parts of the preimage, they will hereafter simply be omitted when listing the equations of the affine charts of $\widetilde{Y}$.

Hence, now disregarding them and focussing on the other irreducible parts, which collectively describe the *proper transform* of $\widetilde{Y}$, we have:

- $\widetilde{Y}_a$: $1 + x(\frac{b}{a})^2\frac{c}{a} + x^3(\frac{c}{a})^3 = 0$. This chart is smooth and does not intersect E (since in $\widetilde{Y}_a \cap E$ we must have $x = 0$ which leaves $1 = 0$ in the equation above. Thus $\widetilde{Y}_a \cap E = \emptyset$).
- $\widetilde{Y}_b$: $(\frac{a}{b})^2 + y\frac{c}{b} + y(\frac{c}{b})^3 = 0$. This chart has a singular point at $(\frac{a}{b}, y, \frac{c}{b}) = (0, 0, 0)$ and intersects E when $a = 0$, i.e. with affine coordinates $(x, y, z, \frac{a}{b}, \frac{c}{b}) = (0, 0, 0, 0, \frac{c}{b})$.
- $\widetilde{Y}_c$: $(\frac{a}{c})^2 + z(\frac{b}{c})^2 + z = 0$. This chart has two singular points at $(\frac{a}{c}, \frac{b}{c}, z) = (0, \pm i, 0)$ and intersects E when $a = 0$, i.e. when $(x, y, z, \frac{a}{c}, \frac{b}{c}) = (0, 0, 0, 0, \frac{b}{c})$.

First of all, the intersection of E with $\widetilde{Y}_b$ is isomorphic to $\mathbb{A}^1$, as it just the line with coordinates $(0, 0, 0, 0, \frac{c}{b})$. Similarly, $\widetilde{Y}_c \cap E$ is another $\mathbb{A}^1$ with coordinates $(0, 0, 0, 0, \frac{b}{c})$. We see then that these two lines are actually just the two affine charts of the same $\mathbb{P}^1$. This is the same idea as used in the A_1 case on page 16. We will denote this particular $\mathbb{P}^1$ by Γ_0.

Now let's look at the singularities in these charts. There are two important observations. Firstly, we have that these three singularities are distinct: the singularity in $\widetilde{Y}_b$ has $\frac{c}{b} = 0$ whereas the singularities in $\widetilde{Y}_c$ have $\frac{b}{c} = \pm i$, which are of course inconsistent with each other. The second observation is that all three of these singularities lie on Γ_0 (where $a = 0$) as they have coordinates $\frac{a}{b} = 0$ and $\frac{a}{c} = 0$ respectively. We will now prove that these singularities are all of type A_1, thus yielding the resolution graph D_4. This will be achieved since each of the three A_1 singularities' blow-ups will contain a single $\mathbb{P}^1$ that will intersect Γ_0 at a distinct point. Observe then that Γ_0 will correspond to the $\mathbb{P}^1$ given by the centre node in the D_4 resolution graph.



So, first let's check the $(\frac{a}{b}, y, \frac{c}{b}) = (0, 0, 0)$ singularity in the $\widetilde{Y}_b$. Change coordinates $\frac{a}{b} \mapsto A, y \mapsto B, \frac{c}{b} \mapsto C$ so that $(\frac{a}{b})^2 + y\frac{c}{b} + y(\frac{c}{b})^3 = 0$ becomes $A^2 + BC + BC^3 = 0$. We blow this up at $(0, 0, 0)$ as normal and, using a new set of projective coordinates $[\alpha : \beta : \gamma]$, get a variety $\widetilde{Y}'$, that is smooth in all three charts, as hoped. Each chart intersects $E = \{(A, B, C) = (0, 0, 0)\} \times \mathbb{P}^2$ as follows:

- $\widetilde{Y}'_\alpha \cap E = \mathbb{V}(1 + \frac{\beta}{\alpha}\frac{\gamma}{\alpha}) \subset \{0\} \times \mathbb{A}^2$
- $\widetilde{Y}'_\beta \cap E = \mathbb{V}((\frac{\alpha}{\beta})^2 + \frac{\gamma}{\beta}) \subset \{0\} \times \mathbb{A}^2$
- $\widetilde{Y}'_\gamma \cap E = \mathbb{V}((\frac{\alpha}{\gamma})^2 + \frac{\beta}{\gamma}) \subset \{0\} \times \mathbb{A}^2$

It's easy to see that these affine equations collectively yield the projective variety $\widetilde{Y}' \cap E = \mathbb{V}(\alpha^2 + \beta\gamma) \subset \{0\} \times \mathbb{P}^2$, a non-degenerate conic in $\mathbb{P}^2$. Now, by Example 15.2 in [10, pp. 39-40], we know that non-degenerate conics in $\mathbb{P}^2$ are isomorphic to $\mathbb{P}^1$. Thus the blow up of this singularity contains exactly one $\mathbb{P}^1$, which is what we wanted.

Where does this $\mathbb{P}^1$ intersect Γ_0? In $\widetilde{Y}_b$, Γ_0 is the $\frac{c}{b}$-axis. Thus it's proper transform in $\widetilde{Y}'$ will be the C-axis. The blow-up variety of $\widetilde{Y}'$ is $X' = \mathbb{V}(A\beta - B\alpha, A\gamma - C\alpha, B\gamma - C\beta)$, so the proper transform of Γ_0 will be the variety $\tilde{\Gamma_0} = \mathbb{V}(A\beta - B\alpha, A\gamma - C\alpha, B\gamma - C\beta, A, B)$. We consider the intersection of $\tilde{\Gamma_0}$ and E when C is left arbitrary, yielding $\alpha = 0 = \beta$. Hence $(\widetilde{Y}' \cap E) \cap (\tilde{\Gamma_0} \cap E)$ contains only the point $[\alpha : \beta : \gamma] = [0 : 0 : 1]$; this is where Γ_0 meets the conic (or $\mathbb{P}^1$) we find by blowing up the A_1 singularity $(\frac{a}{b}, y, \frac{c}{b}) = (0, 0, 0)$.

Now we must check the singularities $(\frac{a}{c}, \frac{b}{c}, z) = (0, \pm i, 0)$ in $\widetilde{Y}_c$. We only need to look at one as the cases will be symmetric. So consider $(0, i, 0)$ and make a linear coordinate change $\frac{a}{c} \mapsto A$, $\frac{b}{c} \mapsto B + i$, $z \mapsto C$ with a new set of projective coordinates $[\alpha : \beta : \gamma]$ so that $(\frac{a}{c})^2 + z(\frac{b}{c})^2 + z = 0$ becomes $A^2 + CB^2 + 2iBC = 0$. This coordinate change translates the $(0, i, 0)$ singularity to $(0, 0, 0)$, which we now blow-up. An important note here is that this blow-up, denoted $\widetilde{Y}''$, is not actually smooth in one of its affine charts. More precisely, $\widetilde{Y}''_\beta$ will contain a singularity at $(\frac{\alpha}{\beta}, B, \frac{\gamma}{\beta}) = (0, -2i, 0)$. We should not be alarmed however; this singularity is just the other singularity $(0, -i, 0) \in \widetilde{Y}_c$ that we didn't blow-up originally. The fact that there are no other singular points in any of the affine charts of $\widetilde{Y}''$ means that we've "smoothed out" $(0, i, 0) \in \widetilde{Y}_c$ as intended. Similar to above, we actually have $\widetilde{Y}'' \cap E = \mathbb{V}(\alpha^2 + 2i\beta\gamma)$, another non-degenerate conic that will give us a $\mathbb{P}^1$ as desired. The intersection with Γ_0 is calculated similarly.

The overall result is the resolution graph given above. We are done. $\Diamond$

3.3.2 D_5

We have $f = x^2 + zy^2 + z^4$ and $\widetilde{Y} = \mathbb{V}(xb - ya, xc - za, yc - zb, x^2 + zy^2 + z^4)$. Looking in each affine chart and disregarding the irreducible part with preimage completely contained in E, we have:

- $\widetilde{Y}_a$: $1 + x\frac{c}{a}(\frac{b}{a})^2 + x^2(\frac{c}{a})^4 = 0$, smooth and does not intersect E.
- $\widetilde{Y}_b$: $(\frac{a}{b})^2 + y\frac{c}{b} + y^2(\frac{c}{b})^4 = 0$. This chart has a singular point at $(\frac{a}{b}, y, \frac{c}{b}) = (0,0,0)$ and intersects E when $a = 0$, i.e. with affine coordinates $(x, y, z, \frac{a}{b}, \frac{c}{b}) = (0,0,0,0,\frac{c}{b})$.
- $\widetilde{Y}_c$: $(\frac{a}{c})^2 + z(\frac{b}{c})^2 + z^2 = 0$. This chart has a singular point at $(\frac{a}{c}, \frac{b}{c}, z) = (0,0,0)$ and intersects E when $a = 0$, i.e. with affine coordinates $(x, y, z, \frac{a}{c}, \frac{b}{c}) = (0,0,0,0,\frac{b}{c})$.

We see that $\widetilde{Y}_b$ and $\widetilde{Y}_c$ each intersect E along $a = 0$ just like in the D_4 case. We know that this corresponds to a $\mathbb{P}^1$ which we will again denote by Γ_0. The singularities in the latter two charts must be distinct as they require $\frac{c}{b} = 0$ and $\frac{b}{c} = 0$ respectively. The singularity $(\frac{a}{b}, y, \frac{c}{b}) = (0,0,0) \in \widetilde{Y}_b$ is of A_1 type and this can be shown in precisely the same manner as for the singularity in the $\widetilde{Y}_b$ chart in the D_4 case (see page 21).

In $\widetilde{Y}_c$ we have a singularity at $(\frac{a}{c}, \frac{b}{c}, z) = (0,0,0)$. We change coordinates $\frac{a}{c} \mapsto A, \frac{b}{c} \mapsto B, z \mapsto C$, use a new set of projective coordinates $[\alpha : \beta : \gamma]$, and denote the blow-up of $A^2 + CB^2 + C^2 = 0$ at $(0,0,0)$ by $\widetilde{Y}'$. Again, throwing away the irreducible part with preimage completely contained in E yields the following:

- $\widetilde{Y}'_\alpha$: $1 + A\frac{\gamma}{\alpha}(\frac{\beta}{\alpha})^2 + (\frac{\gamma}{\alpha})^2 = 0$, smooth and intersects E along $(\frac{\gamma}{\alpha})^2 + 1 = 0$.
- $\widetilde{Y}'_\beta$: $(\frac{\alpha}{\beta})^2 + B\frac{\gamma}{\beta} + (\frac{\gamma}{\beta})^2 = 0$, singular at $(\frac{\alpha}{\beta}, B, \frac{\gamma}{\beta}) = (0,0,0)$ and intersects E along $(\frac{\alpha}{\beta})^2 + (\frac{\gamma}{\beta})^2 = 0$.
- $\widetilde{Y}'_\gamma$: $(\frac{\alpha}{\gamma})^2 + C(\frac{\beta}{\gamma})^2 + 1 = 0$, smooth and intersects E along $(\frac{\alpha}{\gamma})^2 + 1 = 0$.

Firstly, observe that the singularity in $\widetilde{Y}'_\beta$ lies on Γ_0: in $\widetilde{Y}_c$, Γ_0 is the $\frac{b}{c}$-axis. After the coordinate change, this corresponds to the B-axis in $\widetilde{Y}'_\beta$. The singularity at $(\frac{\alpha}{\beta}, B, \frac{\gamma}{\beta}) = (0,0,0)$ lies on this line.

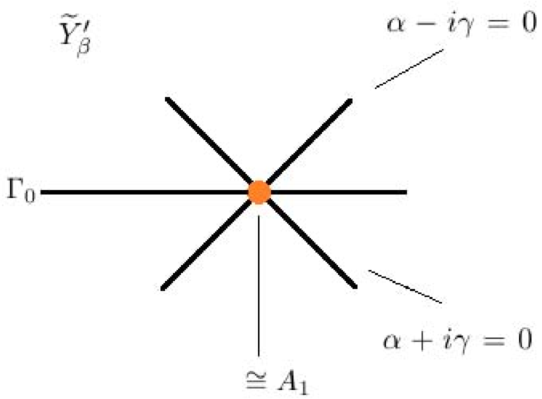
Now, amalgamating the equations where each chart of $\widetilde{Y}'$ intersects E, we have $\widetilde{Y}' \cap E = \mathbb{V}(\alpha^2 + \gamma^2) = \mathbb{V}((\alpha + i\gamma)(\alpha - i\gamma)) = \mathbb{V}(\alpha + i\gamma) \cup \mathbb{V}(\alpha - i\gamma) \subset \{0\} \times \mathbb{P}^2$. This describes a pair of projective lines, namely $\alpha + i\gamma = 0$ and $\alpha - i\gamma = 0$. These lines intersect at the point $(A, B, C; \alpha : \beta : \gamma) = (0,0,0; 0 : 1 : 0)$, which of course we only find in $\widetilde{Y}'_\beta$. In fact, this point of intersection is precisely the singularity in $\widetilde{Y}'_\beta$ mentioned above. It is actually another A_1; denoting its blow-up by $\widetilde{Y}''$ and making the coordinate change $\frac{\alpha}{\beta} \mapsto x, B \mapsto y, \frac{\gamma}{\beta} \mapsto z$ with a fresh set of projective coordinates $[a : b : c]$, we have $\widetilde{Y}'' = \mathbb{V}(xb - ya, xc - za, yc - zb, x^2 + yz + z^2)$. Omitting the calculations we have $\widetilde{Y}'' \cap E = \mathbb{V}(a^2 + bc + c^2)$, a non-degenerate conic that we know is isomorphic to $\mathbb{P}^1$.

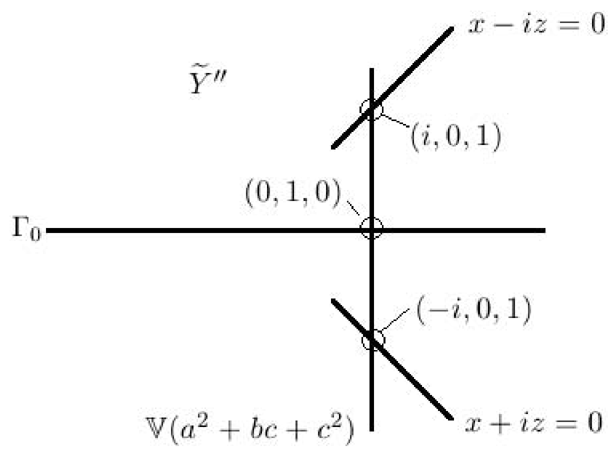
We want to make sure that the two projective lines $\alpha + i\gamma = 0$ and $\alpha - i\gamma = 0$ now intersect this $\mathbb{P}^1$ in $\widetilde{Y}''$ at distinct points. To do this we need to find the proper transform of each of these lines (i.e. blow them up too!) and look where they intersect E, then in turn $\mathbb{V}(a^2 + bc + c^2)$. So taking first the line $\alpha - i\gamma = 0$ which has affine part $x - iz = 0 = y$ after the coordinate change, its blow-up is given by $\mathbb{V}(xb - ya, xc - za, yc - zb, x - iz, y) = \mathbb{V}(xb, xc - za, zb, x - iz, y)$. Hence we have the affine charts listed below. Note that $y = 0$ makes the $b \neq 0$ chart uninteresting as we would only find the origin $(x, y, z) = (0, 0, 0)$ there, which just gives us E.

- $a \neq 0$: $x(1 - i\frac{c}{a}) = 0$
- $c \neq 0$: $z(\frac{a}{c} - i) = 0$

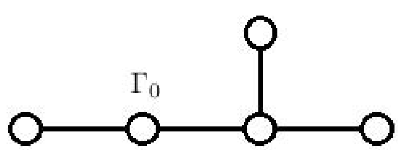
From these two charts, we see that at the intersection with E we must have $a = ic$. Plugging this into $\mathbb{V}(a^2 + bc + c^2)$ yields $bc = 0$. Hence in the $a \neq 0$ and $c \neq 0$ charts, we must have $b = 0$ and so the point of intersection is $[a : b : c] = [i : 0 : 1]$. If we were to repeat this blow-up with the other line $\alpha + i\gamma = 0$, i.e. $x + iz = 0 = y$, we would see that the point of intersection with $\mathbb{V}(a^2 + bc + c^2)$ is $[a : b : c] = [-i : 0 : 1]$, (an almost identical calculation). Hence, we see that each of the two lines intersect the conic (or $\mathbb{P}^1$) at the two distinct points $[\pm i : 0 : 1]$, and so have been "pulled apart" at the singularity that they used to meet in.

Finally we check where Γ_0 hits $\mathbb{V}(a^2 + bc + c^2)$: after the second change of coordinates, Γ_0 in $\widetilde{Y}''$ is the y-axis. Subbing this into the blow-up equations $\mathbb{V}(xb - ya, xc - za, yc - zb)$ and seeing where it intersects E yields the point $[a : b : c] = [0 : 1 : 0]$, which is indeed a point on $\mathbb{V}(a^2 + bc + c^2)$. In summary, we have a situation portrayed by the diagram below. Each bold black line is a $\mathbb{P}^1$ and the coordinates of the intersection points given in the right-hand diagram are in terms of $[a : b : c]$ in $\widetilde{Y}''$.





Finally, adding on the A_1 lying on Γ_0 in the original $\widetilde{Y}_b$ yields the resolution graph for D_5. The node for Γ_0 has been labelled. $\diamondsuit$



3.3.3 D_n

We have $f = x^2 + zy^2 + z^{n-1}$ and $\widetilde{Y} = \mathbb{V}(xb - ya, xc - za, yc - zb, x^2 + zy^2 + z^{n-1})$. Looking in each affine chart and disregarding the irreducible part with preimage completely contained in E, we have:

- $\widetilde{Y}_a$: $1 + x\frac{c}{a}(\frac{b}{a})^2 + x^{n-3}(\frac{c}{a})^{n-1} = 0$, smooth and does not intersect E.
- $\widetilde{Y}_b$: $(\frac{a}{b})^2 + y\frac{c}{b} + y^{n-3}(\frac{c}{b})^{n-1} = 0$. This chart has a singular point at $(\frac{a}{b}, y, \frac{c}{b}) = (0,0,0)$ and intersects E when $a = 0$, i.e. with affine coordinates $(x, y, z, \frac{a}{b}, \frac{c}{b}) = (0,0,0,0,\frac{c}{b})$.
- $\widetilde{Y}_c$: $(\frac{a}{c})^2 + z(\frac{b}{c})^2 + z^{n-3} = 0$. This chart has a singular point at $(\frac{a}{c}, \frac{b}{c}, z) = (0,0,0)$ and intersects E when $a = 0$, i.e. with affine coordinates $(x, y, z, \frac{a}{b}, \frac{b}{c}) = (0,0,0,0,\frac{b}{c})$.

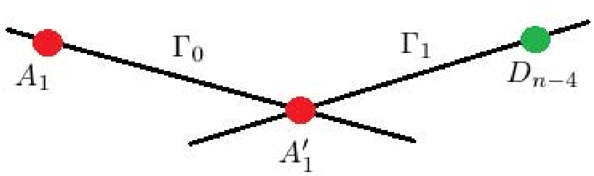
The analysis we need to do here has largely been done before. We see that $\widetilde{Y}_b$ and $\widetilde{Y}_c$ intersect E along $a = 0$, which we already know corresponds to a $\mathbb{P}^1$ that we will again denote by Γ_0. The singularities in the latter two charts must be distinct as they require $\frac{c}{b} = 0$ and $\frac{b}{c} = 0$ respectively. The singularity $(0,0,0) \in \widetilde{Y}_b$ is of A_1 type and this can be shown in precisely the same manner as for the $\widetilde{Y}_b$ chart in the D_4 case (see page 21). Finally, the singularity $(0,0,0) \in \widetilde{Y}_c$ is of D_{n-2} type, which we can see by its defining equation. We just need to make sure that Γ_0 will intersect any $\mathbb{P}^1$s and singularities we find in the first blow-up of this D_{n-2} singularity in the desired manner.

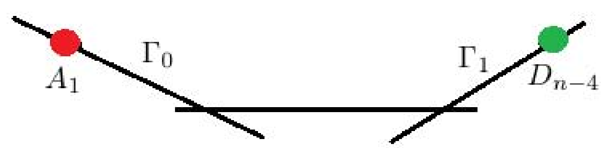
So perform the coordinate change $\frac{a}{c} \mapsto A, \frac{b}{c} \mapsto B, z \mapsto C$ in $\widetilde{Y}_c$ with new projective coordinates $[\alpha : \beta : \gamma]$ and denote the blow-up of this D_{n-2} singularity by $\widetilde{Y}'$. It has the following affine charts:

- $\widetilde{Y}'_\alpha$: $1 + A\frac{\gamma}{\alpha}(\frac{\beta}{\alpha})^2 + A^{n-5}(\frac{\gamma}{\alpha})^{n-3} = 0$, smooth and does not intersect E.
- $\widetilde{Y}'_\beta$: $(\frac{\alpha}{\beta})^2 + B\frac{\gamma}{\beta} + B^{n-5}(\frac{\gamma}{\beta})^{n-3} = 0$. This chart has a singular point at $(\frac{\alpha}{\beta}, B, \frac{\gamma}{\beta}) = (0,0,0)$ and intersects E when $\alpha = 0$, i.e. with affine coordinates $(A, B, C, \frac{\alpha}{\beta}, \frac{\gamma}{\beta}) = (0,0,0,0,\frac{\gamma}{\beta})$.
- $\widetilde{Y}'_\gamma$: $(\frac{\alpha}{\gamma})^2 + C(\frac{\beta}{\gamma})^2 + C^{n-5} = 0$. This chart has a singular point at $(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}, C) = (0,0,0)$ and intersects E when $a = 0$, i.e. with affine coordinates $(A, B, C, \frac{\alpha}{\beta}, \frac{\gamma}{\beta}) = (0,0,0,0,\frac{\beta}{\gamma})$.

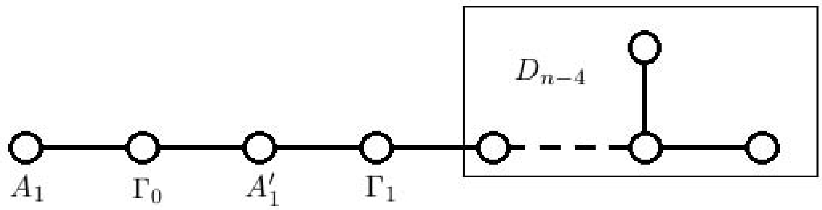
As we would expect, the singularity in $\widetilde{Y}'_\gamma$ is of D_{n-4} type. Denote the $\mathbb{P}^1$ we get from $\alpha = 0$ by Γ_1 and denote the A_1 singularity in the $\widetilde{Y}'_\beta$ chart by A'_1.

In $\widetilde{Y}'$, Γ_0 is the affine line $(A, B, C) = (0, B, 0)$. By studying the blow-up equations $X' = \mathbb{V}(A\beta - B\alpha, A\gamma - C\alpha, B\gamma - C\beta)$, we see that it intersects E at the point $[\alpha : \beta : \gamma] = [0 : 1 : 0]$. Of course, we only find this point in the $\widetilde{Y}'_\beta$ chart; it is in fact the singularity A'_1. Γ_1 also intersects A'_1; it is the point where Γ_1 and Γ_0 meet. We can summarise this with the diagrams below. On the left is the situation described above, and on the right will be what happens when blowing-up A'_1 and thus pulling apart Γ_0 and Γ_0.





This process of course iterates with each blow-up, producing A_1'' and Γ_2 etc. along the way, ending in either the D_4 or D_5 case where we get a branch at the end. Therefore we achieve the desired D_n resolution graph as shown below. We are done.



$\diamondsuit$

3.4 The E_6 Case

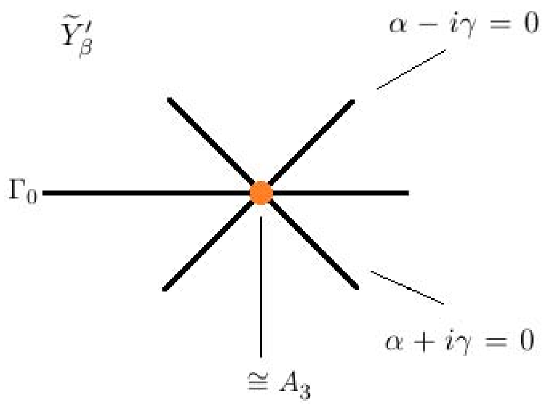
We have $f = x^2 + y^3 + z^4$ and $\widetilde{Y} = \mathbb{V}(xb - ya, xc - za, yc - zb, x^2 + y^3 + z^4)$. Looking in each affine chart and disregarding the irreducible part with preimage completely contained in E, we have:

- $\widetilde{Y}_a$: $1 + x(\frac{b}{a})^3 + x^2(\frac{c}{a})^4 = 0$, smooth and does not intersect E.
- $\widetilde{Y}_b$: $(\frac{a}{b})^2 + y + y^2(\frac{c}{b})^4 = 0$, smooth and intersects E along $a = 0$.
- $\widetilde{Y}_c$: $(\frac{a}{c})^2 + z(\frac{b}{c})^3 + z^2 = 0$, singular at $(\frac{a}{c}, \frac{b}{c}, z) = (0, 0, 0)$ and intersects E along $a = 0$.

We get a $\mathbb{P}^1$ from the intersection of E along $a = 0$ just like in the D_n cases, which we will again denote by Γ_0. This Γ_0 intersects the singularity in $\widetilde{Y}_c$ which we will need to blow up. We change coordinates $\frac{a}{c} \mapsto A, \frac{b}{c} \mapsto B, z \mapsto C$ with a new set of projective coordinates $[\alpha : \beta : \gamma]$ and denote the blow-up of $A^2 + CB^3 + C^2 = 0$ at $(0, 0, 0)$ by $\widetilde{Y}'$. We have:

- $\widetilde{Y}'_\alpha$: $1 + A^2\frac{\gamma}{\alpha}(\frac{\beta}{\alpha})^3 + (\frac{\gamma}{\alpha})^2 = 0$, smooth and intersects E along $(\frac{\gamma}{\alpha})^2 + 1 = 0$.
- $\widetilde{Y}'_\beta$: $(\frac{\alpha}{\beta})^2 + B^2\frac{\gamma}{\beta} + (\frac{\gamma}{\beta})^2 = 0$, singular at $(\frac{\alpha}{\beta}, B, \frac{\gamma}{\beta}) = (0, 0, 0)$ and intersects E along $(\frac{\alpha}{\beta})^2 + (\frac{\gamma}{\beta})^2 = 0$.
- $\widetilde{Y}'_\gamma$: $(\frac{\alpha}{\gamma})^2 + C^2(\frac{\beta}{\gamma})^3 + 1 = 0$, smooth and intersects E along $(\frac{\alpha}{\gamma})^2 + 1 = 0$.

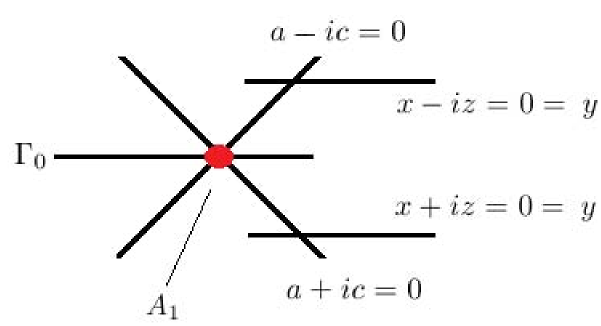
What comes out of all this looks similar to the D_5 case. Sure enough, we have $\widetilde{Y}' \cap E = \mathbb{V}(\alpha^2 + \gamma^2) = \mathbb{V}(\alpha + i\gamma) \cup \mathbb{V}(\alpha - i\gamma) \subset \{0\} \times \mathbb{P}^2$, describing the pair of projective lines $\alpha + i\gamma = 0$ and $\alpha - i\gamma = 0$ that intersect at the point $[\alpha : \beta : \gamma] = [0, 1, 0] \in \widetilde{Y}'_\beta$. This is the point where Γ_0, here the B-axis, also meets the singularity in $\widetilde{Y}'_\beta$. The difference however is that their intersection, occurring at the singularity $(\frac{\alpha}{\beta}, B, \frac{\gamma}{\beta}) = (0, 0, 0)$ is now a $D_3 \cong A_3$ type singularity as can be seen from the equation describing $\widetilde{Y}'_\beta$. So far we have this picture:



Denote the blow up of the A_3 singularity in $\widetilde{Y}'_\beta$ by $\widetilde{Y}''$. Make the coordinate change $\frac{\alpha}{\beta} \mapsto x, B \mapsto y, \frac{\gamma}{\beta} \mapsto z$, so that we are considering the blow-up of $Y'' = \mathbb{V}(x^2 + y^2 z + z^2)$ with the projective coordinates $[a : b : c]$. $\widetilde{Y}''$ has the following charts:

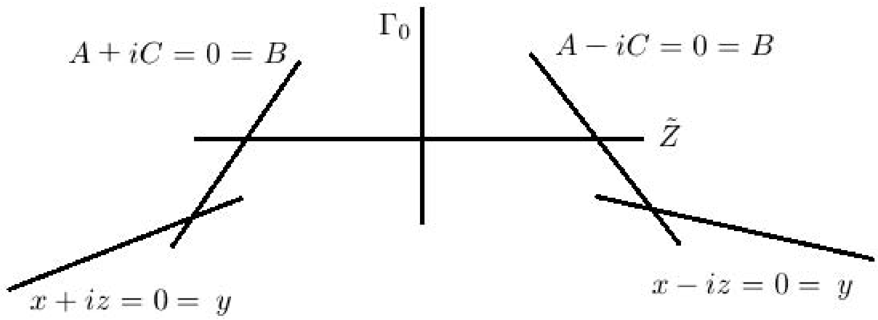
- $\widetilde{Y}''_a$: $1 + x(\frac{b}{a})^2 \frac{c}{a} + (\frac{c}{a})^2 = 0$, smooth and intersects E along $(\frac{c}{a})^2 + 1 = 0$.
- $\widetilde{Y}''_b$: $(\frac{a}{b})^2 + y\frac{c}{b} + (\frac{c}{b})^2 = 0$, singular at $(\frac{a}{b}, y, \frac{c}{b}) = (0, 0, 0)$ and intersects E along $(\frac{a}{b})^2 + (\frac{c}{b})^2 = 0$.
- $\widetilde{Y}''_c$: $(\frac{a}{c})^2 + z(\frac{b}{c})^2 + 1 = 0$, smooth and intersects E along $(\frac{a}{c})^2 + 1 = 0$.

The proper transform of Γ_0 is the y-axis, and the proper transforms of the affine part of the lines $\alpha \pm i\gamma = 0$ from $\widetilde{Y}'_\beta$ are the lines $x \pm iz = 0 = y$. Furthermore we have $\widetilde{Y}'' \cap E = \mathbb{V}(a^2 + c^2) = \mathbb{V}(a + ic) \cup \mathbb{V}(a - ic) \subset \{0\} \times \mathbb{P}^2$, another pair of projective lines $a \pm ic = 0$ meeting at the A_1 singularity in the $\widetilde{Y}''_b$ chart. Consider now only the line $x - iz = 0 = y$. We see from the D_5 case (see page 23) that it intersects E at the point $[a : b : c] = [i : 0 : 1]$. This point cannot lie therefore in the $\widetilde{Y}''_b$ chart and so this line doesn't intersect the A_1 singularity. The line $x - iz = 0 = y$ does however intersect the projective line $a - ic = 0$ at the point $[a : b : c] = [i : 0 : 1]$. Similarly the other line $x + iz = 0 = y$ will intersect the line $a + ic = 0$ at the point $[a : b : c] = [-i : 0 : 1]$. We thus have the following picture:

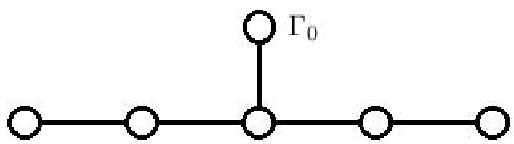


All that's left to do is check that the final blow-up of this A_1 singularity pulls apart Γ_0 and the lines $a \pm ic = 0$ in a manner that will ultimately give us the resolution graph for E_6. Make the coordinate change $\frac{a}{b} \mapsto A, y \mapsto B, \frac{c}{b} \mapsto C$ with new projective coordinates $[\alpha : \beta : \gamma]$ and denote the blow-up of this A_1 singularity by Z. We've done this calculation before: it is $\widetilde{Y}''$ from the D_5 case (see page 23). We know it intersects E in the conic $\tilde{Z} := \mathbb{V}(\alpha^2 + \beta\gamma + \gamma^2)$, giving us our sixth projective line. We've seen from D_5 that Γ_0 intersects $\tilde{Z}$ at the point $[\alpha : \beta : \gamma] = [0 : 1 : 0]$ (see page 23) and that the lines $a \pm ic = 0$, which become $A \pm iC = 0 = B$ after the coordinate change, intersect $\tilde{Z}$ at the points $[\alpha : \beta : \gamma] = [\pm i : 0 : 1]$.

The following picture summarises how all of our lines now intersect (for spatial efficiency it has been rotated from the one above).



Hence we achieve the resolution graph E_6. Γ_0 is labelled.



$\Diamond$

3.5 The E_7 Case

We have $f = x^2 + y^3 + yz^3$ and $\widetilde{Y} = \mathbb{V}(xb - ya, xc - za, yc - zb, x^2 + y^3 + yz^3)$. Looking in each affine chart and disregarding the irreducible part with preimage completely contained in E, we have:

- $\widetilde{Y}_a$: $1 + x(\frac{b}{a})^3 + x^2(\frac{c}{a})^3 \frac{b}{a} = 0$, smooth and does not intersect E.
- $\widetilde{Y}_b$: $(\frac{a}{b})^2 + y + y^2(\frac{c}{b})^3 = 0$, smooth and intersects E along $a = 0$.
- $\widetilde{Y}_c$: $(\frac{a}{c})^2 + z(\frac{b}{c})^3 + z^2 \frac{b}{c} = 0$, singular at $(\frac{a}{c}, \frac{b}{c}, z) = (0, 0, 0)$ and intersects E along $a = 0$.

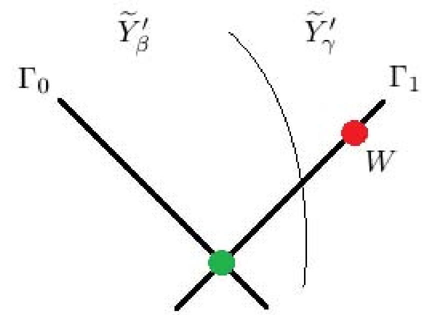
Like the D_n cases, we get a $\mathbb{P}^1$ from the intersection of E along $a = 0$ which we will again denote by Γ_0. This Γ_0 intersects the singularity in $\widetilde{Y}_c$ which we will need to blow-up. We change coordinates $\frac{a}{c} \mapsto A, \frac{b}{c} \mapsto B, z \mapsto C$ with a new set of projective coordinates $[\alpha : \beta : \gamma]$ and denote the blow-up of $A^2 + CB^3 + C^2B = 0$ at $(0, 0, 0)$ by $\widetilde{Y}'$. We have:

- $\widetilde{Y}'_\alpha$: $1 + A^2 \frac{\gamma}{\alpha}(\frac{\beta}{\alpha})^3 + A(\frac{\gamma}{\alpha})^2 \frac{\beta}{\alpha} = 0$, smooth and does not intersect E.
- $\widetilde{Y}'_\beta$: $(\frac{\alpha}{\beta})^2 + B^2 \frac{\gamma}{\beta} + B(\frac{\gamma}{\beta})^2 = 0$, singular at $(\frac{\alpha}{\beta}, B, \frac{\gamma}{\beta}) = (0, 0, 0)$ and intersects E along $\alpha = 0$.

- $\widetilde{Y}'_\gamma$: $(\frac{\alpha}{\gamma})^2 + C^2(\frac{\beta}{\gamma})^3 + C\frac{\beta}{\gamma} = 0$, singular at $(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}, C) = (0,0,0)$ and intersects E along $\alpha = 0$.

Firstly, denote the $\mathbb{P}^1$ gained by $\alpha = 0$ (à la $a = 0$ in previous cases) by Γ_1. Now the singularities in $\widetilde{Y}'_\beta$ and $\widetilde{Y}'_\gamma$ are distinct (as they respectively require $\frac{\gamma}{\beta} = 0$ and $\frac{\beta}{\gamma} = 0$) and both lie on Γ_1. The singularity $(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}, z) = (0,0,0) \in \widetilde{Y}'_\gamma$ can easily shown to be of type A_1; its blow-up, call it W with projective coordinates $[a : b : c]$, contains three smooth charts and we have $W \cap E = \mathbb{V}(a^2 + bc)$, a conic which is isomorphic to $\mathbb{P}^1$ (see D_4 case on page 21).

In $\widetilde{Y}_c$, Γ_0 is the $\frac{b}{c}$-axis. Blowing this up under the new coordinates it is B-axis, and using the blow-up equations $X' = \mathbb{V}(A\beta - B\alpha, A\gamma - C\alpha, B\gamma - C\beta)$, we see that the proper transform of Γ_0 intersects E at the point $[\alpha : \beta : \gamma] = [0 : 1 : 0]$. This intersection therefore only lies in the $\widetilde{Y}'_\beta$ chart, and in fact it is the point at which the singularity in that chart lies on Γ_1 (since Γ_1 in this chart is the $\frac{\gamma}{\beta}$-axis). So far then we have the following picture:

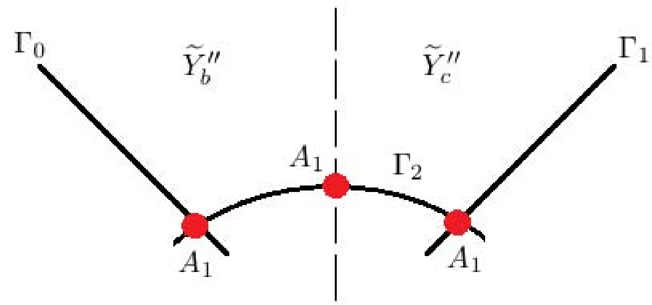


The red dot is the singularity W shown above to be of type A_1. The green dot is the singularity $(\frac{\alpha}{\beta}, B, \frac{\gamma}{\beta}) = (0,0,0) \in \widetilde{Y}'_\beta$ situated at the intersection of Γ_0 and Γ_1. We will blow this up now. Make the change of coordinates $\frac{\alpha}{\beta} \mapsto x, B \mapsto y, \frac{\gamma}{\beta} \mapsto z$ with a new set of projective coordinates $[a : b : c]$ and denote the blow-up of $x^2 + y^2 z + yz^2 = 0$ at $(0,0,0)$ by $\widetilde{Y}''$. We have:

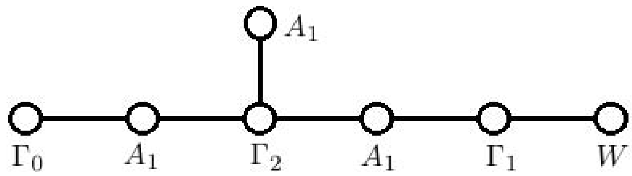
- $\widetilde{Y}''_a$: $1 + x(\frac{b}{a})^2\frac{c}{a} + x(\frac{c}{a})^2\frac{b}{a} = 0$, smooth and does not intersect E.
- $\widetilde{Y}''_b$: $(\frac{a}{b})^2 + y\frac{c}{b} + y(\frac{c}{b})^2 = 0$, singular at two points, $(\frac{a}{b}, y, \frac{c}{b}) = (0,0,0), (0,0,-1)$, and intersects E along $a = 0$.
- $\widetilde{Y}''_c$: $(\frac{a}{c})^2 + z(\frac{b}{c})^2 + z\frac{b}{c} = 0$, singular at two points, $(\frac{a}{c}, \frac{b}{c}, z) = (0,0,0), (0,-1,0)$, and intersects E along $a = 0$.

Again, denote the $\mathbb{P}^1$ gained by the intersection of E along $a = 0$ by Γ_2. The singularities with coordinates $(0,0,0)$ in $\widetilde{Y}''_b$ and $\widetilde{Y}''_c$ must be distinct as they require $\frac{c}{b} = 0$ and $\frac{b}{c} = 0$ respectively. Observe however that $(\frac{a}{b}, y, \frac{c}{b}) = (0,0,-1) \in \widetilde{Y}''_b$ and $(\frac{a}{c}, \frac{b}{c}, z) = (0,-1,0) \in \widetilde{Y}''_c$ are the same singularity. Hence in total we have three distinct singularities that all lie on Γ_2. It is easy to show that all three of these singularities are of A_1 type, just like when studying the three singularities lying on the Γ_0 of the D_4 case (see page 21).

So what do Γ_0 and Γ_1 look like in this blow up? Recall that in $\widetilde{Y}'_\beta$, Γ_0 is the B-axis and Γ_1 is the $\frac{\gamma}{\beta}$-axis. After the coordinate change, these are the y and z axes respectively. Using the blow-up equations $X'' = \mathbb{V}(xb - ya, xc - za, yc - zb)$, their proper transforms intersect E at the points $[a : b : c] = [0 : 1 : 0]$ and $[0 : 0 : 1]$ respectively. We of course only find these points in one of the three, in particular distinct, affine charts. These are actually the coordinates of the respective origins in $\widetilde{Y}''_b$ and $\widetilde{Y}''_c$, where we already know lies an A_1 singularity as explained above. The third singularity with coordinates $(\frac{a}{b}, y, \frac{c}{b}) = (0, 0, -1)$ in $\widetilde{Y}''_b$ for example, lies in both $\widetilde{Y}''_b$ and $\widetilde{Y}''_c$ and doesn't intersect either of Γ_0 or Γ_1. We thus finally arrive at the following picture:



When finally blowing up each of these A_1 singularities and remembering to add on the A_1 denoted by W found in $\widetilde{Y}'_\beta$, we achieve at last the resolution graph representing E_7. Each node has been labelled.



$\Diamond$

3.6 The E_8 Case

The final case relies very heavily on the analysis done in the E_7 case; in fact we will find an E_7 singularity in our first blow-up of E_8. To stay consistent with the names of varieties and chain of coordinate changes used in the E_7 case, we will begin by using the coordinates $(A, B, C; \alpha : \beta : \gamma)$. So instead of $f = x^2 + y^3 + z^5$, consider the variety $\widetilde{Y}^0 = \mathbb{V}(A\beta - B\alpha, A\gamma - C\alpha, B\gamma - C\beta, A^2 + B^3 + C^5)$. Looking in each affine chart and disregarding the irreducible part with preimage completely contained in E, we have:

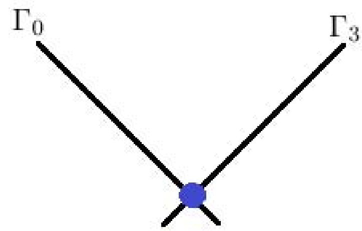
- $\widetilde{Y}^0_\alpha$: $1 + A(\frac{\beta}{\alpha})^3 + A^3(\frac{\gamma}{\alpha})^5 = 0$, smooth and does not intersect E.
- $\widetilde{Y}^0_\beta$: $(\frac{\alpha}{\beta})^2 + B + B^3(\frac{\gamma}{\beta})^5 = 0$, smooth and intersects E along $\alpha = 0$.
- $\widetilde{Y}^0_\gamma$: $(\frac{\alpha}{\gamma})^2 + C(\frac{\beta}{\gamma})^3 + C^3 = 0$, singular at $(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}, C) = (0, 0, 0)$ and intersects E along $\alpha = 0$.

We see from the $\widetilde{Y}^0_\gamma$ equation that the singularity in this chart is of type E_7. Hence we have a $\mathbb{P}^1$ given by $\alpha = 0$ (like in all previous cases) which we will call Γ_3. In $\widetilde{Y}^0_\gamma$ it is the $\frac{\beta}{\gamma}$-axis and intersects the E_7 singularity at the point $[\alpha : \beta : \gamma] = [0 : 0 : 1] \in E$. All we need to do is find the proper transforms of Γ_3 through the blow-ups of E_7 to see where it fits onto the E_7 resolution graph, hopefully giving us the graph for E_8.

So denoting the blow-up of this E_7 singularity by $\widetilde{Y}$ and making the coordinate change $\frac{\alpha}{\gamma} \mapsto x, C \mapsto y, \frac{\beta}{\gamma} \mapsto z$ with projective coordinates $[a : b : c]$, we will get precisely what we saw at the start of the E_7 case:

- $\widetilde{Y}_a$: $1 + x(\frac{b}{a})^3 + x^2(\frac{c}{a})^3\frac{b}{a} = 0$, smooth and does not intersect E.
- $\widetilde{Y}_b$: $(\frac{a}{b})^2 + y + y^2(\frac{c}{b})^3 = 0$, smooth and intersects E along $a = 0$.
- $\widetilde{Y}_c$: $(\frac{a}{c})^2 + z(\frac{b}{c})^3 + z^2\frac{b}{c} = 0$, singular at $(\frac{a}{c}, \frac{b}{c}, z) = (0, 0, 0)$ and intersects E along $a = 0$.

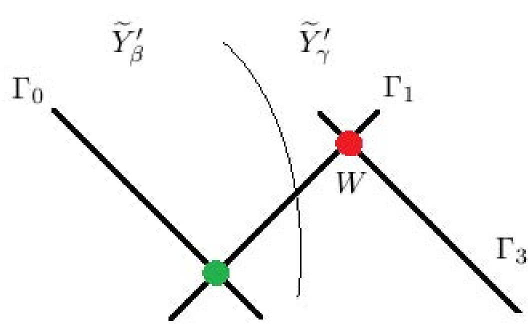
In $\widetilde{Y}^0_\gamma$, Γ_3 was the $\frac{\beta}{\gamma}$-axis so in $\widetilde{Y}$ its proper transform is the z-axis. In the $\widetilde{Y}_c$ chart, it intersects Γ_0, which is the $\frac{b}{c}$-axis here, at the singularity $(\frac{a}{c}, \frac{b}{c}, z) = (0, 0, 0)$. So far then we have this:



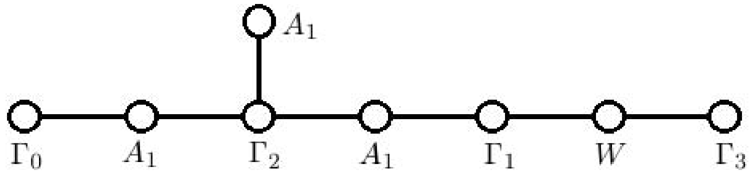
We already know what happens when we blow up this singularity (the blue dot): we get the diagram on page 28. But what happens to Γ_3? Here are the charts from this blow-up for convenience:

- $\widetilde{Y}'_\alpha$: $1 + A^2\frac{\gamma}{\alpha}(\frac{\beta}{\alpha})^3 + A(\frac{\gamma}{\alpha})^2\frac{\beta}{\alpha} = 0$, smooth and does not intersect E.
- $\widetilde{Y}'_\beta$: $(\frac{\alpha}{\beta})^2 + B^2\frac{\gamma}{\beta} + B(\frac{\gamma}{\beta})^2 = 0$, singular at $(\frac{\alpha}{\beta}, B, \frac{\gamma}{\beta}) = (0, 0, 0)$ and intersects E along $\alpha = 0$.
- $\widetilde{Y}'_\gamma$: $(\frac{\alpha}{\gamma})^2 + C^2(\frac{\beta}{\gamma})^3 + C\frac{\beta}{\gamma} = 0$, singular at $(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}, C) = (0, 0, 0)$ and intersects E along $\alpha = 0$.

In $\widetilde{Y}$, Γ_3 was the z-axis, so in $\widetilde{Y}'$ it is the C-axis. Using the blow-up equations $X' = \mathbb{V}(A\beta - B\alpha, A\gamma - C\alpha, B\gamma - C\beta)$, it intersects E at the point $[\alpha : \beta : \gamma] = [0 : 0 : 1]$ which of course we only find in $\widetilde{Y}'_c$. Continuing to use the names from E_7, at this point is the A_1 type singularity denoted W. We hence have the following picture:



Of course, blowing up W will pull apart the lines Γ_1 and Γ_3. We have already seen what happens when blowing up the singularity in the $\widetilde{Y}'_\beta$ chart (the green dot), so we're done! The resolution graph with all nodes labelled, like the E_7 case, is given below.



$\Diamond$

This conclude the blowing-up of all of the Kleinian singularities, which have been shown to have the resolution graphs ADE. $\square$

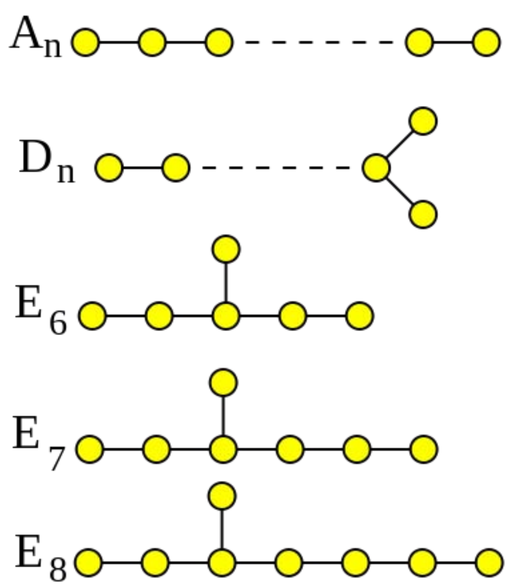


Figure 6: ADE Coxeter-Dynkin diagrams [8]

Interesting Note: It turns out that as well as finite subgroups of $SL(2, \mathbb{C})$, Kleinian surface singularities and simply-laced Dynkin diagrams, ADE can be used to classify several other diverse objects in mathematics that experience seemingly unrelated properties. More examples can be found at [11].

References

[1] Image from http://www.maths.gla.ac.uk/~ajb/3H-WP/Platonic%20solids.html

[2] Hannah Mark, "CLASSIFYING FINITE SUBGROUPS OF SO_3",http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/MarkH.pdf

[3] Image from http://johncarlosbaez.wordpress.com/

[4] Image from http://en.wikipedia.org/wiki/File:Octagonal_prism.png

[5] http://en.wikipedia.org/wiki/Quaternion#Exponential.2C_logarithm.2C_and_power

[6] John Huerta, "Introducing the Quaternions", http://maths-people.anu.edu.au/~huerta/introquaternions.pdf

[7] Igor Dolgachev, "Mckay Correspondence", http://www.math.lsa.umich.edu/~idolga/McKaybook.pdf

[8] Image from http://en.wikipedia.org/wiki/File:Simply_Laced_Dynkin_Diagrams.svg

[9] Robin Hartshorne, "Algebraic Geometry", ISBN-10: 0387902449, ISBN-13: 978-0387902449, Image from page 29.

[10] University of Bath, MA40188 lecture notes 2013-14, Alastair Craw.

[11] http://en.wikipedia.org/wiki/ADE_classification

The following papers were used for various proofs and helped to provide a guideline for the structure of my project.

- Igor Burban, "Du Val Singularities", pages 2-5, http://www.mi.uni-koeln.de/~burban/
- Igor Dolgachev, "Mckay Correspondence", pages 1-13, http://www.math.lsa.umich.edu/~idolga/McKaybook.pdf
- Graham Leuschke and Roger Wiegand, "Cohen-Macaulay Representations", pages 83-89, ISBN-10: 0-8218-7581-7, ISBN-13: 978-0-8218-7581-0
- Robin Hartshorne, "Algebraic Geometry", pages 28-30, ISBN-10: 0387902449, ISBN-13: 978-0387902449

Note: A raised glass to my family, friends, supervisor Alastair Craw and personal tutor David Calderbank for their support during not just this project but my entire four years as an undergraduate. Cheers!