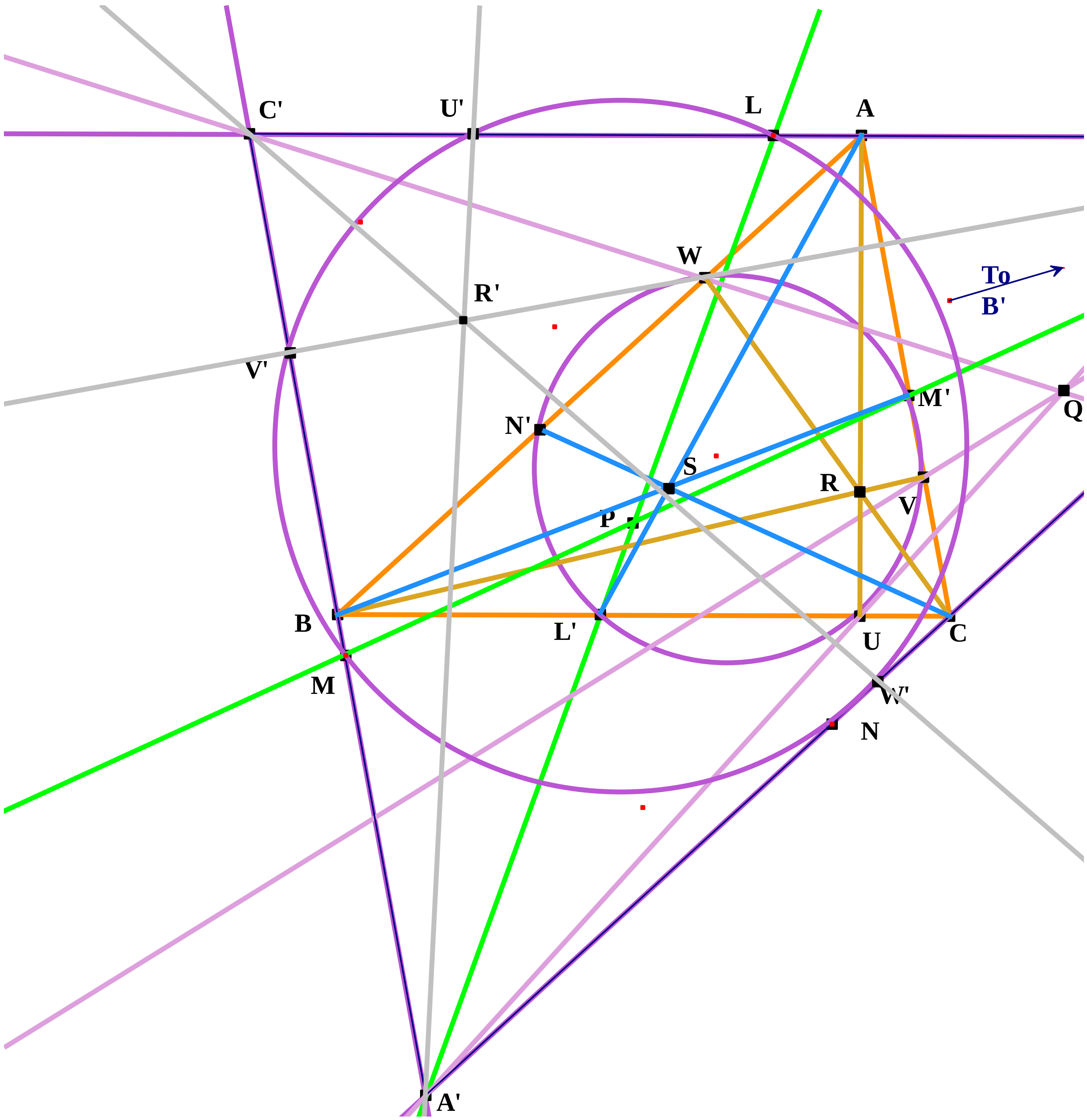
Article: CJB/217/2012

An interesting Perspective in the Anticomplementary Triangle

Christopher Bradley

Abstract: A triangle ABC and its anticomplementary triangle A' B'C' are drawn and a point P is selected through which Cevians A'P, B'P, C'P are drawn meeting the sides of ABC in points L', M', N' and the sides of A', B', C' in points L, M, N. Circles LMN and L'M'N' are drawn meeting the sides of A'B'C' and ABC respectively in points U', V', W' and U, V, W. Several perspectives are created, but the most interesting is that of triangles A'B'C' and UVW with perspector Q.



1. The Anticomplementary Triangle A'B'C' and the Cevians A'P, B'P, C'P

We take ABC to be the triangle of reference so that the line through A parallel to BC has equation $y + z = 0$. This is the line B'C'. Similarly the lines C'A' has equation $z + x = 0$ and A'B' has equation $x + y = 0$.

It follows that A' has co-ordinates A'(– 1, 1, 1) and B', C' have co-ordinates B'(1, – 1, 1) and C'(1, 1, – 1).

We now take P to have co-ordinates P(l, m, n). And it may now be shown that A'P has equation

$$(m - n)x - (n + l)y + (l + m)z = 0. \quad (1.1)$$

Similarly B'P and C'P have equations

$$(n - l)y - (l + m)z + (m + n)x = 0, \quad (1.2)$$

and

$$(l - m)z - (m + n)x + (n + l)y = 0, \quad (1.3)$$

respectively.

The line A'P meets BC at L' and B'C' at L. It may now be verified that L' has co-ordinates L'(0, l + m, n + l) and similarly M', N' have co-ordinates M'(l + m, 0, m + n) and N'(n + l, m + n, 0). The point L has co-ordinates L(2l + m + n, m – n, n – m) and similarly M, N have co-ordinates M(l – n, 2m + n + l, n – l) and N(l – m, m – l, 2n + l + m).

2. The circles L'M'N' and LMN

Circles L'M 'N' and LMN have equations of the form

$$a^2yz + b^2zx + c^2xy + (x + y + z)(ux + vy + wz) = 0. \quad (2.1)$$

We find that for L'M'N'

$u = (1/k)(m + n)(a^2(l + m + 2n)(l + 2m + n) - (2l + m + n)(b^2(l + m + 2n) + c^2(l + 2m + n))),$

$v = (1/k)(n + l)((2l + m + n)(b^2(l + m + 2n) - c^2(l + 2m + n)) - a^2(l + m + 2n)(l + 2m + n)),$

$w = -(1/k)(l + m)(a=(ll + m + 2n)(l + 2m + n) + (2l + m + n)(b^2(l + m + 2n) - c^2(l + 2m + n))),$

where (2.2)

$k = 2(l + m + 2n)(l + 2m + n)(2l + m + n).$

We also find that for LMN we have

$u = (m + n)(m – n)(b^2(l + m)(l + m + 2n) – c^2(n + l)(l + 2m + n))/(l + m)(n + l)(l + m + 2n)(l + 2m + n)$,
$v = (l + n)(l – n)(a^2(l + m)(l + m + 2n) – c^2(m + n)(2l + m + n))/(l + m)(m + n)(l + m + 2n)(2l + m + n)$,
$w = (l + m)(l – m)(a^2(l + n)(l + 2m + n) – b^2(m + n)(2l + m + n))/(l + n)(m + n)(l + 2m + n)(2l + m + n)$. (2.3)

Circle L'M'N meets BC at U with co-ordinates (x, y, z), where
$x = 0$, $y = a^2(l + m + 2n)(l + 2m + n) + (2l + m + n)(b^2(l + m + 2n) – c^2(l + 2m + n))$.
$z = a^2(l + m + 2n)(l + 2m + n) – (2l + m + n)(b^2(l + m + 2n) – c^2(l + 2m + n))$. (2.4)

Circle L'M'N' meets CA at V with co-ordinates (x, y, z), where
$x = a^2(l + m + 2n)(l + 2m + n) + b^2(l + m + 2n)(2l + m + n) – c^2(l + 2m + n)(2l + m + n)$,
$y = 0$, $z = – a^2(l + 2m + n)(2l + m + n) + b^2(l + m + 2n)(l + 2m + n) + c^2(l + m + 2n)(2l + m + n)$.
(2.5)

Circle L'M'N' meets AB at W with co-ordinates (x, y, z), where
$x = – b^2(l + m + 2n)(l + 2m + n) + c^2(2l + m + n)(l + m + 2n) + a^2(2l + m + n)(l + 2m + n)$,
$y = b^2(2l + m + n)(l + m + 2n) + c^2(2l + m + n)(l + 2m + n) – a^2(l + m + 2n)(l + 2m + n)$, $z = 0$.
(2.6)

AL', BM', CN' are concurrent at a point S with co-ordinates

$$S((l + m)(l + n), (m + l)(m + n), (n + m)(n + l)). \quad (2.7)$$

It follows since L'M'N' is a conic that AU, BV, CW are also concurrent at a point R.

Circle LMN meets B'C' at a point U' with co-ordinates (x, y, z), where
$x = a^2(l + m)(l + m + 2n)(n + l)(n + l + 2m)$,
$y = – (m + n)(2l + m + n)(b^2(l + m)(l + m + 2n) – c^2(l + n)(l + 2m + n))$, (2.8)
$z = (m + n)(2l + m + n)(b^2(l + m)(l + m + 2n) – c^2(l + n)(l + 2m + n))$.

Points V' and W' have co-ordinates that may be obtained from those of U' by cyclic change of x, y. Z and a, b, c and l, m, n.

A'U', B'V', C'W' meet at a point R', since A'L, B'M, C'N are concurrent and LMN is a conic.

3. The perspector Q of triangles UVW and A'B'C'

That these two triangles should be in perspective was a surprise and indeed the main reason for performing the calculation and writing the article.

The co-ordinates of Q turn out to be (x, y. z), where

$$x = (2l + m + n)(b^2(l + m + 2n) + c^2(l + 2m + n)) - 3a^2(l^2 + 3l(m + n) + 2m^2 + 5mn + 2n^2),$$
$$y = a^2(l + m + 2n)(l + 2m + n) - (2l + m + n)(3b^2(l + m + 2n) - c^2(l + 2m + n)) \qquad (3.1)$$
$$z = a^2(l^2 + 3l(m + n) + 2m^2 + 5mn + 2n^2) + b^2(2l^2 + l(3m + 5n) + m^2 + 3mn + 2n^2)$$
$$\qquad - 3c^2(2l^2 + l(5m + 3n) + 2m^2 + 3mn + n^2).$$

One cannot be other than mystified at the lack of symmetry of the co-ordinates.

Flat 4,
Terrill Court,
12-14, Apsley Road,
BRISTOL BS8 2SP.