A TAMENESS CONDITION ON EXPANSIONS OF DEFINABLY COMPLETE ORDERED FIELDS

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The following observation is due essentially to Harvey Friedman.

Proposition A. *Let $\mathfrak{R} := (R, <, \dots)$ be an o-minimal expansion of a dense linear order without endpoints. Let $I \subseteq R$ be an open interval, $n \in \mathbb{N}$ and $f \colon I^2 \to R^n$ be definable. Then there is a subinterval J of I such that $f(J^2)$ is not dense. If moreover $\mathfrak{R}$ defines a binary operation $+$ such that $(R, <, +)$ is an ordered group, then for every $r > 0$ there is a subinterval J of I such that $f(J^2)$ avoids an open box of side length r.*

The case of primary interest to Friedman is that $\mathfrak{R}$ be an expansion of the real field. Nevertheless, the proof of the more general statement is straightforward (Monotonicity Theorem and Cell Decomposition). Perhaps less obvious is that o-minimality of $\mathfrak{R}$ is not necessary, say, it is enough that the open core of $\mathfrak{R}$ (that is, the structure generated by its open definable sets) be o-minimal and $\mathfrak{R}$ define no unary function whose graph is somewhere dense. I show here that these conditions are necessary over definably complete expansions of ordered fields. (See [1,2] for definitions, properties and some examples.) For convenience, I give details only over the reals.

Proposition B. *Let $\mathfrak{R}$ be an expansion of the real field. Suppose that for each definable $f \colon \mathbb{R}^2 \to \mathbb{R}$ there is an interval J such that $f(J^2)$ is not dense. Then $\mathfrak{R}$ has o-minimal open core and the graph of each unary definable function is nowhere dense.*

Proof. We show that $\mathfrak{R}$ has o-minimal open core. Let $A \subseteq \mathbb{R}$ be definable and discrete. By [2] (or [1]), it suffices to show that A is finite. Suppose otherwise. By a rational transformation, we may take $A \subseteq (0,1)$ and assume that 0 is a limit point of A. Put $U = (0,1) \setminus \operatorname{cl} A$. Define functions $\lambda, \rho, \mu \colon U \to \mathbb{R}$ by $\lambda(x) = \inf\{\, t \in \mathbb{R} : (t,x) \subseteq U \,\}$, $\rho(x) = \sup\{\, t \in \mathbb{R} : (t,x) \subseteq U \,\}$ and $2\mu(x) = \lambda(x) + \rho(x)$. Define $h \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x,y) = \begin{cases} \dfrac{\mu(x)}{(x - \lambda(x))(\rho(x) - x)}, & x \in U \\ 0, & x \notin U. \end{cases}$$

If $B \subseteq \mathbb{R}^2$ is any open ball that intersects the vertical axis, then $h(B) = \mathbb{R}$. Hence, there is a definable $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(J^2) = \mathbb{R}$ for every interval J, contradicting assumptions.

Let $g \colon \mathbb{R} \to \mathbb{R}$ be definable. Suppose, toward a contradiction, that its graph is somewhere dense; by a rational transformation, we may take its graph to be dense. Define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = g(x)$ if $x = y$ and $f(x,y) = 0$ if $x \neq y$. Then $f(J^2)$ is dense for every interval J, contradicting assumptions. □

References

[1] A. Dolich, C. Miller, and C. Steinhorn, *Structures having o-minimal open core*, Trans. Amer. Math. Soc. **362** (2010), no. 3, 1371–1411. MR2563733

[2] C. Miller and P. Speissegger, *Expansions of the real line by open sets: o-minimality and open cores*, Fund. Math. **162** (1999), no. 3, 193–208. MR1736360

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