Chapter 2

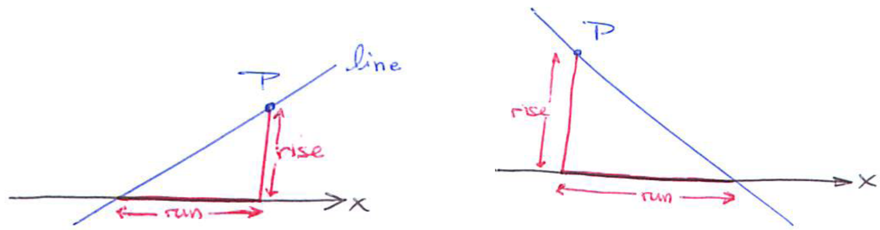
Introduction to Calculus

There are two main concepts in calculus: the derivative and the integral. Underlying both is the concept of a limit. This chapter introduces limits, with an emphasis on developing both your understanding of limits and techniques for finding them.

We start the journey in Section 2.1 where our knowledge about the slope of a line is used to define the slope at a point on a curve. The four limits introduced in Section 2.2 provide the foundation for computing many other limits, particularly the ones needed in Chapter 3. The next few sections present a definition of the limit that pertains to cases other than finding the slope of a tangent line, explores continuous functions (Section 2.4) and three fundamental properties of continuous functions (Section 2.5). We conclude, in Section 2.6, with a first look at graphing functions by hand using intercepts, symmetry, and asymptotes and with the use of technology.

2.1 Slope at a Point on a Curve

The slope of a (straight) line is simply the quotient of "rise over run", as shown in Figure 2.1.1(a).



(a) (b)

Figure 2.1.1: slope $= \frac{\text{rise}}{\text{run}}$; (a) positive slope, (b) negative slope.

It does not matter which point P is chosen on the line. If the line goes down as you move from left to right the "rise" is considered to be negative and the slope is negative. This is the case in Figure 2.1.1(b).

The slope of US Interstates never exceeds 6%=0.06. This means the road can rise (or fall) at most 6 feet in 100 (horizontal) feet, see Figure 2.1.2(a). On the other hand the steepest street in San Francisco is Filbert Street, with a slope of 0.315, see Figure 2.1.2(b).

(a) (b)

Figure 2.1.2: (a) Steepest US interstate has slope 0.06 and (b) Filbert Street has slope 0.315. [EDITOR: Replace with annotated pictures.]

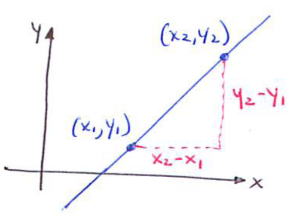


Figure 2.1.3:

Now consider a line L placed in an xy-coordinate system, as in Figure 2.1.3. Since two points determine the line, they also determine its slope.

To find that slope pick any two distinct points on the line, (x_1, y_1) and (x_2, y_2). As Figure 2.1.3 shows, they determine a rise of $y_2 - y_1$ and a run of $x_2 - x_1$, hence

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Note that the run could be negative too; that occurs if x_2 is less than x_1.

EXAMPLE 1 Find the slope of the line through $(4, -1)$ and $(1, 3)$.
SOLUTION Figure 2.1.4 shows two points on a line. Let $(4, -1)$ be (x_1, y_1) and let $(1, 3)$ be (x_2, y_2). So the slope is

$$\frac{3 - (-1)}{1 - 4} = \frac{4}{-3} = -\frac{4}{3}.$$

That the slope is negative is consistent with Figure 2.1.4 which shows that the line descends as you go from left to right. ◇

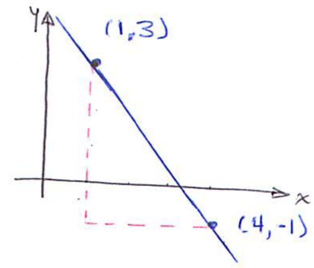


Figure 2.1.4:

Note that the slope in Example 1 does not change if $(4, -1)$ is called (x_2, y_2) and $(1, 3)$ is called (x_1, y_1).

If we know a point on a line and its slope we can draw the line. For instance, say we know a line goes through $(1, 2)$ and has slope 1.4, which is 7/5. We draw a triangle with a vertex at $(1, 2)$ and legs parallel to the axes, as in Figure 2.1.5. The rise and run of the triangle could be 7 and 5, or 1.4 and 1, or any two numbers in the ratio 1.4 : 1.

If we know a point on a line, say (a, b), and the slope of the line, m, we can draw the line and also write its equation. Any point (x, y) on the line, other than (a, b), together with the point (a, b) determine the slope of the line:

$$\text{slope} = \frac{y - b}{x - a} = m.$$

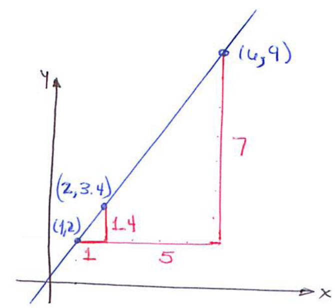


Figure 2.1.5:

The equation can be written as

$$y - b = m(x - a) \qquad \text{or} \qquad y = m(x - a) + b.$$

The slope of a line will be useful when we consider tangents to curves.

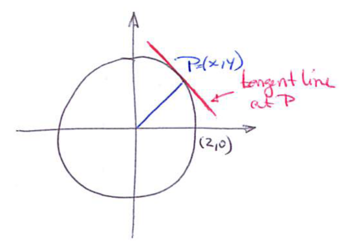


Figure 2.1.6:

Slope at Points on a Circle

Consider a circle with radius 2 and center at the origin $(0, 0)$, as shown in Figure 2.1.6. How do we find the tangent line to the circle at $P = (x, y)$? By "tangent line" we mean, informally, the line that most closely resembles the curve near P. The tangent line is perpendicular to the line OP, and the slope of OP is y/x. Thus the slope of the tangent line at (x, y) is $-x/y$. (Exercise 21 shows that the product of the slopes of perpendicular lines is -1.) For instance, at $(0, 2)$ the slope is $-0/2 = 0$, which records that the tangent line at $(0, 2)$, is horizontal, that is, the tangent line at the top of the circle is parallel to the x-axis.

We say that the slope of the circle at (x, y) is $-y/x$ because that is the slope of the tangent line at this point.

For this special curve we could find the tangent line first, and then its slope. If we had been able to find the slope of the tangent line first, we would then be able to draw the tangent line. That is what we will have to do for other curves, like the three considered next.

The Slope at a Point on the Curve $y = x^2$

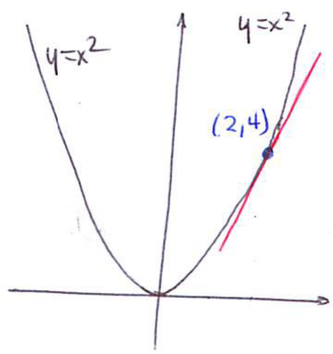
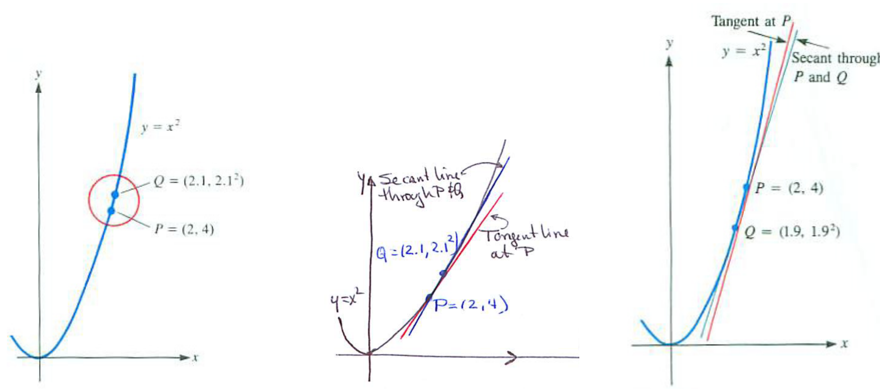


Figure 2.1.7:

Figure 2.1.7 shows the graph of $y = x^2$. How can we find the slope of the tangent line at $(2, 4)$? If we know that slope, we could draw the tangent.

If we knew two points on the tangent, we could calculate its slope. But we know only one point on that line, namely $(2, 4)$. To get around this difficulty we will choose a point Q on the parabola $y = x^2$ near P and compute the slope of the line through P and Q. Such a line is called a **secant**. As Figure 2.1.8 suggests, such a secant line resembles the tangent line at $(2, 4)$. For instance,



(a) (b) (c)

Figure 2.1.8:

choose $Q = (2.1, 2.1^2)$ and compute the slope of the line through P and Q as shown in Figure 2.1.8(b).

$$\text{Slope of secant} \quad = \quad \frac{\text{Change in } y}{\text{Change in } x} = \frac{2.1^2 - 2^2}{2.1 - 2} = \frac{4.41 - 4}{0.1} = \frac{0.41}{0.1} = 4.1.$$

Thus an estimate of the slope of the tangent line is 4.1. If you look at Figure 2.1.8, you will see that this is an overestimate of the slope of the tangent line. So the slope of the tangent line is less than 4.1.

We can also choose the point Q on the parabola to the left of $P = (2, 4)$. For instance, choose $Q = (1.9, 1.9^2)$. (See Figure 2.1.8(c).) Then

$$\text{slope of secant} \quad = \quad \frac{\text{Change in } y}{\text{Change in } x} = \frac{1.9^2 - 2^2}{1.9 - 2} = \frac{3.61 - 4}{-0.1} = \frac{-0.39}{-0.1} = 3.9.$$

Inspecting Figure 2.1.8(c) shows that this underestimates the slope of the tangent line. So the slope of the tangent line is greater than 3.9.

We have trapped the slope of the tangent line between 3.9 and 4.1. To get closer bounds we choose Q even nearer to $(2,4)$.

Using $Q = (2.01, 2.01^2)$ leads to the estimate

$$\frac{2.01^2 - 2^2}{2.01 - 2} = \frac{4.0401 - 4}{0.01} = \frac{0.0401}{0.01} = 4.01$$

and using $Q = (1.99, 1.99^2)$ yields the estimate

$$\frac{1.99^2 - 2^2}{1.99 - 2} = \frac{3.9601 - 4}{-0.01} = \frac{-0.0399}{-0.01} = 3.99.$$

Now we know the slope of the tangent at $(2,4)$ is between 3.99 and 4.01.

To make better estimates we could choose Q even nearer to $(2,4)$, say $(2.0001, 2.0001^2)$. But, still, the slopes we would get would just be estimates.

What we need to know is what happens to the quotient

$$\frac{x^2 - 2^2}{x - 2} \qquad \text{as } x \text{ gets closer and closer to 2.}$$

This chapter is devoted to answering this and other questions of the type: "What happens to the values of a function as the inputs are chosen nearer and nearer to some fixed number?"

The Slope at a Point on the Curve $y = 1/x$

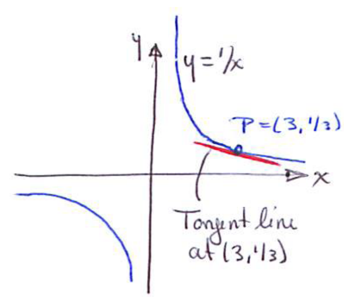


Figure 2.1.9:

Figure 2.1.9 shows the graph of $y = 1/x$. Let us estimate the slope of the tangent line to this curve at $(3, 1/3)$.

It's clear that the slope will be negative. We could draw a run-rise triangle on the tangent and get an estimate for the slope. But let's use the nearby point Q method because we can get better estimates that way.

We pick $Q = (3.1, 1/3.1)$. The points $P = (3, 1/3)$ and Q determine a secant whose slope is

$$\frac{\frac{1}{3} - \frac{1}{3.1}}{3 - 3.1} = \frac{\frac{0.1}{3(3.1)}}{-0.1} = -\frac{1}{3(3.1)} = -\frac{1}{9.3}.$$

That's just an estimate of the slope of the tangent line.

Using $Q = (2.9, 1/2.9)$, we get another estimate:

$$\frac{\frac{1}{3} - \frac{1}{2.9}}{3 - 2.9} = \frac{\frac{-0.1}{3(2.9)}}{0.1} = -\frac{1}{3(2.9)} = -\frac{1}{8.7}.$$

By choosing Q nearer $(3, 1/3)$ we could get better estimates.

The Slope at a Point on the Curve $y = \log_2(x)$

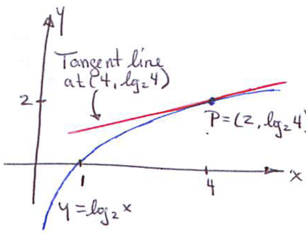


Figure 2.1.10:

Figure 2.1.10 shows the graph of $y = \log_2(x)$. Clearly, its slope is positive at all points.

We will make two estimates of the slope at $(4, \log_2(4))$. Before going any further, observe that $(4, \log_2(4)) = (4, 2)$ (because $\log_2(4) = \log_2(2^2) = 2$).

For the nearby point Q, let us use $(4.001, \log_2(4.001))$. The slope of the secant through $P = (4, 2)$ and Q is

$$\frac{\log_2(4.001) - 2}{4.001 - 4} = \frac{\log_2(4.001) - 2}{0.001}.$$

We use a calculator to estimate $\log_2(4.001)$. First, we have, by Exercise 47 in Section 1.2, to five decimal places,

$$\log_2(4.001) = \frac{\log_{10}(4.001)}{\log_{10}(2)} \approx \frac{0.60217}{0.30103} \approx 2.00036.$$

So the estimate of the slope of the tangent to $y = 1/x$ at $(2, 4)$ is

$$\frac{2.00036 - 2}{0,001} = \frac{0.00036}{0.001} = 0.36.$$

The number 0.36 is an estimate of the slope of the graph of $y = \log_2(x)$ at $P = (4, \log_2(4))$. It is not the slope there, but, even so, it could help us draw the tangent at P.

Summary

We introduced the "nearby point Q" method to estimate the slope of the tangent line to a curve at a given point P on the curve. The closer Q is to P, the better the estimate. We applied the techniques to the curves $y = x^2$, $y = 1/x$, and $y = \log_2(x)$. Note that in no case did we have to draw the curve. Nor did we find the slope of the tangent except in the special cases of a line and a circle. We found only estimates. The rest of this chapter develops methods for finding what happens to a function, such as $f(x) = (x^2 - 4)/(x - 2)$, as the argument gets near and nearer a given number.

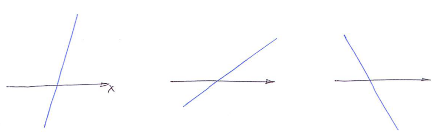
EXERCISES for Section 2.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] Draw an x axis and lines of slope 1/2, 1, 2, 4, 5, −1, and −1/2.

2.[R] Draw an x axis and lines of slope 1/3, 1, 3, −1, and −2/3.

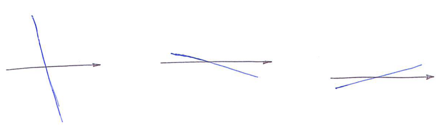
In Exercises 3 to 4 copy the figure and estimate the slope of each line as well as you can. In each case draw a "run–rise" triangle and measure the rise and run with a ruler. (A centimeter ruler is more convenient than one marked in inches.)

3.[R]



(a) (b) (c)

4.[R]



(a) (b) (c)

In Exercises 5 to 8 draw the line determined by the given information and give an equation for the line.

5.[R] through $(1, 2)$ with slope -3

6.[R] through $(1, 4)$ and $(4, 1)$

7.[R] through $(-2, -4)$ and $(0, 4)$

8.[R] through $(2, -1)$ with slope 4

9.[R]

(a) Graph the line whose equation is $y = 2x + 3$.

(b) Find the slope of this line.

10.[R]

(a) Graph the line whose equation is $y = -3x + 1$.

(b) Find the slope of this line.

11.[R] Estimate the slope of the tangent line to $y = x^2$ at $(1, 1)$ using the nearby points $(1.001, 1.001^2)$ and $(0.999, 0.999^2)$.

12.[R] Estimate the slope of the tangent line to $y = x^2$ at $(-3, 9)$ using the nearby points $(-3.01, (-3.01)^2)$ and $(-2.99, (-2.99)^2)$.

13.[R] Estimate the slope of the tangent line to $y = 1/x$ at $(1, 1)$

(a) by drawing a tangent line at $(1, 1)$ and a rise-run triangle.

(b) by using the nearby point $(1.01, 1/1.01)$. (Is the slope of the tangent line smaller or larger than this estimate?)

14.[R] Estimate the slope of the tangent line to $y = 1/x$ at $(0.5, 2)$

(a) by drawing a tangent line at $(0.5, 2)$ and a rise-run triangle.

(b) by using the nearby point $(0.49, 1/0.49)$. (Is the slope of the tangent line smaller or larger than this estimate?)

15.[R] Estimate the slope of the tangent line to $y = \log_2(x)$ at $(2, \log_2(2))$

(a) by drawing a tangent line at $(2, \log_2(2))$ and a rise-run triangle.

(b) by using the nearby point $(2.01, \log_2(2.01))$. (Is the slope of the tangent line smaller or larger than this estimate?)

16.[R] Estimate the slope of the tangent line to $y = \log_2(x)$ at $(4, 2)$

(a) by drawing a tangent line at $(4, 2)$ and a rise-run triangle.

(b) by using the nearby point $(3.99, \log_2(3.99))$. (Is the slope of the tangent line smaller or larger than this estimate?)

17.[R]

(a) Graph $y = x^2$ carefully for x in $[-2, 3]$.

(b) Draw the tangent line to $y = x^2$ at $(1, 1)$ as well as you can and estimate its slope.

(c) Using the nearby poinst $(1.1, 1.1^2)$ and $(0.9, 0.9^2)$, estimate the slope of the tangent line at $(1, 1)$. (Is the slope of the tangent line smaller or larger than this estimate?)

18.[R]

(a) Graph $y = 2^x$ carefully for x in $[0, 2]$.

(b) Draw the tangent line to $y = 2^x$ at $(1, 2)$ as well as you can and estimate its slope.

(c) Using the nearby point $(1.03, 2^{1.03})$, estimate the slope of the tangent line at $(1, 2)$. (Is the slope of the tangent line smaller or larger than this estimate?)

19.[R]

(a) Show that if you compute t
through $P = (1, 2)$ and $Q =$
the same answer with either

(b) Show that in general both w
points P and Q give the sam

20.[R] The angle between a line
axis and the x axis is called its **an**
It is measured counterclockwise f
axis to the line, as shown in Figu
θ denotes both the angle and its m
For a line parallel to the x axis, θ
Show that $\tan(\theta)$ equals the slope

21.[M] (This exercise shows that
slopes of perpendicular lines is -1
have the positive slope m. Let L'
ular to L, of slope m'. For conven
goes through the origin. Note that
on L. (See Figure **??**.)

(a) Use similar triangles ABC
that L' crosses the x-axis at

(b) Show that the slope of L' is
-1.

2.2 Four Special Limits

This section develops the notion of a limit of a function, using four examples that play a key role in Chapter 3.

A Limit Involving x^n

Let a and n be fixed numbers, with n a positive integer.

$$\text{What happens to the quotient } \frac{x^n - a^n}{x - a} \text{ as } x \text{ is chosen nearer and nearer to } a? \tag{2.2.1}$$

To keep the reasoning down-to-earth, let's look at a typical concrete case:

$$\text{What happens to } \frac{x^3 - 2^3}{x - 2} \text{ as } x \text{ gets closer and closer to 2?} \tag{2.2.2}$$

As x approaches 2, the numerator approaches $2^3 - 2^3 = 0$. Because 0 divided by anything (other than 0) is 0 we suspect that the quotient may approach 0. But the denominator approaches $2 - 2 = 0$. This is unfortunate because division by zero is not defined.

That $x^3 - 2^3$ approaches 0 as x approaches 2 may make the quotient small. That the denominator approaches 0 as x approaches 2 may make the quotient very large. How these two opposing forces balance determines what happens to the quotient (2.2.2) as x approaches 2.

We have already seen that it is pointless to replace x in (2.2.2) by 2 as this leads to $(2^3 - 2^3)/(2 - 2) = 0/0$, a meaningless expression.

Instead, let's do some experiments and see how the quotient behaves for specific values of x near 2; some less than 2, some more than 2. Table 2.2.1 shows the results as x increases from 1.9 to 2.1. You are invited to fill in the empty squares in the table below and add to the list with values of x even closer to 2.

Math is not a spectator sport. Check some of the calculations reported in Table 2.2.1.

The cases with $x = 1.99$ and 2.01, being closest to 2, should provide the best estimates of the quotient. This suggests that the quotient (2.2.2) approaches a number near 12 as x approaches 2, whether from below or from above.

While the numerical and graphical evidence is suggestive, this question can be answered once-and-for-all with a little bit of algebra. By the formula for the sum of a geometric series (see (1.4.2) in Section 1.4), $x^3 - 2^3 = (x-2)(x^2 + 2x + 2^2)$. We have

$$\frac{x^3 - 2^3}{x - 2} = \frac{(x-2)(x^2 + 2x + 2^2)}{x - 2} \qquad \text{for all } x \text{ other than 2.} \tag{2.2.3}$$

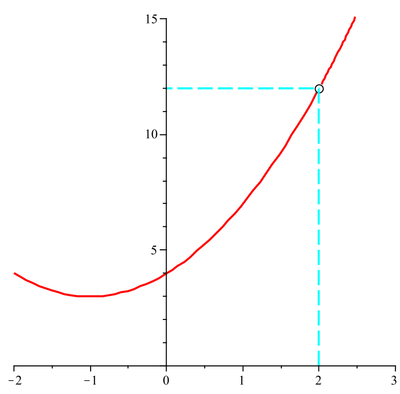
However, when x is not 2, (2.2.3) is meaningful, and we can cancel the $(x - 2)$, showing that

$$\frac{x^3 - 2^3}{x - 2} = x^2 + 2x + 2^2, \qquad x \neq 2.$$

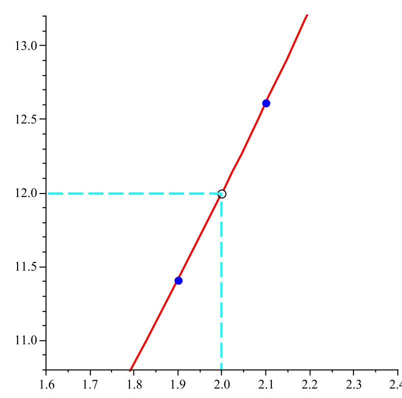
x	x^3	x^3-2^3	$x-2$	$\frac{x^3-2^3}{x-2}$
1.90	6.859	-1.141	-0.1	11.41
1.99	7.8806	-0.1194	-0.01	11.94
1.999				
2.00	8.0000	0.0000	0.00	undefined
2.001				
2.01	8.1206	0.1206	0.01	12.06
2.10	9.261	1.261	0.1	12.61

Table 2.2.1: Table showing the steps in the evaluation of $\frac{x^3-2^3}{x-2}$ for four choices of x near 2.

Recall that a hollow dot on a graph indicates that that point is NOT on the graph.



(a)



(b)

Figure 2.2.1: The graph of a $y=\frac{x^3-2^3}{x-2}$ suggests that the quotient approaches 12 as x approaches 2. In (b), zooming for x near 2 shows how the data in Table 2.2.1 also suggests the quotient approaches 12 as x approaches 2.

It is easy to see what happens to x^2+2x+2^2 as x gets nearer and nearer to 2: x^2+2x+x^2 approaches $4+4+4=12$. This agrees with the calculations (see Table 2.2.1).

We say "the limit of $(x^3-2^3)/(x-2)$ as x approaches 2 is 12" and use the shorthand

$$\begin{aligned}\lim_{x\to 2}\frac{x^3-2^3}{x-2} &= \lim_{x\to 2}(x^2+2x+2^2) \qquad (2.2.4)\\ &= 3\cdot 2^2 = 12. \qquad (2.2.5)\end{aligned}$$

Similar algebra, depending on the formula for the sum of a geometric series, yields

For any positive integer n and fixed number a,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = n\cdot a^{n-1} \qquad (2.2.6)$$

See also Exercises 41 and 42.

A Limit Involving b^x

What happens to $\frac{2^x-1}{x}$ and to $\frac{4^x-1}{x}$ as x approaches 0?

Consider $(2^x-1)/x$ first: As x approaches 0, 2^x-1 approaches $2^0-1=1-1=0$. Since the numerator and denominator in $(2^x-1)/x$ both approach 0 as x approaches 0, we face the same challenge as with $(x^3-2^3)/(x-2)$. There is a battle between two opposing forces.

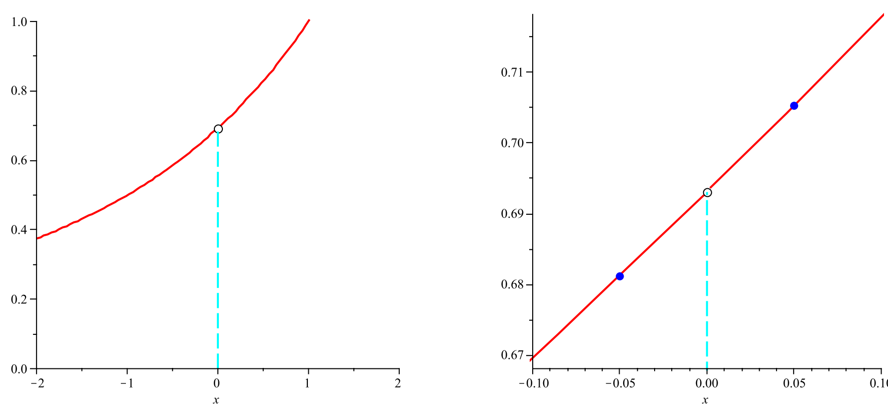
There are no algebraic tricks to help in this case. Instead, we will rely upon numerical data. While this motivation will be convincing, it is not mathematically rigorous. Later, in Appendix D, we will present a way to evaluate these limits that does not depend upon any numerical computations.

Table 2.2.2 records some results (rounded off) for four choices of x. You are invited to fill in the blanks and to add values of x even closer to 0.

> **WARNING** *(Do not believe your eyes!)* The graphs in Figure 2.2.1(b) and Figure 2.2.2(b) are not graphs of straight lines. They look straight only because the viewing windows are so small. Compare the labels on the axes in the two views in each of Figure 2.2.1 and Figure 2.2.2. That the graphs of many common functions look straight as you zoom in on a point will be important in Section 3.1.

x	2^x	$2^x - 1$	$\frac{2^x-1}{x}$
-0.01	0.993093	-0.006907	0.691
-0.001	0.999307	-0.000693	0.693
-0.0001			
0.0001			
0.001	1.000693	0.000693	0.693
0.01	1.006956	0.006956	0.696

Table 2.2.2: Numerical evaluation of $(2^x - 1)/x$ for four different choices of x. The numbers in the last column are rounded to three decimal places. See also Figure 2.2.2.



(a) (b)

Figure 2.2.2: (a) Graph of $y = (2^x - 1)/x$ for x near 0. (b) View for x nearer to 0, with the data points from Table 2.2.2. Note that there is no point for $x = 0$ since the quotient is not defined when x is 0.

It seems that as x approaches 0, $(2^x - 1)/x$ approaches a number whose decimal value begins 0.693. We write

$$\lim_{x\to 0} \frac{2^x - 1}{x} \approx 0.693 \qquad \text{rounded to three decimal places.} \tag{2.2.7}$$

It is then a simple matter to find

$$\lim_{x\to 0} \frac{4^x - 1}{x}.$$

In view of the factoring of the difference of two squares, $a^2 - b^2 = (a-b)(a+b)$, we have $4^x - 1 = (2^x)^2 - 1^2 = (2^x - 1)(2^x + 1)$. Hence

$$\frac{4^x - 1}{x} = \frac{(2^x - 1)(2^x + 1)}{x} = (2^x + 1)\frac{2^x - 1}{x}.$$

As $x \to 0$, $2^x + 1$ approaches $2^0 + 1 = 1 + 1 = 2$ and $(2^x - 1)/x$ approaches (approximately) 0.693. Thus,

$$\lim_{x\to 0} \frac{4^x - 1}{x} \approx 2 \cdot 0.693 \approx 1.386 \qquad \text{rounded to three decimal places.}$$

We now have strong evidence about the values of $\displaystyle\lim_{x\to 0} \frac{b^x - 1}{x}$ for $b = 2$ and $b = 4$. They suggest that the larger b is, the larger the limit is. Since $\lim_{x\to 0} \frac{2^x-1}{x}$ is less than 1 and $\lim_{x\to 0} \frac{4^x-1}{x}$ is more than 1, it seems reasonable that there should be a value of b such that $\lim_{x\to 0} \frac{b^x-1}{x} = 1$. This special number is called e, **Euler's number**. We know that e is between 2 and 4 and that $\lim_{x\to 0} \frac{e^x-1}{x} = 1$. It turns out that e is an irrational number with an endless decimal representation that begins $2.718281828\ldots$. In Chapter 3 we will see that e is as important in calculus as π is in geometry and trigonometry.

Euler named this constant e, but no one knows why he chose this symbol.

In any case we have

Basic Property of e

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1, \qquad \text{and } e \approx 2.71828.$$

In Section 1.2 it was remarked that the logarithm with base b, $\log_b$, can be defined for any base $b > 0$. The logarithm with base $b = e$ deserves special attention. The $\log_e(x)$ is called the **natural logarithm**, and is typically written as $\ln(x)$ or $\log(x)$. Thus, in particular,

$$y = \ln(x) \quad \text{is equivalent to} \quad x = e^y.$$

Note that, as with any logarithm function, the domain of ln is the set of positive numbers $(0, \infty)$ and the range is the set of all real numbers $(-\infty, \infty)$.

In Exercise 40 it is shown that $\lim_{x\to 0} \frac{b^x-1}{x}$ is $\ln(b)$.

Often the exponential function with base e is written as exp. This notation is convenient when the input is complicated:

$$\exp\left(\frac{\sin^3(\sqrt{x})}{\cos(x)}\right) \qquad \text{is easier to read than} \qquad e^{\sin^3(\sqrt{x})/\cos(x)}.$$

Many calculators and computer languages use exp to name the exponential function with base e.

Three Important Bases for Logarithms

While logarithms can be defined for any positive base, three numbers have been used most often: 2, 10, and e. Logarithms to the base 2 are used in information theory, for they record the number of "yes – no" questions needed to pinpoint a particular piece of information. Base 10 was used for centuries to assist in computations. Since the decimal system is based on powers of 10, certain convenient numbers had obvious logarithms; for instance, $\log_{10}(1000) = \log_{10}(10^3) = 3$. Tables of logarithms to several decimal places facilitated the calculations of products, quotients, and roots. To multiply two numbers, you looked up their logarithms, and then searched the table for the number whose logarithm was the sum of the two logarithms. The calculator made the tables obsolete, just as it sent the slide rule into museums. However, a Google search for "slide rule" returns a list of more than 15 million websites full of history, instruction, and sentiment. The number e is the most convenient base for logarithms in calculus. Euler, as early as 1728, used e for the base of logarithms.

A Limit Involving $\sin(x)$

What happens to $\frac{\sin(x)}{x}$ as x gets nearer and nearer to 0?

Appendix E includes a review of radians.

Here x represents an angle, measure in radians. In Chapter 3 we will see that in calculus radians are much more convenient than degrees.

Consider first $x > 0$. Because we are interested in x near 0, we assume that $x < \pi/2$. Figure 2.2.3 identifies both x and $\sin(x)$ on a circle of radius 1, the **unit circle**.

To get an idea of the value of this limit, let's try $x = 0.1$. Setting our calculator in the "radian mode", we find

$$\frac{\sin(0.1)}{0.1} \approx \frac{0.099833}{0.1} = 0.99833. \qquad (2.2.8)$$

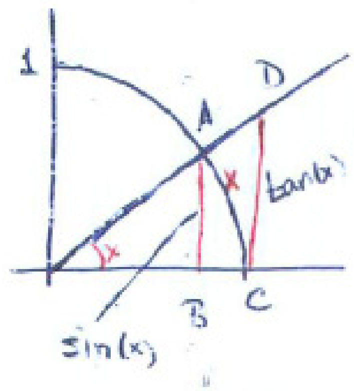


Figure 2.2.3: On the circle with radius 1, (a) x is the arclength subtended by an angle of x radians and $\sin(x) = \overline{AB}$.

Likewise, with $x = 0.01$,

$$\frac{\sin(0.1)}{0.01} \approx \frac{0.0099998}{0.01} = 0.99998. \tag{2.2.9}$$

These results lead us to suspect maybe this limit is 1.

Geometry and a bit of trigonometry show that $\lim_{x\to 0} \frac{\sin(x)}{x}$ is indeed 1. First, using Figure 2.2.3, we show that $\frac{\sin(x)}{x}$ is less than 1 for x between 0 and $\pi/2$. Recall that $\sin(x) = \overline{AB}$. Now, $\overline{AB}$ is shorter than $\overline{AC}$, since a leg of a right triangle is shorter than the hypotenuse. Then $\overline{AC}$ is shorter than the circular arc joining A to C, since the shortest distance between two points is a straight line. Thus,

$$\sin(x) < \overline{AC} < x.$$

So $\sin(x) < x$. Since x is positive, dividing by x preserves the inequality. We have

$$\frac{\sin(x)}{x} < 1. \tag{2.2.10}$$

Next, we show that $\frac{\sin(x)}{x}$ is greater than something which gets near 1 as x approaches 0. Figure 2.2.3 again helps with this step.

The area of triangle OCD is greater than the area of the sector OCA. (The area of a sector of a disk of radius r subtended by an angle θ is $\theta r^2/2$.) Thus

$$\underbrace{\frac{1}{2}\cdot 1 \cdot \tan(x)}_{\text{area of } \Delta OCD} > \underbrace{\frac{x \cdot 1^2}{2}}_{\text{area of sector } OCA}.$$

Multiplying this inequality by 2 simplifies it to

$$\tan(x) > x.$$

In other words,

$$\frac{\sin(x)}{\cos(x)} > x.$$

Now, multiplying by $\cos(x)$ which is positive and dividing by x (also positive) gives

$$\frac{\sin(x)}{x} > \cos(x). \tag{2.2.11}$$

Putting (2.2.10) and (2.2.11) together, we have

$$\cos(x) < \frac{\sin(x)}{x} < 1. \tag{2.2.12}$$

Since $\cos(x)$ approaches 1 as x approaches 0 , $\frac{\sin(x)}{x}$ is squeezed between 1 and something that gets closer and closer to 1, $\frac{\sin(x)}{x}$ must itself approach 1.

We still must look at $\frac{\sin(x)}{x}$ for $x < 0$ as x gets nearer and nearer to 0. Define u to be $-x$. Then u is positive, and

$$\frac{\sin(x)}{x} = \frac{\sin(-u)}{-u} = \frac{-\sin u}{-u} = \frac{\sin u}{u}.$$

As x is negative and approaches zero, u is positive and approaches 0. Thus $\frac{\sin(x)}{x}$ approaches 1 as x approaches 0 through positive or negative values.

In short,

$$\boxed{\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \qquad \text{where the angle, } x\text{, is measured in radians.}}$$

A Limit Involving $\cos(x)$

Knowing that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, we can show that

$$\boxed{\lim_{x\to 0}\frac{1-\cos(x)}{x} = 0.} \tag{2.2.13}$$

All we will say about this limit now is that the numerator, $1 - \cos(x)$ is the length of BC in Figure 2.2.3. Exercises 28 and 29 outline how to establish this limit.

The Meaning of $\lim_{x \to \to 0} \frac{\sin(x)}{x} = 1$

When x is near 0, $\sin(x)$ and x are both small. That their quotient is near 1 tells us much more, namely, that x is a "very good approximation of $\sin(x)$."

That means that the difference $\sin(x) - x$ is small, even in comparison to $\sin(x)$. In other words, the "relative error"

$$\frac{\sin(x) - x}{\sin(x)} \tag{2.2.14}$$

approaches 0 as x approaches 0.
To show that this is the case, we compute

$$\lim_{x \to 0} \frac{\sin(x) - x}{\sin(x)}.$$

We have

$$\begin{aligned} \lim_{x \to 0} \frac{\sin(x) - x}{\sin(x)} &= \lim_{x \to 0} \left(\frac{\sin(x)}{\sin(x)} - \frac{x}{\sin(x)} \right) \\ &= \lim_{x \to 0} \left(1 - \frac{x}{\sin(x)} \right) \\ &= \lim_{x \to 0} \left(1 - \frac{1}{\left(\frac{x}{\sin(x)} \right)} \right) \\ &= 1 - \frac{1}{1} = 0. \end{aligned}$$

As you may check by graphing, the relative error in (2.2.14) stays less than 1 percent for x less than 0.24 radians, just under 14 degrees.

It is often useful to replace $\sin(x)$ by the much simpler quantity x. For instance, the force tending to return a swinging pendulum is proportional to $\sin(\theta)$, where θ is the angle that the pendulum makes with the vertical. As one physics book says, "If the angle is small, $\sin(\theta)$ is nearly equal to θ"; it then replaces $\sin(\theta)$ by θ.

Summary

This section discussed four important limits:

$$\begin{aligned} \lim_{x\to a} \frac{x^n - a^n}{x - a} &= na^{n-1} && (n \text{ a positive integer}) \\ \lim_{x\to 0} \frac{e^x - 1}{x} &= 1 && (e \approx 2.71828) \\ \lim_{x\to 0} \frac{\sin(x)}{x} &= 1 && (\text{angle in radians}) \\ \lim_{x\to 0} \frac{1-\cos(x)}{x} &= 0 && (\text{angle in radians}). \end{aligned}$$

That is, $\lim_{x\to 0} \frac{e^x-1}{x} = 1$ says, informally, that $\frac{\exp(\text{a small number}-1)}{\text{same small number}}$ is near 1.

Each of these limits will be needed in Chapter 3, which introduces the derivative of a function.

The next section examines the general notion of a limit. This is the basis for all of calculus.

EXERCISES for Section 2.2 *Key:* R–routine, M–moderate, C–challenging

In each of Exercises 1 to 10 describe the two opposing forces involved in the limit. If you can figure out the limit on the basis of results in this section, give it. Otherwise, use a calculator to estimate the limit.

1.[R] $\lim_{x\to 2} \frac{x^4-16}{x-2}$

2.[R] $\lim_{x\to 0} \frac{\sin(x)}{x\cos(x)}$

3.[R] $\lim_{x\to 0}(1-x)^{1/x}$

4.[R] $\lim_{x\to 0}(\cos(x))^{1/x}$

5.[R] $\lim_{x\to 0} x^x$, $x>0$

6.[R] $\lim_{x\to 0} \frac{\arcsin(x)}{x}$

7.[R] $\lim_{x\to 0} \frac{\tan(x)}{x}$

HINT: Write $\tan(x) = \sin(x)/\cos(x)$.

8.[R] $\lim_{x\to 0} \frac{\tan(2x)}{x}$

9.[R] $\lim_{x\to 0} \frac{8^x-1}{2^x-1}$

HINT: The numerator is the difference of two cubes; how does b^3-a^3 factor?

10.[R] $\lim_{x\to 0} \frac{9^x-1}{3^x-1}$

Exercises 11 to 15 concern $\lim_{x\to a} \frac{x^n-a^n}{x-a}$.

11.[R] Using the factorization $(x-a)(x+a)=x^2-a^2$ find $\lim_{x\to a} \frac{x^2-a^2}{x-a}$.

12.[R] Using Exercise 11,

(a) find $\lim_{x\to 3} \frac{x^2-9}{x-3}$

(b) find $\lim_{x\to\sqrt{3}} \frac{x^2-3}{x-\sqrt{3}}$

13.[R]

(a) By multiplying it out, show that $(x-a)(x^2+ax+a^2)=x^3-a^3$.

(b) Use (a) to show that $\lim_{x\to a} \frac{x^3-a^3}{x-a}=3a^2$.

(c) By multiplying it out, show that $(x-a)(x^3+ax^2+a^2x+a^3)=x^4-a^4$.

(d) Use (c) to show that $\lim_{x\to a} \frac{x^4-a^4}{x-a}=4a^3$.

14.[R]

(a) What is the domain of $(x^2-9)/(x-3)$?

(b) Graph $(x^2-9)/(x-3)$.

NOTE: Use a hollow dot to indicate an absent point in the graph.

15.[R]

(a) What is the domain of $(x^3-8)/(x-2)$?

(b) Graph $(x^3-8)/(x-2)$.

Exercises 16 to 19 concern $\lim_{x\to 0} \frac{e^x-1}{x}$.

16.[R] What is a definition of the number e?

17.[R] Use a calculator to compute $(2.7^x-1)/x$ and $(2.8^x-1)/x$ for $x=0.001$. NOTE: This suggests that e is between 2.7 and 2.8.

18.[R] Use a calculator to estimate $(2.718^x-1)/x$ for $x=0.1$, 0.01, and 0.001.

19.[R] Graph $y=(e^x-1)/x$ for $x\neq 0$.

Exercises 20 to 30 concern $\lim_{x\to 0} \frac{\sin(x)}{x}$ and $\lim_{x\to 0} \frac{1-\cos(x)}{x}$.

20.[R] Use your calculator to create a graph of $y=\frac{\sin(x)}{x}$.

21.[R] Use your calculator to create a graph of $y=\frac{1-\cos(x)}{x}$.

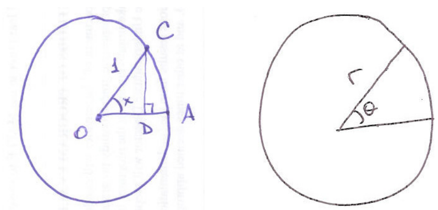
22.[R] Using the fact that $\lim_{x\to 0} \frac{\sin(x)}{x}=1$, find the limits of the following as x approaches 0.

(a) $\frac{\sin(3x)}{3x}$

(b) $\frac{\sin(3x)}{x}$

(c) $\frac{\sin(3x)}{\sin(x)}$

(d) $\frac{\sin^2(x)}{x}$



(a) (b)

Figure 2.2.4:

23.[R] Why is the arc length from A to C in Figure 2.2.4(a) equal to x?

24.[R] Why is the length of CD in Figure 2.2.4(a) equal to $\tan x$?

25.[R] Why is the area of triangle OCD in Figure 2.2.4(a) equal to $(\tan x)/2$?

26.[R] An angle of θ radians in a circle of radius r subtends a sector, as shown in Figure 2.2.4(b). What is the area of this sector? NOTE: For a review of trigonometry, see Appendix E.

27.[R]

(a) Graph $\sin(x)/x$ for x in $[-\pi, 0)$

(b) Graph $\sin(x)/x$ for x in $(0, \pi]$.

(c) How are the graphs in (a) and (b) related?

(d) Graph $\sin(x)/x$ for $x \neq 0$.

28.[R] When $x = 0$, $(1 - \cos(x))/x$ is not defined. Estimate $\lim_{x\to 0} \dfrac{1-\cos(x)}{x}$ by evaluating $(1-\cos(x))/x$ at $x = 0.1$ (radians).

29.[R] To find $\lim_{x\to 0} \dfrac{1-\cos(x)}{x}$ first check this algebra and trigonometry:

$$\frac{1-\cos(x)}{x} = \frac{1-\cos(x)}{x}\frac{1+\cos(x)}{1+\cos(x)} = \frac{1-\cos^2(x)}{x(1+\cos(x))} = \frac{\sin^2(x)}{x(1+\cos(x))} = \frac{\sin(x)}{x}\frac{\sin(x)}{1+\cos(x)}.$$

Then show that

$$\lim_{x\to 0} \frac{\sin(x)}{x}\frac{\sin(x)}{1+\cos(x)} = 0.$$

30.[M] Show that

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\frac{1}{2}.$$

This suggests that, for small values of x, $1-\cos(x)$ is close to $\frac{x^2}{2}$, so that $\cos(x)$ is approximately $1-\frac{x^2}{2}$.

(a) Use a calculator to compare $\cos(x)$ with $1-\frac{x^2}{2}$ for $x = 0.2$ and 0.1 radians. NOTE: 0.2 radians is about 11°.

(b) Use a graphing calculator to compare the graphs of $\cos(x)$ and $1-\frac{x^2}{2}$ for x in $[-\pi,\pi]$.

(c) What is the largest interval on which the values of $\cos(x)$ and $1-\frac{x^2}{2}$ differ by no more than 0.1? That is, for what values of x is it true that $\left|\cos(x)-(1-\frac{x^2}{2})\right| < 0.1$?

NOTE: See Exercise 29.

31.[M] The limit $\lim_{\theta\to 0}\frac{\sin(4\theta)}{\sin(\theta)}$ appears in the design of a water sprinkler in the "Calculus is Everywhere" in Chapter 5 Find that limit.

32.[M]

(a) We examined $(2^x-1)/x$ only for x near 0. When x is large and positive 2^x-1 is large. So both the numerator and denominator of $(2^x-1)/x$ are large. The numerator influences the quotient to become large. The large denominator pushes the quotient toward 0. Use a calculator to see how the two forces balance for large values of x.

(b) Sketch the graph of $f(x) = (2^x-1)/x$ for $x > 0$. (Pay special attention to the behavior of the graph for large values of x.)

33.[M]

(a) When x is negative and $|x|$ is large what happens to $(2^x-1)/x$?

(b) Sketch the graph of $f(x) = (2^x-1)/x$ for $x < 0$. (Pay special attention to the behavior of the graph for large negative values of x.)

34.[M]

(a) Using a calculator, explore what happens to $\sqrt{x^2+x}-x$ for large positive values of x.

(b) Show that for $x>0$, $\sqrt{x^2+x}<x+(1/2)$.

(c) Using algebra, find what number $\sqrt{x^2+x}-x$ approaches as x increases. HINT: Multiply $\sqrt{x^2+x}-x$ by $\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$, an operation that removes square roots from the denominator.

35.[M] Using a calculator, examine the behavior of the quotient $(\theta-\sin(\theta))/\theta^3$ for θ near 0.

36.[M] Using a calculator, examine the behavior of the quotient $\left(\cos(\theta)-1+\dfrac{\theta^2}{2}\right)/\theta^4$ for θ near 0.

Exercises 37 to 40 concern $f(x)=(1+x)^{1/x}$, x in $(-1,0)$ and $(0,\infty)$.

37.[M]

(a) Why is $(1+x)^{1/x}$ not defined when $x=-3/2$ but is defined when $x=-5/3$. Give an infinite number of $x<-1$ for which it is not defined.

(b) For x near 0, $x>0$, $1+x$ is near 1. So we might expect $(1+x)^{1/x}$ to be near 1 then. However, the exponent $1/x$ is very large. So perhaps $(1+x)^{1/x}$ is also large. To see what happens, fill in this table.

x	1	0.5	0.1	0.01	0.001
$1+x$	2				
$1/x$	1				
$(1+x)^{1/x}$	2				

(c) For x near 0 but negative, investigate $(1+x)^{1/x}$ with the use of this table

x	-0.5	-0.1	-0.01	-0.001
$1+x$	0.5			
$1/x$	-2			
$(1+x)^{1/x}$	4			

38.[M] Graph $y=(1+x)^{1/x}$ for x in $(-1,0)$ and $(0,10)$.

Exercises 37 and 38 show that $\lim_{x\to 0}(1+x)^{1/x}$ is about 2.718. This suggests that the number e may equal $\lim_{x\to 0}(1+x)^{1/x}$. In Section 3.2 we show that this is the case. However, the next two exercises give persuasive arguments for this fact. Unfortunately, each argument has a big hole or "unjustified leap," which you are asked to find.

39.[C] Assume that all we know about the number e is that $\lim_{x\to 0}\frac{e^x-1}{x}=1$. We will write this as

$$\frac{e^x-1}{x}\sim 1,$$

and read this as "$(e^x-1)/x$ is close to 1 when x is near 0." Multiplying both sides by x gives

$$e^x-1\sim x.$$

Adding 1 to both sides of this gives

$$e^x\sim 1+x.$$

Finally, raising both sides to the power $1/x$ yields

$$(e^x)^{1/x}\sim(1+x)^{1/x},$$

hence

$$e\sim(1+x)^{1/x}.$$

This suggests that

$$e=\lim_{x\to 0}(1+x)^{1/x}.$$

The conclusion is correct. Most of the steps are justified. Which step is the "big leap"?

40.[C] Assume that $b=\lim_{x\to 0}(1+x)^{1/x}$. We will "show" that

$$\lim_{x\to 0}\frac{b^x-1}{x}=1.$$

First of all, for x near (but not equal to) 0

$$b\sim(1+x)^{1/x}.$$

Then

$$b^x\sim 1+x.$$

Hence

$$b^x-1\sim x.$$

Dividing by x gives

$$\frac{b^x-1}{x}\sim 1.$$

Hence

$$\lim_{x\to 0}\frac{b^x-1}{x}=1.$$

Where is the "suspect step" this time?

41.[C] Let n be a positive integer and define $P_n(x)=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-2}x+a^{n-1}$. This polynomial is equal to the quotient $\frac{x^n-a^n}{x-a}$. That is $(x-a)P_n(x)=x^n-a^n$. (This factorization is justified in Exercise 43 in Section 5.4.)

(a) Verify that $(x-a)P_2(x)=x^2-a^2$. (Compare with Exercise 11)

(b) Verify that $(x-a)P_3(x)=x^3-a^3$. (Compare with Exercise 13(a))

(c) Verify that $(x-a)P_4(x)=x^4-a^4$. (Compare with Exercise 13(c))

(d) Explain why $(x-a)P_n(x)=x^n-a^n$ for all positive integers n.

42.[C] Using the formula for the sum of a geometric progression ((1.4.2) in Section 1.4), show that $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$.

43.[C] An intuitive argument suggested that $\lim_{\theta\to 0}(\sin\theta)/\theta=1$, which turned out to be correct. Try your intuition on another limit associated with the unit circle shown in Figure 2.2.5.

(a) What do you think happens to the quotient

$$\frac{\text{Area of triangle } ABC}{\text{Area of shaded region}}\qquad\text{as } \theta\to 0?$$

More precisely, what does your intuition suggest is the limit of that quotient as $\theta\to 0$?

(b) Estimate the limit in (a) using $\theta=0.01$.

NOTE: This problem is a test of your intuition. This limit, which arose during some research in geometry, is determined in Exercise 54 in Section 5.5. The authors guessed wrong, as has everyone they asked.

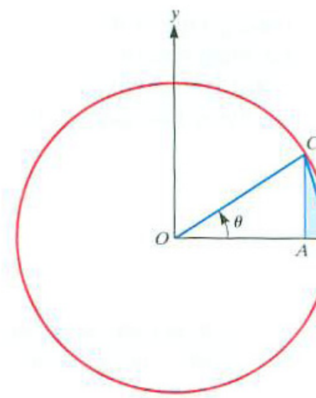


Figure 2.2.5:

2.3 The Limit of a Function: The General Case

Section 2.2 concerned four important limits:

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1},\quad \lim_{x\to 0}\frac{e^x-1}{x}=1,\quad \lim_{x\to 0}\frac{\sin(x)}{x}=1,\quad \lim_{x\to 0}\frac{1-\cos(x)}{x}=0.$$

These are all of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$, in which $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$. However a limit may have a different form, as illustrated in Exercises 39 and 40 in Section 2.2, which concern $\lim_{x\to 0}(1+x)^{1/x}$.

Limits are fundamental to all of calculus. In this section, we pause to discuss the concept of a limit, beginning with the notion of a one-sided limit.

One-Sided Limits

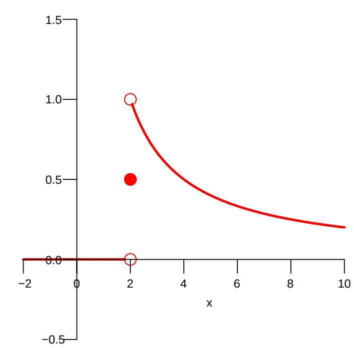


Figure 2.3.1:

The domain of the function shown in Figure 2.3.1 is $(-\infty,\infty)$. In particular, the function is defined when $x=2$ and $f(2)=1/2$. This fact is conveyed by the solid dot at $(2,1/2)$ in the figure. The hollow dots at $(2,0)$ and $(2,1)$ indicate that these points are not on the graph of this function (but some nearby points are on the graph).

Consider the part of the graph for inputs $x>2$, that is, for inputs to the right of 2. As x approaches 2 from the right, $f(x)$ approaches 1. This conclusion can be expressed as

$$\lim_{x\to 2^+} f(x)=1$$

and is read "the limit of f of x, as x approaches 2, from the right, is 1." Similarly, looking at the graph of f in Figure 2.3.1 for x to the left of 2, that is, for $x<2$, the values of $f(x)$ approach a different number, namely, 0. This is expressed with the shorthand

$$\lim_{x\to 2^-} f(x)=0.$$

It might sound strange to say the values of $f(x)$ "approach" 0 since the function values are exactly 0 for all inputs $x<2$. But, it is convenient, and customary, to use the word "approach" even for constant functions.

This illustrates the concept of the "right-hand" and "left-hand" limits, the two **one-sided limits**.

> **DEFINITION** *(Right-hand limit of $f(x)$ at a)* Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (a,c). If, as x approaches a from the right, $f(x)$

approaches a specific number L, then L is called the **right-hand limit** of $f(x)$ as x approaches a. This is written

$$\lim_{x\to a^+} f(x) = L$$

or

$$f(x) \to L \quad \text{as} \quad x \to a^+.$$

The assertion that

$$\lim_{x\to a^+} f(x) = L$$

is read "the limit of f of x as x approaches a from the right is L" or "as x approaches a from the right, $f(x)$ approaches L."

DEFINITION *(Left-hand limit of $f(x)$ at a)* Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (b, a). If, as x approaches a from the left, $f(x)$ approaches a specific number L, then L is called the **left-hand limit** of $f(x)$ as x approaches a. This is written

$$\lim_{x\to a^-} f(x) = L$$

or

$$f(x) \to L \quad \text{as} \quad x \to a^-.$$

Notice that the definitions of the one-sided limits do not require that the number a be in the domain of the function f. If f is defined at a, we do not consider $f(a)$ when examining limits as x approaches a.

The Two-Sided Limit

If the two one-sided limits of $f(x)$ at $x = a$, $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$, exist and are equal to L then we say the limit of $f(x)$ as x approaches a is L.

$$\lim_{x\to a} f(x) = L \qquad \text{means} \qquad \lim_{x\to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^+} f(x) = L.$$

For the function graphed in Figure 2.3.1 we found that $\lim_{x\to 2^+} f(x) = 1$ and $\lim_{x\to 2^-} f(x) = 0$. Because they are different, the two-sided limit of $f(x)$ at 2, $\lim_{x\to 2} f(x)$, does not exist.

EXAMPLE 1 Figure 2.3.2 shows the graph of a function f whose domain is the closed interval $[0, 5]$.

(a) Does $\lim_{x\to 1} f(x)$ exist?

(b) Does $\lim_{x\to 2} f(x)$ exist?

(c) Does $\lim_{x\to 3} f(x)$ exist?

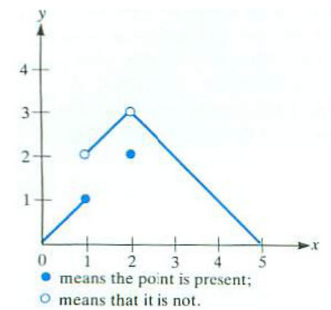


Figure 2.3.2:

SOLUTION

(a) Inspection of the graph shows that

$$\lim_{x\to 1^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 1^+} f(x) = 2.$$

Although the two one-sided limits exist, they are not equal. Thus, $\lim_{x\to 1} f(x)$ does not exist. In short, "f does not have a limit as x approaches 1."

(b) Inspection of the graph shows that

$$\lim_{x\to 2^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x\to 2^+} f(x) = 3.$$

Thus $\lim_{x\to 2} f(x)$ exists and is 3. That $f(2) = 2$, as indicated by the solid dot at $(2, 2)$, plays no role in our examination of the limit of $f(x)$ as $x \to 2$ (either one-sided or two-sided).

(c) Inspection, once again, shows that

$$\lim_{x\to 3^-} f(x) = 2 \qquad \text{and} \qquad \lim_{x\to 3^+} f(x) = 2.$$

Thus $\lim_{x\to 3} f(x)$ exists and is 2. Incidentally, the fact that $f(3) = 2$ is irrelevant in determining $\lim_{x\to 3} f(x)$.

◇

We now define the (two-sided) limit without referring to one-sided limits.

DEFINITION *(Limit of $f(x)$ at a.)* Let f be a function and a some fixed number. Assume that the domain of f contains open intervals (b, a) and (a, c), as shown in Figure 2.3.3. If there is a number L such that as x approaches a, from both the right and the left, $f(x)$ approaches L, then L is called the **limit** of $f(x)$ as x approaches a. This is expressed as either

$$\lim_{x\to a} f(x) = L \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to a.$$

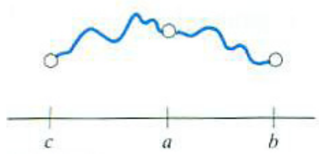


Figure 2

Figure 2.3.3: The function f is defined on open intervals on both sides of a.

EXAMPLE 2 Let f be the function defined by by $f(x) = \dfrac{x^n - a^n}{x - a}$ where n is a positive integer. This function is defined for all x except a. How does it behave for x near a?

SOLUTION In Section 2.2 and its Exercises we found that as x gets closer and closer to a, $f(x)$ gets closer and closer to na^{n-1}. This is summarized with the shorthand

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1},$$

read as "the limit of $\frac{x^n-a^n}{x-a}$ as x approaches a is na^{n-1}." ◇

EXAMPLE 3 Investigate the one-sided and two-sided limits for the square root function at 0.

SOLUTION The function $\sqrt{x}$ is defined only for x in $[0, \infty)$. We can say that the right-hand limit at 0 exists since $\sqrt{x}$ approaches 0 as $x \to 0$ through positive values of x; that is, $\lim_{x \to 0^+} \sqrt{x} = 0$. Because $\sqrt{x}$ is not defined for any negative values of x, the left-hand limit of $\sqrt{x}$ at 0 does not exist. Consequently, the two-sided limit of $\sqrt{x}$ at 0, $\lim_{x \to 0} \sqrt{x}$, does not exist. ◇

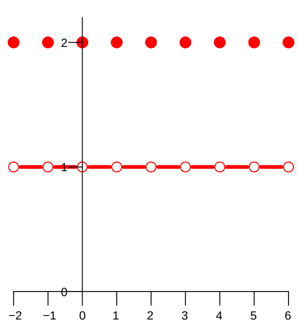


Figure 2.3.4:

EXAMPLE 4 Consider the function f defined so that $f(x) = 2$ if x is an integer and $f(x) = 1$ otherwise. For which a does $\lim_{x \to a} f(x)$ exist?

SOLUTION The graph of f, shown in Figure 2.3.4, will help us decide. If a is not an integer, then for all x sufficiently near a, $f(x) = 1$. So $\lim_{x \to a} f(x) = 1$. Thus the limit exists for all a that are not integers.

Now consider the case when a is an integer. In deciding whether $\lim_{x \to a} f(x)$ exists we never consider the value of f at a, namely $f(a) = 2$. For all x sufficiently near an integer a, $f(x) = 1$. Thus, once again, $\lim_{x \to a} f(x) = 1$. The limit exists but is not $f(a)$.

Thus, $\lim_{x \to a} f(x)$ exists and equals 1 for every number a. ◇

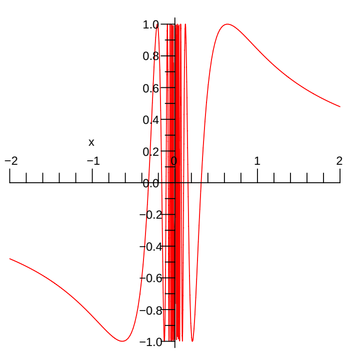


Figure 2.3.5: $y = g(x) = \sin(1/x)$.

EXAMPLE 5 Let $g(x) = \sin(1/x)$. For which a does $\lim_{x \to a} g(x)$ exist?

SOLUTION To begin, graph the function. Notice that the domain of g consists of all x except 0. When x is very large, $1/x$ is very small, so $\sin(1/x)$ is small. As x approaches 0, $1/x$ becomes large. For instance, when $x = \frac{1}{2n\pi}$, for a non-zero integer n, $1/x = 2n\pi$ and therefore $\sin(1/x) = \sin(2n\pi) = 0$. Thus, the graph of $y = g(x)$ for x near 0 crosses the x-axis infinitely often. Similarly, $g(x)$ takes the values 1 and -1 infinitely often for x near 0. The graph is shown in Figure 2.3.5.

Does $\lim_{x\to 0} g(x)$ exist? Does $g(x)$ tend toward one specific number as $x \to 0$? No. The function oscillates, taking on all values from -1 to 1 (repeatedly) for x arbitrarily close to 0. Thus $\lim_{x\to 0}\sin(1/x)$ does not exist.

At all other values of a, $\lim_{x\to a} g(x)$ does exist and equals $g(a) = \sin(1/a)$.
◇

Infinite Limits at a

A function may assume arbitrarily large values as x approaches a fixed number. One important example is the tangent function. As x approaches $\pi/2$ from the left, $\tan(x)$ takes on arbitrarily large positive values. (See Figure 2.3.6.) We write

$$\lim_{x\to \frac{\pi}{2}^-} \tan(x) = +\infty.$$

However, as $x \to \frac{\pi}{2}$ from inputs larger than $\pi/2$, $\tan(x)$ takes on negative values of arbitrarily large absolute value. We write

$$\lim_{x\to \frac{\pi}{2}^+} \tan(x) = -\infty.$$

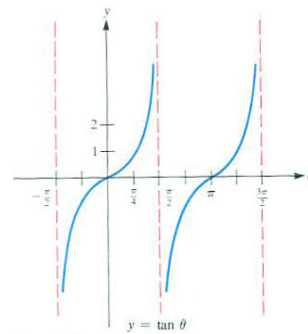


Figure 2.3.6:

DEFINITION *(Infinite limit of $f(x)$ at a)* Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (a, c). If, as x approaches a from the right, $f(x)$ becomes <u>and remains</u> arbitrarily large and positive, then the limit of $f(x)$ as x approaches a is said to be positive infinity. This is written

$$\lim_{x\to a^+} f(x) = +\infty$$

or sometimes just

$$\lim_{x\to a^+} f(x) = \infty.$$

If, as x approaches a from the left, $f(x)$ becomes and remains arbitrarily large and positive, then we write

$$\lim_{x\to a^-} f(x) = +\infty.$$

Similarly, if $f(x)$ assumes values that are negative and these values remain arbitrarily large in absolute value, we write either

$$\lim_{x\to a^+} f(x) = -\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = -\infty,$$

depending upon whether x approaches a from the right or from the left.

Limits as $x \to \infty$

Sometimes it is useful to know how $f(x)$ behaves when x is a very large positive number (or a negative number of large absolute value).

EXAMPLE 6 Determine how $f(x) = 1/x$ behaves for

(a) large positive inputs

(b) negative inputs of large absolute value

(c) small positive inputs

(d) negative inputs of small absolute value

SOLUTION

x	$1/x$
10	0.1
100	0.01
1000	0.001

(a) To get started, make a table of values as shown in the margin. As x becomes arbitrarily large, $1/x$ approaches 0: $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$. This conclusion would be read as "as x approaches ∞, $f(x)$ approaches 0."

(b) This is similar to (a), except that the reciprocal of a negative number with large absolute value is a negative number with a small absolute value. Thus, $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$.

(c) For inputs that are positive and approaching 0, the reciprocals are positive and large: $\lim_{x\to 0^+} \frac{1}{x} = +\infty$.

(d) Lastly, the reciprocal of inputs that are negative and approaching 0 from the left are negative and arbitrarily large in absolute value: $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$.

◇

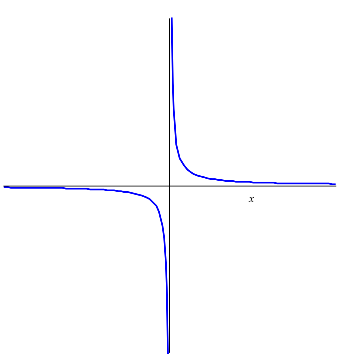


Figure 2.3.7:

More generally, for any fixed positive exponent p,

$$\lim_{x\to\infty} \frac{1}{x^p} = 0.$$

Limits of the form $\lim_{x\to\infty} P(x)$ and $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$, where P and Q are polynomials are easy to treat, as the following examples show.

Keep in mind that ∞ is not a number. It is just a symbol that tells us that something — either the inputs or the outputs of a function — become arbitrarily large.

EXAMPLE 7 Find $\lim\limits_{x\to\infty} (2x^3 - 5x^2 + 6x + 5)$.

SOLUTION When x is large, x^3 is much larger than either x^2 or x. With this in mind, we use a little algebra to determine the limit:

$$2x^3 - 5x^2 + 6x + 5 = x^3\left(2 - \frac{5}{x} + \frac{6}{x^2} + \frac{5}{x^3}\right).$$

The expression in parentheses approaches 2, while x^3 gets arbitrarily large. Thus

$$\lim_{x\to\infty} \frac{2x^3 - 5x^2 + 6x + 5}{x^3} = \infty.$$

◇

EXAMPLE 8 Find $\lim_{x\to\infty} \frac{2x^3-5x^2+6x+5}{7x^4+3x+2}$.

SOLUTION We use the same technique as in Example 7.

$$\begin{array}{rrcl} & 2x^3 - 5x^2 + 6x + 5 & = & x^3\left(2 - \frac{5}{x} + \frac{6}{x^2} + \frac{5}{x^3}\right) \\ \text{and} & 7x^4 + 3x + 2 & = & x^4\left(7 + \frac{3}{x^3} + \frac{2}{x^4}\right) \\ \text{so that} & \frac{2x^3-5x^2+6x+5}{7x^4+3x+2} & = & \frac{x^3\left(2-\frac{5}{x}+\frac{6}{x^2}+\frac{5}{x^3}\right)}{x^4\left(7+\frac{3}{x^3}+\frac{2}{x^4}\right)} \\ & & = & \frac{1}{x}\,\frac{2-\frac{5}{x}+\frac{6}{x^2}+\frac{5}{x^3}}{7+\frac{3}{x^3}+\frac{2}{x^4}}. \end{array}$$

As x gets arbitrarily large, $\frac{1}{x}$ approaches 0, $2 - \frac{5}{x} + \frac{6}{x^2} + \frac{5}{x^3}$ approaches 2, and $7 + \frac{3}{x^3} + \frac{2}{x^4}$ approaches 7. Thus,

$$\lim_{x\to\infty} \frac{2x^3 - 5x^2 + 6x + 5}{7x^4 + 3x + 2} = 0.$$

◇

As these two examples suggest, the limit of a quotient of two polynomials, $\frac{P(x)}{Q(x)}$, is completely determined by the limit of the quotient of the highest degree term in $P(x)$ and in $Q(x)$.

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

where a_n and b_m are not 0. Then

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}.$$

In particular, if $m = n$, the limit is a_n/b_m. If $m > n$, the limit is 0. If $n > m$, the limit is infinite, either ∞ or $-\infty$, depending on the signs of a_n and b_n.

Summary

This section introduces the concept of a limit and notations for the various types of limits. One-sided limits are the foundation for the two-sided limit as well as for infinite limits and limits at infinity.

It is important to keep in mind that when deciding whether $\lim_{x \to a} f(x)$ exists, you never consider $f(a)$. Perhaps a isn't even in the domain of the function. Even if a is in the domain, the value $f(a)$ plays no role in deciding whether $\lim_{x \to a} f(x)$ exists.

EXERCISES for Section 2.3 *Key:* R–routine
M–moderate, C–challenging

In Exercises 1 to 8 the limits exist. Find them.

1.[R] $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

2.[R] $\lim_{x \to 4} \frac{x^2 - 9}{x - 3}$

3.[R] $\lim_{x \to 0} \frac{\sin(x)}{x}$

4.[R] $\lim_{x \to \frac{\pi}{2}} \frac{\sin(x)}{x}$

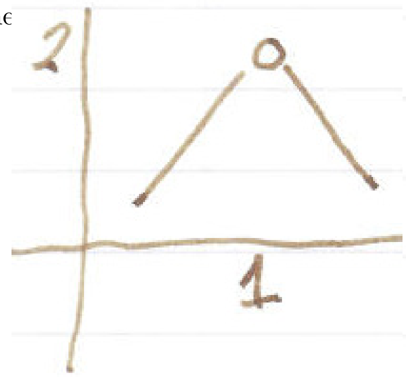
5.[R] $\lim_{x \to 0} \frac{e^x - 1}{2x}$

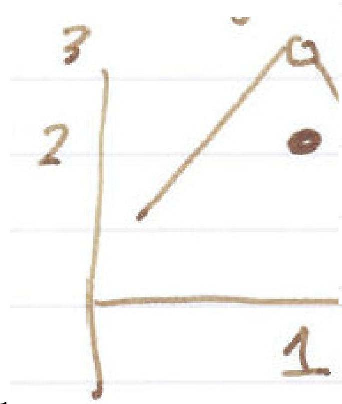
6.[R] $\lim_{x \to 2} \frac{e^x - 1}{2x}$

7.[R] $\lim_{x \to 0} \frac{1 - \cos(x)}{3x}$

8.[R] $\lim_{x \to \pi} \frac{1 - \cos(x)}{3x}$

In Exercises 9 to 12 the graph of a function $y = f(x)$ is given. Decide whether $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 1^-} f(x)$, and $\lim_{x \to 1} f(x)$ exist. If they do exist, give their values.





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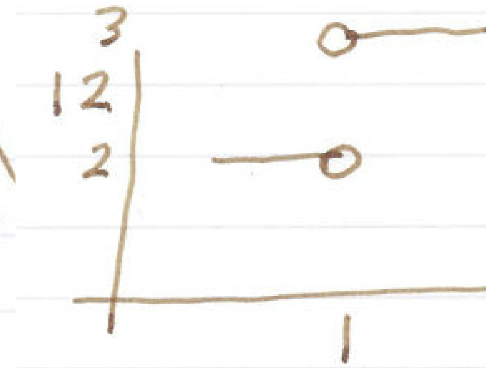


ARTIST: Please remove the "12" from the vertical axis in Figure 12

13.[R]

(a) Sketch the graph of $y = \log_2(x)$.

(b) What are $\lim_{x \to \infty} \log_2(x)$, $\lim_{x \to 4} \log_2(x)$, and $\lim_{x \to 0^+} \log_2(x)$?

14.[R]

(a) Sketch the graph of $y = 2^x$.

(b) What are $\lim_{x\to\infty} 2^x$, $\lim_{x\to 4} 2^x$, and $\lim_{x\to-\infty} 2^x$?

15.[R] Find $\lim_{x\to a} \frac{x^3-8}{x-2}$ for $a = 1$, 2, and 3.

16.[R] Find $\lim_{x\to a} \frac{x^4-16}{x-2}$ for $a = 1$, 2, and 3.

17.[R] Examine $\lim_{x\to a} \frac{e^x-1}{x-2}$ for $a = -1$, 0, 1, and 2.

18.[R] Find $\lim_{x\to a} \frac{\sin(x)}{x}$ for $a = \frac{\pi}{6}$, $\frac{\pi}{4}$, and 0.

In Exercises 19 to 24, find the given limit (if it exists).

19.[R] $\lim_{x\to\infty} 2^{-x}\sin(x)$

20.[R] $\lim_{x\to\infty} 3^{-x}\cos(2x)$

21.[R] $\lim_{x\to\infty} \frac{3x^5+2x^2-1}{6x^5+x^4+2}$

22.[R] $\lim_{x\to\infty} \frac{13x^5+2x^2+1}{2x^6+x+5}$

23.[R] $\lim_{x\to\infty} \frac{10x^6+x^5+x+1}{x^6}$

24.[R] $\lim_{x\to\infty} \frac{25x^5+x^2+1}{x^3+x+2}$

In Exercises 25 to 27, information is given about functions f and g. In each case decide whether the limit asked for can be determined on the basis of that information. If it can, give its value. If it cannot, show by specific choices of f and g that it cannot.

25.[M] Given that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 1$, discuss

(a) $\lim_{x\to\infty} (f(x)+g(x))$

(b) $\lim_{x\to\infty} (f(x)/g(x))$

(c) $\lim_{x\to\infty} (f(x)g(x))$

(d) $\lim_{x\to\infty} (g(x)/f(x))$

(e) $\lim_{x\to\infty} (g(x)/|f(x)|)$

26.[M] Given that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, discuss

(a) $\lim_{x\to\infty} (f(x)+g(x))$

(b) $\lim_{x\to\infty} (f(x)-g(x))$

(c) $\lim_{x\to\infty} (f(x)g(x))$

(d) $\lim_{x\to\infty} (g(x)/f(x))$

27.[M] Given that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} g(x) = \infty$, discuss

(a) $\lim_{x\to\infty} (f(x)/g(x))$

(b) $\lim_{x\to\infty} (f(x)g(x))$

(c) $\lim_{x\to\infty} (f(x)-1)g(x)$

28.[M] Let $f(x) = \cos(1/x)$.

(a) What is the domain of f?

(b) Does $\lim_{x\to 0} \cos(1/x)$ exist?

(c) Graph $f(x) = \cos(1/x)$.

29.[M] Let $f(x) = x\sin(1/x)$.

(a) What is the domain of f?

(b) Graph the lines $y = x$ and $y = -x$.

(c) For which x does $f(x) = x$? When does $f(x) = -x$? (Notice that the graph of $y = f(x)$ goes back and forth between these lines.)

(d) Does $\lim_{x\to 0} f(x)$ exist? If so, what is it?

(e) Does $\lim_{x\to\infty} f(x)$ exist? If so, what is it?

(f) Graph $y = f(x)$.

30.[M] Let $f(x) = \frac{|x|}{x}$, which is defined except at $x = 0$.

(a) What is $f(3)$?

(b) What is $f(-2)$?

(c) Graph $y = f(x)$.

(d) Does $\lim_{x \to 0^+} f(x)$ exist? If so, what is it?

(e) Does $\lim_{x \to 0^-} f(x)$ exist? If so, what is it?

(f) Does $\lim_{x \to 0} f(x)$ exist? If so, what is it?

In Exercises 31 to 33, find $\lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ for the following functions.

31.[M] $f(x) = 5x$

32.[M] $f(x) = x^2$

33.[M] $f(x) = e^x$

34.[M] Figure 2.3.8 shows a circle of radius a. Find

(a) $\lim_{\theta \to 0^+} \dfrac{\overline{AB}}{\widehat{CB}}$ NOTE: $\widehat{CB}$ is the length of the arc of the circle with radius a.

(b) $\lim_{\theta \to 0^+} \dfrac{\overline{AB}}{\overline{CD}}$

(c) $\lim_{\theta \to 0} \dfrac{\text{area of } ABC}{\text{area of } ABCD}$.

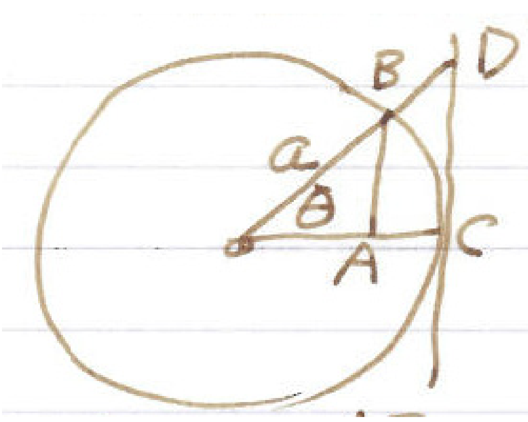


Figure 2.3.8: Exercise 34

35.[M] Let $f(x)$ be the diameter of the largest circle that fits in a 1 by x rectangle.

(a) Find a formula for $f(x)$.

(b) Graph $y = f(x)$.

(c) Does $\lim_{x\to 1} f(x)$ exist?

36.[M] I am thinking of two numbers near 0. What, if anything, can you say about their

(a) product?

(b) quotient?

(c) difference?

(d) sum?

37.[M] I am thinking about two large positive numbers. What, if anything, can you say about their

(a) product?

(b) quotient?

(c) difference?

(d) sum?

38.[C] Find $\lim_{h\to 0} \dfrac{f(\theta+h)-f(\theta)}{h}$ for $f(x) = \sin(x)$. HINT: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.

39.[C] Find $\lim_{h\to 0} \dfrac{f(\theta+h)-f(\theta)}{h}$ for $f(x) = \cos(x)$. HINT: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

40.[C] Find $\lim_{x\to 0} \dfrac{e^{2x}-1}{x}$.

41.[C] Sam and Jane are discussing

$$f(x) = \frac{3x^2 + 2x}{x + 5}.$$

Sam: For large x, $2x$ is small in comparison to $3x^2$, and 5 is small in comparison to x. So the quotient $\frac{3x^2+2x}{x+5}$ behaves like $\frac{3x^2}{x} = 3x$. Hence, the graph of $y = f(x)$ is very close to the graph of the line $y = 3x$ when x is large.

Jane: "Nonsense. After all,

$$\frac{3x^2 + 2x}{x + 5} = \frac{3x + 2}{1 + (5/x)}$$

which clearly behaves like $3x+2$ for large x. Thus the graph of $y = f(x)$ stays very close to the line $y = 3x + 2$ when x is large.

Settle the argument.

42.[C] Sam, Jane, and Wilber are arguing about limits in a case where $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = \infty$.

Sam: $\lim_{x\to\infty} f(x)g(x) = 0$, since $f(x)$ is going toward 0.

Jane: Rubbish! Since $g(x)$ gets large, it will turn out that $\lim_{x\to\infty} f(x)g(x) = \infty$.

Wilber: You're both wrong. The two influences will balance out and you will see that $\lim_{x\to\infty} f(x)g(x)$ is near 1.

Settle the argument.

43.[C] Sam and Jane are arguing about limits in a case where $f(x) \geq 1$ for $x > 0$, $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0} g(x) = \infty$. What can be said about $\lim_{x\to 0^+} f(x)^{g(x)}$?

Sam: That's easy. Multiply a bunch of numbers near 1 and you get a number near 1. So the limit will be 1.

Jane: Rubbish! Since $f(x)$ may be bigger than 1 and you are multiplying it lots of times, you will get a really large number. There's no doubt in my mind: $\lim_{x\to 0} f(x)^{g(x)} = \infty$.

Settle the argument.

44.[C] An urn contains n marbles. One is green and the remaining $n-1$ are red. When picking one marble at random without looking, the probability is $1/n$ of getting the green marble, and $(n-1)/n$ of getting a red marble. If you do this experiment n times, each time putting the chosen marble back, the probability of <u>not</u> getting the green marble on any of the n experiments is $((n-1)/n)^n$.

(a) Let $p(n) = \left(\frac{n-1}{n}\right)^n$. Comp
$p(4)$ to at least three decima
of the decimal point).

(b) Show that as $n \to \infty$, $p(n)$ a
rocal of $\lim_{x\to 0} (1+x)^{1/x}$.

2.4 Continuous Functions

This section introduces the notion of a continuous function. While almost all functions met in practice are continuous, we must always remain alert that a function might not be continuous. We begin with an informal description and then give a more useful working definition.

An Informal Introduction to Continuous Functions

When we draw the graph of a function defined on some interval, we usually do not have to lift the pencil off the paper. Figure 2.4.1 shows this typical situation.

A function is said to be **continuous** if, when considered on any interval in its domain, its graph can he traced without lifting the pencil off the paper. (The domain may consist of several intervals.) According to this definition any polynomial is continuous. So is each of the basic trigonometric functions, including $y = \tan(x)$, whose graph is shown in Figure 2.3.6 of Section 2.3.

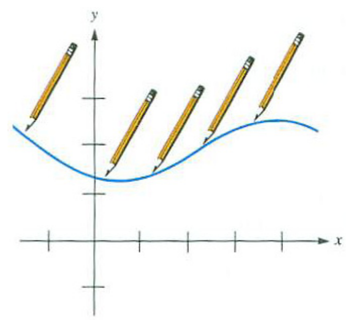


Figure 2.4.1:

You may be tempted to say "But $\tan(x)$ blows up at $x = \pi/2$ and I have to lift my pencil off the paper to draw the graph." However, $x = \pi/2$ is not in the domain of the tangent function. *On every interval in its domain,* $\tan(x)$ *behaves quite decently;* on such an interval we can sketch its graph without lifting the pencil from the paper. That is why $\tan(x)$ is continuous. The function $1/x$ is also continuous, since it "explodes" only at a number not in its domain, namely at $x = 0$. The function whose graph is shown in Figure 2.4.2 is not continuous. It is defined throughout the interval $[-2, 3]$, but to draw its graph you must lift the pencil from the paper near $x = 1$. However, when you consider the function *only for x in $[1, 3]$, then it is continuous.* By the way, a formula for the piecewise-defined function given graphically in Figure 2.4.2 is:

$$f(x) = \begin{cases} x + 1 & \text{for } x \text{ in } [-2, 1) \\ x & \text{for } x \text{ in } [1, 2) \\ -x + 4 & \text{for } x \text{ in } [2, 3]. \end{cases}$$

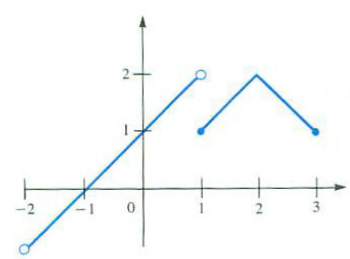


Figure 2.4.2:

It is pieced together from three different continuous functions.

The Definition of Continuity

Our informal "moving pencil" notion of a continuous function requires drawing a graph of the function. Our working definition does not require such a graph. Moreover, it easily generalizes to functions of more than one variable in later chapters.

To get the feeling of this second definition, imagine that you had the information shown in the table in the margin about some function f. What would

x	$f(x)$
0.9	2.93
0.99	2.9954
0.999	2.9999997

you expect the output $f(1)$ to be?

It would be quite a shock to be told that $f(1)$ is, say, 625. A reasonable function should present no such surprise. The expectation is that $f(1)$ will be 3. More generally, we expect the output of a function at the input a to be closely connected with the outputs of the function at inputs near a. The functions of interest in calculus usually behave that way. In short, "What you expect is what you get." With this in mind, we define the notion of *continuity at a number* a. We first assume that the domain of f contains an open interval around a.

> **DEFINITION** *(Continuity at a number a)* Assume that $f(x)$ is defined in some open interval that contains the number a. Then the function f is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$. This means that
>
> 1. $f(a)$ is defined (that is, a is in the domain of f).
> 2. $\lim_{x\to a} f(x)$ exists.
> 3. $\lim_{x\to a} f(x)$ equals $f(a)$.

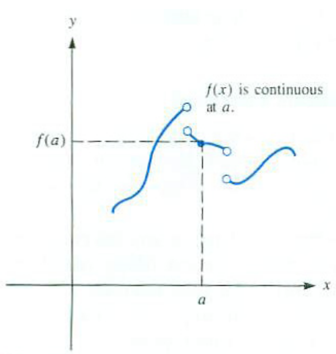


Figure 2.4.3:

As Figure 2.4.3 shows, whether a function is continuous at a depends on its behavior both at a and at inputs near a. Being continuous at a is a local matter, involving perhaps very tiny intervals about a.

To check whether a function f is continuous at a number a, we ask three questions:

Question 1: Is a in the domain of f?

Question 2: Does $\lim_{x\to a} f(x)$ exist?

Question 3: Does $f(a)$ equal $\lim_{x\to a} f(x)$?

If the answer is "yes" to each of these questions, we say that f is continuous at a.

If a is in the domain of f and the answer to Question 2 or to Question 3 is "no," then f is said to be **discontinuous at** a. If a is not in the domain of f, we do not speak of it being continuous or discontinuous there.

We are now ready to define a continuous function.

> **DEFINITION** *(Continuous function)* Let f be a function whose domain is the x-axis or is made up of open intervals. Then f is a **continuous function** if it is continuous at each number a in its domain. A function that is not continuous is called a **discontinuous function**.

EXAMPLE 1 Use the definition of continuity to decide whether $f(x) = 1/x$ is continuous.

SOLUTION This function f is continuous at every point a for which the answers to Questions 1, 2, and 3 are all "yes".

If a is not 0, it is in the domain of f. So, for a not 0, the answer to Question 1 is "yes." Since

$$\lim_{x \to a} \frac{1}{x} = \frac{1}{a},$$

the answer to Question 2 is "yes." Because

$$f(a) = \frac{1}{a},$$

the answer to Question 3 is also "yes." Thus $f(x) = 1/x$ is continuous at every number in its domain. Hence f is a continuous function. Note that the conclusion agrees with the "moving pencil" picture of continuity. ◇

Not every important function is continuous. Let $f(x)$ be the greatest integer that is less than or equal to x. For instance, $f(1.8) = 1$, $f(1.9) = 1$, $f(2) = 2$, and $f(2.3) = 2$. This function is often used in number theory and computer science, where it is denoted $[x]$ or $\lfloor x \rfloor$ and called the **floor** of x. People use the floor function every time they answer the question, "How old are you?" The next example examines where the floor function fails to be continuous.

EXAMPLE 2 Let f be the floor function, $f(x) = \lfloor x \rfloor$. Graph f and find where it is continuous. Is f a continuous function?

SOLUTION We begin with the following table to show the behavior of $f(x)$ for x near 1 or 2.

x	0	0.5	0.8	1	1.1	1.99	2	2.01
$\lfloor x \rfloor$	0	0	0	1	1	1	2	2

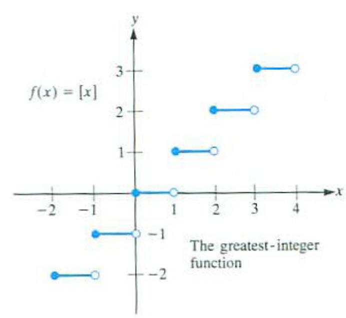


Figure 2.4.4:

For $0 \le x < 1$, $\lfloor x \rfloor = 0$. But at the input $x = 1$ the output jumps to 1 since $\lfloor 1 \rfloor = 1$. For $1 \le x < 2$, $\lfloor x \rfloor$ remains at 1. Then at 2 it jumps to 2. More generally, $\lfloor x \rfloor$ has a jump at every integer, as shown in Figure 2.4.4.

Let us show that f is not continuous at $a = 2$ by seeing which of the three conditions in the definition are not satisfied. First of all, Question 1 is answered "yes" since 2 lies in the domain of the function; indeed, $f(2) = 2$.

What is the answer to Question 2? Does $\lim_{x \to 2} f(x)$ exist? We see that

$$\lim_{x \to 2^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = 2.$$

Since the left-hand and right-hand limits are not equal, $\lim_{x\to 2} f(x)$ does not exist. Question 2 is answered "no."

Already we know that the function is not continuous at $a = 2$. Since the limit does not exist there is no point in considering Question 3. Because there is one point in the domain where $\lfloor x \rfloor$ is not continuous, this is a discontinuous function. More specifically, the floor function is discontinuous at a whenever a is an integer.

Is f continuous at a if a is not an integer? Let us take the case $a = 1.5$, for instance.

Question 1 is answered "yes," because $f(1.5)$ is defined.
(In fact, $f(1.5) = 1$.)

Question 2 is answered "yes," since $\lim_{x\to 1.5} f(x) = 1$.

Question 3 is answered "yes," since $\lim_{x\to 1.5} f(x) = f(1.5)$.
(Both values are 1.)

The floor function is continuous at $a = 1.5$. Similarly, f is continuous at every number that is not an integer.

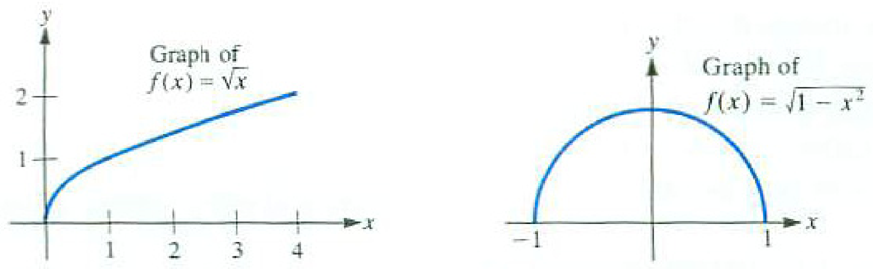
Note that $\lfloor x \rfloor$ is continuous on any interval that does not include an integer. For instance, if we consider the function only on the interval $(1.1, 1.9)$, it is continuous there. ◇

Continuity at an Endpoint

The functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x^2}$ are graphed in Figures 2.4.5(a) and (b), respectively. We would like to call both of these functions continuous. However, there is a slight technical problem. The number 0 is in the domain of f, but there is no open interval around 0 that lies completely in the domain, as our definition of continuity requires. Since $f(x) = \sqrt{x}$ is not defined for x to the left of 0, we are not interested in numbers x to the left of 0. Similarly, $g(x) = \sqrt{1-x^2}$ is defined only when $1 - x^2 \geq 0$, that is, for $-1 \leq x \leq 1$. To cover this type of situation we utilize one-sided limits to define **one-sided continuity**.

DEFINITION *(Continuity from the right at a number.)* Assume that $f(x)$ is defined in some closed interval $[a, c]$. Then the function f **is continuous from the right at** a if

1. $f(a)$ is defined
2. $\lim_{x\to a^+} f(x)$ exists
3. $\lim_{x\to a^+} f(x)$ equals $f(a)$



(a) (b)

Figure 2.4.5:

Figure 2.4.6 illustrates this definition, which also takes care of the continuity of $g(x) = \sqrt{1-x^2}$ at -1 in Figure 2.4.5(b). The next definition takes care of the right-hand endpoints.

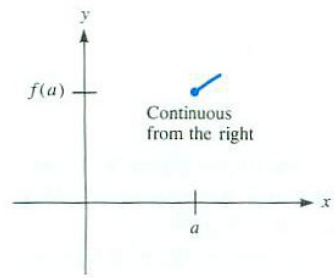


Figure 2.4.6:

DEFINITION *(Continuity from the left at a number a.)* Assume that $f(x)$ is defined in some closed interval $[b, a]$. Then the function f **is continuous from the left at** a if

1. $f(a)$ is defined
2. $\lim_{x \to a^-} f(x)$ exists
3. $\lim_{x \to a^-} f(x)$ equals $f(a)$

Figure 2.4.7 illustrates this definition.

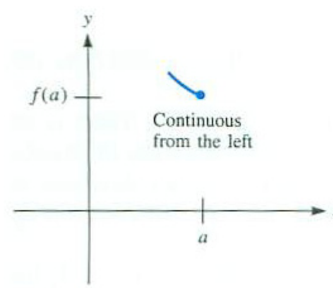


Figure 2.4.7:

With these two extra definitions to cover some special cases in the domain, we can extend the definition of continuous function to include those functions whose domains may contain endpoints. We say, for instance, that $\sqrt{1-x^2}$ is continuous because it is continuous at any number in $(-1, 1)$, is continuous from the right at -1, and continuous from the left at 1.

These special considerations are minor matters that will little concern us in the future. The key point is that $\sqrt{1-x^2}$ and $\sqrt{x}$ are both continuous functions. So are practically all the functions studied in calculus.

The following example reviews the notion of continuity.

EXAMPLE 3 Figure 2.4.8 is the graph of a certain (piecewise-defined) function $f(x)$ whose domain is the interval $(-2, 6]$. Discuss the continuity of $f(x)$ at (a) 6, (b) 4, (c) 3, (d) 2, (e) 1, and (f) -2.

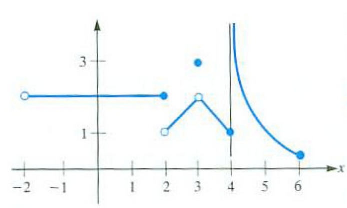


Figure 2.4.8:

SOLUTION

(a) Since $\lim\limits_{x \to 6-} f(x)$ exists and equals $f(6)$, f is continuous from the left at 6.

(b) Since $\lim_{x\to 4} f(x)$ does not exist, f is not continuous at 4.

(c) Inspection of the graph shows that $\lim_{x\to 3} f(x) = 2$. However, Question 3 is answered "no" because $f(3) = 3$, which is *not* equal to $\lim_{x\to 3} f(x)$. Thus f is not continuous at 3.

(d) Though $\lim_{x\to 2-} f(x)$ and $\lim_{x\to 2+} f(x)$ both exist, they are not equal. (The left-hand limit is 2; the right-hand limit is 1.) Thus $\lim_{x\to 2} f(x)$ does not exist, the answer to Question 2 is "no," and f is discontinuous at $x = 2$.

(e) At 1, "yes" is the answer to all three questions: $f(1)$ is defined, $\lim_{x\to 1} f(x)$ exists (it equals 2) and, finally, it equals $f(1)$. f is continuous at $x = 1$.

(f) Since -2 is not even in the domain of this function, we do not speak of continuity or discontinuity of f at -2.

$\diamond$

As Example 3 shows, a function can fail to be continuous at a given number a in its domain for either of two reasons:

1. $\lim_{x\to a} f(x)$ might not exist
2. when, $\lim_{x\to a} f(x)$ does exist, $f(a)$ might not be equal to that limit.

Continuity and Limits

Some limits are so easy that you can find them without any work; for instance, $\lim_{x\to 2} 5^x = 5^2 = 25$. Others offer a challenge; for instance, $\lim_{x\to 2} \frac{x^3-2^3}{x-2}$.

If you want to find $\lim_{x\to a} f(x)$, and you know f is a continuous function with a in its domain, then you just calculate $f(a)$. In such a case there is no challenge and the limit is called *determinate*.

The interesting case for finding $\lim_{x\to a} f(x)$ occurs when f is not defined at a. That is when you must consider the influences operating on $f(x)$ when x is near a. You may have to do some algebra or computations. Such limits are called *indeterminate*.

The four limits encountered in Section 2.2, $\lim_{x\to a} \dfrac{x^n - a^n}{x - a}$, $\lim_{x\to 0} \dfrac{b^x - 1}{x}$, $\lim_{x\to 0} \dfrac{\sin(x)}{x}$, and $\lim_{x\to 0} \dfrac{1-\cos(x)}{x}$ all required some work to find their value. These types of limits will be discussed in detail in Section 5.5.

We list the properties of limits which are helpful in computing limits.

Each of these properties remains valid when the two-sided limit is replaced with a one-sided limit.

Theorem 2.4.1 (Properties of Limits)**.** *Let g and h be two functions and assume that $\lim_{x\to a} g(x) = A$ and $\lim_{x\to a} h(x) = B$. Then*

Sum $\lim_{x\to a}(g(x)+h(x)) = \lim_{x\to a} g(x) + \lim_{x\to a} h(x) = A + B$
the limit of the sum is the sum of the limits

Difference $\lim_{x\to a}(g(x)-h(x)) = \lim_{x\to a} g(x) - \lim_{x\to a} h(x) = A - B$
the limit of the difference is the difference of the limits

Product $\lim_{x\to a}(g(x)h(x)) = \left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a} h(x)\right) = AB$
the limit of the product is the product of the limits

Constant Multiple $\lim_{x\to a}(kg(x)) = k\left(\lim_{x\to a} g(x)\right) = kA$, *for any constant* k
special case of Product

Quotient $\lim_{x\to a}\left(\frac{g(x)}{h(x)}\right) = \frac{(\lim_{x\to a} g(x))}{(\lim_{x\to a} h(x))} = \frac{A}{B}$, *provided* $B \neq 0$
the limit of the quotient is the quotient of the limits, provided the denominator is not 0

Power $\lim_{x\to a}\left(g(x)^{h(x)}\right) = \left(\lim_{x\to a} g(x)\right)^{(\lim_{x\to a} h(x))} = A^B$, *provided* $A > 0$
the limit of a varying base to a varying power

EXAMPLE 4 Find $\lim_{x\to 0} \frac{(x^4-16)\sin(5x)}{x^2-2x}$.

SOLUTION Notice that the denominator can be factored to obtain

$$\frac{(x^4-16)\sin(5x)}{x^2-2x} = \frac{x^4-2^4}{x-2}\cdot\frac{\sin(5x)}{x}.$$

This allows the limit to be rewritten as

$$\lim_{x\to 0}\frac{x^4-2^4}{x-2}\cdot\lim_{x\to 0}\frac{\sin(5x)}{x}$$

where we have also used $16 = 2^4$. Now, $\lim_{x\to 0}\frac{x^4-2^4}{x-2} = 4\cdot 2^{4-1} = 32$. Also,

$$\lim_{x\to 0}\frac{\sin(5x)}{x} = \lim_{x\to 0} 5\frac{\sin(5x)}{5x} = 5\lim_{x\to 0}\frac{\sin(5x)}{5x} = 5\cdot 1 = 5.$$

We conclude that

$$\lim_{x\to 0}\frac{(x^4-16)\sin(5x)}{x^2-2x} = \lim_{x\to 0}\frac{x^4-2^4}{x-2}\cdot\lim_{x\to 0}\frac{\sin(5x)}{5x} = 32\cdot 5 = 160.$$

◇

Summary

This section opened with an informal view of continuous functions, expressed in terms of a moving pencil. It then gave the definition, phrased in terms of limits, which we will use throughout the text.

The development concludes in the next section, which describes three important properties of continuous functions.

EXERCISES for Section 2.4 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 12, which of these limits can be found at a glance and which require some analysis? That is, decide in each case whether the limit is determinate or indeterminate. Do not evaluate the limit.

1.[R] $\lim_{x\to 0}(2^x - 1)$

2.[R] $\lim_{x\to\infty}\left(\left(\frac{1}{2}\right)2^x - 1\right)$

3.[R] $\lim_{x\to 1}\frac{3^x - 1}{2^x - 1}$

4.[R] $\lim_{x\to 2}\frac{3^x - 1}{2^x - 1}$

5.[R] $\lim_{x\to\infty}\frac{x}{2^x}$

6.[R] $\lim_{x\to 0}\frac{x}{2^x}$

7.[R] $\lim_{x\to 0^+}\frac{x^2}{e^x - 1}$

8.[R] $\lim_{x\to\frac{\pi}{2}^-}(\sin(x))^{\tan(x)}$

9.[R] $\lim_{x\to 0^+} x\log_2(x)$

10.[R] $\lim_{x\to 0^+}(2+x)^{3/x}$

11.[R] $\lim_{x\to\infty}(2+x)^{3/x}$

12.[R] $\lim_{x\to 0^-}\frac{(2+x)^3}{x}$

In Exercises 13 to 16, evaluate the limit.

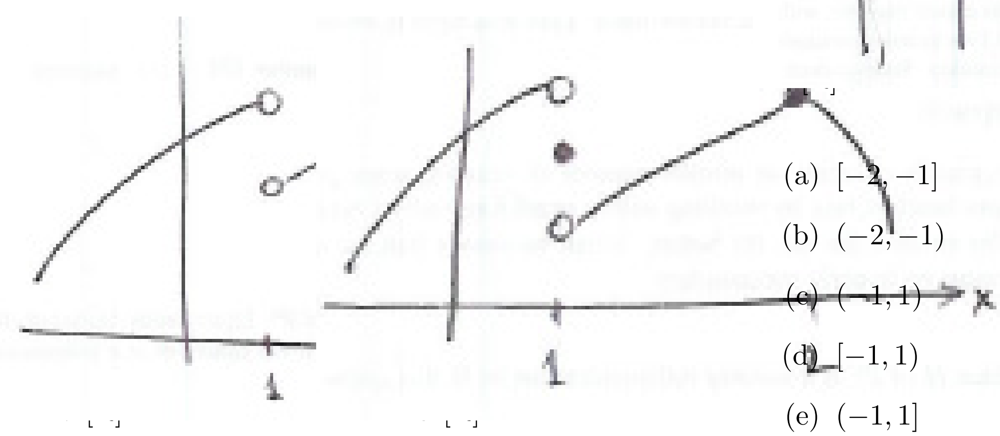
13.[R] $\lim_{x\to\frac{\pi}{2}}\sin(x)\frac{e^x - 1}{x}$

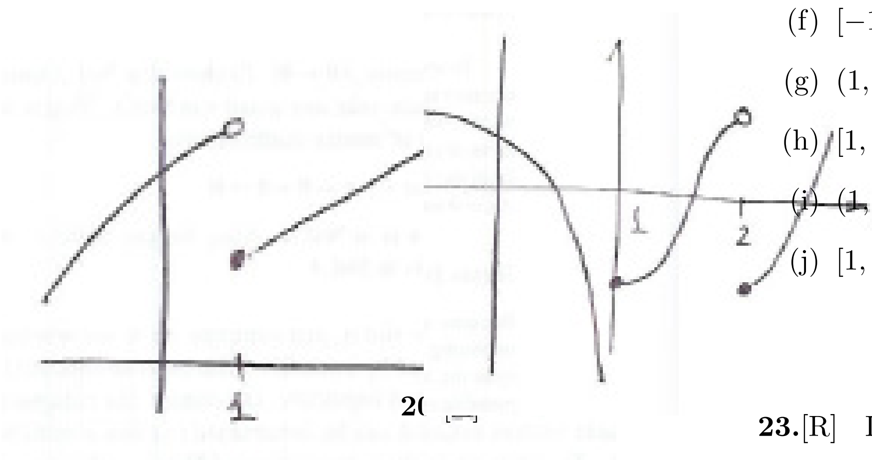
14.[R] $\lim_{x\to 0}\frac{\cos(x)\,(e^x - 1)}{x}$

15.[R] $\lim_{x\to 0}\frac{\sin(2x)}{x(\cos(3x))^2}$

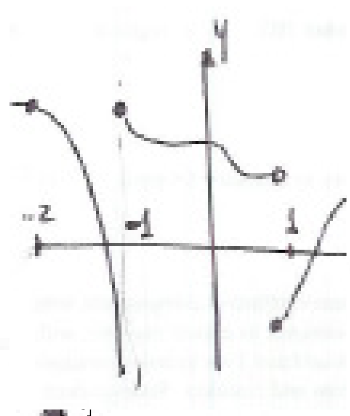
16.[R] $\lim_{x\to 1}\frac{(x-1)\cos(x)}{x^3 - 1}$

In Exercises 17 to 20 the graph of a function $y = f(x)$ is given. Determine all numbers c for which $\lim_{x\to c} f(x)$ does not exist.

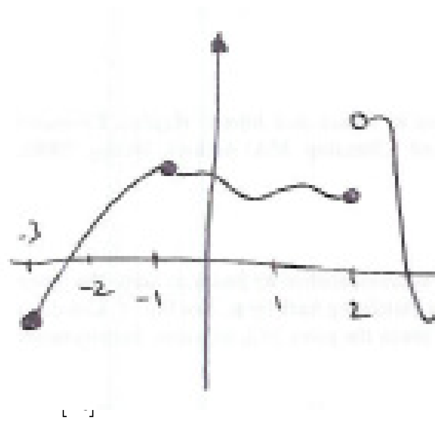




In Exercises 21 and 22 the graph of a function $y = f(x)$ and several intervals are given. For each interval, decide if the function is continuous on that interval.



(a) $[-2,-1]$
(b) $(-2,-1)$
(c) $(-1,1)$
(d) $[-1,1)$
(e) $(-1,1]$
(f) $[-1,1]$
(g) $(1,2)$
(h) $[1,2)$
(i) $(1,2]$
(j) $[1,2]$



(a) $[-3,2]$
(b) $(-1,3)$
(c) $(-1,2)$
(d) $[-1,2)$
(e) $(-1,2]$
(f) $[-1,2]$
(g) $(2,3)$
(h) $[2,3)$
(i) $(2,3]$
(j) $[2,3]$

23.[R] Let $f(x) = x + |x|$.

(a) Graph f.

(b) Is f continuous at -1?

(c) Is f continuous at 0?

24.[M] Let $f(x) = 2^{1/x}$ for $x \neq 0$.

(a) Find $\lim_{x\to\infty} f(x)$.

(b) Find $\lim_{x\to-\infty} f(x)$.

(c) Does $\lim_{x\to 0^+} f(x)$ exist?

(d) Does $\lim_{x\to 0^-} f(x)$ exist?

(e) Graph f, incorporating the information from parts *(a)* to *(d)*.

(f) Is it possible to define $f(0)$ in such a way that f is continuous throughout the x-axis?

25.[M] Let $f(x) = x\sin(1/x)$ for $x \neq 0$.

(a) Find $\lim_{x\to\infty} f(x)$.

(b) Find $\lim_{x\to-\infty} f(x)$.

(c) Find $\lim_{x\to 0} f(x)$.

(d) Is it possible to define $f(0)$ in such a way that f is continuous throughout the x-axis?

(e) Sketch the graph of f.

In Exercises 26 to 28 find equations that the numbers k, p, and/or m must satisfy to make each function continuous.

26.[M] $f(x) = \begin{cases} \frac{\sin(x)}{2x} & x \neq 0 \\ p & x = 0 \end{cases}$

27.[M] $f(x) = \begin{cases} k & x \leq 0 \\ \arcsin(x) & 0 < x \leq \frac{\pi}{2} \\ p & x > \frac{\pi}{2} \end{cases}$

28.[M] $f(x) = \begin{cases} \ln(x) & x > 1 \\ k - m\sqrt{x} & 0 < x \leq 1 \\ pe^{-x} & x \leq 0 \end{cases}$

29.[M]

(a) Let f and g be two functions defined for all numbers. If $f(x) = g(x)$ when x is not 3, must $f(3) = g(3)$?

(b) Let f and g be two continuous functions defined for all numbers. If $f(x) = g(x)$ when x is not 3, must $f(3) = g(3)$?

Explain your answers.

30.[C] *The reason* 0^0 *is not de* hoped that if the positive number x are both close to 0, then b^x mig fixed number. If that were so, it w inition for 0^0. Experiment with and x near 0 and on the basis of paragraph on the theme, "Why 0^0

2.5 Three Important Properties of Continuous Functions

Continuous functions have three properties important in calculus: the "extreme-value" property, the "intermediate-value" property, and the "permanence" property. All three are quite plausible, and a glance at the graph of a typical continuous function may persuade us that they are obvious. No proofs will be offered: they depend on the precise definitions of limits given in Sections 3.8 and 3.9 and are part of an advanced calculus course.

We will say that a function has a **local or relative maximum** at a point $(c, f(c))$ when $f(c) \geq f(x)$ for x near c. More precisely, there is an open interval I containing c such that if x is in I, and $f(x)$ is defined, then $f(x) \leq f(c)$. Likewise, a function has a **local or relative minimum** at a point $(c, f(c))$ when $f(c) \leq f(x)$ for x near c. Each maximum or minimum is referred to as an **extreme value** or **extremum** of the function.

The plural of extremum is extrema.

Extreme-Value Property

The first property is that a function continuous throughout the closed interval $[a, b]$ takes on a largest value somewhere in the interval.

Theorem (Maximum-Value Property). *Let f be continuous throughout a closed interval $[a, b]$. Then there is at least one number in $[a, b]$ at which f takes on a maximum value. That is, for some number c in $[a, b]$, $f(c) \geq f(x)$ for all x in $[a, b]$.*

To persuade yourself that this is plausible, imagine sketching the graph of a continuous function. (See Figure 2.5.1.)

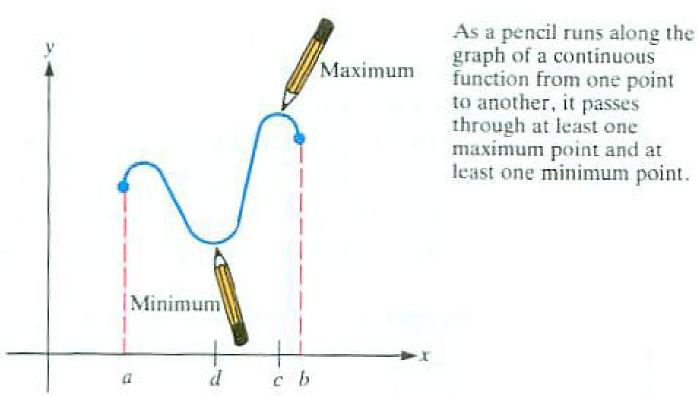


Figure 2.5.1:

The maximum-value property guarantees that a maximum value exists, but it does *not tell how* to find it. The problem of finding it is addressed in Chapter 4.

There is also a **minimum-value property** that states that every continuous function on a closed interval takes on a smallest value somewhere in this interval. See Figure 2.5.1 for an illustration of this property. Combining the two properties, we have:

Theorem (Extreme-Value Property). *Let f be continuous throughout the closed interval $[a, b]$. Then there is at least one number in $[a, b]$ at which f takes on a minimum value and there is at least one number in $[a, b]$ at which f takes on a maximum value. That is, for some numbers c and d in $[a, b]$, $f(d) \leq f(x) \leq f(c)$ for all x in $[a, b]$.*

EXAMPLE 1 Find all numbers in $[0, 3\pi]$ at which the cosine function, $f(x) = \cos(x)$, takes on a maximum value. Also, find all numbers in $[0, 3\pi]$ at which f takes on a minimum value.

SOLUTION Figure 2.5.2 is a graph of $f(x) = \cos(x)$ for x in $[0, 3\pi]$. Inspection of the graph shows that the maximum value of $\cos(x)$ for $0 \leq x \leq 3\pi$ is 1, and it is attained twice: when $x = 0$ and when $x = 2\pi$. The minimum value is -1, which is also attained twice: when $x = \pi$ and when $x = 3\pi$. ◇

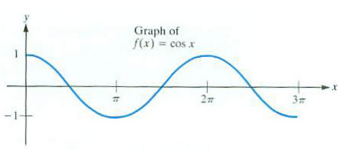


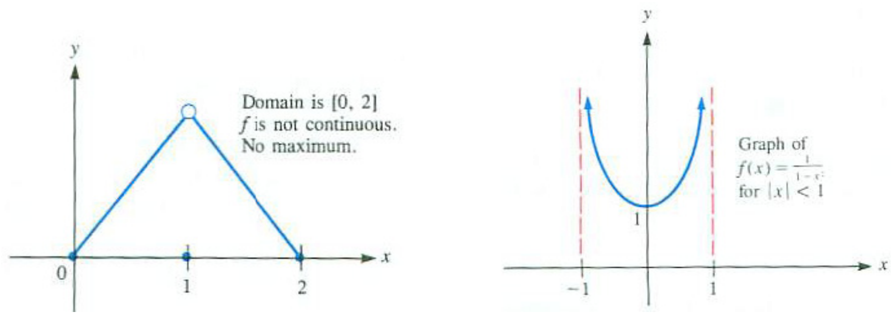
Figure 2.5.2:

The Extreme-Value Property has two assumptions: "f is continuous" and "the domain is a closed interval." If either of these conditions is removed, the conclusion need not hold.

Figure 2.5.3(a) shows the graph of a function that is *not* continuous, is defined on a closed interval, but has no maximum value. On the other hand $f(x) = \frac{1}{1-x^2}$ is continuous on $(-1, 1)$. It has no maximum value, as a glance at Figure 2.5.3(b) shows. This does not violate the Extreme-Value Property, since the domain $(-1, 1)$ is not a closed interval.

Intermediate-Value Property

Imagine graphing a continuous function f defined on the closed interval $[a, b]$. As your pencil moves from the point $(a, f(a))$ to the point $(b, f(b))$ the y-coordinate of the pencil point goes through all values between $f(a)$ and $f(b)$. (Similarly, if you hike all day, starting at an altitude of 5,000 feet and ending at 11,000 feet, you must have been, say, at 7,000 feet at least once during the day. In mathematical terms, not in terms of a pencil (or a hike), "a function that is continuous throughout an interval takes on all values between any two of its values".



(a) (b)

Figure 2.5.3:

Theorem (**Intermediate-Value Property**). *Let f be continuous throughout the closed interval $[a, b]$. Let m be any number such that $f(a) \leq m \leq f(b)$ or $f(a) \geq m \geq f(b)$. Then there is at least one number c in $[a, b]$ such that $f(c) = m$.*

Pictorially, the Intermediate-Value Property asserts that, if m is between $f(a)$ and $f(b)$, a horizontal line of height m must meet the graph of f at least once, as shown in Figure 2.5.4.

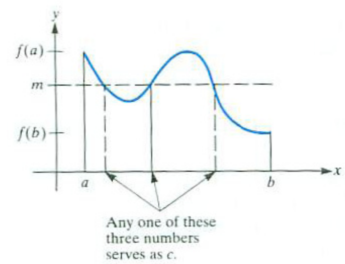


Figure 2.5.4:

Even though the property guarantees the existence of a certain number c, it does *not tell how* to find it. To find c we must be able to solve an equation, namely, the equation $f(x) = m$.

EXAMPLE 2 Use the Intermediate-Value Property to show that the equation $2x^3 + x^2 - x + 1 = 5$ has a solution in the interval $[1, 2]$.

SOLUTION Let $P(x) = 2x^3 + x^2 - x + 1$. Then

$$\begin{aligned} P(1) &= 2 \cdot 1^3 + 1^2 - 1 + 1 = 3 \\ \text{and} \qquad P(2) &= 2 \cdot 2^3 + 2^2 - 2 + 1 = 19. \end{aligned}$$

Since P is continuous (on $[1, 2]$) and $m = 5$ is between $P(1) = 3$ and $P(2) = 19$, the Intermediate-Value Property says there is at least one number c between 1 and 2 such that $P(c) = 5$.

To get a more accurate estimate for a number c such that $P(c) = 5$, find a shorter interval for which the Intermediate-Value Property can be applied. For instance, $P(1.2) = 4.696$ and $P(1.3) = 5.784$. By the Intermediate-Value Property, there is a number c in $[1.2.1.3]$ such that $P(c) = 5$. ◇

EXAMPLE 3 Show that the equation $-x^5 - 3x^2 + 2x + 11 = 0$ has at least one real root. In other words, the graph of $y = -x^5 - 3x^2 + 2x + 11$ crosses the x-axis.

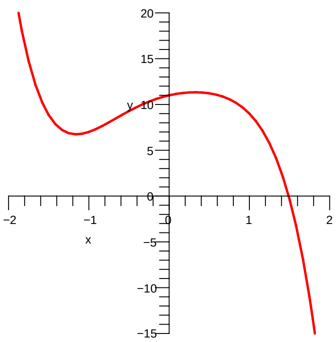


Figure 2.5.5:

SOLUTION Let $f(x) = -x^5 - 3x^2 + 2x + 11$. We wish to show that there is a number c such that $f(c) = 0$. In order to use the Intermediate-Value Property, we need an interval $[a, b]$ for which 0 is between $f(a)$ and $f(b)$, that is, one of $f(a)$ and $f(b)$ is positive and the other is negative. Then we could apply that property, using $m = 0$.

We show that there are numbers a and b with $a < b$, $f(a) > 0$ and $f(b) < 0$. Because $\lim_{x\to\infty} f(x) = -\infty$, for x large and positive, $f(x)$ is negative for x large and positive. Thus, there is a positive number b such that $f(b) < 0$. Similarly, $\lim_{x\to-\infty} f(x) = \infty$, means that when x is negative and of large absolute value, $f(x)$ is positive. So there is a negative number a such that $f(a) > 0$. Thus there are numbers a and b, with $a < b$, such that $f(a) > 0$ and $f(b) < 0$. For instance, $f(-1) = 7$ and $f(2) = -29$.

The number 0 is between $f(a)$ and $f(b)$. Since f is continuous on the interval $[a, b]$, there is a number c in $[a, b]$ such that $f(c) = 0$. (In particular there is a number c in $[-1, 2]$. This number c is a solution to the equation $-x^5 - 3x^2 + 2x + 11 = 0$. ◇

Note that the argument in Example 3 shows that any polynomial of odd degree has a real root. The argument does not hold for polynomials of even degree; the equation $x^2 + 1 = 0$, for instance, has no real solutions.

EXAMPLE 4 Use the Intermediate-Value Property to show that there is a negative number such that $\ln(x + 4) = x^2 - 3$.

SOLUTION We wish to show that there is a negative number c where the function $\ln(x + 4)$ has the same value as the function $x^2 - 3$. The equation $\ln(x+4) = x^2-3$ is equivalent to $\ln(x+4)-x^2+3 = 0$. The problem reduces to showing that the function $f(x) = \ln(x+4)-x^2+3$ has the value 0 for some input c (with $c < 0$).

We will proceed, as we did in the previous example. We want to find numbers a and b (both in $(-\infty, 0)$) such that $f(a)$ and $f(b)$ have opposite signs.

Before beginning the search for a and b, note that $\ln(x + 4)$ is defined only for $x + 4 > 0$, that is, for $x > -4$. To complete the search for a and b, make a table of values of $f(x)$ for some sample arguments in $(-4, 0)$.

x	-3	-2	-1	0
$f(x)$	-6	-0.307	3.099	4.386

We see that $f(-2)$ is negative and $f(-1)$ is positive. Since $m = 0$ lies between $f(-2)$ and $f(-1)$, and f is continuous on $[-2, -1]$, the Intermediate-Value Property asserts that there is a number in $[-2, -1]$ such that $f(c) = 0$. It follows that $\ln(c + 4) = c^2 - 3$. ◇

In Example 4 the Intermediate-Value Property does not tell what c is. The graphs of $\ln(x+1)$ and x^2-3 in Figure 2.5.6 suggest that there are two points of intersection, but only one with a negative input. The graph, and the table of values, suggest that the intersection point occurs when the input is close to -2. Calculations on a calculator or computer show that $c \approx -1.931$.

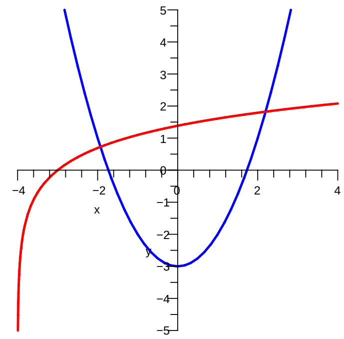


Figure 2.5.6:

Permanence Property

The extrema property as well as the intermediate-value property involve the behavior of a continuous function throughout an interval. The next property concerns the "local" behavior of a continuous function.

Consider a continuous function f on an open interval that contains the number a. Assume that $f(a) = p$ is positive. Then it seems plausible that f remains positive in some open interval that contains a. We can say something stronger:

Theorem 2.5.1 (The Permanence Property)**.** *Assume that the domain of a function f contains an open interval that includes the number a. Assume that f is continuous at a and that $f(a) = p$ is positive. Let q be any number less than p. Then there is an open interval including a such that $f(x) \geq q$ for all x in that interval.*

To persuade yourself that the permanence principle is plausible, imagine what the graph of $y = f(x)$ looks like near $(a, f(a))$, as in Figure 2.5.7.

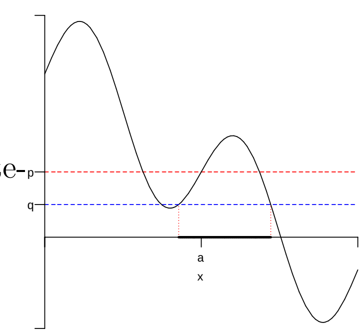


Figure 2.5.7:

Summary

This section stated, without proofs, the Extreme-Value Property, the Intermediate-Value Property, and the Permanence Property. Each will be used several times in later chapters.

EXERCISES for Section 2.5 *Key:* R–routine, M–moderate, C–challenging

1.[R] For each of the given intervals, find the maximum value of $\cos(x)$ over that interval and the value of x at which it occurs.

(a) $[0, \pi/2]$

(b) $[0, 2\pi]$

2.[R] Does the function $\frac{x^3+x^4}{1+5x^2+x^6}$ have (a) a maximum value for x in $[1, 4]$? (b) a minimum value for x in $[1, 4]$? If so, use a graphing device to determine the extreme values.

3.[R] Does the function $2^x - x^3 + x^5$ have (a) a maximum value for x in $[-3, 10]$? (b) a minimum value for x in $[-3, 10]$? If so, use a graphing device to determine the extreme values.

4.[R] Does the function x^3 have a maximum value for x in (a) $[2, 4]$? (b) $[-3, 5]$? (c) $(1, 6)$? If so, where does the maximum occur and what is the maximum value?

5.[R] Does the function x^4 have a minimum value for x in (a) $[-5, 6]$? (b) $(-2, 4)$? (c) $(3, 7)$? (d) $(-4, 4)$? If so, where does the minimum occur and what is the minimum value?

6.[R] Does the function $2 - x^2$ have (a) a maximum value for x in $(-1, 1)$? (b) a minimum value for x in $(-1, 1)$? If so, where?

7.[R] Does the function $2 + x^2$ have (a) a maximum value for x in $(-1, 1)$? (b) a minimum value for x in $(-1, 1)$? If so, where?

8.[R] Show that the equation $x^5 + 3x^4 + x - 2 = 0$ has at least one solution in the interval $[0, 1]$.

9.[R] Show that the equation $x^5 - 2x^3 + x^2 - 3x = -1$ has at least one solution in the interval $[1, 2]$.

In Exercises 10 to 14 verify the Intermediate-Value Property for the specified function f, the interval $[a, b]$, and the indicated value m. Find all c's in each case.

10.[R] $f(x) = 3x + 5$, $[a, b] = [1, 2]$, $m = 10$.

11.[R] $f(x) = x^2 - 2x$, $[a, b] = [-1, 4]$, $m = 5$.

12.[R] $f(x) = \sin(x)$, $[a, b] = [\frac{\pi}{2}, \frac{11\pi}{2}]$, $m = -1$.

13.[R] $f(x) = \cos(x)$, $[a, b] = [0, 5\pi]$, $m = \frac{\sqrt{3}}{2}$.

14.[R] $f(x) = x^3 - x$, $[a, b] = [-2, 2]$, $m = 0$.

15.[R] Use the Intermediate-Value Property to show that the equation $3x^3 + 11x^2 - 5x = 2$ has a solution.

16.[M] Show that the equation $2^x = 3x$ has a solution in the interval $[0, 1]$.

17.[M] Does the equation $x + \sin(x) = 1$ have a solution?

18.[M] Does the equation $x^3 = 2^x$ have a solution?

19.[M] Let $f(x) = 1/x$, $a = -1$, $b = 1$, $m = 0$. Note that $f(a) \leq 0 \leq f(b)$. Is there at least one c in $[a, b]$ such that $f(c) = 0$? If so, find c; if not, does this imply the Intermediate-Value Property sometimes does not hold?

20.[M] Use the Intermediate-Value Property to show that there is a positive number such that $\ln(x + 4) = x^2 + 3$.

Exercises 21 and 22 illustrate the Permanence Property.

21.[M] Let $f(x) = 5x$. Then $f(1) = 5$. Find an interval (a, b) containing 1 such that $f(x) \geq 4.9$ for all x in (a, b).

22.[M] Let $f(x) = x^2$. Then $f(2) = 4$. Find an interval (a, b) containing 2 such that $f(x) \geq 3.8$ for all x in (a, b).

23.[C] Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of odd degree n and with positive leading coefficient a_n. Show that there is at least one real number r such that $P(r) = 0$.

24.[C] (This continues Exercise 23.) The **factor theorem** from algebra asserts that the number r is a root of a polynomial $P(x)$ if and only if $x - r$ is a factor of $P(x)$. For instance, 2 is a root of the polynomial $x^2 - 3x + 2$ and $x - 2$ is a factor of it: $x^2 - 3x + 2 = (x - 2)(x - 1)$. NOTE: See also Exercise 47 in Section 8.4.

(a) Use the factor theorem and Exercise 23 to show that every polynomial of odd degree has a factor of degree 1.

(b) Show that none of the polynomials x^2+1, x^4+1, or $x^{100} + 1$ has a first-degree factor.

(c) Verify that $x^4+1 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$. (It can be shown using complex numbers that every polynomial with real coefficients is the product of polynomials with real coefficients of degrees at most 2.)

25.[C] Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ where a_n and a_0 have opposite signs.

(a) Show that the $f(x)$ has a positive root, that is, the equation $f(x) = 0$ has a positive solution.

(b) Show that if each of the roots in (a) is simple, there are an odd number of them. HINT: Use a picture. NOTE: A number c is a simple root of $f(x)$ when $x - c$ is a factor of $f(x)$ but $(x - c)^2$ is not a factor.

(c) If the roots in (a) are not simple, what would be the corresponding statement? HINT: Use a picture.

(d) What can you say about the roots of $f(x)$ if a_n and a_0 have the same sign?

Convex Sets and Curves

A set in the plane bounded by a curve is **convex** if for any two points P and Q in the set the line segment joining them also lies in the set. (See Figure 2.5.8(a).) The boundary of a convex set we will call a **convex curve**. (These ideas generalize to a solid and its boundary surface.) The notion of convexity dates back to Archimedes.

Disks, triangles, and parallelograms are convex sets. The quadrilateral shown in Figure 2.5.8(b) is not convex. Convex sets will be referred to in the following exercises and occasionally in the exercises in later chapters.

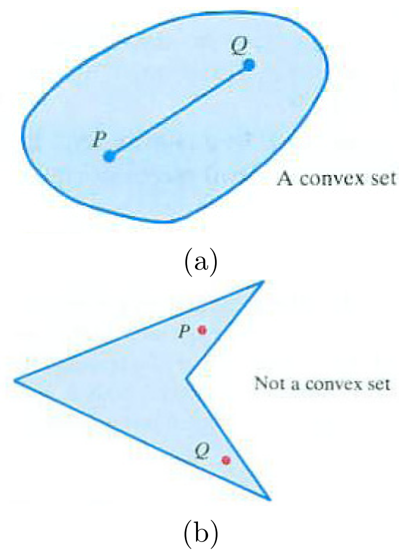


Figure 2.5.8: (a) There are no dents in the boundary of a convex set. (b) Not a convex set.

Exercises 26 to 32 concern convex sets and show how the Intermediate-Value Property gives geometric information. In these exercises you will need to define various functions geometrically. *You may assume these functions are continuous.*

26.[C] Let L be a line in the plane and let K be a convex set. Show that there is a line parallel to L that cuts K into two pieces with equal areas.

Follow these steps.

(a) Introduce an x-axis perpendicular to L with its origin on L. Each line parallel to L and meeting K crosses the x-axis at a number x. Label the line L_x. Let a be the smallest and b the largest of these numbers x. (See Figure 2.5.9.) Let the area of K be A.

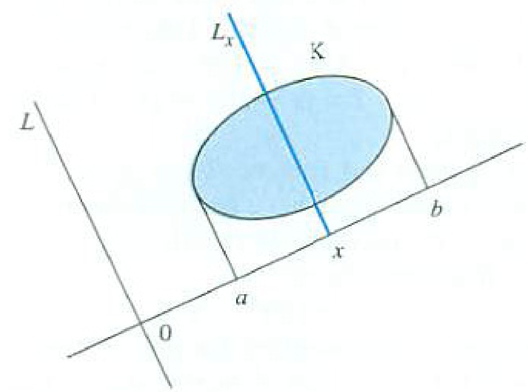


Figure 2.5.9:

(b) Let $A(x)$ be the area of K situated to the left of the line L_x corresponding to x. What is $A(a)$? $A(b)$?

(c) Use the Intermediate-Value Property to show that there is an x in $[a, b]$ such that $A(x) = \frac{A}{2}$.

(d) Why does *(c)* show that there is a line parallel to L that cuts K into two pieces of equal areas?

27.[C] Solve the preceding exercise by applying the Intermediate-Value Property to the function $f(x) = A(x) - B(x)$, where $B(x)$ is the area to the right of L_x.

28.[C] Let P be a point in the plane and let K be a convex set. Is there a line through P that cuts K into two pieces of equal areas?

29.[C] Let K_1 and K_2 be two convex sets in the plane. Is there a line that simultaneously cuts K_1 into two pieces of equal areas and cuts K_2 into two pieces of equal areas? NOTE: This is known as the "two pancakes" question.

30.[C] Let K be a convex set in the plane. Show that there is a line that simultaneously cuts K into two pieces of equal area and cuts the boundary of K into two pieces of equal length.

31.[C] Let K be a convex set in the plane. Show that there are two perpendicular lines that cut K into four pieces of equal areas. (It is not known whether it is always possible to find two perpendicular lines that divide K into four pieces whose areas are $\frac{1}{8}$, $\frac{1}{8}$, $\frac{3}{8}$, and $\frac{3}{8}$ of the area of K, with the parts of equal area sharing an edge, as in Figure 2.5.10.) What if the parts of equal areas are to be opposite each other, instead?

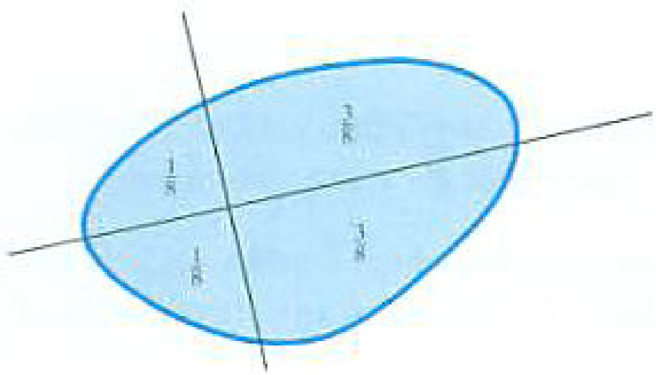


Figure 2.5.10:

32.[C] Let K be a convex set in the plane whose boundary contains no line segments. A polygon is said to **circumscribe** K if each edge of the polygon is tangent to the boundary of K.

(a) Is there necessarily a circumscribing equilateral triangle? If so, how many?

(b) Is there necessarily a circumscribing rectangle? If so, how many?

(c) Is there necessarily a circumscribing square?

33.[C] Let f be a continuous function whose domain is the x-axis and has the property that

$$f(x+y) = f(x) + f(y) \qquad \text{for all numbers } x \text{ and } y.$$

For any constant c, $f(x) = cx$ satisfies this equation since $c(x+y) = cx + cy$. This exercise shows that f must be of the form $f(x) = cx$ for some constant c.

(a) Let $f(1) = c$. Show that $f(2) = 2c$.

(b) Show that $f(0) = 0$.

(c) Show that $f(-1) = -c$.

(d) Show that that for any positive integer n, $f(n) = cn$.

(e) Show that that for any negative integer n, $f(n) = cn$.

(f) Show that $f(\frac{1}{2}) = \frac{c}{2}$.

(g) Show that that for any non-zero integer n, $f(\frac{1}{n}) = \frac{c}{n}$.

(h) Show that that for any intger m and any positive integer n, $f(\frac{m}{n}) = \frac{m}{n}c$.

(i) Show that for any irrational number x, $f(x) = cx$. This is where the continuity of f enters. Parts *(h)* and *(i)* together complete the solution.

34.[C]

(a) Let f be a continuous function defined for all real numbers. Is there necessarily a number x such that $f(x) = x$?

(b) Let f be a continuous function with domain $[0,1]$ such that $f(0) = 1$ and $f(1) = 0$. Is there necessarily a number x such that $f(x) = x$?

35.[C] Let f be a continuous function defined on $(-\infty, \infty)$ such that $f(0) = 1$ and $f(2x) = f(x)$ for all numbers x.

(a) Give an example of such a function f.

(b) Find all functions satisfying these conditions.

Explain your answers.

2.6 Techniques for Graphing

One way to graph a function $f(x)$ is to compute $f(x)$ at several inputs x, plot the points $(x, f(x))$ that you get, and draw a curve through them. This procedure may be tedious and, if you happen to choose inputs that give misleading information, may result in an inaccurate graph.

Another way is to use a calculator that has a graphing routine built in. However, only a portion of the graph is displayed and, if you have no idea what to expect, you may have asked it to display a part of the graph that is misleading or of little interest. At points with large function values, the graph may be distorted by the calculator's choice of scale.

So it pays to be able to get some idea of the general shape of a graph quickly, without having to compute lots of values. This section describes some shortcuts.

Intercepts

The x-coordinates of the points where the graph of a function meets the x-axis are the **x-intercepts** of the function. The y coordinates of the points where a graph meets the y-axis are the **y-intercepts** of the function.

EXAMPLE 1 Find the intercepts of the graph of $y = x^2 - 4x - 5$.

SOLUTION To find the x-intercepts, set $y = 0$, obtaining

$$0 = x^2 - 4x - 5.$$

Fortunately, this quadratic factors nicely:

$$0 = x^2 - 4x - 5 = (x - 5)(x + 1).$$

The equation is satisfied when $x = 5$ or $x = -1$. There are two x-intercepts, 5 and -1. (If the equation did not factor easily, the quadratic formula could be used.)

To find y-intercepts, set $x = 0$, obtaining

$$y = 0^2 - 4 \cdot 0 - 5 = -5.$$

There is only one y-intercept, namely -5.

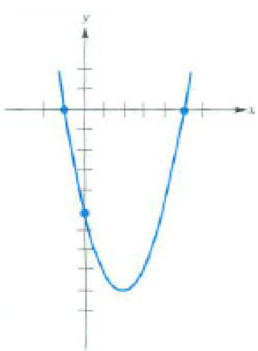


Figure 2.6.1: The graph of $y = x^2 - 4x - 5$, with intercepts.

The intercepts in this case give us three points on the graph. Tabulating a few more points gives the parabola in Figure 2.6.1, where the intercepts are shown as well. ⋄

If $f(x)$ is not defined when $x = 0$, there is no y-intercept. If $f(x)$ is defined when $x = 0$, then it's easy to get the y-intercept; just evaluate $f(0)$. While there is at most one y-intercept, there may be many x-intercepts. To find them, solve the equation $f(x) = 0$. In short,

Finding Asymptotes
To find the y-intercept, compute $f(0)$.
To find the x-intercepts, solve the equation $f(x) = 0$.

Symmetry of Odd and Even Functions

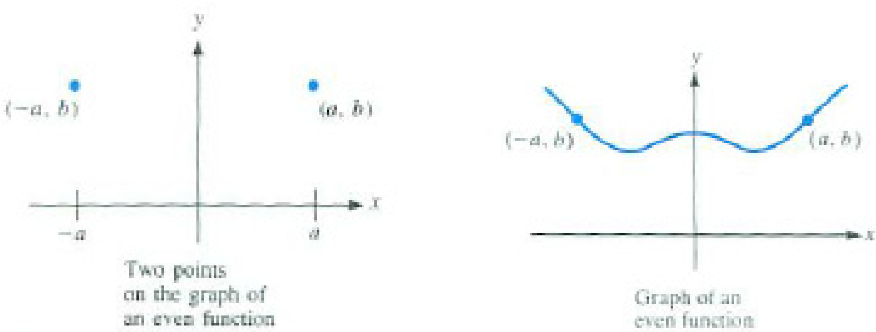
Some functions have the property that when you replace x by $-x$ you get the same value of the function. For instance, the function $f(x) = x^2$ has this property since

$$f(-x) = (-x)^2 = x^2 = f(x).$$

So does the function $f(x) = x^n$ for any *even* integer n. There are fancier functions, such as $3x^4 - 5x^2 + 6x$, $\cos(x)$, and $e^x + e^{-x}$, that also have this property.

DEFINITION *(Even function.)* A function f such that $f(-x) = f(x)$ is called an **even function**.

For an even function f, if $f(a) = b$, then $f(-a) = b$ also. In other words, if the point (a, b) is on the graph of f, so is the point $(-a, b)$, as indicated by Figure 2.6.2(a).



(a) (b)

Figure 2.6.2:

This means that the graph of f is symmetric with respect to the y-axis, as shown in Figure 2.6.2(b). So if you notice that a function is even, you can save half the work in finding its graph. First graph it for positive x and then get the part for negative x free of charge by reflecting across the y-axis. If you wanted to graph $y = x^4/(1 - x^2)$, for example, first stick to $x > 0$, then reflect the result.

DEFINITION *(Odd function.)* A function f with $f(-x) = -f(x)$ is called an **odd function**.

The function $f(x) = x^3$ is odd since

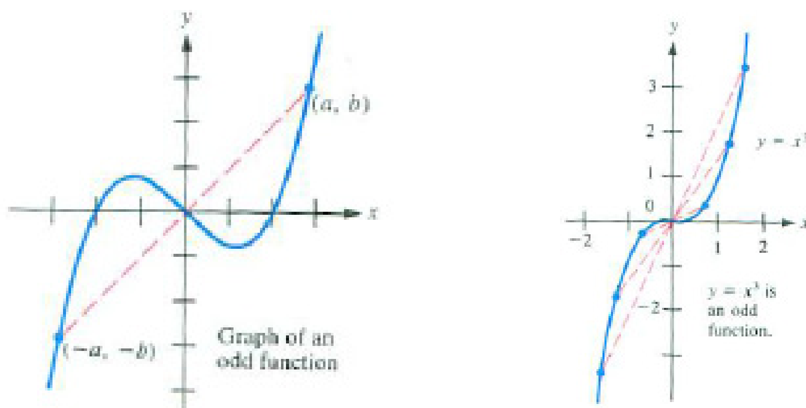
$$f(-x) = (-x)^3 = -(x^3) = -f(x).$$

For any odd integer n, $f(x) = x^n$ is an odd function. The sine function is also odd, since $\sin(-x) = -\sin(x)$.

If the point (a, b) is on the graph of an odd function, so is the point $(-a, -b)$, since

$$f(-a) = -f(a) = -b.$$

(See Figure 2.6.3(a).) Note that the origin $(0, 0)$ is the midpoint of the segment whose ends are (a, b) and $(-a, -b)$. The graph is said to be "symmetric with respect to the origin."



(a) (b)

Figure 2.6.3:

If you work out the graph of an odd function for positive x, you can obtain the graph for negative x by reflecting it point by point through the origin. For example, if you graph $y = x^3$ for $x \geq 0$, as in Figure 2.6.3(b), you can complete the graph by reflection with respect to the origin, as indicated by the dashed lines.

Most functions are neither even nor odd. For instance, $x^3 + x^4$ is neither even nor odd since $(-x)^3 + (-x)^4 = -x^3 + x^4$, which is neither $x^3 + x^4$ nor $-(x^3 + x^4)$.

Asymptotes

If $\lim_{x\to\infty} f(x) = L$ where L is a real number, the graph of $y = f(x)$ gets arbitrarily close to the horizontal line $y = L$ as x increases. The line $y = L$ is called a **horizontal asymptote** of the graph of f. (See Figure 2.6.4.)

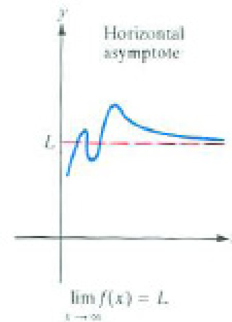


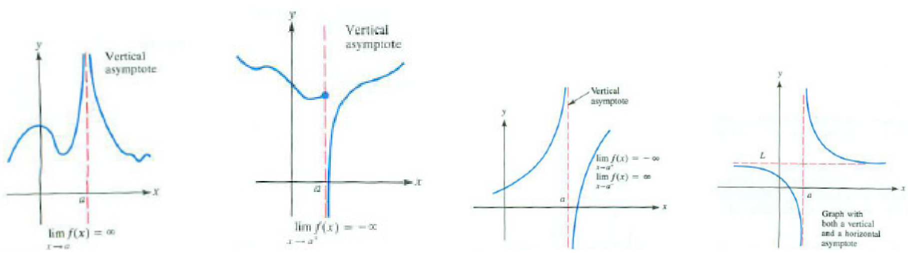
Figure 2.6.4:

If a graph has an asymptote, we can draw it and use it as a guide in drawing the graph.

If $\lim_{x\to a} f(x) = \infty$, then the graph resembles the vertical line $x = a$ for x near a. The line $x = a$ is called a **vertical asymptote** of the graph of $y = f(x)$. The same term is used if

$$\lim_{x\to a} f(x) = -\infty, \qquad \lim_{x\to a^+} f(x) = \infty \text{ or } -\infty, \quad \text{or} \quad \lim_{x\to a^-} f(x) = \infty \text{ or } -\infty.$$

Figure 2.6.5 illustrates these situations.



(a) (b) (c) (d)

Figure 2.6.5:

EXAMPLE 2 Graph $f(x) = 1/(x-1)^2$.
SOLUTION To see if there is any *symmetry*, check whether $f(-x)$ is $f(x)$ of $-f(x)$. We have

$$f(-x) = \frac{1}{(-x-1)^2} = \frac{1}{(x+1)^2}.$$

Since $1/(x+1)^2$ is neither $1/(x-1)^2$ nor $-1/(x-1)^2$, the function $f(x)$ is neither even nor odd. Therefore the graph is *not* symmetric with respect to the y-axis or with respect to the origin.

To determine the y-intercept compute $f(0) = 1/(0-1)^2 = 1$. The y-intercept is 1. To find any x-intercepts, solve the equation $f(x) = 0$, that is,

$$\frac{1}{(x-1)^2} = 0.$$

Since no number has a reciprocal equal to zero, there are no x-intercepts.

To search for a *horizontal asymptote* examine

$$\lim_{x\to\infty} 1/(x-1)^2 \qquad \text{and} \qquad \lim_{x\to-\infty} 1/(x-1)^2.$$

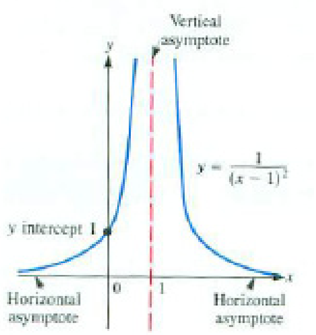


Figure 2.6.6:

Both limits are 0. The line $y = 0$, that is, the x-axis, is an asymptote both to the right and to the left. Since $1/(x-1)^2$ is positive, the graph lies above the asymptote.

To discover any *vertical asymptotes*, find where the function $1/(x-1)^2$ "blows up" — that is, becomes arbitrarily large (in absolute value). This happens when the denominator $(x-1)^2$ becomes zero. Solving $(x-1)^2 = 0$ we find $x = 1$. The function is not defined for $x = 1$. The line $x = 1$ is a vertical asymptote.

To determine the shape of the graph near the line $x = 1$, we examine the one-sided limits: $\lim_{x\to 1^+} 1/(x-1)^2$ and $\lim_{x\to 1^-} 1/(x-1)^2$. Since the square of a nonzero number is always positive, we see that $\lim_{x\to 1^+} 1/(x-1)^2 = \infty$ and $\lim_{x\to 1^-} 1/(x-1)^2 = \infty$. All this information is displayed in Figure 2.6.6. ◇

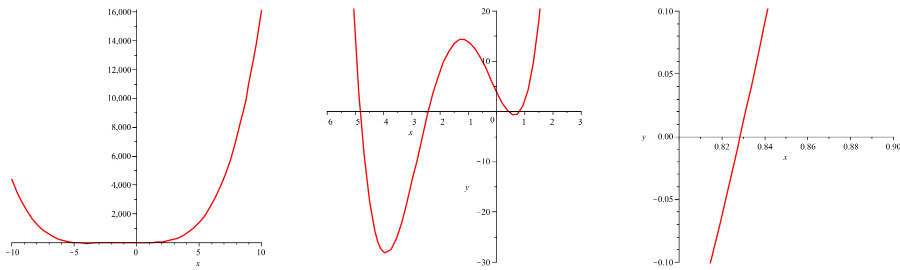
Technology-Assisted Graphing

A graphing utility needs to "know" the function and the viewing window. We will show by three examples some of the obstacles you may run into and how to avoid them. More techniques to help overcome these challenges will be presented in Chapter 4.

The **viewing window** is the portion of the xy-plane to be displayed. We will say the viewing window is $[a, b] \times [c, d]$ when the window extends horizontally from $x = a$ to $x = b$ and vertically from $y = c$ to $y = d$. The graph of a function $y = f(x)$ is created by evaluating $f(x)$ for a sample of numbers x between a and b. The point $(x, f(x))$ is added to the plot. It is customary to connect these points to form the graph of $y = f(x)$. The examples in the remainder of this section demonstrate some of the unpleasant messes that can happen, and how you can avoid them.

EXAMPLE 3 Find a viewing window that shows the general shape of the graph of $y = x^4 + 6x^3 + 3x^2 - 12x + 4$. Use graphs to estimate the location of the rightmost x intercept.

SOLUTION Figure 2.6.7(a) is typical of the first plot of a function. Choose a fairly wide x interval, here $[-10, 10]$, and let the graphing software choose an appropriate vertical range. While this view is useless for estimating any specific x intercept, it is tempting to say that any x intercepts will be between $x = -6$ and $x = 3$. Figure 2.6.7(b) is the graph of this function on the viewing window $[-6, 3] \times [-30, 30]$. Now four x intercepts are visible. The rightmost one occurs around $x = 0.8$. Figure 2.6.7(c) is the result of zooming in on this part of the graph. From this view we estimate that the rightmost x intercept is about 0.83.



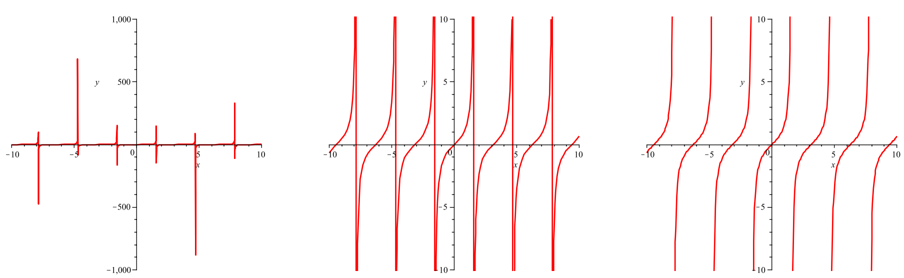
(a) (b) (c)

Figure 2.6.7:

In fact, using a CAS, the four x intercepts for this function are found to occur at 0.8284, 0.4142, -2.4142, and -4.8284 (to four decimal places). $\diamond$

Generating a collection of points and connecting the dots can sometimes lead to ridiculous results, as in Example 4.

EXAMPLE 4 Find a viewing window that clearly shows the general shape and periodicity of the graph of $y = \tan(x)$.
SOLUTION A computer-generated plot of $y = \tan(x)$ for x between -10 and 10 with no vertical height of the viewing window is shown in Figure 2.6.8(a). This graph is not periodic; it looks more like an echocardiogram than the graph of one of the trigonometric functions.



(a) (b) (c)

Figure 2.6.8:

Notice that the default vertical height is very long: $[-1000, 1000]$. Reducing this by a factor of 100, that is, to $[-10, 10]$, yields Figure 2.6.8(b). This graph is periodic and exhibits the expected behavior.

To understand this plot you must realize that the software selects a sample of input values from the domain, computes the value of tangent of each input, then connects the points in order of the input values. The tangent of the last input smaller than $\pi/2$ is large and positive and the tangent of the first input larger than $\pi/2$ is large but negative. Neither of these points is in the viewing window, but the line segment connecting these points does pass through the viewing window and appears as the "vertical" line at $x = \pi/2$ in Figure 2.6.8(b). Because the tangent is not defined for every odd multiple of $\pi/2$, similar reasoning explains the other "vertical" lines at every odd multiple of $\pi/2$

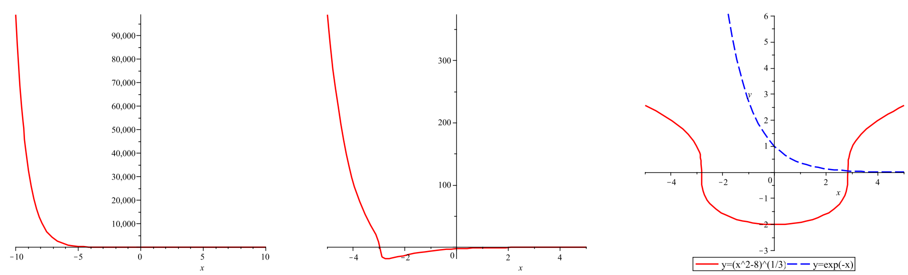
These segments are not really a part of the graph. Figure 2.6.8(c) shows the graph of $y = \tan(x)$ with these extraneous segments removed. ◇

Example 4 illustrates why we must remain alert when using technology. We have to check that the results are consistent with what we already know.

The next example shows that sometimes it is not possible to show all of the important features of a function in a single graph.

EXAMPLE 5 Use one or more graphs to show all major features of the graph $y = e^{-x}\sqrt[3]{x^2-8}$.

SOLUTION The graph of this function on the x interval $[-10, 10]$ with the vertical window chosen by the software is shown in Figure 2.6.9(a). In this window, the exponential function dominates the graph.



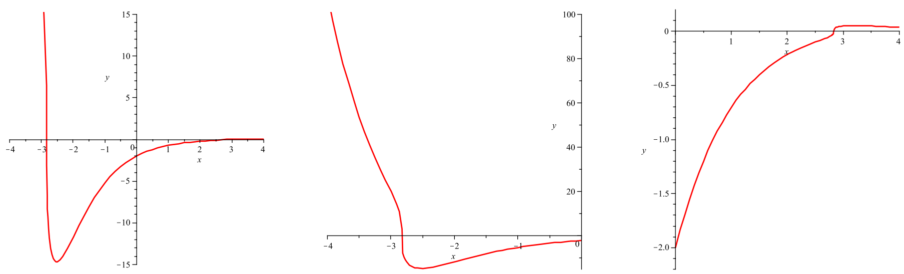
(a) (b) (c)

Figure 2.6.9:

At $x = 0$ the value of the function is $(0-8)^{1/3}e^0 = -2$. To get enough detail to see both the positive and negative values of the function, zoom in by reducing the x interval to $[-5, 5]$. The result is Figure 2.6.9(b). Reducing the x interval to $[-4, 4]$ and specifying the y interval as $[-15, 15]$ gives Figure 2.6.9(c).

We could continue to adjust the viewing window until we find suitable views. A more systematic approach is to look at the graphs of $y = \sqrt[3]{x^2-8}$ and $y = e^{-x}$ separately, but on the same pair of axes. (See Figure 2.6.10(a).) The exponential growth of e^{-x} for negative values of x stretches (vertically) the graph of $y = \sqrt[3]{x^2-8}$ to the left of the y-axis while the exponential decay for $x > 0$ (vertically) compresses the graph of $y = \sqrt[3]{x^2-8}$ to the right of the y-axis.

It is prudent to produce two separate plots to represent the sketch of this function. To the left of the y-axis, with a viewing window of $[-4, 0] \times [-15, 100]$, the graph of the function is shown in Figure 2.6.10(b). To the right of the y-axis, with a much shorter viewing window of $[0, 4] \times [-2.2, 0.2]$, the graph is as shown in Figure 2.6.10(c). $\diamond$



(a) (b) (c)

Figure 2.6.10:

Summary

The first half of this section presents three tools for making a quick sketch of the graph of $y = f(x)$ by hand.

1. *Check for intercepts.* Find $f(0)$ to get the y-intercept. Solve $f(x) = 0$ to get the x-intercepts.

2. *Check for symmetry.* Is $f(-x)$ equal to $f(x)$ or $-f(x)$?

3. *Check for asymptotes.* If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ (where L is some real number), then the line $y = L$ is a horizontal asymptote. If $\lim_{x\to a} f(x) = +\infty$ of $-\infty$, then the line $x = a$ is a vertical asymptote. This is also the case whenever $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ is $+\infty$ or $-\infty$.

The second half of the section provides some pointers for using an automatic graphing utility. The key to their use for graphing is to specify an appropriate viewing window.

Computer-Based Mathematics

Graphing calculators provide an easy way to graph a function. Computer algebra systems (CAS) such as Maple, Mathematica, and Derive can perform symbolic operations on mathematical expressions: for example, they can factor a polynomial:

$$x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2 = (x-1)^2(x+1)^2(x-2),$$

express the quotient of two polynomials as the sum of simpler quotients:

$$\frac{36}{x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2} = \frac{-3}{(x+1)^2} - \frac{9}{(x-1)^2} - \frac{4}{x+1} + \frac{4}{x-2},$$

and solve equations, such as

$$\arctan(x^2+1) = \pi/3 \qquad \text{and} \qquad \sin\left(\frac{\pi}{x}\right) - \frac{\pi}{x}\cos\left(\frac{\pi}{x}\right) = 0.$$

Some of these symbolic features are now available on calculators, PDAs, telephones, and other handheld devices.

These tools will continue to develop and you need to be aware that they do exist, and can do much more than graph functions. As they become more common, and easier to use, they will change the way mathematics is used in the real world. The ability to factor a polynomial or to solve an equation will be less important than the ability to apply basic principles of mathematics and science to set up and to analyze the equations.

EXERCISES for Section 2.6 *Key:* R–routine, M–moderate, C–challenging

1.[R] Show that these are even functions.

(a) x^2+2

(b) $\sqrt{x^4+1}$

(c) $1/x^2$

2.[R] Show that these are even functions.

(a) $5x^4-x^2$

(b) $\cos(2x)$

(c) $7/x^6$

3.[R] Show that these are odd functions.

(a) x^3-x

(b) $x+1/x$

(c) $\sqrt[3]{x}$

4.[R] Show that these are odd functions.

(a) $2x+\frac{1}{2}x$

(b) $\tan(x)$

(c) $x^{5/3}$

5.[R] Show that these functions are neither odd nor even.

(a) $3+x$

(b) $(x+2)^2$

(c) $\frac{x}{x+1}$

6.[R] Show that these functions are neither odd nor even.

(a) $2x-1$

(b) e^x

(c) x^2+1/x

7.[R] Label each function as even, odd, or neither.

(a) $x+x^3+5x^4$

(b) $7x^4-5x^2$

(c) e^x-e^{-x}

8.[R] Label each function as even, odd, or neither.

(a) $\frac{1+x}{1-x}$

(b) $\ln(x^2+1)$

(c) $\sqrt[3]{x^2+1}$

In Exercises 9 to 18 find the x- and y-intercepts, if there are any.

9.[R] $y=2x+3$

10.[R] $y = 3x-7$

11.[R] $y = x^2+3x+2$

12.[R] $y = 2x^2+5x+3$

13.[R] $y = 2x^2+1$

14.[R] $y = x^2+x+1$

15.[R] $y = \sin(x+1)$

16.[R] $y = \ln(x^2+1)$

17.[R] $y = \frac{x^2-1}{(}x^2+1)$

18.[R] $y = e^{\cos(x)}$

In Exercises 19 to 24 find all the horizontal and vertical asymptotes.

19.[R] $y=\frac{x+2}{x-2}$

20.[R] $y=\frac{x-2}{x^2-9}$

21.[R] $y=\frac{x}{x^2+1}$

22.[R] $y=\frac{2x+3}{x^2+4}$

23.[R] $y=\frac{x^2+1}{x^2-3}$

24.[R] $y=\frac{x}{x^2+2x+1}$

In Exercises 25 to 32 graph the function.

25.[R] $y = \frac{1}{x-2}$

26.[R] $y = \frac{1}{x+3}$

27.[R] $y = \frac{1}{x^2-1}$

28.[R] $y = \frac{x}{x^2-2}$

29.[R] $y = \frac{x^2}{1+x^2}$

30.[R] $y = \frac{1}{x^3+x^{-1}}$

31.[R] $y = \frac{1}{x(x-1)(x+2)}$

32.[R] $y = \frac{x+2}{x^3+x^2}$

Use a graphing utility to sketch a graph of the functions in Exercise 33 to 51. Be sure to indicate the viewing window used to generate your graph.

33.[R] $(x^2+x-6)\ln(x+2)$

34.[R] $(x^2-x+6)\ln(x+2)$

35.[R] $(x^2+4)\ln(x+1)$

36.[R] $(x^2-4)\ln(x+1)$

37.[R] $\frac{x^3}{x^2-4}\arctan\left(\frac{x}{5}\right)$

38.[R] $\frac{(x^2-4)}{x^3}\arctan\left(\frac{x}{5}\right)$

39.[R] $\frac{x^3-3x}{x^2-4}$

40.[R] $\frac{x^3-2x}{x^2-4}$

41.[R] $\frac{\sin(x)}{x}$

42.[R] $\frac{\sin(2x)}{x}$

43.[R] $\frac{\sin(2x)}{3x}$

44.[R] $\frac{\sin(x)}{3x}$

45.[R] $\frac{x-\arctan(x)}{x^3}$

46.[R] $\frac{x-\arctan(x)}{x^3+x}$

47.[R] $\frac{x-\arctan(x)}{x^3-1}$

48.[R] $\frac{x-\arctan(x)}{x^3+1}$

49.[R] $\frac{5x^3+x^2+1}{7x^3+x+4}$

50.[R] $\frac{x^3-3x}{x^2-4}\arctan\left(\frac{x}{4}\right)$

51.[R] $\frac{x^3-2x}{x^2-4}\arctan\left(\frac{x}{4}\right)$

Exercises 52 to 58 concern even and odd functions.

52.[M] If two functions are odd, what can you say about

(a) their sum?

(b) their product?

(c) their quotient?

53.[M] If two functions are even, what can you say about

(a) their sum?

(b) their product?

(c) their quotient?

54.[M] If f is odd and g is even, what can you say about

(a) $f + g$?

(b) fg?

(c) f/g?

55.[M] What, if anything, can you say about $f(0)$ if

(a) f is an even function?

(b) f is an odd function?

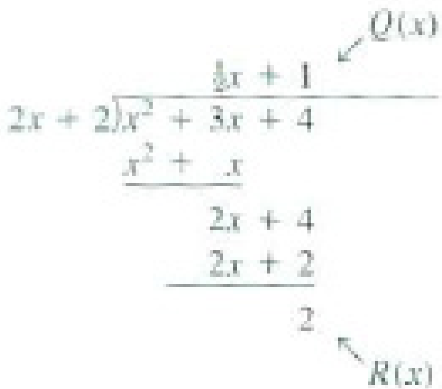
NOTE: Assume 0 is in the domain of f.

56.[M] Which polynomials are even? Explain.

57.[M] Which polynomials are odd? Explain.

58.[M] Is there a function that is both odd and even? Explain.

Exercises 59 to 62 concern tilted asymptotes. Let $A(x)$ and $B(x)$ be polynomials such that the degree of $A(x)$ is equal to 1 more than the degree of $B(x)$. Then when you divide $B(x)$ into $A(x)$, you get a quotient $Q(x)$, which is a polynomial of degree 1, and a remainder $R(x)$, which is a polynomial of degree less than the degree of $B(x)$.
For example, if $A(x) = x^2 + 3x + 4$ and $B(x) = 2x + 2$,



Thus

$$x^2 + 3x + 4 = \left(\frac{1}{2}x + 1\right)(2x+2) + 2)$$

This tells us that

$$\frac{x^2+3x+4}{2x+2} = \frac{1}{2}x + 1 + \frac{2}{2x+2}.$$

When x is large, $2/(2x+2) \to 0$. Thus the graph of $y = \frac{x^2+3x+4}{2x+2}$ is asymptotic to the line $y = \frac{1}{2}x + 1$. (See Figure 2.6.11.)

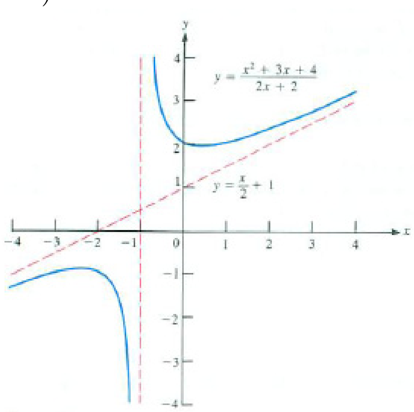


Figure 2.6.11:

Whenever the degree of $A(x)$ exceeds the degree of $B(x)$ by exactly 1, the graph of $y = A(x)/B(x)$ has a tilted asymptote. You find it as we did in the example, by dividing $B(x)$ into $A(x)$, obtaining a quotient $Q(x)$ and a remainder $R(x)$. Then

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)}.$$

The asymptote is $y = Q(x)$. In each exercise graph the function, showing all asymptotes.

59.[M] $y = \frac{x^2}{x-1}$

60.[M] $y = \frac{x^3}{x^2-1}$

61.[M] $y = \frac{x^2-4}{x+4}$

62.[M] $y = \frac{x^2+x+1}{x-2}$

A **piecewise-defined function** is a function that is given by different formulas on different pieces of the domain.
Read the directions for your graphing software to learn how to graph a piecewise-defined function. Then use your graphing utility to sketch a graph of the functions in Exercises 63 and 64.

63.[M] $y = \begin{cases} x^2 - x & x < 1 \\ \sqrt{x-1} & x \geq 1 \end{cases}$

64.[M] $y = \begin{cases} \frac{\sin(x)}{x} & x < 0 \\ \sin x & 0 \leq x \geq \pi \\ x - 2 & x > \pi \end{cases}$

Some graphing utilities have trouble plotting functions with fractional exponents. General rules when graphing $y = x^{p/q}$ where p/q is a positive fraction in lowest terms are:

- If p is even and q is odd, then graph $y = |x|^{p/q}$.
- If p and q are both odd, then graph $y = \frac{|x|}{x}|x|^{p/q}$.

Use that advice and a calculator to sketch the graph of each function in Exercises 65 to 68.

65.[M] $y = x^{1/3}$

66.[M] $y = x^{2/3}$

67.[M] $y = x^{4/7}$

68.[M] $y = x^{3/7}$

69.[C] Let $P(x)$ be a polynomial of degree m and $Q(x)$ a polynomial of degree n. For which m and n does the graph of $y = P(x)/Q(x)$ have a horizontal asymptote?

70.[C] Assume you already have drawn the graph of a function $y = f(x)$. How would you obtain the graph of $y = g(x)$ from that graph if

(a) $g(x) = f(x) + 2$?

(b) $g(x) = f(x) - 2$?

(c) $g(x) = f(x - 2)$?

(d) $g(x) = f(x + 2)$?

(e) $g(x) = 2f(x)$?

(f) $g(x) = 3f(x - 2)$?

71.[C] Is there a function f defined for all x such that $f(-x) = 1/f(x)$? If so, how many? If not, explain why there are no such functions.

72.[C] Is there a function f defined for all x such that $f(-x) = 2f(x)$? If so, how many? If not, explain why there are no such functions.

73.[C] Is there a constant k such that the function

$$f(x) = \frac{1}{3^x - 1} + k$$

is odd? even?

2.S Chapter Summary

One concept underlies calculus: the limit of a function. For a function defined near a (but not necessarily at a) we ask, "What happens to $f(x)$ as x gets nearer and nearer to a." If the values get nearer and nearer one specific number, we call that number the limit of the function as x approaches a. This concept, which is not met in arithmetic or algebra or trigonometry, distinguishes calculus.

For instance, when $f(x) = (2^x - 1)/x$, which is not defined at $x = 0$, we conjectured on the basis of numerical evidence that $f(x)$ approaches 0.693 (to three decimals). Later we will see that this limit is a certain logarithm. With that information we found that $(4^x - 1)/x$ must approach 2(0.693), which is larger than 1. We then defined e as that number (between 2 and 4) such that $(e^x - 1)/x$ approaches 1 as x approaches 0. The number e is as important in calculus as π is in geometry or trigonometry. The number e is about 2.718 (again to three decimals) and is called Euler's number. That is why a scientific calculator has a key for e^x, the most convenient exponential for calculus, as will become clear in the next chapter.

When angles are measured in radians,

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$$

These two limits will serve as the basis of the calculus of trigonometric functions developed in the next chapter. The simplicity of the first limit is one reason that in calculus and its applications angles are measured in radians. If angles were measured in degrees, the first limit would be $\pi/180$, which would complicate computations.

Most of the functions of interest in later chapters are "continuous." The value of such a function at a number a in its domain is the same as the limit of the function as x approaches a. However, we will be interested in a few functions that are not continuous.

A continuous function has three properties, which will be referred to often:

Extreme-Value Property

- On a closed interval it attains a maximum value and a minimum value.

Intermediate -Value Property

- On a closed interval it takes on all values between its values at the end points of the interval.

Permanence Property

- If it is positive at some number and defined at least on an open interval containing that number, then it remains positive at least on some open interval containing that number. More generally, if $f(a) = p > 0$, and q is less than p, then $f(x)$ remains larger than q, at least on some open interval containing a. A similar statement holds when $f(a)$ is negative.

A quick sketch of the graph of a generic continuous function makes the three properties plausible. In advanced calculus they are all established using only the precise definition of continuity and properties of the real numbers — but no pictures. Such strictness is necessary because there are some pretty wild continuous functions. For instance, there is one such that when you zoom in on its graph at any point, the parts of the graph nearer and nearer the point do not look like straight line segments.

The initial steps in the analysis of a function utilize intercepts, symmetry, and asymptotes. The same ideas are also helpful when selecting an appropriate viewing window when using an electronic graphing utility. Additional techniques will be added in Chapter 4, particularly Section 4.3.

EXERCISES for 2.S *Key:* R–routine, M–moderate, C–challenging

1.[R] Define Euler's constant, e, and give its decimal value to five places.

In Exercises 2 to 4 state the given property in your own words, using as few mathematical symbols as possible.

2.[R] The Maximum-Value Property.

3.[R] The Intermediate-Value Property.

4.[R] The Permanence Property.

5.[R]

(a) Verify that $x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$.

(b) Use (a) to find $\lim_{x\to a} \frac{x^5-a^5}{x-a}$.

6.[R] Let $f(x) = \frac{1}{x+2}$ for x not equal to -2. Is there a continuous function $g(x)$, defined for all x, that equals $f(x)$ when x is not -2? Explain your answer.

7.[R] Let $f(x) = \frac{2^x-1}{x}$ for x not equal to 0. Is there a continuous function $g(x)$, defined for all x, that equals $f(x)$ when x is not 0? Explain your answer.

8.[R] Let $f(x) = \sin(1/(x-1))$ for x not equal to 1. Is there a continuous function $g(x)$, defined for all x, that equals $f(x)$ when x is not 1? Explain your answer.

9.[R] Let $f(x) = x\sin(1/x)$ for x not equal to 0. Is there a continuous function $g(x)$, defined for all x, that equals $f(x)$ when x is not 0? Explain your answer.

10.[M] Show that $\lim_{x\to 1} \frac{x^{1/3}-1}{x-1} = \frac{1}{3}$ by first writing the denominator as $\left(x^{1/3}\right)^3 - 1$ and using the factorization $u^3 - 1 = (u-1)(u^2+u+1)$.

11.[M] Use the factorization in Exercise 5 to find $\lim_{x\to a} \frac{x^{-5}-a^{-5}}{x-a}$.

12.[M] Assume $b > 1$. If $\lim_{x\to 0} \frac{b^x-1}{x} = L$, find $\lim_{x\to 0} \frac{(1/b)^x-1}{x}$

13.[M] By sketching a graph, show that if a function is not continuous it may not

(a) have a maximum even if its domain is a closed interval,

(b) satisfy the Intermediate-Value Theorem, even if its domain is a closed interval,

(c) have the Permanence Property, even if its domain is an open interval.

14.[M] Let g be an increasing function such that $\lim_{x\to a} g(x) = L$.

(a) Sketch the graph of a function f whose domain includes an open interval around L such that

$$f\left(\lim_{x\to a} g(x)\right) \text{ and } \lim_{x\to a} f(g(x))$$

both exist but the are not equal

(b) What property of f would assure us that the two limits in (a) would be equal?

We obtained $\lim_{x\to a} \frac{x^n-a^n}{x-a}$ be exploiting the factorization of $x^n - a^n$. Calling $x - a$ simply h, that limit can be written as $\lim_{h\to 0} \frac{(a+h)^n-a^n}{h}$. This limit can be evaluated, but by different algebra, as Exercises 15 and 16 show.

15.[M]

(a) Show that $(a + h)^2 = a^2 + 2ah + h^2$.

(b) Use (a) to evaluate $\lim_{h\to 0} \frac{(a+h)^2-a^2}{h}$.

16.[M]

(a) Show that $(a + h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$.

(b) Use (a) to evaluate $\lim_{h\to 0} \frac{(a+h)^3-a^3}{h}$.

17.[M] If you are familiar with the Binomial Theorem, use it to show that for any positive integer n, $\lim_{h\to 0} \frac{(a+h)^n-a^n}{h} = na^{n-1}$. NOTE: The Binomial Theorem expresses $(a + b)^n$, when multiplied out, as the sum of $n + 1$ terms. Using calculus, we will develop it in Section 5.4 (Exercise 31).

In Exercises 18 to 21 find each limit.

18.[M] $\lim_{x\to\infty} \frac{\ln(5x)}{\ln(4x^2)}$

19.[M] $\lim_{x\to\infty} \frac{\ln(5x)}{\ln(4x)}$

20.[M] $\lim_{x\to\infty} \frac{\log_2(x^2)}{\log_4(x)}$

21.[M] $\lim_{x\to\infty} \frac{\log_3(x^5)}{\log_9(x)}$

22.[M] Find $\lim_{h\to 0} \frac{(e^2)^h-1}{h}$ by factoring the numerator.

23.[M] Define $f(x) = \begin{cases} \frac{h(x)}{x-3} & x \neq 3 \\ p & x = 3 \end{cases}$ What conditions on h must be satisfied to make f continuous?

24.[M] Assuming that $\lim_{x\to 0^+} x^x = 1$ and that $\lim_{x\to\infty} \ln(x) = \infty$, deduce each of the following limits:

(a) $\lim_{x\to 0} x \ln(x)$

(b) $\lim_{x\to\infty} \frac{\ln(x)}{x}$ HINT: Use (a).

(c) $\lim_{x\to\infty} x^{1/x}$

(d) $\lim_{x\to\infty} \frac{\ln(x)}{x^k}$, k a positive constant

(e) $\lim_{x\to\infty} \frac{x}{e^x}$

(f) $\lim_{x\to\infty} \frac{x^n}{e^x}$, n a positive integer

(g) $\lim_{x\to\infty} \frac{\ln(x)^n}{x}$, n a positive integer

25.[M] Define $f(x) = \begin{cases} \frac{x^3-3x^2-4x+k}{x-3} & x \neq 3 \\ p & x = 3 \end{cases}$

(a) For what values of k and p is f continuous? (Justify your answer.)

(b) For these values of k and p, is f an even or odd function? (Justify your answer.)

26.[M] Two points on a circle or sphere will be called "opposite" if they are the ends of a diameter of the circle or sphere.

(a) Assuming the temperature is continuous, show that there are opposite points on the equator that have the same temperatures.

(b) Show that there may not be opposite points on the equator where the temperatures are equal and also the barometric pressures are equal.

NOTE: The Borsuk-Ulam theorem in topology implies that there are opposite points on the earth where the temperatures are equal and the pressures are equal.

27.[M] Let $f = g + h$, where g is an even function and f is an odd function. Express g and h in terms of f.

28.[M]

(a) Show that any function f can be written as the sum of an even function and an odd function.

(b) In how many ways can a given function be written that way?

29.[M] If f is an odd function and g is an even function, what, if anything, can be said about (a) fg, (b) f^2, (c) $f + g$, (d) $f + f$, and (e) f/g? Explain.

30.[C] The graph of some function f whose domain is $[2, 4]$ and range is $[1, 3]$ is shown in Figure 2.S.1(a). Sketch the graphs of the following functions and state their domain and range.

(a) $g(x) = -3f(x)$,

(b) $g(x) = f(x + 1)$,

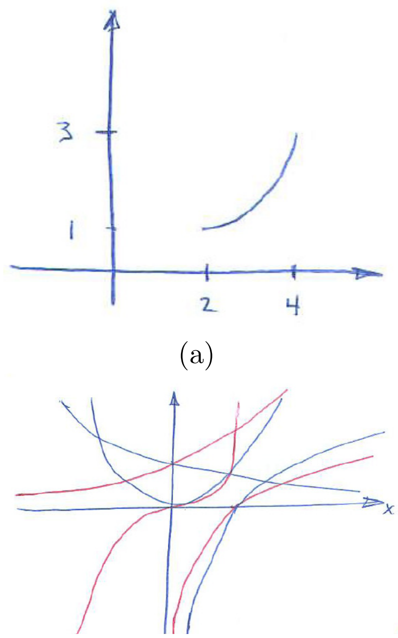
(c) $g(x) = f(x - 1)$,

(d) $g(x) = 3 + f(x)$,

(e) $g(x) = f(2x)$,

(f) $g(x) = f(x/2)$,

(g) $g(x) = f(2x - 1)$.



(a)

(b)

Figure 2.S.1:

31.[C] For a constant k, find $\lim_{h\to 0} \frac{(e^k)^h - 1}{h}$. HINT: Replace h in the denominator by hk, but do it legally.

32.[C]

(a) Calculate $(0.99999)^x$ for various large values of x.

(b) Using the evidence gathered in (a), conjecture the value of $\lim_{x\to\infty}(0.99999)^x$.

(c) Why is $\lim_{x\to\infty}(0.99999)^{x+1}$ the same as $\lim_{x\to\infty}(0.99999)^x$?

(d) Denoting the limit in (b) as L, show that $0.99999L = L$.

(e) Using (d), find L.

33.[C] (Contributed by G. D. Chakerian) This exercise obtains $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$ without using areas. Figure 2.S.2 shows a circle C of radius 1 with center at the origin and a circle $C(r)$ of radius $r > 1$ that passes through the center of C. Let $S(r)$ be the part of $C(r)$ that lies within C. Its ends are P and Q. Let θ be the angle subtended by the top half of $S(r)$ at the center of $C(r)$. Note that as $r \to \infty$, $\theta \to 0$. Define $A(\theta)$ to be the length of the arc $S(r)$ as a function of θ.

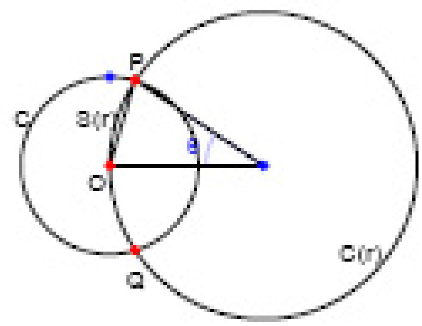


Figure 2.S.2:

(a) Looking at Figure 2.S.2, det
HINT: What happens to P a

(b) Show that $A(\theta)$ is $\frac{\theta/2}{\sin(\theta/2)}$.

(c) Combining (a) and (b), show
1.

Calculus is Everywhere # 2
Bank Interest and the Annual Percentage Yield

The Truth in Savings Act, passed in 1991, requires a bank to post the Annual Percentage Yield (APY) on deposits. That yield depends on how often the bank computes the interest earned, perhaps as often as daily or as seldom as once a year. Imagine that you open an account on January 1 by depositing \$1000. The bank pays interest monthly at the rate of 5 percent a year. How much will there be in your account at the end of the year? For simplicity, assume all the months have the same length. To begin, we find out how much there is in the account at the end of the first month. The account then has the initial amount, \$1000, plus the interest earned during January. Because there are 12 months, the interest rate in each month is 5 percent divided by 12, which is 0.05/12 percent per month. So the interest earned in January is \$1000 times 0.05/12. At the end of January the account then has

$$\$1000 + \$1000(0.05/12) = \$1000(1 + 0.05/12).$$

The initial deposit is "magnified" by the factor $(1 + 0.05/12)$.

The amount in the account at the end of February is found the same way, but the initial amount is $\$1000(1 + 0.05/12)$ instead of \$1000. Again the amount is magnified by the factor $1 + 0.05/12$, to become

$$\$1000(1 + 0.05/12)^2.$$

The amount at the end of March is

$$\$1000(1 + 0.05/12)^3.$$

At the end of the year the account has grown to

$$\$1000(1 + 0.05/12)^{12},$$

which is about \$1051.16.

The deposit earned \$51.16. If instead the bank computed the interest only once, at the end of the year, so-called "simple interest," the deposit would earn only 5 percent of \$1000, which is \$50. The depositor benefits when the interest is computed more than once a year, so-called "compound interest." A competing bank may offer to compute the interest every day. In that case, the account would grow to

$$\$1000(1 + 0.05/365)^{365},$$

n	$(1+1/n)^n$	$(1+1/n)^n$
1	$(1+1/1)^1$	2.00000
2	$(1+1/2)^2$	2.25000
3	$(1+1/3)^3$	2.37037
10	$(1+1/10)^{10}$	2.59374
100	$(1+1/100)^{100}$	2.70481
1000	$(1+1/1000)^{1000}$	2.71692

Table C.2.1:

which is about \$1051.27, eleven cents more than the first bank offers. More generally, if the initial deposit is A, the annual interest rate is r, and interest is computed n times a year, the amount at the end of the year is

$$A(1+r/n)^n. \tag{C.2.1}$$

In the examples, A is \$1000, r is 0.05, and n is 12 and then 365. Of special interest is the case when A is 1 and r is a generous 100 percent, that is, $r = 1$. Then (C.2.1) becomes

$$(1+1/n)^n. \tag{C.2.2}$$

How does (C.2.2) behave as n increase? Table C.2.1 shows a few values of (C.2.2), to five decimal places. The base, $1+1/n$, approaches 1 as n increases, suggesting that (C.2.2) may approach a number near 1. However, the exponent gets large, so we are multiplying lots of numbers, all of them larger than 1. It turns out that as n increases $(1+1/n)^n$ approaches the number e defined in Section 2.2. One can write

$$\lim_{x\to 0^+}(1+x)^{1/x} = e.$$

Note that the exponent, $1/x$, is the reciprocal of the "small number" x.

With that fact at our disposal, we can figure out what happens when an account opens with \$1000, the annual interest rate is 5 percent, and the interest is compounded more and more often. In that case we would be interested in

$$1000 \lim_{n\to\infty}\left(1+\frac{0.05}{n}\right)^n.$$

Unfortunately, the exponent n is not the reciprocal of the small number $0.05/n$. But a little algebra can overcome that nuisance, for

$$\left(1+\frac{0.05}{n}\right)^n = \left(\left(1+\frac{0.05}{n}\right)^{\frac{n}{0.05}}\right)^{0.05}. \tag{C.2.3}$$

The expression in parentheses has the form "(1 + small number) raised to the reciprocal of that small number." Therefore, as n increases, (C.2.3)) approaches $e^{0.05}$, which is about 1.05127. No matter how often interest is compounded, the \$1000 would never grow beyond \$1051.27.

The definition of e given in Section 2.2 has no obvious connection to the fact that $\lim_{x\to 0^+}(1+x)^{1/x}$ equals the number e. It seems "obvious," by thinking in terms of banks, that as n increases, so does $(1+1/n)^n$. Without thinking about banks, try showing that it does increase. (This limit will be evaluated in Section 3.4.)

EXERCISES

1.[R] A dollar is deposited at the beginning of the year in an account that pays an interest rate r of 100% a year. Let $f(t)$, for $0 \le t \le 1$, be the amount in the account at time t. Graph the function if the bank pays

(a) only simple interest, computed only at $t = 1$.

(b) compound interest, twice a year computed at $t = 1/2$ and 1.

(c) compound interest, three times a year computed at $t = 1/3$, $2/3$, and 1.

(d) compound interest, four times a year computed at $t = 1/4$, $1/2$, $3/4$, and 1.

(e) Are the functions in (a), (b), (c), and (d) continuous?

(f) One could expect the account that is compounded more often than another would always have more in it. Is that the case?