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Heat Content Asymptotics with Inhomogeneous Neumann and Dirichlet Boundary Conditions

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Abstract. Let M be a compact manifold with smooth boundary. We establish the existence of an asymptotic expansion for the heat content asymptotics of M with inhomogeneous Neumann and Dirichlet boundary conditions. We prove all the coefficients are locally determined and determine the first several terms in the asymptotic expansion.

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Let M be a smooth compact Riemannian manifold of dimension m with smooth boundary ∂M. We assume given a decomposition of $\partial M = C_N \cup C_D$ as the union of two disjoint closed sets. We will take Neumann boundary conditions on C_N and Dirichlet boundary conditions on C_D. We permit $C_N = \emptyset$ or $C_D = \emptyset$.

We let the initial temperature be given by a smooth function Φ on M. We suppose given an auxiliary smooth function Ψ which is defined on ∂M. On the Neumann boundary component C_N, we pump heat into the manifold across the boundary at a constant rate determined by Ψ; on the Dirichlet boundary component C_D, we keep the boundary at temperature Ψ. These boundary conditions are independent of t. Let ∂_N be the inward unit normal. The resulting temperature function $H_{\Phi,\Psi}(x;t)$ is the solution to the equations:

$$\begin{aligned}
&(\partial_t + \Delta) H_{\Phi,\Psi}(x;t) = 0 \text{ for } x \in M, t > 0 && \text{(heat equation)},\\
&\lim_{t\downarrow 0} H_{\Phi,\Psi}(x;t) = \Phi(x) \text{ for } x \in M && \text{(initial condition)},\\
&\partial_N H_{\Phi,\Psi}(y;t) = \Psi(y) \text{ for } y \in C_N && \text{(Neumann boundary condition)},\\
&H_{\Phi,\Psi}(y;t) = \Psi(y) \text{ for } y \in C_D && \text{(Dirichlet boundary condition)}.
\end{aligned} \tag{1}$$

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We refer to Carslaw and Jaeger [5] for a discussion of the physical motivation of this problem. Let F represent the specific heat at a point $x \in M$. We define the weighted heat content energy function

$$\beta(\Phi, \Psi, F)(t) := \int_M H_{\Phi,\Psi}(x; t) F(x)\,\mathrm{d}x.$$

THEOREM 1. *As $t \downarrow 0$, there is an asymptotic expansion of the form*

$$\beta(\Phi, \Psi, F)(t) \sim \sum_{n=0}^{\infty} \beta_n(\Phi, \Psi, F) t^{n/2}.$$

Let L be the second fundamental form on the boundary and let R be the Riemann curvature tensor of M with the sign convention that $R_{1221} = +1$ for the standard sphere in 3 space. Let ';' and ':' denote covariant differentiation with respect to the Levi–Civita connections of M and of ∂M respectively. We choose a local orthonormal frame $\{e_1, \ldots, e_m\}$ for the tangent bundle of M restricted to the boundary so that e_m is the inward unit normal. Thus, for example, the inward unit normal derivative of F is given by $F_{;m}$. Let indices a, b, c etc. range from 1 through $m-1$. We adopt the Einstein convention and sum over repeated indices. We have the following combinatorial formulae

THEOREM 2.

(1) $\beta_0(\Phi, \Psi, F) = \int_M \Phi F$.
(2) $\beta_1(\Phi, \Psi, F) = -\frac{2}{\sqrt{\pi}} \int_{C_D} (\Phi - \Psi) F$.
(3) $\beta_2(\Phi, \Psi, F) = -\int_M F\Delta\Phi + \frac{1}{2}\int_{C_D}(FL_{aa} - 2F_{;m})(\Phi - \Psi) + \int_{C_N}(\Phi_{;m} - \Psi)F$.
(4) $\beta_3(\Phi, \Psi, F) = -\frac{1}{6\sqrt{\pi}} \int_{C_D} \big\{(\Phi - \Psi)F(L_{aa}L_{bb} - 2L_{ab}L_{ab} - 2R_{amma}) - 8F\Delta\Phi$
$+4F(\Phi - \Psi)_{:aa} + 8(F_{;mm} - L_{aa}F_{;m})(\Phi - \Psi)\big\} + \frac{4}{3\sqrt{\pi}} \int_{C_N} (\Phi_{;m} - \Psi) F_{;m}$.
(5) $\beta_4(\Phi, \Psi, F) = \frac{1}{2}\int_M (\Delta\Phi \cdot \Delta F) + \frac{1}{32}\int_{C_D}\{16(\Delta\Phi)_{;m}F + 16(\Phi - \Psi)(\Delta F)_{;m}$
$-8L_{aa}[F\Delta\Phi + (\Phi - \Psi)\Delta F] + (\Phi - \Psi)F(-2L_{ab}L_{ab}L_{cc} + 4L_{ab}L_{ac}L_{bc}$
$-2R_{ambm}L_{ab} + 2R_{abcb}L_{ac} + R_{ijji;m} + 2L_{ab:ab}) - 8L_{ab}(\Phi_{:a} - \Psi_{:a})F_{:b}\}$
$-\frac{1}{4}\int_{C_N}\{2(\Phi_{;m} - \Psi)\Delta F + 2F_{;m}\Delta\Phi - L_{aa}(\Phi_{;m} - \Psi)F_{;m}\}$.

If there are no Dirichlet components, we can also compute β_5 and β_6.

THEOREM 3. *Assume C_D is empty. Then*

(1) $\beta_5(\Phi, \Psi, F) = \frac{1}{15\sqrt{\pi}} \int_{C_N}\{-8(\Delta\Phi)_{;m}F_{;m} - 8(\Phi_{;m} - \Psi)(\Delta F)_{;m}$
$-4(\Phi_{;m} - \Psi)_{:a}F_{:a} + (L_{aa}L_{bb} + 2L_{ab}L_{ab} + 2R_{amma})(\Phi - \Psi_{;m})F\}$.
(2) $\beta_6(\Phi, \Psi, F) = -\frac{1}{6}\int_M \Delta^2\Phi \cdot \Delta F + \frac{1}{96}\int_{C_N}\{16(\Delta\Phi)_{;m}\Delta F + 16F_{;m}\Delta^2\Phi +$
$16(\Phi_{;m} - \Psi)\Delta^2 F - 8L_{aa}(\Delta\Phi)_{;m}F_{;m} - 8L_{aa}(\Phi_{;m} - \Psi)(\Delta F)_{;m} + (2L_{ab}L_{ab}L_{cc}$
$+ 4L_{ab}\,L_{ac}L_{bc} - 2R_{ambm}L_{ab} + 2R_{abcb}L_{ac} + 4R_{amma}L_{bb} + R_{ijji;m} + 2L_{ab:ab})$
$(\Phi_{;m} - \Psi)F_{;m} - 8L_{aa}(\Phi_{;m} - \Psi)_{:a}(F_{;m})_{:a} - 8L_{ab}(\Phi_{;m} - \Psi)_{:a}(F_{;m})_{:b}$.

The coefficients in Theorem 2 and Theorem 3 are given by integral formulae. This holds true for general n.

THEOREM 4. *The coefficients* $\beta_n(\Phi, \Psi, F)$ *are locally computable for all* n.

Proofs of Theorems 1–4. Since we will be varying the manifold M and the sets C_N and C_D, we change our notational convention slightly and denote the heat content function by $\beta(\Phi, \Psi, F; M, C_N, C_D)$. The equations given in display (1) decouple so

$$\begin{aligned} &H_{\Phi,\Psi}(x;t) = H_{\Phi,0}(x;t) + H_{0,\Psi}(x;t),\\ &\beta(\Phi, \Psi, F; M, C_N, C_D)(t) = \beta(\Phi, 0, F; M, C_N, C_D)(t)\\ &\qquad\qquad + \beta(0, \Psi, F; M, C_N, C_D)(t). \end{aligned} \tag{2}$$

Thus in the proof of Theorems 1–4, we may consider the cases $(\Phi, 0, F)$ and $(0, \Psi, F)$ separately. Since C_N and C_D are disjoint closed sets, we can choose smooth functions F_N and F_D on M so that $F = F_N + F_D$, so that F_D is supported near C_D, and so that F_N vanishes near C_D. We may then decompose

$$\begin{aligned} \beta(\Psi, 0, F; M, C_N, C_D)(t) \;=\; &\beta(\Psi, 0, F_N; M, C_N, C_D)(t)\\ &+\beta(\Psi, 0, F_D; M, C_N, C_D)(t). \end{aligned} \tag{3}$$

The principal of not feeling the boundary shows there exist errors $\mathcal{E}$ which vanish to infinite order as $t \downarrow 0$ so that

$$\begin{aligned} &\beta(\Psi, 0, F_N; M, C_N, C_D)(t) = \beta(\Psi, 0, F_N; M, \partial M, \emptyset)(t) + \mathcal{E}_N(t),\\ &\beta(\Psi, 0, F_D; M, C_N, C_D)(t) = \beta(\Psi, 0, F_D; M, \emptyset, \partial M)(t) + \mathcal{E}_N(t); \end{aligned} \tag{4}$$

the heat which flows across the additional boundary components is weighted with zero near these components. We refer to Grubb [7] for further details concerning elliptic boundary value problems and to Hsu [8] for a discussion of the principle of not feeling the boundary from a probabilistic point of view. Theorems 1–4 for the functions $\beta(\Phi, 0, F_N; M, \partial M, \emptyset)$ and $\beta(\Phi, 0, F_D; M, \emptyset, \partial M)$ follow from results in [1, 2, 6]. In these papers, we considered the heat content of a manifold with homogeneous Neumann or homogeneous Dirichlet boundary conditions. We now use equations (3) and (4) to see that Theorems 1–4 hold for $\beta(\Phi, 0, F; M, C_N, C_D)$.

We study the heat content function $\beta(0, \Psi, F; M, C_N, C_D)$; to complete the proof of Theorems 1–4. This corresponds to zero initial condition and inhomogeneous boundary conditions. Suppose for the moment that the average value of Ψ is zero on ∂M. Then there exists a harmonic function U so that $U|_{C_D} = \Psi$ and so that $(\partial_N U)|_{C_N} = \Psi$. The defining relations given in display (1) then show that $U(x) - H_{U,0}(x;t) = H_{0,\Psi}(x;t)$. Let $\mathcal{U} := \int_M U$. Then

$$\beta(0, \Psi, F; M, C_N, C_D)(t) = \mathcal{U} - \beta(U, 0, F; M, C_N, C_D)(t). \tag{5}$$

Theorems 1–3 now follow from the corresponding assertions for the heat content function $\beta(U, 0, F; M, C_N, C_D)$.

We now remove the assumption that $\int_{\partial M} \Psi = 0$. Choose a point x_0 in the interior of M and choose $\varepsilon > 0$ so that the disk $D_\varepsilon(x_0)$ of radius ε about x_0 is disjoint from ∂M. Choose a smooth function $\tilde{F}$ so that $\tilde{F}$ agrees with F near ∂M and so that $\tilde{F}$ is identically zero near $\partial D_\varepsilon(x_0)$. Let $\tilde{M} = M - D_\varepsilon(x_0)$. Choose $\tilde{\Psi}$ to agree with Ψ on ∂M. Extend $\tilde{\Psi}$ to $\partial D_\varepsilon(x_0)$ so that $\int_{\tilde{M}} \tilde{\Psi} = 0$. Let $\tilde{C}_N = C_N \cup \partial D_\varepsilon(x_0)$, and let $\tilde{C}_D = C_D$. The principal of not feeling the boundary shows that there is an error $\mathcal{E}$ which vanishes to infinite order as $t \downarrow 0$ so that

$$\beta(0, \Psi, F; M, C_N, C_D)(t) = \beta(0, \tilde{\Psi}, \tilde{F}; \tilde{M}, \tilde{C}_N, \tilde{C}_D)(t) + \mathcal{E}(t), \tag{6}$$

the heat which flows across the extra boundary component is weighted with weight 0. Since the average value of $\tilde{\Psi}$ vanishes, Theorems 1–3 hold for the modified heat content function $\beta(0, \tilde{\Psi}, \tilde{F}; \tilde{M}, \tilde{C}_N, \tilde{C}_D)$. We use equation (6) to see Theorems 1–3 also hold for $\beta(0, \Psi, F; M, C_N, C_D)$. This completes the proof of Theorems 1–3.

In the proof of Theorem 4, the argument given above shows that we need only deal with the case $(0, \Psi, F)$ and that we may assume either $C_D = \emptyset$ or that $C_N = \emptyset$. The case $C_N = \emptyset$ involving pure Dirichlet boundary conditions was discussed in [3] so we assume $C_D = \emptyset$, i.e. that we are dealing with pure Neumann boundary conditions. We also note that the argument given above shows that we may assume without loss of generality that the average value of Ψ vanishes and consequently U is well defined. Although we have shown that $\beta_n(U, 0, F; -)$ is locally computable, this does not imply that $\beta_n(0, \Psi, F; -)$ is locally computable. The construction of U from Ψ is global in nature; a-priori one would expect both the values of U and the normal derivatives of U to enter into the local formulae. This would involve the Neumann to Dirichlet problem which is known to be non-local. The local formulae given in Theorems 2 and 3 show that this does not happen for $n \leqslant 6$. There are no undesirable extra terms and the coefficients are in fact locally determined if $n \leqslant 6$. To complete the proof of Theorem 4, we must show this is true for all n.

Let $\mathcal{P}$ be the vector space of all bilinear partial differential operators $P(\Psi, F)$ which are invariantly defined and which only involve tangential derivatives of Ψ on ∂M. Let $\mathcal{Q}^0$ and $\mathcal{Q}^1$ be the vector spaces of all operators of the form

$$Q_0 := \sum_p P_p^0(\Delta^p \Psi, F),\ Q_1 := \sum_p P_p^1((\Delta^p \Psi)_{;m}, F) \text{ for } P_p^i \in \mathcal{P}.$$

The analysis of [1] shows that there exists $Q_n^i \in \mathcal{Q}^i$ so that

$$\beta_{2k-1}(\Theta, 0, F) = \int_{\partial M} (Q_{2k-1}^0 + Q_{2k-1}^1)(\Theta, F),$$

$$\beta_{2k}(\Theta, 0, F) = \int_{\partial M} (Q_{2k}^0 + Q_{2k}^1)(\Theta, F) + \frac{(-1)^k}{k!} \int_M \Delta^k \Phi \cdot f \tag{7}$$

for any smooth function Θ defined on M. Furthermore, if $\Theta_{;m} = 0$, then

$$\beta_n(\Theta, 0, F) = -2\beta_{n-2}(\Delta\Theta, O, F). \tag{8}$$

Choose a smooth function Θ on M so that for all k we have $(\Delta^k\Theta)_{;m}|_{\partial M} = 0$ and $(\Delta^k\Theta)|_{\partial M} = (\Delta^k U)|_{\partial M}$. We then have $Q_n^1(\Theta, F) = 0$ on ∂M. We use equations (7) and (8) to see

$$\int_{\partial M} Q_n^0(\Theta, F) = -\frac{2}{n}\int_{\partial M} Q_{n-2}^0(\Delta\Theta, F),$$

for all n; the interior boundary integrals which appear if n is even cancel. Since we have $\beta_0 = 0$ and $\beta_1 = 0$, this recursion relation shows

$$0 = \int_{\partial M} Q_n^0(\Theta, F) = \int_{\partial M} Q_n^0(U, f).$$

Since U is harmonic, we have $Q_n^1(U, F) = P_n(U_{;m}, F) = P_n(\Psi, N)$ for some P_n in $\mathcal{P}$. Consequently $\beta_n(0, \Psi, F)$ is computable in terms of the tangential derivatives of Ψ and the total derivatives of F. This completes the proof of Theorem 4.□

REMARK 5. An essential technical consideration in our proof was the creation of an additional Neumann boundary component to ensure that the average value of Ψ vanished. This is not necessary if C_D is non-empty; there is no obstruction to finding a harmonic function with given boundary data if there is at least one Dirichlet component. This can be seen as follows. Suppose that there is at least one Dirichlet boundary component. Choose Ψ_1 so that $\Psi_1 = \Psi$ on C_N and so that $\int_{\partial M} \Psi_1 = 0$. We find U_1 harmonic so that $\partial_N U_1|_{\partial M} = \Psi_1$. Using probabilistic methods, we can find U_2 harmonic so that $\partial_N U_2 = 0$ on C_N and so that $U_2 = \Psi - U_1$ on C_D. We then note $U = U_1 + U_2$ is a harmonic function with the required boundary data. We could also argue physically to see that we could take $U = \lim_{t\to\infty} H_{0,\Psi}(x; t)$ since the presence of at least one Dirichlet component eliminates infinite heat build up and establishes the existence of a stationary solution.

REMARK 6. An essential feature of our analysis is that the Neumann and the Dirichlet boundary conditions are imposed on disjoint closed subsets of the boundary. One could consider the following simple situation where this does not hold. Let M be the unit ball in 3 space. Take homogeneous zero Neumann boundary conditions on the upper hemisphere of the bounding sphere and homogeneous zero Dirichlet boundary conditions on the lower hemisphere of the bounding sphere; this corresponds to insulating the upper hemisphere and immersing the lower hemisphere in ice water. This problem, where the Neumann and Dirichlet boundaries intersect in a smooth surface of codimension two is not tractable by present methods as far as we know.

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