Bremsstrahlung radiation

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- G. Ghisellini: "Radiative Processes in High Energy Astrophysics"
- M. Longair: "High Energy Astrophysics" § 6.3 – 6.6
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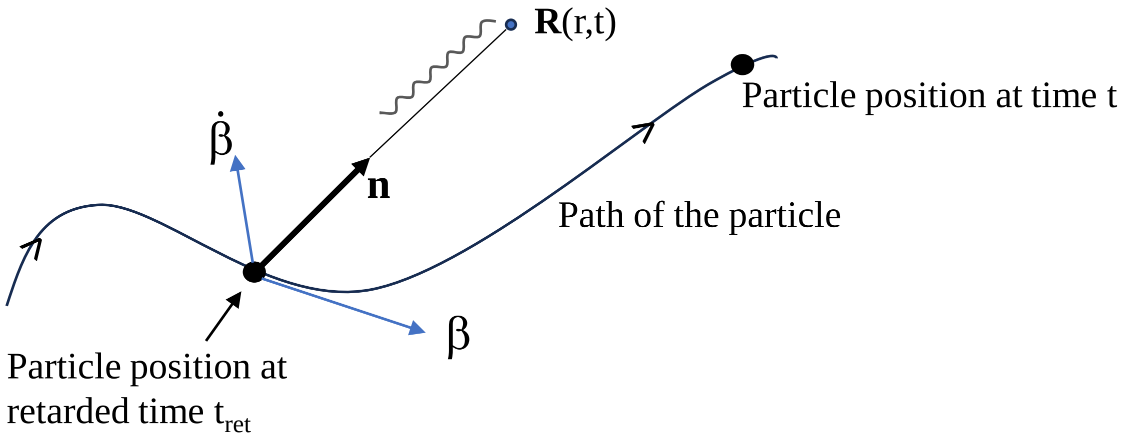
Electromagnetic field from moving particles

Consider a particle of charge q that moves along a trajectory r(t). The problem of solving the **Maxwell equations** reduces to the differentiation of the retarded potentials (Liénard-Wiechart potentials): $\mathbf{A} = \left[\frac{q\mathbf{u}}{c\kappa R}\right]_{t_{ret}}, \phi = \left[\frac{q}{\kappa R}\right]_{t_{ret}}$ with $\kappa \equiv 1 - \mathbf{n}\cdot\beta$ and $\beta \equiv \frac{\mathbf{v}}{c}$;

n is the unit vector of the (arbitrary) observation direction (the line of sight).

The electromagnetic field it generates at given position at time t is given by the **E,B** fields evaluated at the **retarded time** $t_{ret} = t - \frac{R}{c}$.



$$\mathbf{B}(\boldsymbol{r},t) = [\boldsymbol{n} \times \boldsymbol{E}(\boldsymbol{r},t)]$$

$$\mathbf{E}(\boldsymbol{r},t) = q\left[\frac{(\mathbf{n}-\beta)(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R} \times \{(\mathbf{n}-\beta)\} \times \dot{\beta}\right]$$

Velocity field , falling off as $1/R^2$.

For v<<c, it is the Coulomb law.

Acceleration field, falls off as 1/R.

Perpendicular to n.

At large distance from the charge, the acceleration field becomes the dominant one.

Radiation field

At large distances, we can then consider only the acceleration field, called «radiation field».

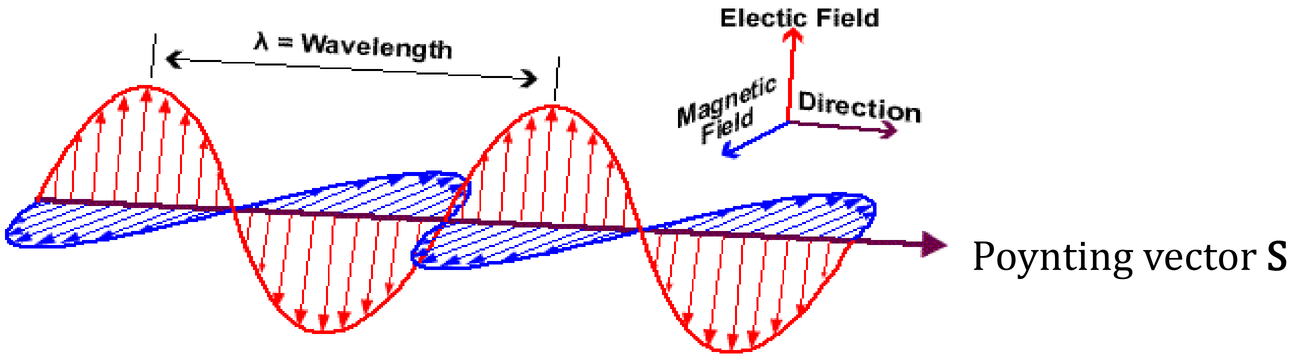
$$\mathbf{B}_{rad}(\boldsymbol{r},t) = [\boldsymbol{n} \times \boldsymbol{E}(\boldsymbol{r},t)] \qquad \mathbf{E}(\boldsymbol{r},t)_{rad} = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 \mathrm{R}} \times \{(\mathbf{n} - \beta)\} \times \dot{\beta}\right]$$

All evaluated at retarded time.

These terms, which add to the velocity fields, arise from differentiating the retarded potentials w.r.t. the retarded time, which implicitely depends on the position. This has very important consequences.

The Poynting vector $\mathbf{S} = \frac{c}{8\pi}\mathbf{E} \times \mathbf{B}$ has the direction of $\mathbf{n}$ ($\perp$to $\mathbf{E}$ and $\mathbf{B}$) and magnitude: $\mathrm{S} = \frac{c}{8\pi}\, \mathrm{E}_{rad}^2 = \frac{c}{8\pi}\, \mathrm{B}_{rad}^2$

Since $\mathrm{S} \propto \mathrm{R}^{-2}$, its **flux** doesn't vanishes even at large distances. This allows the radiation to flow to infinite distances! The existence of the radiation field is thus a consequence of the retardation, which, in turn, is a consequence of the finite value of the light speed.



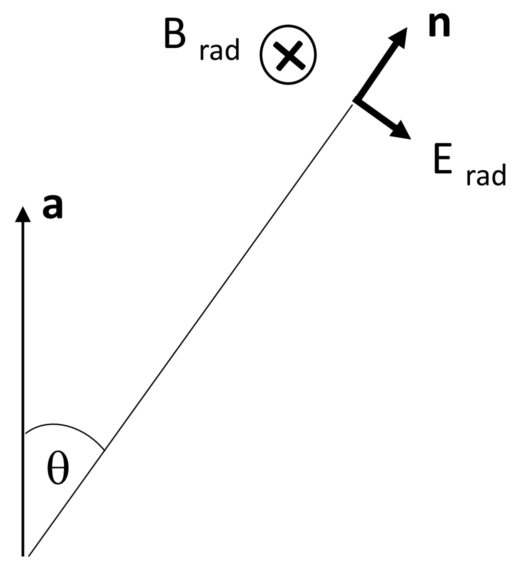
Radiation from accelerated particles

- A charged particle undergoing acceleration **radiates** photons. E.g. electrons moving back and forth in antennae (oscillating dipole) produce electromagnetic radiation.

- The power emitted by a charge q with an acceleration $\boldsymbol{a}$ is given by the Larmor's formula:

$dP/d\Omega = (q^2 a^2/4\pi c^3)\sin^2\theta$ which, integrated over solid angles, gives:

$$P = \frac{2\,q^2\,a^2}{3c^3}$$

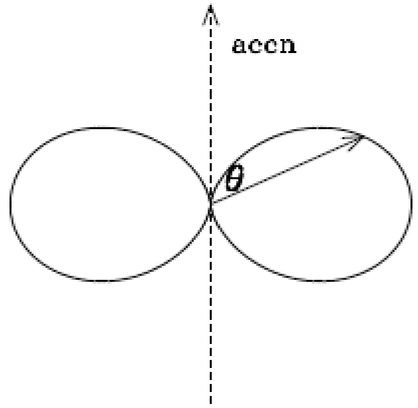


Properties:

-The emitted power is proportional to the square of q and a;

-The photons are emitted in a characteristic dipolar form $\propto \sin^2\theta$; no emission along the direction of **a**, maximum emission ⊥ to **a**.

- If the particle accelerates along a line, in the non-relativistic case, the radiation is 100% linearly polarised on the plane of **a** and **n**.



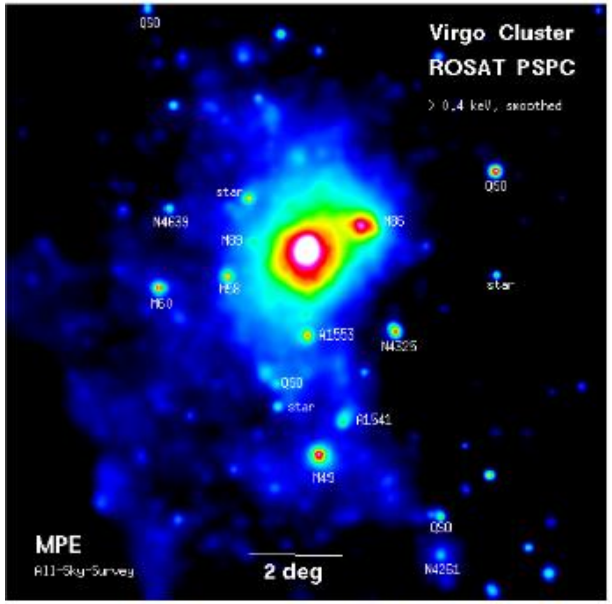
Dipolar emission by an accelerated non relativistic charge. See also RL Chapter 3

Bremsstrahlung radiation

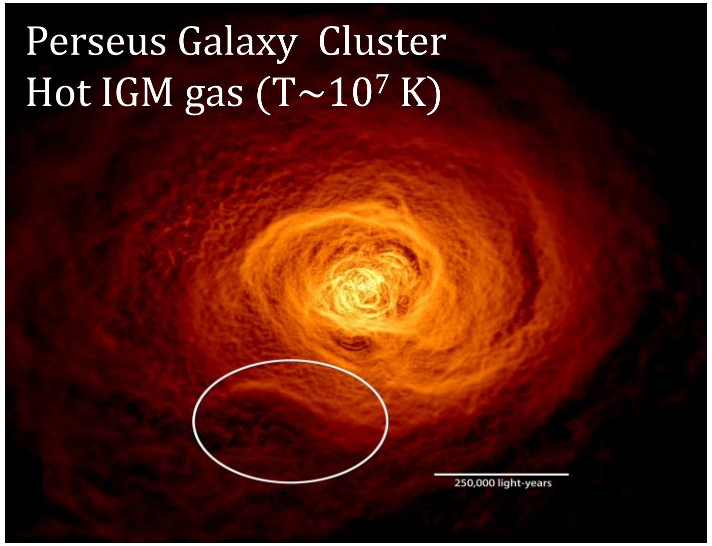
- The Bremsstrahlung is radiation emitted from **charges accelerated** (deflected) in the Coulomb field of another charge. It is also called free-free radiation. Energy is emitted at the expenses of kinetic energy («braking radiation»), thus free-free radiation can cool the plasma. In fact, Bremsstrahlung is the main cooling process for high T ($> 10^7$ K) plasma.
- Detailed treatment requires quantum electrodynamics However, a classical treatment is justified in some (most of) regimes, and the formulas so obtained have the correct functional dependence for most of the physical parameters. Therefore, we first give a classical treatment and then state the quantum results as corrections (Gaunt factors) to the classical formulas.
- In a system of charges, be τ the typical time scale for changes within the system. If the charge distribution has a size L smaller than the distance traveled by the light in the time τ, we can we can neglect the difference between the retarded times of the $\mathbf{E}_i$ emitted by each particle, and just sum them all to have the total electric field.
- For such a system of charges, $\mathrm{P} = \frac{2\, q^2 \ddot{d}^2}{3 c^3}$ where $\mathbf{d} = \sum_i q_i \boldsymbol{r}_i$ = dipole of the system.
- Bremsstrahlung due to collisions of **like particles** is zero in the dipole approximation, because the dipole moment $\mathbf{d} = \sum q_i \boldsymbol{r}_i$ is, in that case, proportional to the center of mass, $\sum m_i \boldsymbol{r}_i$, which is a constant of motion: $\ddot{\mathbf{d}} = 0$, no radiation according to Larmor law. We then consider electron-ion collisions, where the main radiatiors are the electrons (lower mass, larger acceleration).

Bremsstrahlung radiation- astrophysical examples:

- HII regions: $n_e \sim 10^2 - 10^3\ cm^{-3}$; $T \sim 10^4$ K
- 2. Galactic bulges & hot-coronae (10^7 K)
- 3. Intergalactic gas in clusters/groups : $n_e \sim 10^{-3}$ cm -3 ; $T \sim 10^7 - 10^8$ K



Virgo clusters is filled with hot gas strongly radiating via bremsstrahlung at 2keV (several billion degrees). Optically thin. Here, seen by ROSAT in the X rays. ↔ 8 Mpc

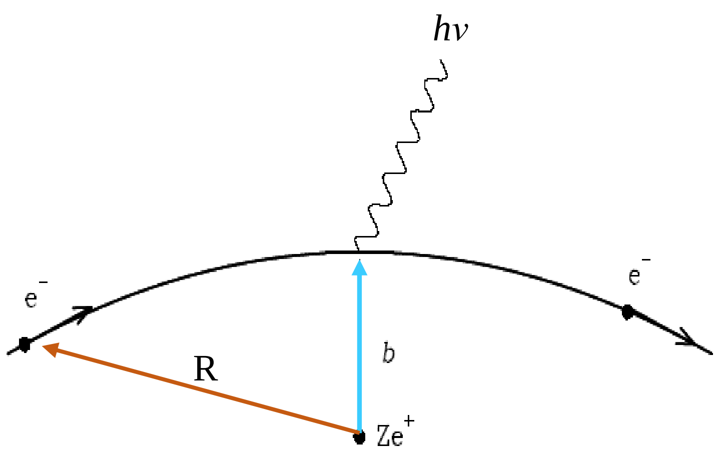


X-ray image of the hot gas in the Perseus Cluster , made from 16 days of Chandra observations. An oval highlights the location of an enormous wave found to be rolling through the gas.
Image credit: NASA's Goddard Space Flight Center / Stephen Walker *et al*.

HII regions: ionized by young hot stars and filled with hot gas, denser but cooler than a galaxy cluster.

Bremsstrahlung radiation from single electron

Consider an electron, e^-, with velocity v, passing , with impact parameter b, a charge of Z protons, total charge Ze^+ .



R=distance electron -ion

b=impact parameter

The electron is accelerated during the interaction. We use classical analogue of Born approximation: path of electrons not influenced by the nucleus, thus compute the motion along a straight line with given impact parameter. The acceleration of the electron in the Coulomb field is:

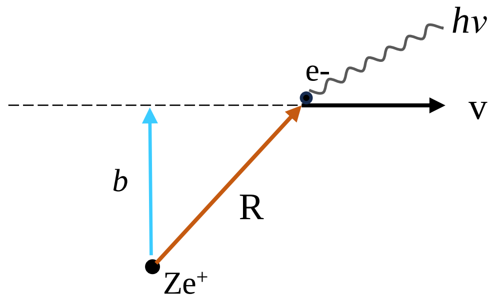
$$a = \frac{Z\,e^2}{m_e(b^2+v^2t^2)} = a(t,b)$$

Using Larmor's formula, $$P = \frac{2\,q^2\,a^2}{3c^3} = \frac{2\,z^2e^6}{3c^3\,m_e^2(b^2+v^2t^2)^2}$$

- The emitted power is maximum for the lightest particles (electrons) and at the closest distance (R~ b)
- The interaction is very quick; we define «collision time» for close interaction the time interval $\Delta t \sim 2b/v$
- The emitted energy in a single collision, **considering R ~b** , is:

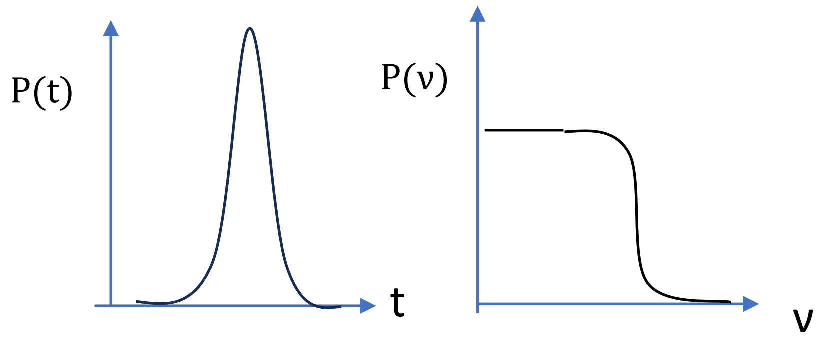
$$dW = P\ \Delta t \sim \frac{4\,z^2e^6}{3c^3\,m_e^2\ b^3 v}$$

Born approximation



Bremsstrahlung radiation from single electron

Radiation comes in pulses Δt long, slightly asymmetric because of the decrease of velocity. The acceleration is not uniform, so photons are produced with a range of frequencies, i.e. a spectrum, given by the Fourier analysis of the pulse.



$$\nu_{cut} = \omega_{cut} / 2\pi = v/(2\pi b)$$

The result is a flat spectrum in frequency, with an upper cutoff ($\omega_{cut} = v/2b$) related to the interaction time, $\Delta t \sim 2b/v$

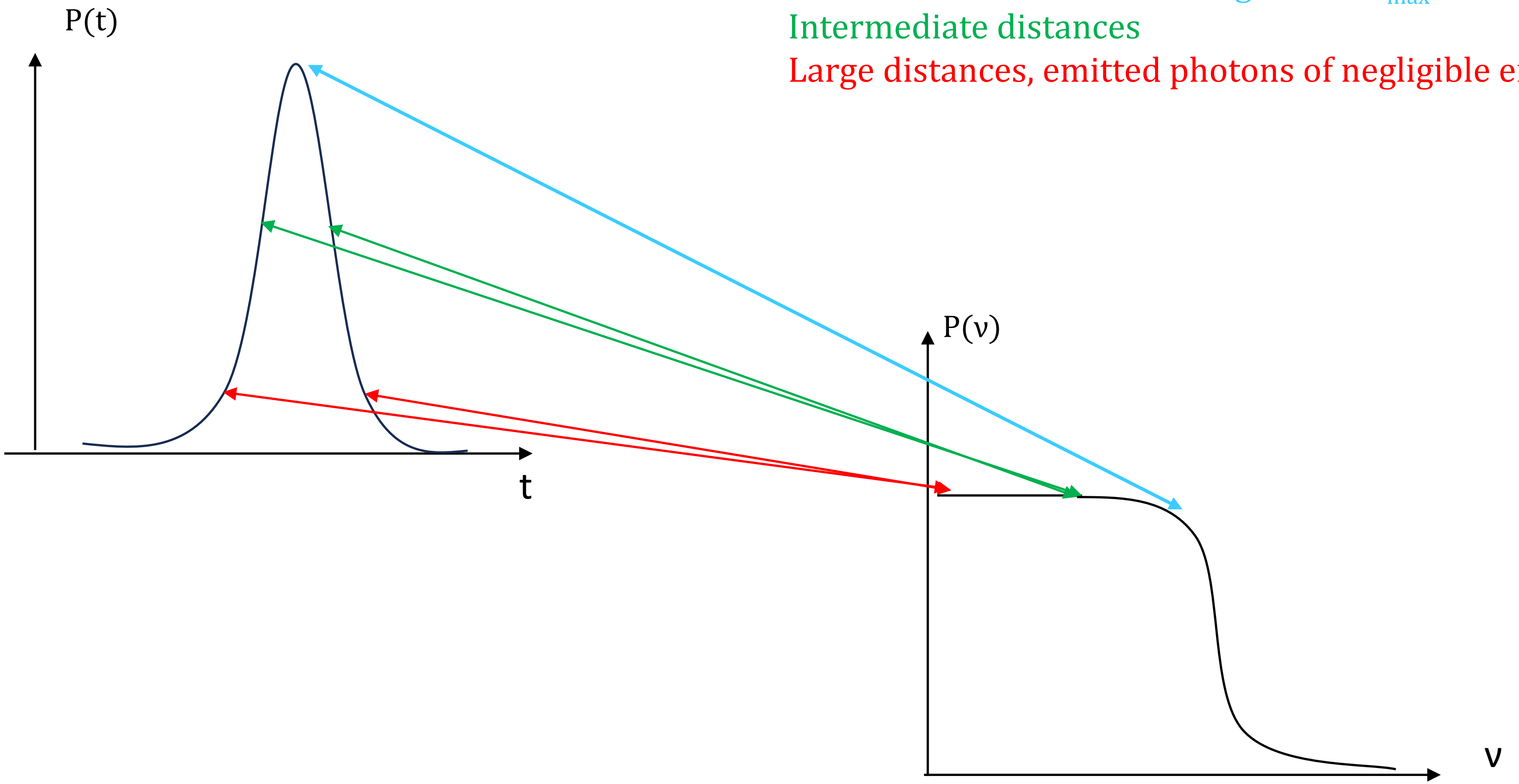
($\omega_{cut} = 2\pi\, \nu_{cut}$) In the sigle event, the intensity (radiated energy dW per unit frequency) in the flat part of the spectrum ($\omega < \omega_{cut}$) is given by:

$$\frac{P\Delta t}{\Delta\nu} \sim \frac{P\Delta t}{\nu_{max}} = \frac{16\, z^2 e^6}{3c^3\, m_e^2\, v^2 b^2} = \frac{dW(b)}{d\nu}; \qquad \frac{dW(b)}{d\omega} = \frac{8\, z^2 e^6}{3\,\pi\, c^3\, m_e^2\, v^2 b^2} \text{ for } b \ll \frac{v}{\omega}, \quad 0 \text{ otherwise.}$$

Energy emitted per unit frequency. At $\omega > \omega_{cut}$, the power drops as as $\exp(-2\,\omega\, b/v)$.

Bremsstrahlung radiation from single electron

Closest distance from the target ⟶ ν_{max}
Intermediate distances
Large distances, emitted photons of negligible energy (frequency)



Maximum and minimum impact parameter

Let us define b_{max} and b_{min} the extrema of the impact parameter values which can contribute to the emitted power at a given frequency.

- b_{max}: the single-electron emission drops exponentially as exp(-2 ω b/v), significant signal only for $b \ll v/\omega \equiv b_{max}$.
- b_{min}: we may take the classical value at which the straight-line approximation ceases to be valid, i.e. when $\Delta v \sim v$ (the change in velocity is to the path)

$$\Delta v \simeq a \, \Delta t = (Ze^2/ m_e b^2) \cdot 2b/v \longrightarrow b_{min,1} = \frac{2Ze^2}{mv^2} \quad \text{(classical)} ;$$

Or, we can consider the limit dictated by the quantum uncertainty principle: $\Delta x \, \Delta p \geq \hbar$

Taking $\Delta x \sim b$ and $\Delta p \sim mv$, we have $b_{min,2} = \dfrac{\hbar}{mv}$ (quantistic)

When $b_{min,1} \gg b_{min,2}$ a classical decription of the scattering is valid and we can use $b_{min} = b_{min,1}$. Otherwise, if $b_{min,1} \ll b_{min,2}$ the uncertainty principle plays an important role and we use $b_{min} = b_{min,2}$.

We then consider the largest between $b_{min,1}$ and $b_{min,2}$. Their ratio is $\dfrac{b_{min,2}}{b_{min,1}} = \dfrac{\hbar/mv}{2Ze^2/mv^2} \sim \dfrac{137\,v}{Zc} = \dfrac{137}{Zc}\sqrt{\dfrac{3kT}{m_e}} > 1$ when $v \geq 0.01c$

In any case, **the total energy radiated by one electron cannot exceed its kinetc energy!!!** Total radiated energy $= P \, \Delta t \leq \frac{1}{2} m_e v^2$

Problem 11. For a single electron-ion interaction, and for a fixed electron velocity, the maximum total radiated energy corresponds to the minimum allowed impact parameter.

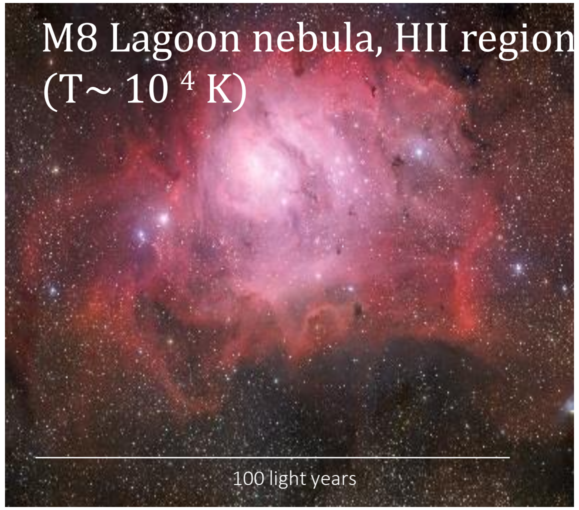
Show that this cannot exceed the kinetic energy of the electron, either for $b_{min} = b_{min,1}$ and for $b_{min} = b_{min,2}$

(useful to remember that $2\pi\, e^2 / 2m_e c = \alpha = 1/137$)

Problem 12 . Consider an electron moving at a speed of v =1000 km $s^{-1.}$

a) Calculate the frequency of the emitted radiation in case it would radiate all its kinetic energy in a single interaction (a single photon).

b) In case this electron belongs to a population of particles at thermal equilibrium for which the typical speed is $v = 1000$ km s^{-1}, determine which is the temperature of the plasma.



Problem. 13. Evaluate b_{min} for the warm plasma in the Lagoon nebula observed at 1 GHz, selecting if we have to use the classical or quantum value.

Gaunt factor

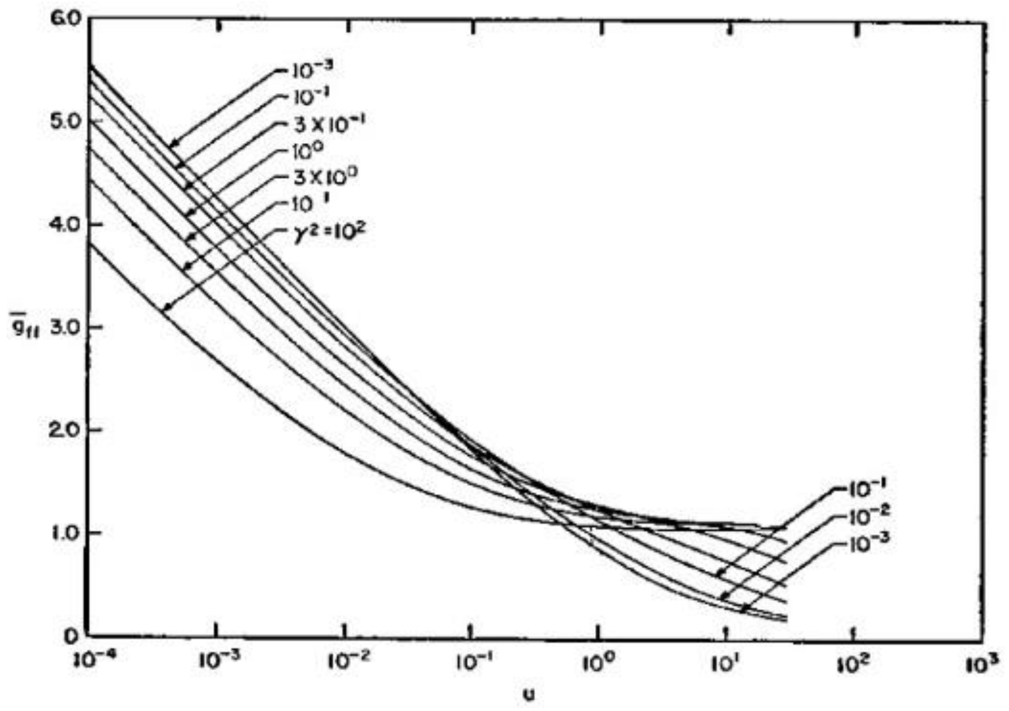
When dealing with a gas cloud hosting multiple electron-ion interactions, we have to, integrate the single-electron spectum over the emitters velocity distribution and the possible values of the impact parameter in order to obtain the total emitted power per unit volume and unit frequency. The power emitted in each collision event is, indeed, dependent on b. For a plamsa of electrons **all at the same speed** v, we end up with the calculation of the integral:

$$\int_{b_{min}}^{b_{max}} \frac{db}{b} = \ln\left(\frac{b_{max}}{b_{min}}\right)$$

It is convenient to enclose **all the quantum correction** to our classical treatment in the so-called **Gaunt factor** g_{ff}, function of electron energy and emission frequency.

It is defined as $$g_{ff}(v, \omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{max}}{b_{min}}\right).$$

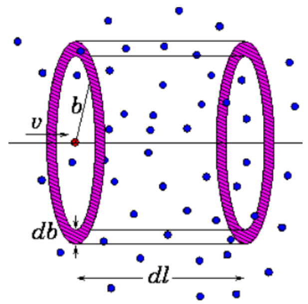
(Values are tabulated, e.g. in Bressaard and Van del Hulst 1962).



Gaunt factor, $\bar{g}_{ff}$, from Karzas & Latter, Fig. 5. $u = h\nu/kT$, for various values of $\gamma^2 = Z^2\,\mathrm{Ry}/kT$, where 1 Ry = 13.6 eV.

Thermal Bremsstrahlung emissivity

Generalise to a population of electrons in a (partially) ionized gas cloud . An astrophysically useful case is that of a population of electrons with uniform temperature (kinetic) T. Their total emission in this case is named thermal bremsstrahlung.



Number of electron-ions collision per unit time with impact parameter b,b+db: in a unit time: $2\pi b v n_i db$

For a ionized gas cloud at T, the velocity distribution is the Maxwell distribution:

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right)$$

The typical impact parameter is set by the number densities of electrons and ions, n_e and n_i and by f(v). Integrating the single-electron spectum over the velocity distribution and the possible values of the impact parameter, we obtain the total emitted power per unit volume and unit frequency:

$$\varepsilon_\nu^{ff} \equiv \frac{dW}{dV\, dt\, d\nu} = \frac{2^5 \pi e^6}{3 m_e c^3} \left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-\frac{1}{2}} Z^2 n_e n_i e^{-\left(\frac{h\nu}{kT}\right)} \bar{g}_{ff} \qquad (\mathrm{erg\ s^{-1}\ cm^{-3}\ Hz^{-1}})$$

Here, $\bar{g}_{ff}$(T,ω) is a velocity averaged Gaunt factor; it is of order unity over a wide range of temperature and densities (values around 10-15 in the radio domain, slightly higher than 1 at higher energies).

In CGS units, the power emitted per unit frequency is $(\mathrm{erg\ s^{-1}\ cm^{-3}\ Hz^{-1}})$:

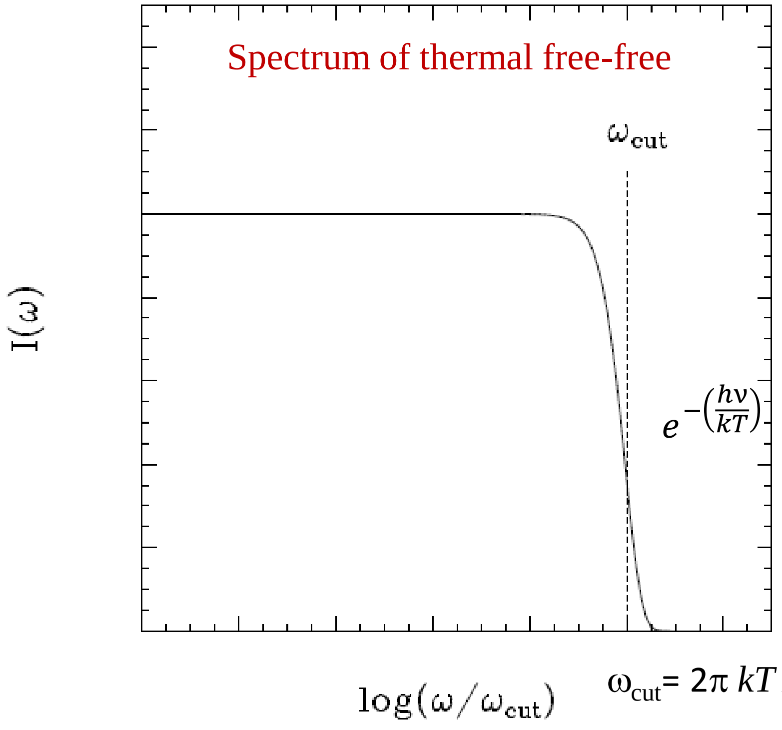
$$\varepsilon_\nu^{ff} \equiv \frac{dW}{dV\, dt\, d\nu} = 6.8 \times 10^{-38} T^{-\frac{1}{2}} Z^2 n_e n_i e^{-\left(\frac{h\nu}{kT}\right)} \bar{g}_{ff}$$

Flat spectrum with cutoff

Note that $\varepsilon_\nu^{ff} = 4\pi j_\nu$ (emission coefficient integrated over the solid angle, for an isotropically emitting plasma, from lecture 1: $\varepsilon_\nu = \int j_\nu \, d\Omega$)

Thermal Bremsstrahlung emissivity

This spectrum is flat in frequencies, up to a cut off at approximately $\nu_{max} = kT/h$, so the cutoff can be used to dermine the cloud **temperature**. The cutoff is indeed due to the Maxwellian distribution of velocities. Integrating over the whole spectrum up to ν_{cut}, one has



$\omega_{cut} = 2\pi\, kT/h$

$$\varepsilon^{ff} = \frac{dW}{dV\ dt} = \frac{2^5\pi e^6}{3hm_e c^3}\left(\frac{2\pi kT}{3m_e}\right)^{1/2} Z^2\, n_e n_i\ \bar{g}_{ff}$$

or, in CGS, (erg s^{-1} cm^{-3}, radiated power per cubic cm)

$$\varepsilon^{ff}(T) = 1.4\ \times 10^{-27}\, T^{\frac{1}{2}}\, Z^2\, n_e n_i\ \bar{g}_B \equiv \text{ total (bolometric) emissivity}$$

Here, $\bar{g}_B$ is a frequency average of the velocity averaged Gaunt factor, which is in the range 1.1 to 1.5.

In an optically **thin** medium, any internally generated radiation is essentially free to escape from the emitting region without further interaction with the medium.

For optically **thick** medium, radiation is only moving a short distance within the medium (w.r.t. its size) before being absorbed again. The final spectrum is shaped by the balance between emission and absorption. In a thick region, the spectrum is constrained to be not more efficient than a black body.

Note. While the individual interaction produces polarized radiation (fixed direction of E and B fields of the e-m wave wrt the acceleration) all the interactions produced in a hot plasma originate **unpolarized emission** (random orientation of the plane of interaction).

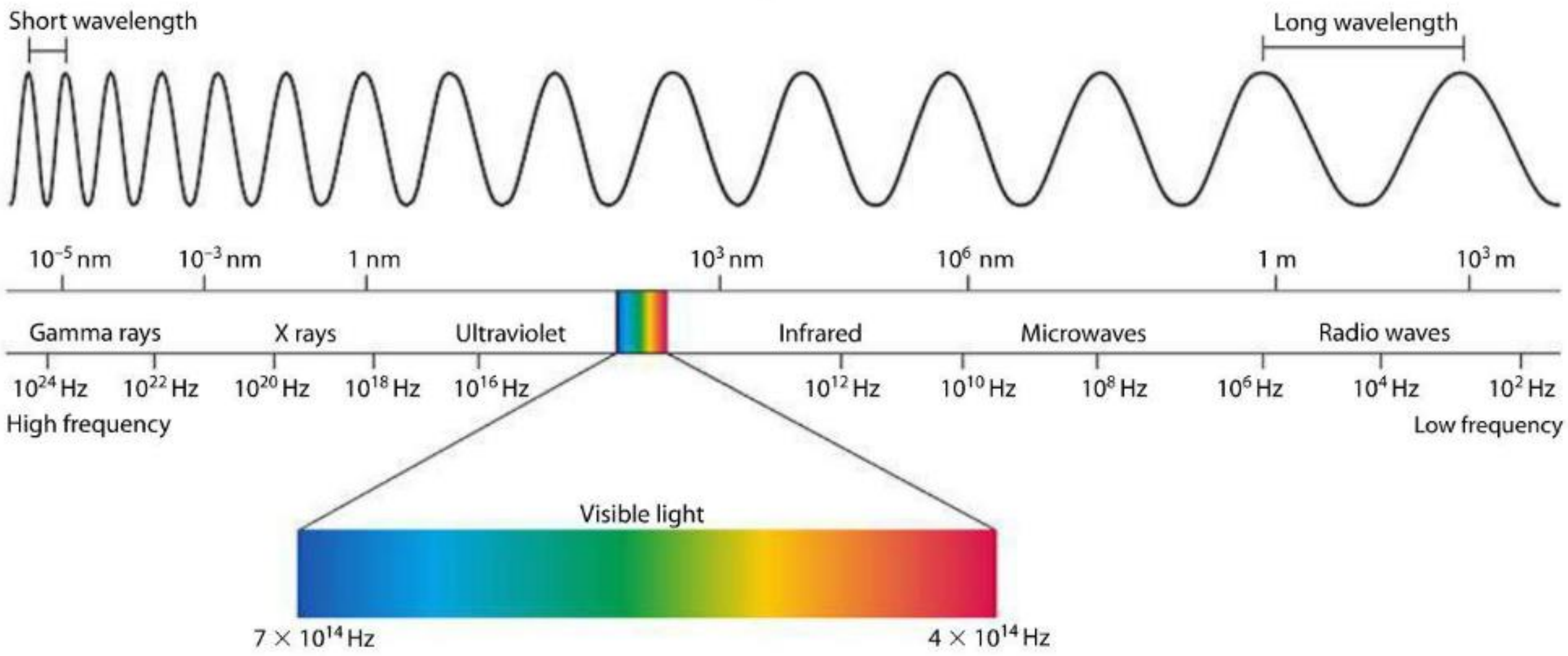
Thermal Bremsstrahlung emissivity

The cut-off frequency in the thermal bremsstrahlung spectrum depends on the temperature only, and is conventionally set when the exponential equals 1/e : $\nu_{max} = kT/h$

$\nu_{cut=} \frac{kT}{h} = 2.08\times 10^{10} \left(\frac{T}{K}\right)$ Hz Observationally used to determine T plasma!

For a **warm plasma** (HII region, $n_e \sim 10^2 - 10^3 cm^{-3}$ T ~ 10 ^{4} K), $\nu_{cut=}$ 10^{14} Hz (optical)

For a **hot plasma** (IGM in clusters of galaxies, $n_e \sim 10^{-3} cm^{-3}$, T ~ 10 ^{8} K), $\nu_{cut=}$ 10^{18} Hz (X-rays)



Thermal Bremsstrahlung self absorption

Now we want to determine the absorption coefficient. For thermal emission, we can use Kirchhoff's law:

$$\frac{\varepsilon_\nu^{ff}}{4\pi} = j_\nu^{ff} = \alpha_\nu^{ff}\, B_\nu(T)$$

and, using the Planck function,

$$\alpha_\nu^{ff} = \frac{4e^6}{3hm_e c}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2} Z^2\, n_e n_i\, \nu^{-3}\left(1 - e^{-h\nu/kT}\right)\bar{g}_{ff}$$

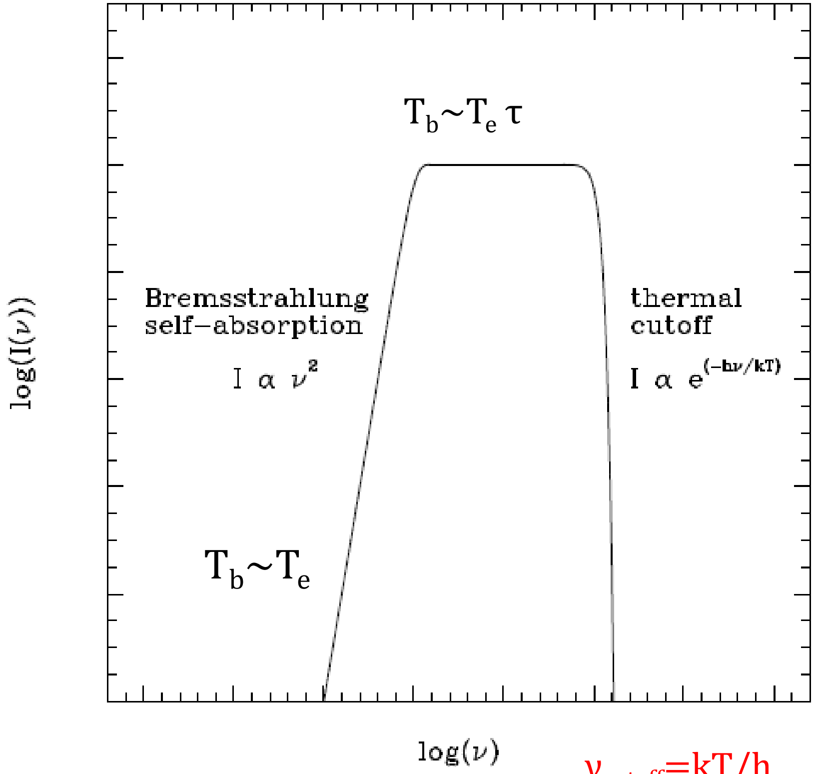
For $h\nu \gg kT$, $\alpha_\nu^{ff} \propto \nu^{-3}$ and absorption is negligible.

For $h\nu \ll kT$ (low frequencies) , Rayleigh-Jeans regime:

$$\alpha_\nu^{ff} = \frac{4e^6}{3m_e kc}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-3/2} Z^2\, n_e n_i\, \nu^{-2}\bar{g}_{ff}$$

Since $\bar{g}_{ff}(T) \approx 10$, in CGS $\alpha_\nu^{ff} \approx 0.2\ T^{-3/2} Z^2\, n_e n_i\, \nu^{-2}$ cm^{-1}

Self absorption becomes increasingly important at low frequencies.



$\nu_{cutoff} = kT/h$

Thermal Bremsstrahlung self absorption

Transfer equation gives the specific intensity, or Brightness, of a thermal free-free emitting cloud :

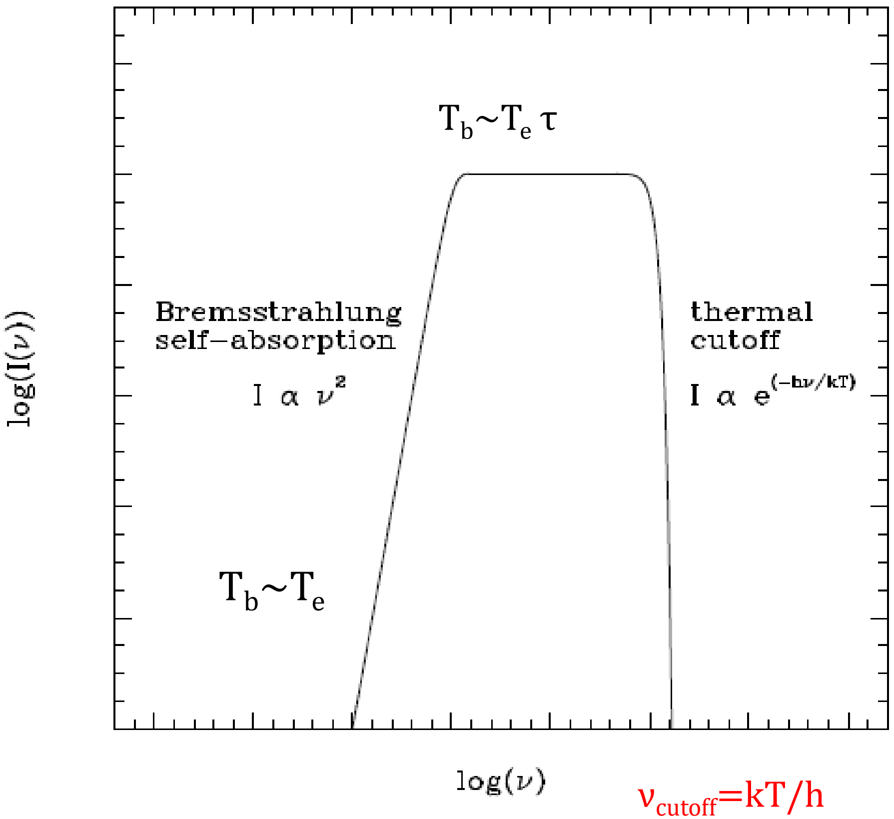
$$I_\nu(\tau) = \frac{j_\nu}{\alpha_\nu}(1 - e^{-\tau_\nu})$$

The **opacity** determines the shape of the spectrum.

In the low frequencies range, we can consider different values of the optical depth.

For $h\nu \ll kT$, when absorption is important,

$$\frac{j_\nu}{\alpha_\nu} \sim \frac{T^{-\frac{1}{2}} Z^2\, n_e n_i\, \bar{g}_{\ ff}}{T^{-3/2} Z^2\, n_e n_i\, \nu^{-2} \bar{g}_{\ ff}} \longrightarrow I_\nu(\tau) \approx T\, \nu^2\, (1 - e^{-\tau_\nu})$$



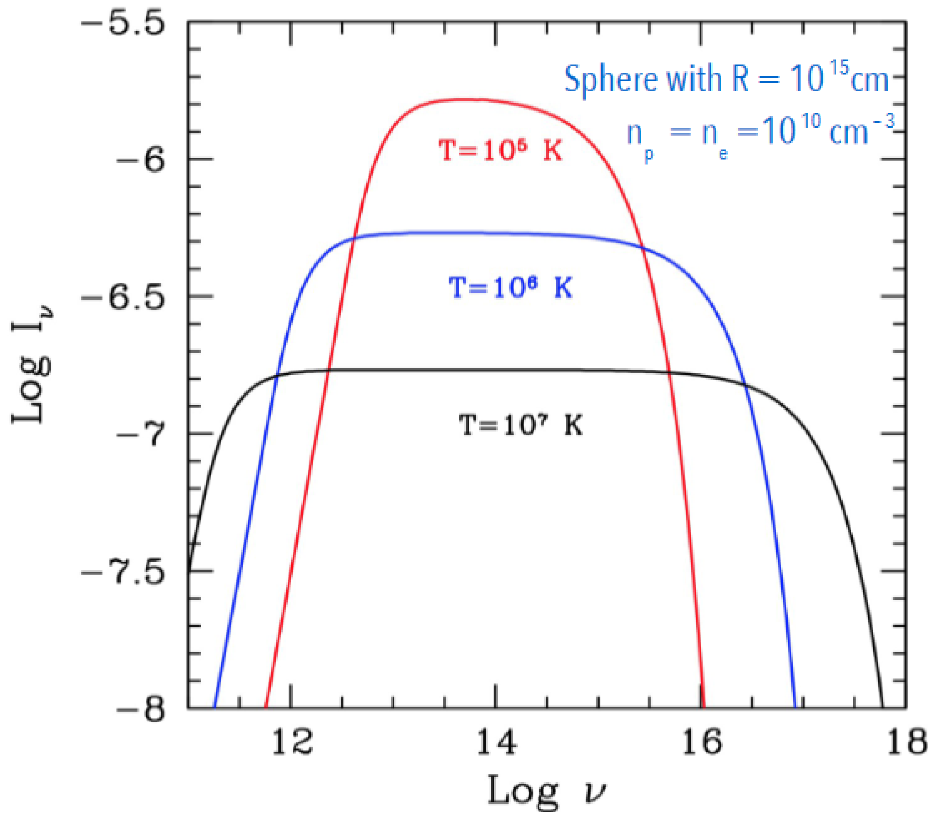
$\nu_{cutoff} = kT/h$

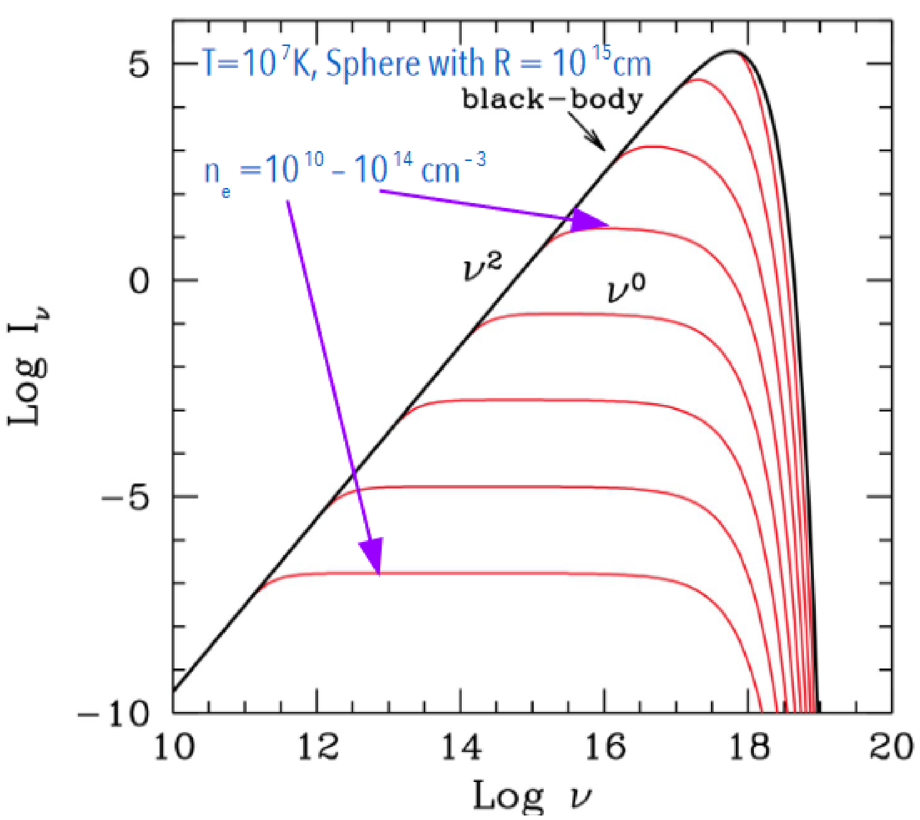
Possible values of optical depth in this regime:

- $\tau >> 1$, then $(1 - e^{-\tau_\nu}) \approx 1$ $\rightarrow I_\nu(\tau) \approx T\, \nu^2$ (Rayleigh-Jeans tail of Planck function) and $T_b \sim T$ ($\equiv T_e$) : the brightness temperature approaches the electron temperature. here the spectrum is blackbody-like because photons thermalize with electrons due to frequent absorption-emission (Opacity region of the spectrum).
- $\tau << 1$, then $(1 - e^{-\tau_\nu}) = (1 - 1 + \tau - O(\tau^2)) \approx \tau$ $I_\nu(\tau) \approx T\, \nu^2\, \tau$ and $T_b \sim T_e\, \tau$: the brightness temperature is much lower than the electron temperature. Many photons escape from the cloud before thermalizing with electrons. Large mean free path.

Thermal Bremsstrahlung spectrum

(Figures from Ghisellini)- Plasma sphere of **fixed size,** with $n_e = n_i$.





Fixed density: self-absorption is more effective at low temperatures.

$\varepsilon_\nu^{ff} \propto T^{-1/2}$; $\alpha_\nu^{ff} \propto T^{-1/2}$; $\nu_{max} \propto T$

Fixed temperature: increasing density, the plasma becomes transparent at increasing frequencies, approaching a black-body (completely opaque at all frequencies).

$\varepsilon_\nu^{ff} \propto n_e^2$; $\alpha_\nu^{ff} \propto n_e^2$; $\nu_{max} = kT/h =$ constant

Relativistic bremsstrahlung

It is a non thermal bremmstrahlung. In a distribution of relativistic electrons, one has to use relativistic kinetic energy. Not a Mawellian velocity distribution (*) .

RELATIVISTIC DOPPLER SHIFT

The main energy radiated in a collision (in the electron rest frame) is $dW' \sim \gamma \frac{Z^2 e^6}{m^2 c^4 b^3}$ and the energy in the observer frame is dW= γ dW' ,

$$dW \sim \gamma^2 \left(\frac{Ze}{b^2}\right)^2 \left(\frac{e^2}{mc^2}\right)^2 b \quad \Rightarrow dW \sim \gamma^2 \, b \, \sigma_T \, U_{field} \qquad \sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = \frac{8\pi}{3} r_0^2$$

, r_0 =classical electron radius

U_{field} is the energy density of the external field (in the rest frame of the ion) , thus we can estimate the bremsstrahlung process as though the electron passing close to the ion knocks out the energy density in a volume $b\,\sigma_T$, and in the process boosts this by a factor γ^2 .

The typical velocities are now relativistic and the energy

distribution is described by a power law.

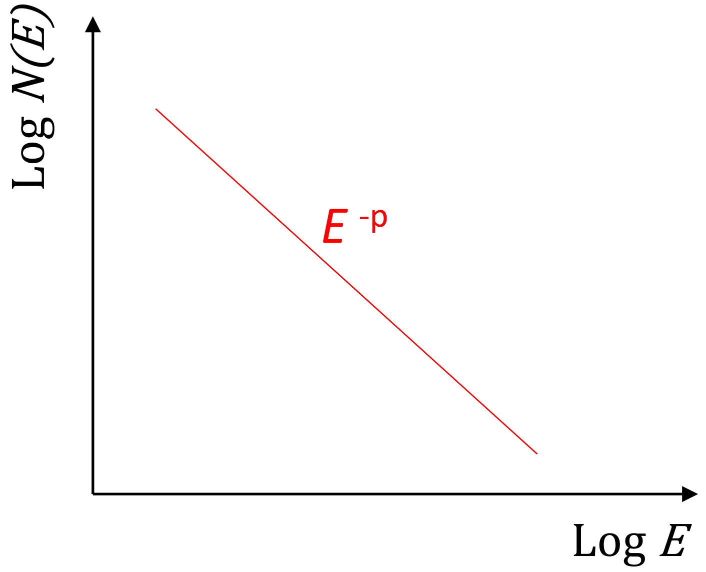
Number density of particles between E and E+dE :

$$N(E)dE = C\,E^{-p}\,\mathrm{dE} \text{ , or } N(\gamma)dE = C\,\gamma^{-p}\,d\gamma$$

Integrating over the energies (Lorentz factors),

$$\varepsilon^{ff}_{\nu,\, relativistic} \propto \int_{h\nu}^{\infty} E^{-p}\, dE \approx \nu^{-p+1}$$

The resulting spectrum is itself a power law.



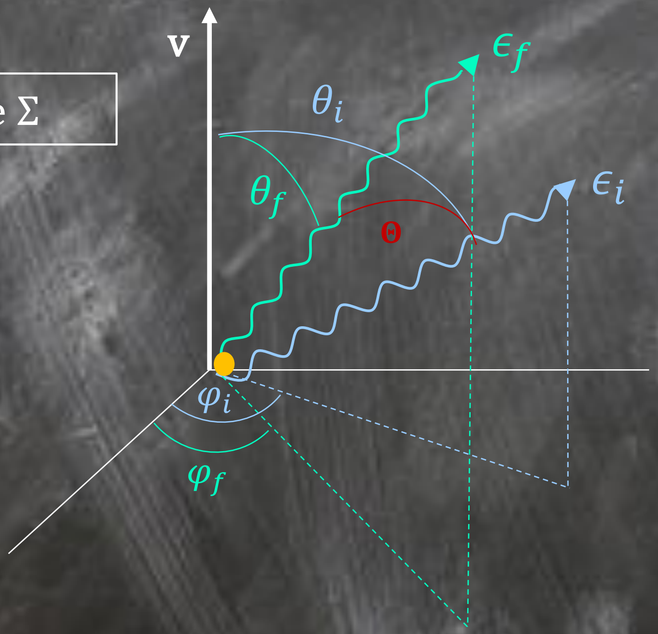
$\theta =$ angle between electron velocity and photon direction

$$\epsilon' = \gamma\epsilon(1 - \beta\cos\theta).$$

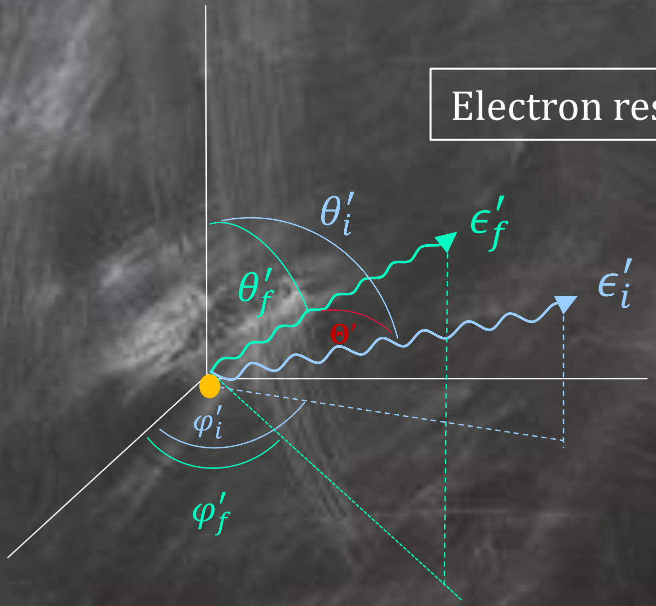
$$\epsilon = \gamma\epsilon'(1 + \beta\cos\theta')$$

Relativistic Doppler shift

Laboratory frame Σ



Electron rest frame Σ'



(*) **Relativistic velocity distribution.**

The Maxwelll Boltzmann distribution gives the probability for a single particle to have a speed in the interval [v,v+dv]. This probability is not zero for speeds v>c , in conflict with special relativity.

The distribution of speeds of particles in a hypothetical gas of relativistic particles is the **Maxwell–Jüttner** distribution

Similar to the Maxwell–Boltzmann distribution, the Maxwell–Jüttner distribution considers a classical ideal gas where the particles are dilute and do not significantly interact with each other. The distinction from Maxwell–Boltzmann's case is that effects of special relativity are taken into account. In the limit of low temperatures $T \ll m\, c^2/K_B$(where m is the mass of gas particles), this distribution becomes identical to the Maxwell–Boltzmann distribution.

The relativistic correction requires to replace the Newtonian kinetic energy ½ mv^2 with the relativistic kinetic energy $(\gamma-1)mc^2$ everywhere it appears in the distribution..

The relativistic energy of a particle is $E_r = \sqrt{(m_0 c^2)^2 + (pc)^2}$ and the relativistic kinetic energy is $E_k = \frac{mc^2}{\sqrt{1-(v/c)^2}} - mc^2 = mc^2(\gamma-1)$.

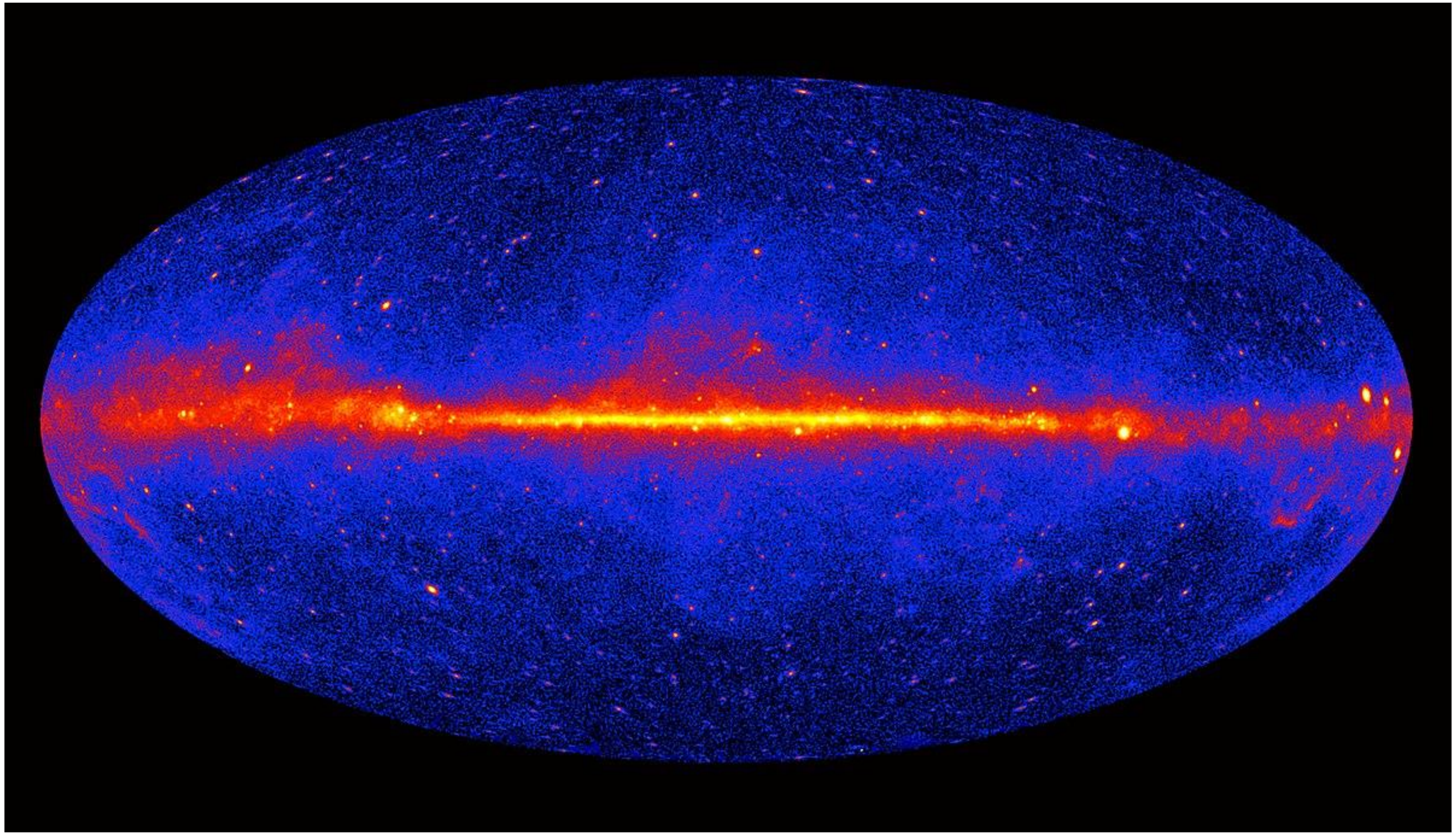
The **Maxwell–Jüttner** distribution gives the distribution for the Lorentz factors:

$$f(\gamma) = \frac{\gamma^2 \beta(\gamma)}{\theta \mathbf{K}_2\left(\frac{1}{\theta}\right)} e^{-\gamma/\theta}$$

where $\theta = \sqrt{\frac{K_B T}{mc^2}}$, $\beta = v/c$ and $\mathbf{K}_2$ is the modified Bessel function of the second kind.

Relativistic bremsstrahlung

Gamma-ray emission is detected from our Galaxy which is thought to arise from relativistic Bremsstrahlung from high energy electrons. The radiative energy is carried by photons with energies in the range 30-100MeV, suggesting many relativistic electrons with $\gamma \sim 100$



The Fermi Large Area Telescope detects gamma rays with energies ranging from 20 million electron volts (MeV) to more than 300 billion (GeV).

Discrete gamma-ray sources include **pulsars** and **supernova remnants** within our galaxy and distant galaxies powered by **supermassive black holes.**

NASA / DOE / Fermi LAT Collaboration

Bremsstrahlung cooling

A cloud of ionized gas emitting bremsstrahlung radiation loses energy , so the gas is cooling.

Cooling time : $t = \frac{E^{tot}}{\varepsilon_{ff}(T)} = \frac{3/2(n_e+n_p)kT}{\varepsilon_{ff}(T)}$.

We can assume $n_e = n_p$ (fully ionized H), so $t = \frac{3\, n_e kT}{2.4 \cdot 10^{-27} T^{1/2} n_e^2 \bar{g}_{ff}} \sim \frac{1.8 \cdot 10^{11}}{n_e\, \bar{g}_{ff}} T^{1/2}$ sec $= \frac{6.3 \cdot 10^{3}}{n_e\, \bar{g}_{ff}} T^{1/2}$ yr

Thermal bremsstrahlung is the main cooling process at temperatures above 10^{7} K.

Problem 14. An HII region is generated by an O-type star. Suppose it has a spherical volume of radius 1pc filled by pure Hydrogen plasma, at a temperature T= 1.6×10^4 K and numerical density n_e=10 cm^{-3}. Calculate the specific emissivity and the bolometric one. If the star was removed, after how much time will the region stop to emit? If the region is at a distance 250 pc, calculate the flux observed by a radiotelescope at 10 GHz. .(Assume the emission is optically thin at this frequency).

Useful formulae: flux $S_\nu = \frac{\varepsilon_\nu(T) \cdot V}{4\pi D^2}$ $\quad$ $L = \varepsilon(T) \cdot V$

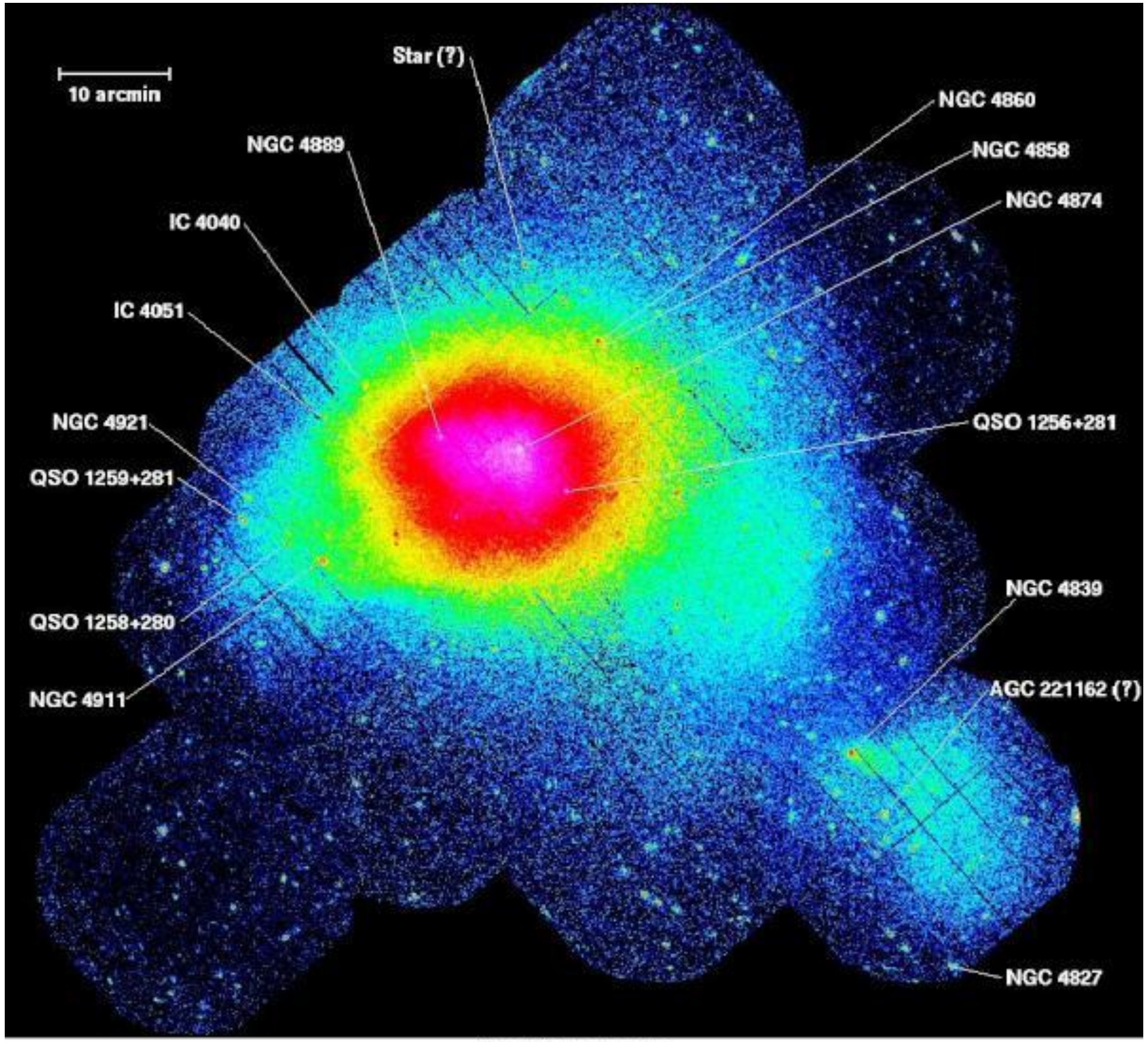
Bremsstrahlung cooling

Typical values of IC hot gas have radiative cooling time exceeding 10 Gyr .

All galaxy clusters are X–ray emitters.

Coma cluster (z=0.0232), size ~ 1 Mpc

ESA/XMM-Newton, the image covers the 0.3 to 2.0 keV energy range



Coma Cluster of galaxies

Image courtesy of U. Briel, MPE Garching, Germany

Credit: ESA/XMM-Newton, CC BY-SA 3.0 IGO

European Space Agency

Worked example.

Suppose we take the matter in our own Galaxy and heat it to T=10^6 K in a spherical region of order the Sun's distance from the Galactic center, R≈ 10 Kpc. Assume the cloud to be **fully ionized H**, with uniform density and total mass M=10^{11} $M_\odot$.

Electron density:

The total number of protons and electrons is $N_p = N_e = \frac{M}{m_p} = \frac{10^{11} \times 2 \times 10^{33}}{1.7 \times 10^{-24}} = 1.2 \times 10^{68}$ (neglect m_e) .

The volume of the cloud is $V = \frac{4\pi}{3} R^3 = 1.1 \times 10^{68}$ cm^3 , so the number density is $n_e = \frac{N_e}{V} \cong 1\ e^- \text{cm}^{-3}$

Hot thin Plasma!

Optical depth:

In a ionized medium, the primary source of opacity is due to Thomson (elastic) scattering of photons by free electrons.

In a uniform medium, the optical depth of the cloud is $\tau = \frac{1}{n_e \sigma_T}$, where $\sigma_T = 0.67 \times 10^{-24}$ cm^2 is the Thomson cross section.

In our example, then, $\tau = \frac{1\ cm^3}{0.67 \times 10^{-24}\ cm^2} = 1.49 \times 10^{24}$ cm ≈ 500 kpc (1 pc = 3 × 10^{18} cm).

τ =500 kpc ≫ physical dimension of the cloud (10 kpc) : the cloud is optically **thin**. Low density, radiation escapes easily.

Problem 15. What electron density is required to make the cloud optically thick?

The cutoff frequency of this cloud will be at $h\nu \approx kT$, so that ν≈ 2 × 10^{16} Hz (UV region) → source of UV photons.

Problem 16. The mean density of free electrons in the Universe is ~ 10^{-5} cm $^{-3}$. Show the Universe is optically thin to electron scattering. UV photons from a distant cloud like the one above would be nevertheless very hard to see. Why?

Worked example (continued)

Cooling rate

In CGS units, the power emitted per cubic cm is $\varepsilon^{ff}(T) = 1.4 \times 10^{-27}\, T^{\frac{1}{2}}\, Z^2\, n_e n_i\; \bar{g}_B$

The frequency averaged gaunt factor for thermal emission is usually adopted to be $\bar{g}_B$ =1.2.

We have Z= 1 for an Hydrogen plasma. Then

$$\varepsilon^{ff}(T) = 1.7 \times 10^{-27} \text{ erg cm}^{-3}\text{sec}^{-1}$$

From the volume of the cloud, V≈ 10^{68} cm^3 , we compute the total radiation loss:

$$\frac{dE}{dt} = 1.7 \times 10^{-27} \times 10^{68} = 2 \times 10^{44} \text{ erg sec}^{-1}$$

Total energy of the cloud

Each electron has kinetic energy $\mathrm{E_{kin}} = \frac{3}{2}\mathrm{kT} = 2.1 \times 10^{-10}$ erg ; there are ≈ 1.2×10^{68} electrons, so the total kinetic energy is

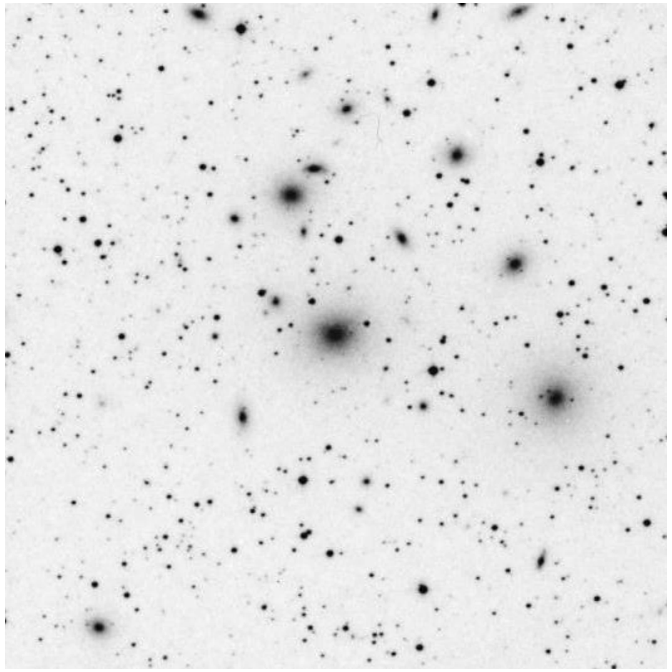
$$\mathrm{E_{tot}} = 2.1 \times 10^{-10} \times 1.2 \times 10^{68} \text{ erg} = 2.5 \times 10^{58} \text{ erg.}$$

So the cooling time is $\mathrm{t} = \frac{E_{tot}}{dE/dt} = \frac{2.5 \times 10^{58} \text{ erg}}{2 \times 10^{44} \text{ erg sec}^{-1}} = 1.5 \times 10^{14} \text{ sec} \sim 10^{6} \text{ years}$: hot thin plasma cools very rapidly!

Problem 17 Derive a general expression for the cooling time for a thin plasma, via thermal bremsstrahlung, in terms of the electron density n_e , the ion density n_i , ion charge Z, radius R and temperature T.

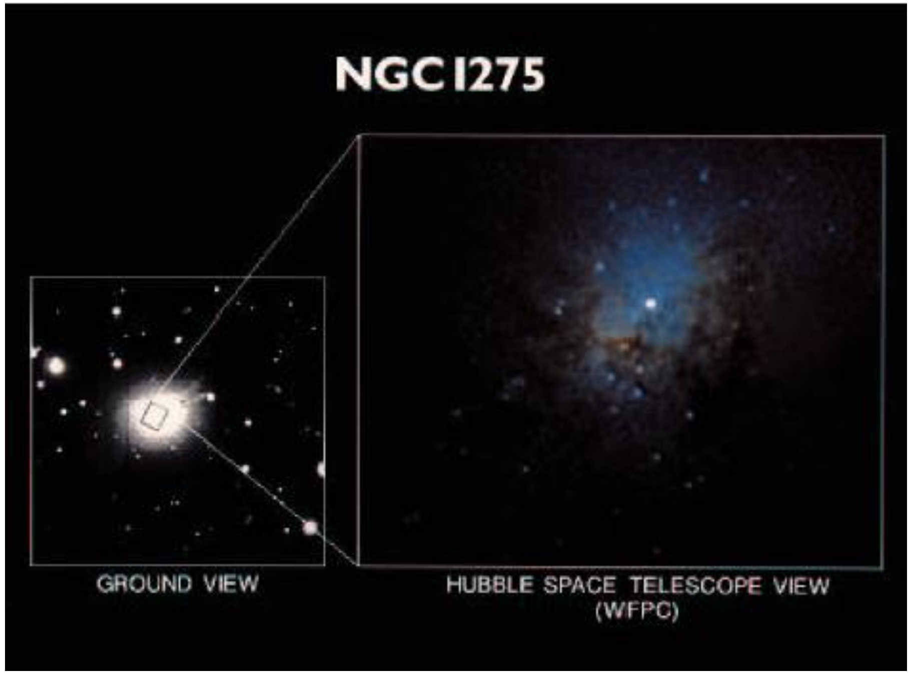
Bremsstrahlung case study- The Perseus cluster.

Perseus is a large cluster of galaxies in the Northern emisphere. It is about 100 Mpc away and contains thousands of galaxies embedded in a hot halo of gas. The central galaxy appears to be in the process of collision with a spiral galaxy, and is a well studied source of X-rays. The Perseus is one of the brightest X-ray cluster sources. Optical images at various scales are shown below:

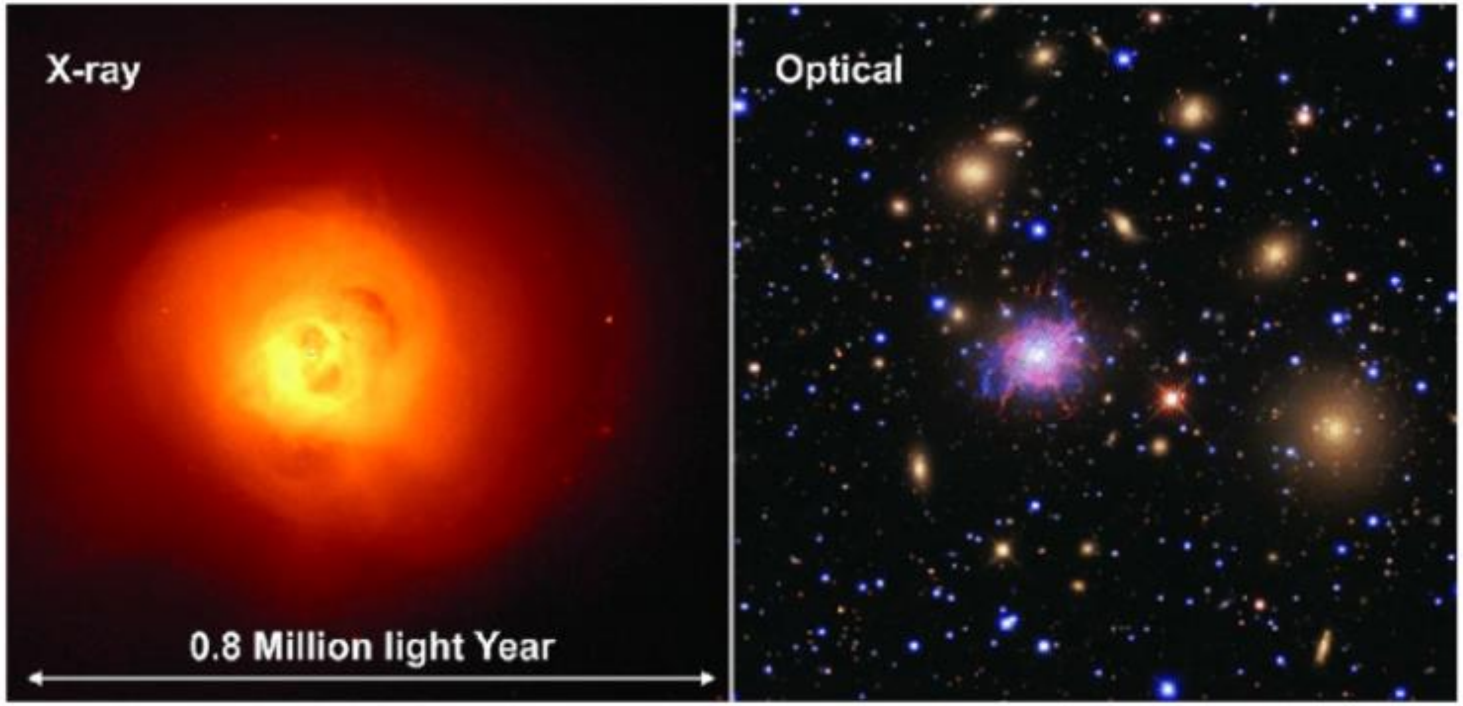


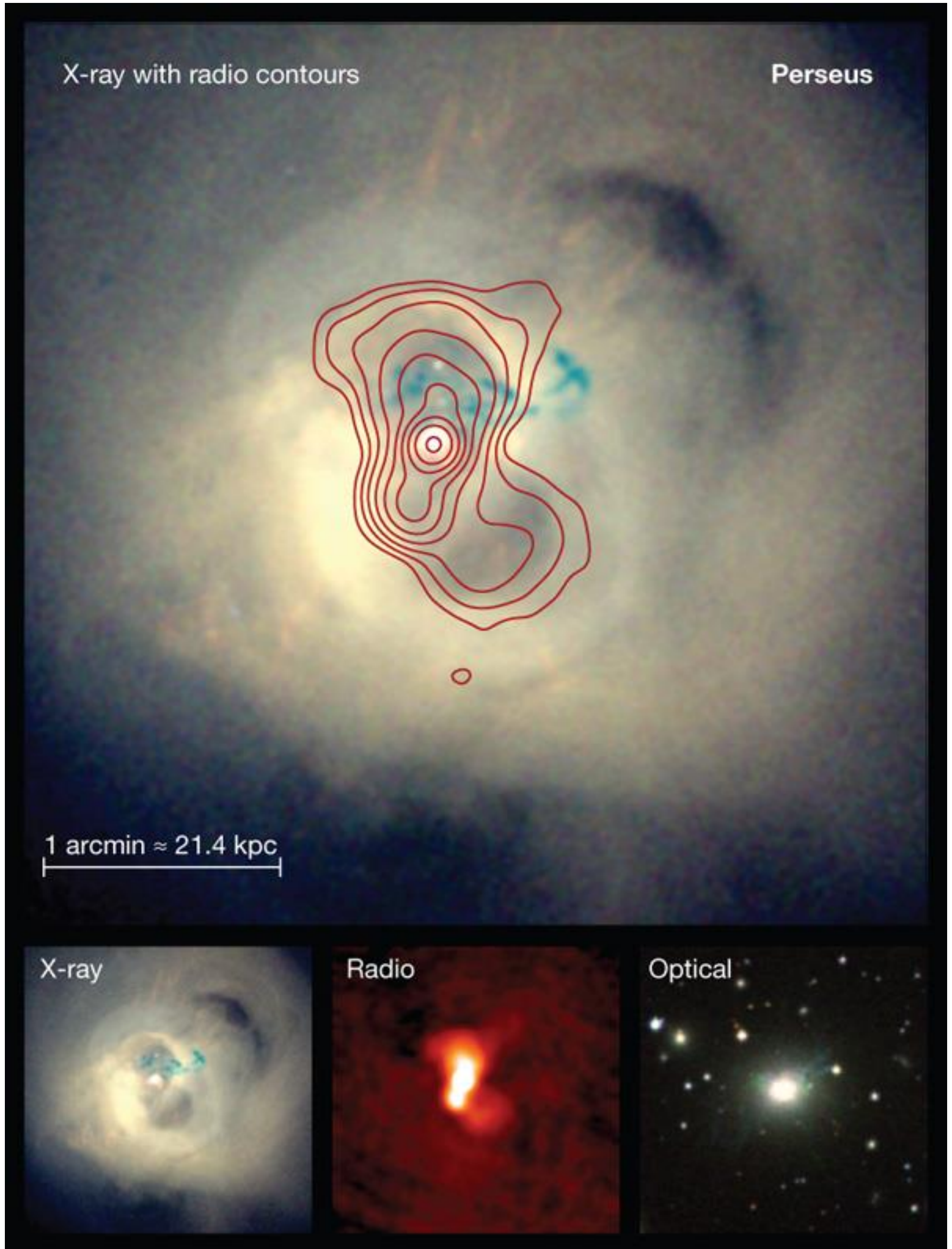
Perseus cluster from Palomar Sky Survey

Zoomed in view of the very central region, Hubble Space Telescope



Bremsstrahlung case study- The Perseus cluster.





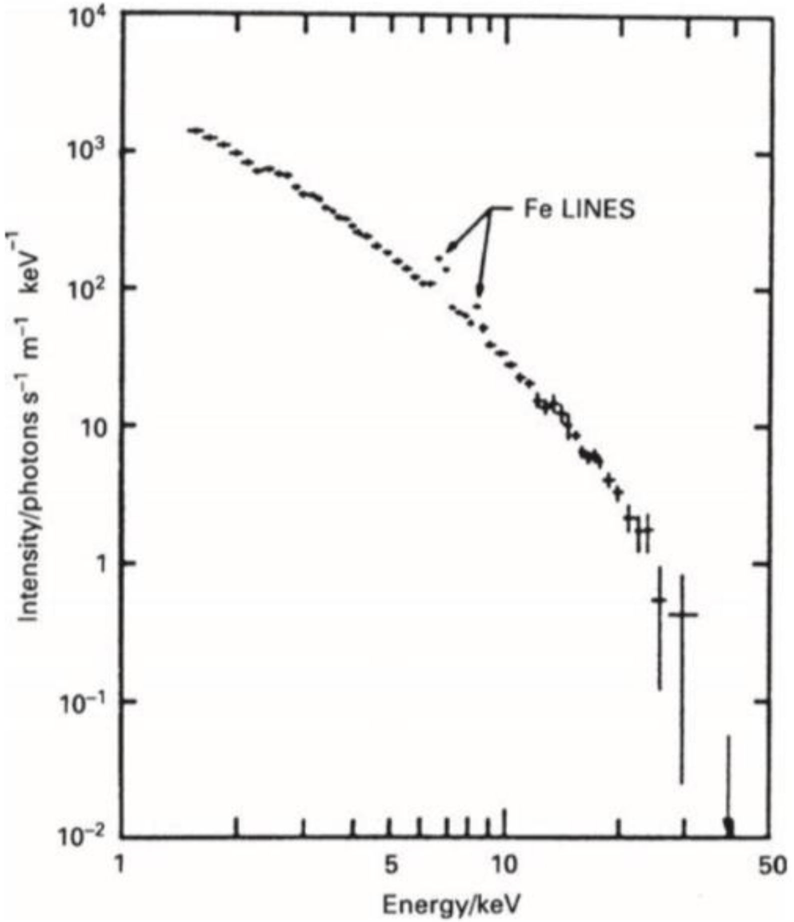
X ray and optical Perseus clusterer. Optical image shows stars and galaxies, whereas X image shows a huge amount of plasma distrubuted across a wide region (Fabian et al. 2011, Gabany 2009).

Bremsstrahlung case study- The Perseus cluster.

The X-ray spectrum of Perseus cluster observed by the HEAO-A2 instrument. The continuum emission can be accounted for the thermal bremsstrahlung of Hot intracluster gas at a temperature of kT= 06.5 eV, that is $T = 7.5 \times 10^{7}$ K. The thermal nature of the radiation is confirmed by the observation of the Lyα and Lyβ emission lines of highly ionized iron, Fe^{+25}, at energies 6.7 and 7.9 Kev, respectively. The ionization potential of Fe^{+24} is 8.825 keV, hence the gas must be very hot.

An estimate of the cooling time turn out to be relatively **short** compared to the age of the universe!

Computations indicate that hundreds of solar masses per year are being deposited into the core of the system (Fabian et al. 1981).



Note: it is common practice in X-and γ- ray astronomy to show spectra in terms of the number of photons per unit energy interval rather than intensity and so a flat intensity spectrum, $I(\nu)\, d\nu \propto \nu^0\, d\nu$, corresponds to a photon number intensity $N(\varepsilon) d\varepsilon \propto \varepsilon^{-1}\, d\varepsilon$, where $\varepsilon = h\nu$.