Two "Styles" of Axiomatization: Rules versus Axioms. A Constructive View of Theories.

Logic Colloquium 2017

August 17, 2017

Idea:

To use HoTT and Univalent Foundations as a paradigm for thinking about & representing & building theories in mathematics and science. This requires a serious revision of the standard axiomatic representation of theories both at the syntactic, semantic and pragmatic levels.

Euclid's *Elements*

Axioms (Common Notions) and Postulates in Euclid are *rules* but not sentences that admit truth-values, i.e., not axioms in the modern sense.

Many of Euclid's "Propositions" are Problems followed by Constructions while some other are Theorems followed by Proofs.

Problems and Theorems in Euclid share a common structure (an ancient prototype of Curry-Horward corespondence) and make part of a single deductive system, which is not adequately represented in standard modern axiomatic reconstructions of Euclid's geometry such as Hilbert's.

Is the missing element epistemically important and valuable? Well, given the remarkable endurance of Euclid's pattern of mathematical reasoning during many centuries and the fact that it survives today in the mainstream *non-formalized* mathematics there are certain reasons to think so.

Newton's *Principia*

Mathematical and experimental *methods* play a crucial role in the theoretical structure of the *Principia*. The title of the first Section of the first Book of Newton's *Principia* reads:

Of the Method of First and Last Ratios of Quantities

Quantum Field Theory

comprises both mathematical methods (such as Renormalization methods) and very sophisticated experimental methods used, in particular in ATLAS and CMS experiments at CERN's LHC in 2012.

Do such experimental methods play a role in the *logical* structure of QFT? Yes, because they provide crucial *evidences* aka *proofs* for claims of this theory. Any reasonable logical analysis and any logical reconstruction of theories involves an analysis and reconstruction of its proofs.

Standard approaches to formalization / axiomatization of mathematical and scientific theories leave mathematical and scientific *methods* (including methods of verification of established results) outside the formal axiomatic architecture of theories.

Syntactic View

Who and When: Logical positivists in 1930-ies: Carnap, Hempel, E. Nagel.

What: Old-fashioned Hilbert-style axiomatization of physical, biological and other theories.
Formal theories are given direct empirical interpretations in the same way in which such theories can be given geometrical or arithmetical interpretations.

Since empirical theories may involve mathematical theories one should decide which part of the non-logical syntax is interpreted empirically ("observational terms") and which part is interpreted mathematically ("theoretical terms"):

Semantic View

Who and When: E. Beth, P. Suppes and others since 1950-ies

What: To use Tarski's set-theoretic semantics of logic and his set-theoretic Model theory as an intermediate level of representation between the syntax of formal theories and the empirical and mathematical contents of the represented theories.

Slogan: A theory is a class of models but not a system of sentences expressed with some formal language.

Important idea: To make formal representation invariant with respect to possible syntactic choices.

Constructive View (disederata for a formal constructive framework

- supports representation of scientific methods including extra-logical methods;
- combines logical rules with constructive rules (i.e., rules for non-propositional objects);
- combines the representation of knowledge-that and knowledge-how;
- supports thought-experimentation and the experimental design (van Fraassen);
- involves a theory of formal semantics (rather than interprets syntax directly).

ProtoType: UniMath Library

Hilbert-style axiomatic theory

- A theory is a system of formal sentences (= sentential forms), which are satisfied in a model;
- Semantics of *logical* terms is rigidly fixed, semantics of *non-logical* terms is variable (cf. the standard concept of signature as the full list of non-logical terms).
  Interpretation: non-logical terms of formal sentences are given semantic values; under Tarski set-theoretic semantics these values are sets.
- The logical part of a theory comprises a fixed short list of syntactic rules, which are always interpreted as rules of logical inference.

Hilbert-style axiomatic theory (continued)

- An axiomatized theory comprises a distinguished subset of formal sentences called axioms, which generates other sentences (called theorems) via the application of the rules. An interpretation of the axioms that makes them into true statements also makes true all sentences generated by these axioms, so the theorems "logically follow from" the axioms.
- The theoretical content of a given theory is captured by axioms, theorems and their interpretations. Rules reflect the relation of logical consequence between these sentences and thus are not specific for any given theory.
- Axioms serve as "implicit definitions" (of classes of structures satisfying the axioms).

Gentzen-style axiomatic (?) theory

- “The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs.” (Gentzen:1935, p. 68). Natural Deduction and Sequent Calculus
- Syntactic rules serve as “implicit definitions” (of terms which are subject to the given rules). How a rule can be “satisfied”? A rule does not admit truth-values at the first place!

Gentzen-style axiomatic (?) theory

- "Meaning explanation" (cf. a program compiler) and Proof-Theoretic Semantics.
- Gentzen's proposal is an alternative to Hilbert's axiomatization of *logical* theories (via a long list of distinguished tautologies and a short list of rules, which generate all other tautologies from the distinguished ones). Can the same rule-based method of formal representation work outside the pure logic?

What is a Genzen-style theory? Syntax

We represent a given theory as a system of rules (including specific non-logical rules) rather than a system of sentences. Since a rule with the empty set of premises is an axiom this syntactic representation is more general than Hilbert's. We don't distinguish in advance between logical and extra-logical rules; we don't decide in advance which formulas represent sentences and which represent non-propositional objects.

Such distinctions are obviously semantic but they are often treated as a part of syntax. As we shall see HoTT provides the needed distinctions internally.

When a Hilbert-style and a Gentzen-style representation are deductively equivalent?

Definition

A propositional axiomatic theory is called *Hilbertian* when it comprises as theorems all formulae of the form $K_{A,B}$ and $S_{A,B,C}$ where

$$\begin{array}{ll} K_{A,B} \doteq & A \rightarrow (B \rightarrow A) \\ S_{A,B,C} \doteq & (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \end{array}$$

and has exactly one rule, namely MP.

When a Hilbert-style and a Gentzen-style representation are deductively equivalent?

Definition

Theory T is said to have the *Deduction Property* (DP for short) if $\Gamma, F \vdash G$ entails $\Gamma \vdash F \to G$ for all Γ, F and G.

When a Hilbert-style and a Gentzen-style representation are deductively equivalent?

Theorem

An axiomatic propositional theory is Hilbertian if and only if it has the Deduction Property.

(Vladimir Krupski)

When a Hilbert-style and a Gentzen-style representation are deductively equivalent?

The above observation allows us to specify certain sufficient but not necessary conditions of the translatability of Gentzen-style systems into the Hilbertian form. Many deductive systems of interest do not have the Deduction Property.

The difference between Tarski's model theoretic semantic, which is associated with the Hilbert style both historically and conceptually, and the PST motivated by Gentzen's work remains essential even when Hilbert- and Gentzen-style systems are deductively equivalent. Syntactic translations between such systems do not translate the intended semantics of derivations.

Martin-Löf 1983: Proof = Evidence

MLTT is a system of logic that is designed for managing proofs=evidences rather than only managing truth-values:

"[P]roof and knowledge are the same. Thus, if proof theory is construed not in Hilbert's sense, as metamathematics, but simply as a study of proofs in the original sense of the word, then proof theory as the same as theory of knowledge, which, in turn, is the same as logic in the original sense of the word, as the study of reasoning, or proof, not as metamathematics." (Martin-Löf 1983)

explanation of $t : T$ in Martin-Löf 1984

- t is an element of set T (Curry-Howard)
- t is a proof (construction) of proposition T
- t is a method of fulfilling (realizing) the intention (expectation) T (Heyting)
- t is a method of solving the problem (doing the task) T (Kolmogorov)

explanation of $t : T$

If we take seriously , the idea that a proposition is defined by laying down how its canonical proofs are and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to unnecessary duplication to keep the notions of proposition and set [. . .] apart. Instead, we simply identify them.

HoTT

"Types are Homotopy Types / Spaces."

This is obviously an *informal* interpretation, a mere "way of thinking of" and imagining elements of a formal syntactic system. Notice that unlike the case of Hilbert-style formal theories all (but not only non-logical) symbols and expressions of MLTT are interpreted here

One more item to the above list of informal interpretations? NOT just that.

h-stratification in MLTT

- (i) Given space A is called *contractible* (aka space of *h*-level -2) when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- (ii) We say that A is a space of *h*-level $n+1$ if for all its points x, y path spaces $paths_A(x, y)$ are of *h*-level n.

h-hierarchy

(0) : single point pt;

(1) : the empty space $\emptyset$ and the point pt : truth values aka *classical* or "mere" propositions

(2) : sets aka *intuitionistic*propositions aka theorems

(3) : (flat) groupoids

(4) : 2-groupoids

-
-

(n) $n-2$-groupoids

- ...

(ω) ω-groupoids

HoTT semantics for $t : T$ for (1)-types

propositions and truth-values

HoTT semantics for $t : T$ for (2)-types

theorems and their proofs / sets and their elements

HoTT semantics for $t : T$ for higher -types

(also valid for lower types):

spaces and points, which support higher-order structures from elements of some other spaces (viz. map spaces);

objects are points;
constructions are points provided with additional higher-order structures: paths, surfaces (homotopies), etc.

The above stratification of types is a robust mathematical structure in MLTT discovered via the homotopic interpretation of MLTT syntax. MLTT intended semantic does nottake this structure into account. HoTT semantics does.

HoTT semantics (or the version thereof that I defend) does not license the idea that *every* type is a proposition and that sets and propositions are the same.

Instead it recovers the distinction between propositional (in the Classical sense) and non-propositional (higher) types and the distinction between logical inferences and extra-logical constructions. Logic belongs to the level (1) of the *h*-hierarchy. Set theory belongs to level 2, etc. Every extra-logical (= non-propositional) construction serves here as a proof / evidence for a proposition obtained by its (1)-truncation. Thus the schematic rules of MLTT are applied under this semantics both at the propositional level and at the higher *h*-levels.

Voevodsky on Univalent Foundations

The main idea of the Univalent Foundations of Mathematics — to use constructive type theory together with the intuition coming from its univalent homotopy-theoretic semantics to write and to prove theorems about mathematical objects of all "levels" formally. (Voevodsky 2015)

Venus Homotopically

Such a straightforward application of HoTT in sciences may probably work in some special cases:

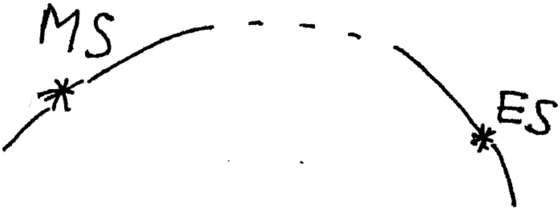

Figure: A proof that Morning Star and Evening Star are the same

Open problem: a formal procedural meaning explanation for UF

but in order to build with it a powerful tool for representing scientific theories it should be supplied by a formal mathematical procedural semantics / meaning explanation

(rather than a model theory based on the standard Tarskian satisfaction relation)

which could replace Set theory in its role of the intermediate level of representation in Suppes-Tarski "semantic" approach. We need a procedural intermediate language or "compiler" that translates MLTT or similar syntax into the procedural language of scientific theories that expresses commands and prescriptions of how to make measurements, conduct observations and experiments, etc.

Open problem: a formal procedural meaning explanation

An obvious candidate is Voevodsky's constructions on C-systems, which are formalizable in UniMath. However I don't know at the moment how to use them in the role of such compilers. The distinction between model theory (that involves truth-evaluation of formal sentences under a given interpretation) and procedural meaning explanation remains unclear to me in this case.

The concept of logical inference that emerges in this framework is not Tarsk's semantic consequence. It admits the existing PTS-style semi-formal explanations but its full procedural semantic explanation remains an open problem.

Conclusion 1

HoTT and Univalent Foundations suggest a novel approach in formal representation of scientific theories. The constructive view of theories motivated by this approach better fits the modern conception of science based on methods than the standard syntactic and semantic approaches, which rely on a more traditional Aristotelian image of science as a bulk of propositional knowledge structured by the relation of logical consequence.

Conclusion 2

While Suppes-style semantic representation of theories assumes a set-theoretic ontology, which allows one to think about all objects studied by sciences as elements of certain sets, the HoTT-based constructive approach assumes that universes of Homotopy types may serve as universal representational schemes of geometrical nature, which can play in modern science a role similar to which Euclidean geometrical representations play in the Early Modern science.

Conclusion 3

Last but not least such HoTT-based constructive representations of theories have an important pragmatic advantage of being ready for realization on computers.

thank you!

philomatica.org