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Extendibility, stable extendibility and span of some vector bundles over lens spaces mod 3

Dedicated to Professor Takao Matumoto on his sixtieth birthday

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ABSTRACT. Let $L^n(3)$ be the $(2n+1)$-dimensional standard lens space mod 3 and let ν denote the normal bundle associated to an immersion of $L^n(3)$ in the Euclidean $(4n+3)$-space. In this paper we obtain a theorem on stable unextendibility of R-vector bundles over $L^n(3)$ improving some results in [5] and [6], and study relations between stable extendibility and span of vector bundles over $L^n(3)$. Furtheremore, we prove that $c\nu$ is extendible to $L^m(3)$ for every $m > n$ if and only if $0 \le n \le 5$, and prove that $c(\nu \otimes \nu)$ is extendible to $L^m(3)$ for every $m > n$ if and only if $0 \le n \le 13$ or $n = 15$, where c stands for the complexification and $\otimes$ denotes the tensor product.

1. Introduction

Let (X, A) be a pair of spaces. A t-dimensional F-vector bundle ζ over A is said to be extendible (respectively stably extendible) to X if and only if there is a t-dimensional F-vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to ζ, where F is either the real number field R or the complex number field C (cf. [9] and [2]).

Let η_n be the canonical C-line bundle over the mod 3 standard lens space $L^n(3)$ of dimension $2n+1$, and $r\eta_n$ its real restriction.

For positive integers ℓ and t, define an integer $S(t, \ell)$ as follows.

$$S(t, \ell) = \min\{j \mid \lfloor t/2 \rfloor < j \text{ and } {}_{\lfloor t/2 \rfloor + \ell}\mathrm{C}_j \not\equiv 0 \mod 3\},$$

where $\lfloor x \rfloor$ denotes the largest integer s with $s \le x$, and ${}_k\mathrm{C}_r$ denotes the binomial coefficient $k!/(r!(k-r)!)$. Clearly, $\lfloor t/2 \rfloor < S(t, \ell) \le \lfloor t/2 \rfloor + \ell$. We obtain

THEOREM 1. *Let ℓ, n and t be positive integers with $\lfloor t/2 \rfloor + \ell < 3^{\lfloor n/2 \rfloor}$ and let ζ be a t-dimensional R-vector bundle over $L^n(3)$ which is stably equivalent to*

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$(\lfloor t/2 \rfloor + \ell) r\eta_n$. *Then* $n < 2S(t, \ell)$ *and* ζ *is not stably extendible to* $L^m(3)$ *for any* $m \geq 2S(t, \ell)$.

For an F-vector bundle ζ, $\mathrm{span}_F\, \zeta$ stands for the maximum number of linearly independent cross-sections of ζ, where F is R or C. Let n, h and t be positive integers with $t < 2h$. Consider the condition

$$C(m, a, c) : \mathrm{span}_R\{(a3^{\lfloor n/2 \rfloor} + h) r\eta_m \oplus c\} \geq 2(a3^{\lfloor n/2 \rfloor} + h) - t + c$$

for some integer m with $m > n$ and some non-negative integers a and c, where $\oplus$ denotes the Whitney sum.

For R-vector bundles over $L^n(3)$, we have

Theorem 2. *Let* n, h *and* t *be positive integers, and let* ζ *be a* t*-dimensional* R*-vector bundle over* $L^n(3)$ *which is stably equivalent to* $hr\eta_n$.

(i) *Suppose that* $t \geq 2h$. *Then* ζ *is stably extendible to* $L^m(3)$ *for every* $m > n$.

(ii) *Suppose that* $t < 2h$. *Then, for any* $m > n$, ζ *is stably extendible to* $L^m(3)$ *if and only if the condition* $C(m, a, c)$ *holds for some non-negative integers* a *and* c.

Let n and t be positive integers, let h and k be non-negative integers with $t < h + k$. Consider the condition

$$\begin{aligned} C(m, a, b, c) : \mathrm{span}_C\{&(a3^{\lfloor n/2 \rfloor} + h)\eta_m \oplus (b3^{\lfloor n/2 \rfloor} + k)\eta_m^2 \oplus c\} \\ &\geq a3^{\lfloor n/2 \rfloor} + h + b3^{\lfloor n/2 \rfloor} + k - t + c \end{aligned}$$

for some integer m with $m > n$ and some non-negative integers a, b and c, where η_m^2 denotes the tensor product $\eta_m \otimes \eta_m$.

For C-vector bundles over $L^n(3)$, we have

Theorem 3. *Let* n *and* t *be positive integers, let* h *and* k *be non-negative integers, and let* ζ *be a* t*-dimensional* C*-vector bundle over* $L^n(3)$ *which is stably equivalent to* $h\eta_n \oplus k\eta_n^2$. *Then*

(i) *Suppose* $t \geq h + k$. *Then* ζ *is stably extendible to* $L^m(3)$ *for every* $m > n$.

(ii) *Suppose* $t < h + k$. *Let* m *be any integer with* $m > n$. *Then, if* ζ *is stably extendible to* $L^m(3)$, *the condition* $C(m, a, b, c)$ *holds for some non-negative integers* a, b *and* c.

Moreover, in the case (ii) *we have the following necessary and sufficient condition:*

(iia) *Let* n *be even. Then, for any* $m > n$, ζ *is stably extendible to* $L^m(3)$ *if and only if the condition* $C(m, a, b, c)$ *holds for some non-negative integers* a, b *and* c.

(iib) *Let n be odd. Then, for any $m > n$, ζ is stably extendible to $L^m(3)$ if and only if the condition $C(m, a, b, c)$ holds for some non-negative integers a, b and c such that $a \equiv b \equiv 0 \mod 3$.*

Next, we have

THEOREM 4. *Let $c\nu$ be the complexification of the normal bundle ν associated to an immersion of $L^n(3)$ in the Euclidean $(4n+3)$-space R^{4n+3}. Then $c\nu$ is extendible to $L^m(3)$ for every $m > n$ if and only if $0 \leq n \leq 5$.*

Finally, we have

THEOREM 5. *Let $c\nu^2$ be the complexification of the square $\nu^2 (= \nu \otimes \nu)$ of the normal bundle ν associated to an immersion of $L^n(3)$ in R^{4n+3}. Then $c\nu^2$ is extendible to $L^m(3)$ for every $m > n$ if and only if $0 \leq n \leq 13$ or $n = 15$.*

This paper is arranged as follows. In Section 2 we recall the known results that are used for our proofs. In Sections 3, 4 and 5 we prove Theorems 1, 2 and 3, respectively. By complexifying the results obtained in the previous papers [6] and [5], we prove Theorems 4 and 5 in Sections 6 and 7, respectively. Some examples are shown in corresponding sections. The authors wish to express sincere thanks to the referee for valuable suggestions.

2. Known results

Let $L_0^n(3)$ denote the $2n$-skeleton of the CW-decomposition of $L^n(3)$. The ring structure of the reduced Grothendieck ring $\tilde{K}_R(L^n(3))$ is determined in [3] as follows (cf. [4] and [7]).

THEOREM 2.1 (cf. [3, Theorem 2] and [4, Proposition 2.11]).

$$\tilde{K}_R(L^n(3)) \cong \begin{cases} \tilde{K}_R(L_0^n(3)) + Z/2 & \text{for } n \equiv 0 \mod 4, \\ \tilde{K}_R(L_0^n(3)) & \text{otherwise}, \end{cases}$$

where + denotes the direct sum. The group $\tilde{K}_R(L_0^n(3))$ is isomorphic to the cyclic group $Z/3^{\lfloor n/2 \rfloor}$ of order $3^{\lfloor n/2 \rfloor}$ and is generated by $r\sigma_n (= r\eta_n - 2)$. Moreover, the ring structure is given by

$$(r\sigma_n)^2 = -3r\sigma_n, \quad \text{namely } (r\eta_n)^2 = r\eta_n + 2, \qquad \text{and} \qquad (r\sigma_n)^{\lfloor n/2 \rfloor + 1} = 0.$$

The ring structure of the reduced Grothendieck ring $\tilde{K}_C(L^n(3))$ is determined in [3] as follows (cf. [4] and [7]).

THEOREM 2.2 (cf. [3, Theorem 1] and [4, Lemma 2.4]).

$$\tilde{K}_C(L^n(3)) \cong \tilde{K}_C(L_0^n(3)) \cong Z/3^{\lfloor (n+1)/2 \rfloor} + Z/3^{\lfloor n/2 \rfloor}$$

The first summand is generated by $\sigma_n = \eta_n - 1$ *and the second summand is generated by* σ_n^2. *Moreover, the ring structure is given by*

$$\sigma_n^3 = -3\sigma_n^2 - 3\sigma_n, \quad \textit{namely } \eta_n^3 = 1, \qquad \textit{and} \qquad \sigma_n^{n+1} = 0.$$

Let $d = 1$ or 2 according as $F = R$ or C, and, for a real number x, let $\lceil x \rceil$ denote the smallest integer s with $x \leq s$. The following two results are well-known.

THEOREM 2.3 (cf. [1, Theorem 1.2, p. 99]). *Let* $m = \lceil (n+1)/d - 1 \rceil$. *Then each t-dimensional F-vector bundle over an n-dimensional CW-complex X is equivalent to* $\alpha \oplus (t - m)$ *for some m dimensional F-vector bundle* α *over X if* $m \leq t$.

THEOREM 2.4 (cf. [1, Theorem 1.5, p. 100]). *Let* $m = \lceil (n+2)/d - 1 \rceil$. *Then two t-dimensional F-vector bundles over an n-dimensional CW-complex which are stably equivalent are equivalent if* $m \leq t$.

COROLLARY 2.5. *Let X be a finite CW-complex and A its subcomplex and let* ζ *be an F-vector bundle over A such that* $\lceil (\dim A + 2)/d - 1 \rceil \leq \dim \zeta$. *Then* ζ *is extendible to X if and only if* ζ *is stably extendible to X.*

PROOF. The "only if" part is clear. We prove the "if" part. Suppose that ζ is stably equivalent to $i^*\alpha$ for some F-vector bundle α over X, where $i : A \to X$ is the inclusion. If $\lceil (\dim A + 2)/d - 1 \rceil \leq \dim \zeta$, then ζ is equivalent to $i^*\alpha$ by Theorem 2.4. □

The following is due to [8].

THEOREM 2.6 (cf. [8, Theorem 1.7]). *Let p be an odd prime and let* $L^m(p)$ *denote the* $(2m+1)$*-dimensional standard lens space* mod p. *Assume that* ξ *is any* $(2r+1)$*-dimensional R-vector bundle over* $L^m(p)$ *satisfying*

$$p_s(\xi) = 0, \ p_{s+1}(\xi) = 0, \dots, p_r(\xi) = 0$$

for an integer s with $1 \leq s \leq r$, *where* $p_j(\xi) \in H^{4j}(L^m(p), Z)$ *is the j-th Pontrjagin class of* ξ. *Then*

$$\operatorname{span}_R \xi \geq 2r - 2s + 2$$

if $p > m - 2s + 2$, *and if* $\operatorname{span}_R \pi^*\xi \geq 2r - 2s + 2$, *where* $\pi : S^{2m+1} \to L^m(p)$ *is the projection.*

On stable unextendibility of C-vector bundles over $L^n(p)$, we have

THEOREM 2.7 (cf. [6, Theorem 4.5]). *Let p be a prime and let* ℓ, *n and t be positive integers with*

$$t + \ell < p^{\lfloor n/(p-1) \rfloor}.$$

Assume that ζ is any t-dimensional C-vector bundle over $L^n(p)$ which is stably equivalent to a sum of $t+\ell$ non-trivial C-line bundles. Then $n < t+\ell$ and ζ is not stably extendible to $L^m(p)$ for every $m \geq t+\ell$.

Let p be a prime. For calculations of the binomial coefficients mod p, the following is useful.

Lemma 2.8 (cf. [10, Lemma 2.6, p. 5]). *Let p be a prime and let $a = \sum_{0 \leq i \leq m} a(i) p^{a(i)}$ and $b = \sum_{0 \leq i \leq m} b(i) p^{b(i)}$ $(0 \leq a(i),\ b(i) < p)$. Then*

$$ {}_a\mathrm{C}_b \equiv \prod_{0 \leq i \leq m} {}_{a(i)}\mathrm{C}_{b(i)} \mod p.$$

3. Proof of Theorem 1

Proof of Theorem 1. Since ζ is stably equivalent to $(\lfloor t/2 \rfloor + \ell) r\eta_n$, the total Pontrjagin class $p(\zeta)$ of ζ is given by

$$p(\zeta) = p((\lfloor t/2 \rfloor + \ell) r\eta_n) = p(r\eta_n)^{\lfloor t/2 \rfloor + \ell} = (1 + z_n^2)^{\lfloor t/2 \rfloor + \ell},$$

where z_n is the generator of $H^2(L^n(3), Z) = Z/3$. Suppose that $2S(t,\ell) \leq n$. Then $p_{S(t,\ell)}(\zeta) = {}_{\lfloor t/2 \rfloor + \ell}\mathrm{C}_{S(t,\ell)} z_n^{2S(t,\ell)} \neq 0$ by the definition of $S(t,\ell)$. On the other hand, $p_{S(t,\ell)}(\zeta) = 0$, since ζ is t-dimensional and $\lfloor t/2 \rfloor < S(t,\ell)$. This is a contradiction. Hence we have $n < 2S(t,\ell)$.

Next, suppose that ζ is stably extendible to $L^m(3)$ for $m > n$. Then ζ is stably extendible to the $2m$-skeleton $L_0^m(3)$ of $L^m(3)$ and there is a t-dimensional R-vector bundle α over $L_0^m(3)$ such that ζ is stably equivalent to $i^*\alpha$, where $i : L^n(3) \to L_0^m(3)$ is the standard inclusion. By Theorem 2.1, there is an integer q such that $\alpha - t = qr\sigma_m$. Hence

$$\zeta - t = i^*(\alpha - t) = i^*(qr\sigma_m) = qr\sigma_n.$$

On the other hand, by the assumption,

$$\zeta - t = (\lfloor t/2 \rfloor + \ell) r\eta_n - 2(\lfloor t/2 \rfloor + \ell) = (\lfloor t/2 \rfloor + \ell) r\sigma_n.$$

Thus, in $\tilde{K}_R(L^n(3))$, $\{q - (\lfloor t/2 \rfloor + \ell)\} r\sigma_n = 0$. So, by Theorem 2.1, $q - (\lfloor t/2 \rfloor + \ell) \equiv 0 \mod 3^{\lfloor n/2 \rfloor}$. Hence there is an integer a such that $q = a3^{\lfloor n/2 \rfloor} + \lfloor t/2 \rfloor + \ell$. Here, taking q sufficiently large, we may assume that a is non-negative. Since $\lfloor t/2 \rfloor + \ell < 3^{\lfloor n/2 \rfloor}$ by the assumption, ${}_q\mathrm{C}_j \equiv {}_{\lfloor t/2 \rfloor + \ell}\mathrm{C}_j \mod 3$ for $\lfloor t/2 \rfloor < j \leq \lfloor t/2 \rfloor + \ell$. Therefore

$p(\alpha) = (1 + z_m^2)^q$, and

$p_{S(t,\ell)}(\alpha) = {}_q\mathrm{C}_{S(t,\ell)} z_m^{2S(t,\ell)} = {}_{\lfloor t/2 \rfloor + \ell}\mathrm{C}_{S(t,\ell)} z_m^{2S(t,\ell)} \neq 0$

for $2S(t,\ell) \leq m$. On the other hand, $p_{S(t,\ell)}(\alpha) = 0$, since α is t-dimensional and $\lfloor t/2 \rfloor < S(t,\ell)$. This is a contradiction. □

In [6, Theorem B], we have proved that the normal bundle $\nu(f_n)$ associated to an immersion $f_n : L^n(3) \to R^{4n+3}$ is extendible to $L^m(3)$ for every $m \geq n$ if and only if $0 \leq n \leq 5$. For $n = 6$, we have

Example 3.1. *$\nu(f_6)$ is not stably extendible to $L^{18}(3)$.*

Proof. By (3.2) of [6], $\nu(f_6) = 20r\eta_6 - 26$. Putting $n = 6$ and $t = 14$, we have $\lfloor t/2 \rfloor = 7 = 2 \cdot 3 + 1$, $\lfloor t/2 \rfloor + \ell = 20 = 2 \cdot 3^2 + 2$ and $\ell = 13$. Hence $S(t, \ell) = 3^2 = 9$ by Lemma 2.8. So it follows from Theorem 1 that $\nu(f_6)$ is not stably extendible to $L^{18}(3)$. □

This example is an improvement of Example 3.4 of [6] which states that $\nu(f_6)$ is not stably extendible to $L^{40}(3)$.

In [5, Theorem 6], we have proved that the square $\nu(f_n)^2$ of the normal bundle $\nu(f_n)$ associated to an immersion $f_n : L^n(3) \to R^{4n+3}$ is extendible to $L^m(3)$ for every $m \geq n$ if and only if $0 \leq n \leq 13$ or $n = 15$. For $n = 14$, we have

Example 3.2. *$\nu(f_{14})^2$ is not stably extendible to $L^{972}(3)$.*

Proof. By Theorem 5.2 of [5], $\nu(f_{14})^2 = 612r\eta_{14} - 324$. Putting $n = 14$ and $t = 30^2 = 900$, we have $\lfloor t/2 \rfloor = 450 = 3^5 + 2 \cdot 3^4 + 3^3 + 2 \cdot 3^2$, $\lfloor t/2 \rfloor + \ell = 612 = 2 \cdot 3^5 + 3^4 + 3^3 + 2 \cdot 3^2$ and $\ell = 162$. Hence $S(t, \ell) = 2 \cdot 3^5 = 486$ by Lemma 2.8. So it follows from Theorem 1 that $\nu(f_{14})^2$ is not stably extendible to $L^{972}(3)$. □

This example is an improvement of the former part of Corollary 5.4 of [5] which states that $\nu(f_{14})^2$ is not stably extendible to $L^{1224}(3)$. For $n = 16$, we have

Example 3.3. *$\nu(f_{16})^2$ is not stably extendible to $L^{4374}(3)$.*

Proof. By Theorem 5.2 of [5], $\nu(f_{16})^2 = 4358r\eta_{16} - 7920$. Putting $n = 16$ and $t = 34^2 = 1156$, we have $\lfloor t/2 \rfloor = 578 = 2 \cdot 3^5 + 3^4 + 3^2 + 2$, $\lfloor t/2 \rfloor + \ell = 4538 = 2 \cdot 3^7 + 2 \cdot 3^4 + 2$ and $\ell = 3960$. Hence $S(t, \ell) = 3^7 = 2187$ by Lemma 2.8. So it follows from Theorem 1 that $\nu(f_{16})^2$ is not stably extendible to $L^{4374}(3)$. □

This example is an improvement of the latter part of Corollary 5.4 of [5] which states that $\nu(f_{16})^2$ is not stably extendible to $L^{9076}(3)$.

4. Proof of Theorem 2

Proof of Theorem 2. (i) By the assumption $\zeta - t = hr\eta_n - 2h$ in $\tilde{K}_R(L^n(3))$. If $t \geq 2h$, $\zeta = hr\eta_n \oplus (t - 2h) = i^*(hr\eta_m \oplus (t - 2h))$ in $K_R(L^n(3))$

for every $m > n$, where $i : L^n(3) \to L^m(3)$ is the standard inclusion. So ζ is stably equivalent to $i^*(hr\eta_m \oplus (t-2h))$, and hence ζ is stably extendible to $L^m(3)$.

(ii) Suppose that $t < 2h$ and that m is any integer with $m > n$. If ζ is stably extendible to $L^m(3)$, then ζ is stably extendible to the $2m$-skeleton $L_0^m(3)$ of $L^m(3)$ and there is a t-dimensional R-vector bundle α over $L_0^m(3)$ such that ζ is stably equivalent to $i^*\alpha$, where $i : L^n(3) \to L_0^m(3)$ is the standard inclusion. By Theorem 2.1, there is an integer p such that $\alpha - t = pr\sigma_m$. Hence

$$\zeta - t = i^*(\alpha - t) = i^*(pr\sigma_m) = pr\sigma_n.$$

On the other hand, by the assumption,

$$\zeta - t = hr\eta_n - 2h = hr\sigma_n.$$

Therefore $(p-h)r\sigma_n = 0$ in $\tilde{K}_R(L^n(3))$. So, by Theorem 2.1, we have $p - h \equiv 0 \mod 3^{\lfloor n/2 \rfloor}$. Hence there is an integer a such that $p - h = a3^{\lfloor n/2 \rfloor}$. Here, taking p sufficiently large, we may assume that a is non-negative. Therefore, in $\tilde{K}_R(L_0^m(3))$,

$$\alpha - t = (a3^{\lfloor n/2 \rfloor} + h)r\sigma_m = (a3^{\lfloor n/2 \rfloor} + h)(r\eta_m - 2),$$

and so, in $K_R(L_0^m(3))$,

$$(a3^{\lfloor n/2 \rfloor} + h)r\eta_m = \{2(a3^{\lfloor n/2 \rfloor} + h) - t\} \oplus \alpha,$$

where $2(a3^{\lfloor n/2 \rfloor} + h) - t > 0$ since $2h > t$. If we take p so large that $\dim\{(a3^{\lfloor n/2 \rfloor} + h)r\eta_m\} = 2(a3^{\lfloor n/2 \rfloor} + h) \geq \lceil (2m+1+2) - 1 \rceil = 2m+2$, we have the equality above of R-vector bundles by Theorem 2.4. This implies

$$\mathrm{span}_R\{(a3^{\lfloor n/2 \rfloor} + h)r\eta_m \oplus c\} \geq 2(a3^{\lfloor n/2 \rfloor} + h) - t + c$$

for any non-negative integer c. So the condition $C(m, a, c)$ holds.

Conversely, suppose that the condition $C(m, a, c)$ holds for some non-negative integers a and c. Then there is a t-dimensional R-vector bundle β over $L^m(3)$ such that

$$(a3^{\lfloor n/2 \rfloor} + h)r\eta_m \oplus c = \{2(a3^{\lfloor n/2 \rfloor} + h) - t + c\} \oplus \beta.$$

Let $i : L^n(3) \to L^m(3)$ be the standard inclusion. Then, applying i^* to the both sides of the equality above, we have, in $K_R(L^n(3))$,

$$hr\eta_n = (2h - t) \oplus i^*\beta,$$

since $3^{\lfloor n/2 \rfloor}(r\eta_n - 2) = 0$ in $\tilde{K}_R(L^n(3))$ by Theorem 2.1. Thus $\zeta - t = hr\eta_n - 2h = i^*\beta - t$ in $\tilde{K}_R(L^n(3))$. So ζ is stably equivalent to $i^*\beta$, and hence ζ is stably extendible to $L^m(3)$. □

Let $\nu(f_n)$ be the normal bundle associated to an immersion f_n of $L^n(3)$ in R^{4n+3} and $\nu(f_n)^2$ its square. The following are applications of Theorem 2(ii).

Example 4.1. *$\nu(f_6)$ is extendible to $L^{14}(3)$.*

Proof. By (3.2) of [6], $\nu(f_6) = 20r\eta_6 - 26$. Put $n = 6$, $t = 14$, $h = 20$, $m = 14$, $a = 0$ and $c = 1$ in Theorem 2(ii). Then $t < 2h$ and it suffices to prove

$$\mathrm{span}_R(20r\eta_{14} \oplus 1) \geq 27(= 2h - t + c).$$

The total Pontrjagin class and the 7-th Pontrjagin class of $20r\eta_{14} \oplus 1$ are as follows: $p(20r\eta_{14} \oplus 1) = p(20r\eta_{14}) = p(r\eta_{14})^{20} = (1 + z_{14}^2)^{20}$, and $p_7(20r\eta_{14} \oplus 1) = {}_{20}C_7 z_{14}^{14} = 0$, since ${}_{20}C_7 \equiv 0 \mod 3$ by Lemma 2.8, where z_m is the generator of $H^2(L^m(3), Z) = Z/3$. Putting $p = 3$, $m = 14$, $r = 20$, $\xi = 20r\eta_{14} \oplus 1$ and $s = 7$ in Theorem 2.6, we have $p_s(\xi) = 0$, $p_{s+1}(\xi) = 0, \dots, p_r(\xi) = 0$ and $p > m - 2s + 2$. Hence

$$\mathrm{span}_R(20r\eta_{14} \oplus 1) \geq 28(= 2r - 2s + 2).$$

We therefore obtain the result by Theorem 2(ii) and Corollary 2.5. □

Example 4.2. *$\nu(f_{14})^2$ is extendible to $L^{900}(3)$.*

Proof. By Theorem 5.2 of [6], $\nu(f_{14})^2 = 612r\eta_{14} - 324$. Put $n = 14$, $t = 30^2 = 900$, $h = 612$, $m = 900$, $a = 0$ and $c = 1$ in Theorem 2(ii). Then $t < 2h$ and it suffices to prove

$$\mathrm{span}_R(612r\eta_{900} \oplus 1) \geq 325(= 2h - t + c).$$

The total Pontrjagin class and the 450-th Pontrjagin class of $612r\eta_{900} \oplus 1$ are as follows: $p(612r\eta_{900} \oplus 1) = p(612r\eta_{900}) = (1 + z_{900}^2)^{612}$, and $p_{450}(612r\eta_{900} \oplus 1) = {}_{612}C_{450} z_{900}^{900} = 0$, since ${}_{612}C_{450} \equiv 0 \mod 3$ by Lemma 2.8. (Note $612 = 2 \cdot 3^5 + 3^4 + 3^3 + 2 \cdot 3^2$, $450 = 3^5 + 2 \cdot 3^4 + 3^3 + 2 \cdot 3^2$.) Putting $p = 3$, $m = 900$, $r = 612$, $\xi = 612r\eta_{900} \oplus 1$ and $s = 450$ in Theorem 2.6, we have $p_s(\xi) = 0$, $p_{s+1}(\xi) = 0, \dots, p_r(\xi) = 0$ and $p > m - 2s + 2$. Hence

$$\mathrm{span}_R(612r\eta_{900} \oplus 1) \geq 326(= 2r - 2s + 2).$$

We therefore obtain the result by Theorem 2(ii) and Corollary 2.5. □

Example 4.3. *$\nu(f_{16})^2$ is extendible to $L^{1156}(3)$.*

Proof. By Theorem 5.2 of [6], $\nu(f_{16})^2 = 4538r\eta_{16} - 7920$. Put $n = 16$, $t = 34^2 = 1156$, $h = 4538$, $m = 1156$, $a = 0$ and $c = 1$ in Theorem 2(ii). Then $t < 2h$ and it suffices to prove

$$\mathrm{span}_R(4538r\eta_{1156} \oplus 1) \geq 7921(= 2h - t + c).$$

The total Pontrjagin class and the 578-th Pontrjagin class of $4538r\eta_{1156} \oplus 1$ are as follows: $p(4538r\eta_{1156} \oplus 1) = p(4538r\eta_{1156}) = (1 + z_{1156}^2)^{4538}$, and $p_{578}(4538r\eta_{1156} \oplus 1) = {}_{4538}\mathrm{C}_{578} z_{1156}^{1156} = 0$, since ${}_{4538}\mathrm{C}_{578} \equiv 0 \bmod 3$ by Lemma 2.8. (Note $4538 = 2 \cdot 3^7 + 2 \cdot 3^4 + 2$, $578 = 2 \cdot 3^5 + 3^4 + 3^2 + 2$.) Putting $p = 3$, $m = 1156$, $r = 4538$, $\xi = 4538r\eta_{1156} \oplus 1$ and $s = 578$ in Theorem 2.6, we have $p_s(\xi) = 0$, $p_{s+1}(\xi) = 0, \dots, p_r(\xi) = 0$ and $p > m - 2s + 2$. Hence

$$\mathrm{span}_R(4538r\eta_{1156} \oplus 1) \geq 7922 (= 2r - 2s + 2).$$

We therefore obtain the result by Theorem 2(ii) and Corollary 2.5. □

5. Proof of Theorem 3

Proof of Theorem 3. (i) By the assumption $\zeta - t = h\eta_n \oplus k\eta_n^2 - (h + k)$ in $\tilde{K}_C(L^n(3))$. If $t \geq h + k$, $\zeta = h\eta_n \oplus k\eta_n^2 + (t - h - k) = i^*(h\eta_m \oplus k\eta_m^2 + (t - h - k))$ in $K_C(L^n(3))$ for every $m > n$, where $i : L^n(3) \to L^m(3)$ is the standard inclusion. So ζ is stably equivalent to $i^*(h\eta_m \oplus k\eta_m^2 + (t - h - k))$, and hence ζ is stably extendible to $L^m(3)$.

(ii) Suppose that $t < h + k$ and that m is any integer with $m > n$. If ζ is stably extendible to $L^m(3)$, then there is a t-dimensional C-vector bundle α over $L^m(3)$ such that ζ is stably equivalent to $i^*\alpha$, where $i : L^n(3) \to L^m(3)$ is the standard inclusion. By Theorem 2.2, there are integers p and q such that $\alpha - t = p(\eta_m - 1) + q(\eta_m^2 - 1)$ in $\tilde{K}_C(L^m(3))$. Hence

$$\zeta - t = i^*(\alpha - t) = p(\eta_n - 1) + q(\eta_n^2 - 1).$$

On the other hand, by the assumption,

$$\zeta - t = h\eta_n \oplus k\eta_n^2 - (h + k) = h(\eta_n - 1) + k(\eta_n^2 - 1).$$

Therefore, in $\tilde{K}_C(L^n(3))$, $(p - h)(\eta_n - 1) + (q - k)(\eta_n^2 - 1) = 0$, namely

$$\{(p - h) + 2(q - k)\}\sigma_n + (q - k)\sigma_n^2 = 0.$$

So, by Theorem 2.2, we have $q - k \equiv 0 \bmod 3^{\lfloor n/2 \rfloor}$ and $p - h \equiv 0 \bmod 3^{\lfloor n/2 \rfloor}$. Hence there are integers a and b such that $p = a3^{\lfloor n/2 \rfloor} + h$ and $q = b3^{\lfloor n/2 \rfloor} + k$. Here, taking p and q sufficiently large, we may assume that a and b are non-negative. Therefore, in $\tilde{K}_C(L^m(3))$,

$$\alpha - t = (a3^{\lfloor n/2 \rfloor} + h)(\eta_m - 1) + (b3^{\lfloor n/2 \rfloor} + k)(\eta_m^2 - 1),$$

and so, in $K_C(L^m(3))$,

$$(a3^{\lfloor n/2 \rfloor} + h)\eta_m \oplus (b3^{\lfloor n/2 \rfloor} + k)\eta_m^2 = (a3^{\lfloor n/2 \rfloor} + h + b3^{\lfloor n/2 \rfloor} + k - t) \oplus \alpha,$$

where $a3^{\lfloor n/2 \rfloor} + h + b3^{\lfloor n/2 \rfloor} + k - t > 0$ since $h + k > t$. If we take p and q so large that $\dim\{(a3^{\lfloor n/2 \rfloor} + h)\eta_m \oplus (b3^{\lfloor n/2 \rfloor} + k)\eta_m^2\} = a3^{\lfloor n/2 \rfloor} + h + b3^{\lfloor n/2 \rfloor} + k \geq$

$\lceil (2m+1+2)/2-1 \rceil = m+1$, we have the equality above of C-vector bundles by Theorem 2.4. This implies

$$\mathrm{span}_C\{(a3^{\lfloor n/2 \rfloor}+h)\eta_m \oplus (b3^{\lfloor n/2 \rfloor}+k)\eta_m^2 \oplus c\} \geq a3^{\lfloor n/2 \rfloor}+h+b3^{\lfloor n/2 \rfloor}+k-t+c$$

for any non-negative integer c. So the condition $C(m,a,b,c)$ holds.

(iia) Let n be even and let m be any integer with $m > n$. To prove the "if" part, suppose that the condition $C(m,a,b,c)$ holds for some non-negative integers a, b and c. Then there is a t-dimensional C-vector bundle β over $L^m(3)$ such that

$$(a3^{n/2}+h)\eta_m \oplus (b3^{n/2}+k)\eta_m^2 \oplus c = (a3^{n/2}+h+b3^{n/2}+k-t+c) \oplus \beta.$$

Let $i : L^n(3) \to L^m(3)$ be the standard inclusion. Then, applying i^* to the both sides of the equality above, we have, in $K_C(L^n(3))$,

$$(a3^{n/2}+h)\eta_n \oplus (b3^{n/2}+k)\eta_n^2 = (a3^{n/2}+h+b3^{n/2}+k-t) \oplus i^*\beta.$$

In case n is even, the equalities $3^{n/2}\eta_n = 3^{n/2}$ and $3^{n/2}\eta_n^2 = 3^{n/2}$ hold in $K_C(L^n(3))$ by Theorem 2.2. Hence we have, in $K_C(L^n(3))$,

$$h\eta_n \oplus k\eta_n^2 = (h+k-t) \oplus i^*\beta.$$

Thus $\zeta - t = h\eta_n \oplus k\eta_n^2 - (h+k) = i^*\beta - t$ in $\tilde{K}_C(L^n(3))$. So ζ is stably equivalent to $i^*\beta$, and hence ζ is stably extendible to $L^m(3)$.

(iib) Let n be odd and let m be any integer with $m > n$. To prove the "if" part, suppose that the condition $C(m,a,b,c)$ holds for some non-negative integers a, b and c such that $a \equiv b \equiv 0 \bmod 3$. Then $a3^{\lfloor n/2 \rfloor}\eta_n = a3^{\lfloor n/2 \rfloor}$ and $b3^{\lfloor n/2 \rfloor}\eta_n^2 = b3^{\lfloor n/2 \rfloor}$ by Theorem 2.2. Hence we have the result in the way similar to the proof of (iia). □

Let $cv(f_n)$ be the complexfication of the normal bundle $v(f_n)$ associated to an immersion f_n of $L^n(3)$ in R^{4n+3} and $cv(f_n)^2$ its square. The following are applications of Theorem 3(iia).

Example 5.1. *$cv(f_6)$ is extendible to $L^{14}(3)$.*

Proof. Complexifying the both sides of the equality in the proof of Example 3.1 and using the equality $cr\eta_6 = \eta_6 + \eta_6^2$, we have $cv(f_6) = 20\eta_6 + 20\eta_6^2 - 26$. Put $n = 6$, $\zeta = cv(f_6)$, $t = 14$, $h = k = 20$, $m = 14$, $a = b = c = 0$ in Theorem 3(iia). Then $t < h + k$ and it suffices to prove

$$\mathrm{span}_C(20\eta_{14} \oplus 20\eta_{14}^2) \geq 26 (= h + k - t).$$

In fact, $\dim(20\eta_{14} \oplus 20\eta_{14}^2) = 40$ and $\lceil (\dim L^{14}(3)+1)/2 - 1 \rceil = 14$. Hence, by Theorem 2.3, the inequality above holds. Thus we have the result. □

Example 5.2. *$cv(f_{14})^2$ is extendible to $L^{900}(3)$.*

Proof. Complexifying the both sides of the equality in the proof of Example 3.2 and using the equality $cr\eta_{14} = \eta_{14} + \eta_{14}^2$, we have $cv(f_{14})^2 = 612\eta_{14} + 612\eta_{14}^2 - 324$. Put $n = 14$, $\zeta = cv(f_{14})^2$, $t = 900$, $h = k = 612$, $m = 900$, $a = b = c = 0$ in Theorem 3(iia). Then $t < h + k$ and it suffices to prove

$$\mathrm{span}_C(612\eta_{900} \oplus 612\eta_{900}^2) \geq 324(= h + k - t).$$

In fact, $\dim(612\eta_{900} \oplus 612\eta_{900}^2) = 1224$ and $\lceil(\dim L^{900}(3) + 1)/2 - 1\rceil = 900$. Hence, by Theorem 2.3, the inequality above holds. Thus we have the result. □

Example 5.3. *$cv(f_{16})^2$ is extendible to $L^{1156}(3)$.*

Proof. Complexifying the both sides of the equality in the proof of Example 3.3 and using the equality $cr\eta_{16} = \eta_{16} + \eta_{16}^2$, we have $cv(f_{16})^2 = 4538\eta_{16} + 4538\eta_{16}^2 - 7920$. Put $n = 16$, $\zeta = cv(f_{16})^2$, $t = 1156$, $h = k = 4538$, $m = 1156$, $a = b = c = 0$ in Theorem 3(iia). Then $t < h + k$ and it suffices to prove

$$\mathrm{span}_C(4538\eta_{1156} \oplus 4538\eta_{1156}^2) \geq 7920(= h + k - t).$$

In fact, $\dim(4538\eta_{1156} \oplus 4538\eta_{1156}^2) = 9076$ and $\lceil(\dim L^{1156}(3) + 1)/2 - 1\rceil = 1156$. Hence, by Theorem 2.3, the inequality above holds. Thus we have the result. □

6. Proof of Theorem 4

On the complexification of the normal bundle associated to an immersion of $L^n(3)$ in R^{4n+3} for $0 \leq n \leq 5$, we have

Theorem 6.1. *Let $cv(f_n)$ be the complexification of the normal bundle $v(f_n)$ associated to an immersion $f_n : L^n(3) \to R^{4n+3}$. Then*

$$cv(f_0) = 2, \qquad cv(f_1) = 4, \qquad cv(f_2) = 6,$$

$$cv(f_3) = 2\eta_3 \oplus 2\eta_3^2 \oplus 4, \qquad cv(f_4) = 4\eta_4 \oplus 4\eta_4^2 \oplus 2,$$

$$cv(f_5) = 3\eta_5 \oplus 3\eta_5^2 \oplus 6,$$

Proof. Complexifying the both sides of the equalities in Theorem 3.1 of [6] and using the equality $cr\eta_n = \eta_n + \eta_n^2$, we obtain the equalities above in $K_C(L^n(3))$. Since $\lceil(\dim L^n(3) + 2)/2 - 1\rceil = n + 1 \leq 2n + 2 = \dim cv(f_n)$, these equalities hold as C-vector bundles by Theorem 2.4. □

Lemma 6.2. *Let ζ be a C-vector bundle over $L^n(q)$, where q is an integer > 1, and let $m \geq n$. Then ζ is extendible (respectively stably extendible)* *to $L^m(q)$ if and only if $\zeta \otimes \eta_n$ is extendible (respectively stably extendible) to $L^m(q)$.*

PROOF. If ζ is extendible (respectively stably extendible) to $L^m(q)$, there is a C-vector bundle ξ over $L^m(q)$ such that the equality $i^*\xi = \zeta$ holds as C-vector bundles over $L^n(q)$ (respectively in $K_C(L^n(q))$), where $i : L^n(q) \to L^m(q)$ is the standard inclusion. Hence $i^*(\xi \otimes \eta_m) = \zeta \otimes \eta_n$ holds as C-vector bundles over $L^n(q)$ (respectively in $K_C(L^n(q))$). Therefore $\zeta \otimes \eta_n$ is extendible (respectively stably extendible) to $L^m(q)$.

Conversely, if $\zeta \otimes \eta_n$ is extendible (respectively stably extendible) to $L^m(q)$ for $m \geq n$, so is $\zeta \otimes (\eta_n^r)$ for any $r \geq 1$ by the argument above inductively. Since $\eta_n^q = 1$ (cf. [7, Proposition 4.1, p. 198]), $\zeta(= \zeta \otimes (\eta_n^q))$ is extendible (respectively stably extendible) to $L^m(q)$. □

THEOREM 6.3. *If $n \geq 6$, $cv(f_n)$ is not stably extendible to $L^m(3)$ for some $m > n$. More precisely, the following hold.*

(i) *$cv(f_6)$ is not stably extendible to $L^{21}(3)$.*

(ii) *$cv(f_7)$ is not stably extendible to $L^{24}(3)$.*

(iii) *If n is even and $n \geq 8$, $cv(f_n)$ is not stably extendible to $L^m(3)$ for any $m \geq 3^{n/2} - q(n+1)$, where q is an integer satisfying $1 \leq q < 3^{n/2}/\{4(n+1)\}$.*

(iv) *If n is odd and $n \geq 9$, $cv(f_n)$ is not stably extendible to $L^m(3)$ for any $m \geq 3^{(n-1)/2} - q(n+1)$, where q is an integer satisfying $1 \leq q < 3^{(n-1)/2}/\{4(n+1)\}$.*

PROOF. (i) We have the equality $v(f_6) = 28 - 7r\eta_6$ by the equality in (3.2) for $n = 6$ of [6]. Complexifying the both sides and using the equality $cr\eta_6 = \eta_6 + \eta_6^2$, we have $cv(f_6) = 28 - 7\eta_6 - 7\eta_6^2$. Then, multiplying them by η_6, we obtain

$$cv(f_6) \otimes \eta_6 = 28\eta_6 - 7\eta_6^2 - 7 = \eta_6 \oplus 20\eta_6^2 - 7,$$

since $\eta_6^3 = 1$, $27(\eta_6 - 1) = 0$ and $27(\eta_6^2 - 1) = 0$ in $K_C(L^6(3))$ by Theorem 2.2. Put $p = 3$, $n = 6$, $\zeta = cv(f_6) \otimes \eta_6$ and $t = 14$ in Theorem 2.7. Then $\ell = 7$ and $t + \ell = 21 < 27 = 3^3$. Hence $cv(f_6) \otimes \eta_6$ is not stably extendible to $L^{21}(3)$, and so is $cv(f_6)$ by Lemma 6.2.

(ii) We have the equality $v(f_7) = 32 - 8r\eta_7$ by the equality in (3.2) for $n = 7$ of [6]. Complexifying the both sides and using the equality $cr\eta_7 = \eta_7 + \eta_7^2$, we have $cv(f_7) = 32 - 8\eta_7 - 8\eta_7^2$. Then $i^*(cv(f_7)) = 32 - 8\eta_6 - 8\eta_6^2$, where $i : L^6(3) \to L^7(3)$ is the standard inclusion. Multiplying the both sides of the equality above η_6, we obtain

$$i^*(cv(f_7)) \otimes \eta_6 = 32\eta_6 - 8\eta_6^2 - 8 = 5\eta_6 \oplus 19\eta_6^2 - 8,$$

since $\eta_6^3 = 1$, $27(\eta_6 - 1) = 0$ and $27(\eta_6^2 - 1) = 0$ in $K_C(L^6(3))$ by Theorem 2.2. Put $p = 3$, $n = 6$, $\zeta = i^*(cv(f_7)) \otimes \eta_6$ and $t = 16$ in Theorem 2.7. Then $\ell = 8$

and $t+\ell=24<27=3^3$. Hence $i^*(cv(f_7))\otimes\eta_6$ is not stably extendible to $L^{24}(3)$, and so is $cv(f_7)$ by Lemma 6.2.

(iii) Clearly there is an integer q satisfying the inequalities $1\le q<3^{n/2}/\{4(n+1)\}$ if $n\ge 8$.

We have the equality $v(f_n)=4(n+1)-(n+1)r\eta_n$ in (3.2) of [6]. Complexifying the both sides, using the equality $cr\eta_n=\eta_n+\eta_n^2$ and multiplying them by any integer q, we obtain the equality

$$(*)\qquad qcv(f_n)=-q(n+1)\eta_n-q(n+1)\eta_n^2+4q(n+1).$$

Then, adding the both sides of the equality $(*)$ by $q(n+1)\eta_n^2$ and using the equality $3^{n/2}(\eta_n-1)=0$ in $K_C(L^n(3))$ obtained from Theorem 2.2, we have

$$qcv(f_n)\oplus q(n+1)\eta_n^2=\{3^{n/2}-q(n+1)\}\eta_n+4q(n+1)-3^{n/2}.$$

Suppose that q satisfies the inequalities $1\le q<3^{n/2}/\{4(n+1)\}$ and put $p=3$, $\zeta=qcv(f_n)\oplus q(n+1)\eta_n^2$ and $t=3q(n+1)$ in Theorem 2.7. Then $\ell=3^{n/2}-4q(n+1)>0$ and $t+\ell=3^{n/2}-q(n+1)<3^{n/2}$. Hence $qcv(f_n)\oplus q(n+1)\eta_n^2$ is not stably extendible to $L^m(3)$ for any $m\ge 3^{n/2}-q(n+1)$, and so is $cv(f_n)$.

(iv) Clearly there is an integer q satisfying the inequalities $1\le q<3^{(n-1)/2}/\{4(n+1)\}$ if $n\ge 9$.

Applying i^* to the equality $(*)$, where $i:L^{n-1}(3)\to L^n(3)$ is the standard inclusion, adding the resulting equality by $q(n+1)\eta_{n-1}^2$ and using the equality $3^{(n-1)/2}(\eta_{n-1}-1)=0$ in $K_C(L^{n-1}(3))$ obtained from Theorem 2.2, we have

$$i^*qcv(f_n)\oplus q(n+1)\eta_{n-1}^2=\{3^{(n-1)/2}-q(n+1)\}\eta_{n-1}+4q(n+1)-3^{(n-1)/2}.$$

Suppose that q satisfies the inequalities $1\le q<3^{(n-1)/2}/\{4(n+1)\}$ and put $p=3$, $\zeta=i^*qcv(f_n)\oplus q(n+1)\eta_{n-1}^2$ and $t=3q(n+1)$ in Theorem 2.7. Then $\ell=3^{(n-1)/2}-4q(n+1)>0$ and $t+\ell=3^{(n-1)/2}-q(n+1)<3^{(n-1)/2}$. Hence $i^*qcv(f_n)\oplus q(n+1)\eta_{n-1}^2$ is not stably extendible to $L^m(3)$ for any $m\ge 3^{(n-1)/2}-q(n+1)$, and so is $cv(f_n)$. □

Now, we are ready to prove Theorem 4.

Proof of Theorem 4. Since η_n, η_n^2 and trivial C-vector bundles over $L^n(3)$ are extendible to $L^m(3)$ for every $m\ge n$, the "if" part follows from Theorem 6.1.

The "only if" part follows from Theorem 6.3. □

7. Proof of Theorem 5

On the complexification of the square of the normal bundle associated to an immersion of $L^n(3)$ in R^{4n+3} for $0\le n\le 13$ and $n=15$, we have

THEOREM 7.1. *Let $cv(f_n)^2$ be the complexification of the square of the normal bundle $v(f_n)$ associated to an immersion $f_n : L^n(3) \to R^{4n+3}$. Then*

$$cv(f_0)^2 = 4, \qquad cv(f_1)^2 = 16, \qquad cv(f_2)^2 = 36,$$

$$cv(f_3)^2 = 2\eta_3 \oplus 2\eta_3^2 \oplus 60, \qquad cv(f_4)^2 = 5\eta_4 \oplus 5\eta_4^2 \oplus 90,$$

$$cv(f_5)^2 = 144, \qquad cv(f_6)^2 = 8\eta_6 \oplus 8\eta_6^2 \oplus 180,$$

$$cv(f_7)^2 = 11\eta_7 \oplus 11\eta_7^2 \oplus 234, \qquad cv(f_8)^2 = 324,$$

$$cv(f_9)^2 = 29\eta_9 \oplus 29\eta_9^2 \oplus 342, \qquad cv(f_{10})^2 = 125\eta_{10} \oplus 125\eta_{10}^2 \oplus 234,$$

$$cv(f_{11})^2 = 207\eta_{11} \oplus 207\eta_{11}^2 \oplus 162, \qquad cv(f_{12})^2 = 275\eta_{12} \oplus 275\eta_{12}^2 \oplus 126,$$

$$cv(f_{13})^2 = 86\eta_{13} \oplus 86\eta_{13}^2 \oplus 612, \qquad cv(f_{15})^2 = 395\eta_{15} \oplus 395\eta_{15}^2 \oplus 234.$$

PROOF. Complexifying the both sides of the equalities in Theorem 5.1 of [5], we obtain the equalities above in the way similar to the proof of Theorem 6.1. □

THEOREM 7.2. *If $n = 14$ or $n \geq 16$, $cv(f_n)^2$ is not stably extendible to $L^m(3)$ for some $m > n$. More precisely, the following hold.*

(i) *$cv(f_{14})^2$ is not stably extendible to $L^{1224}(3)$.*

(ii) *If n is even and $n \geq 16$, $cv(f_n)^2$ is not stably extendible to $L^m(3)$ for any $m \geq 3^{n/2} - 7s(n+1)^2$, where s is an integer satisfying $1 \leq s < 3^{n/2}/\{18(n+1)^2\}$.*

(iii) *If n is odd and $n \geq 17$, $cv(f_n)^2$ is not stably extendible to $L^m(3)$ for any $m \geq 3^{(n-1)/2} - 7s(n+1)^2$, where s is an integer satisfying $1 \leq s < 3^{(n-1)/2}/\{18(n+1)^2\}$.*

PROOF. We have the equality $v(f_n)^2 = -7(n+1)^2 r\eta_n + 18(n+1)^2$ in the proof of Theorem 5.1 of [5]. Complexifying the both sides and using the equality $cr\eta_n = \eta_n + \eta_n^2$, we obtain the equality

$$(**) \qquad cv(f_n)^2 = -7(n+1)^2\eta_n - 7(n+1)^2\eta_n^2 + 18(n+1)^2.$$

On the other hand, complexifying the equality $3^{\lfloor n/2 \rfloor}(r\eta_n - 2) = 0$ in $K_R(L^n(3))$ obtained from Theorem 2.1, we have the equality $3^{\lfloor n/2 \rfloor}(\eta_n + \eta_n^2 - 2) = 0$ in $K_C(L^n(3))$. Adding the right-hand side of the equality $(**)$ by this equality, we obtain

$$cv(f_n)^2 = \{3^{\lfloor n/2 \rfloor} - 7(n+1)^2\}\eta_n + \{3^{\lfloor n/2 \rfloor} - 7(n+1)^2\}\eta_n^2 + 18(n+1)^2 - 2 \cdot 3^{\lfloor n/2 \rfloor}.$$

(i) Consider the equality above for $n = 14$

$$cv(f_{14})^2 = (3^7 - 7\cdot 15^2)\eta_{14} + (3^7 - 7\cdot 15^2)\eta_{14}^2 + 18\cdot 15^2 - 2\cdot 3^7$$
$$= 612\eta_{14} + 612\eta_{14}^2 - 324.$$

Put $p = 3$, $n = 14$, $\zeta = cv(f_{14})^2$ and $t = 30^2 = 900$ in Theorem 2.7. Then $\ell = 324$ and $t + \ell = 1224 < 2187 = 3^7$. Hence $cv(f_{14})^2$ is not stably extendible to $L^{1224}(3)$.

(ii) Clearly there is an integer s satisfying the inequalities $1 \le s < 3^{n/2}/\{18(n+1)^2\}$ if $n \ge 16$.

Multiplying the equality $(**)$ by any integer s, we obtain the equality

$$(***) \qquad scv(f_n)^2 = -7s(n+1)^2\eta_n^2 - 7s(n+1)^2\eta_n + 18s(n+1)^2.$$

Then, adding the equality $(***)$ by $7s(n+1)^2\eta_n^2$ and using the equality $3^{n/2}(\eta_n - 1) = 0$ in $K_C(L^n(3))$ obtained from Theorem 2.2, we have

$$scv(f_n)^2 \oplus 7s(n+1)^2\eta_n^2 = \{3^{n/2} - 7s(n+1)^2\}\eta_n + 18s(n+1)^2 - 3^{n/2}.$$

Suppose that s satisfies the inequalities $1 \le s < 3^{n/2}/\{18(n+1)^2\}$ and put $p = 3$, $\zeta = scv(f_n)^2 \oplus 7s(n+1)^2\eta_n^2$ and $t = 11s(n+1)^2$ in Theorem 2.7. Then $\ell = 3^{n/2} - 18s(n+1)^2 > 0$ and $t + \ell = 3^{n/2} - 7s(n+1)^2 < 3^{n/2}$. Hence $scv(f_n)^2 \oplus 7s(n+1)^2\eta_n^2$ is not stably extendible to $L^m(3)$ for any $m \ge 3^{n/2} - 7s(n+1)^2$, and so is $cv(f_n)^2$.

(iii) Clearly there is an integer s satisfying the inequalities $1 \le s < 3^{(n-1)/2}/\{18(n+1)^2\}$ if $n \ge 17$.

Applying i^* to the equality $(***)$, where $i : L^{n-1}(3) \to L^n(3)$ is the standard inclusion, adding the resulting equality by $7s(n+1)^2\eta_{n-1}^2$ and using the equality $3^{(n-1)/2}(\eta_{n-1} - 1) = 0$ in $K_C(L^{n-1}(3))$ obtained from Theorem 2.2, we have

$$i^*scv(f_n)^2 \oplus 7s(n+1)^2\eta_{n-1}^2 = \{3^{(n-1)/2} - 7s(n+1)^2\}\eta_{n-1}$$
$$+ 18s(n+1)^2 - 3^{(n-1)/2}.$$

Suppose that s satisfies the inequalities $1 \le s < 3^{(n-1)/2}/\{18(n+1)^2\}$ and put $p = 3$, $\zeta = i^*scv(f_n)^2 \oplus 7s(n+1)^2\eta_{n-1}^2$ and $t = 11s(n+1)^2$ in Theorem 2.7. Then $\ell = 3^{(n-1)/2} - 18s(n+1)^2 > 0$ and $t + \ell = 3^{(n-1)/2} - 7s(n+1)^2 < 3^{(n-1)/2}$. Hence $i^*scv(f_n)^2 \oplus 7s(n+1)^2\eta_{n-1}^2$ is not stably extendible to $L^m(3)$ for every $m \ge 3^{(n-1)/2} - 7s(n+1)^2$, and so is $cv(f_n)^2$. □

Now, we are ready to prove Theorem 5.

Proof of Theorem 5. The "if" part follows from Theorem 7.1 in the way similar to that of Theorem 4. The "only if" part follows from Theorem 7.2. □

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