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Existence of global solutions in time for reaction-diffusion systems with inhomogeneous terms in cones

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Abstract. We consider nonnegative solutions of the initial-boundary value problems in cone domains for the reaction-diffusion systems with inhomogeneous terms dependent on space-time coordinates. In this paper we show the condition for the existence of global solutions. Our conditions for the global existence are optimal in view of our nonexistence results in 2009.

1. Introduction

We consider nonnegative solutions of the initial-boundary value problem for the reaction-diffusion systems of the form

$$\begin{cases} u_t = \Delta u + K_1(x,t)v^{p_1}, & x \in D,\ t > 0, \\ v_t = \Delta v + K_2(x,t)u^{p_2}, & x \in D,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in D, \\ v(x,0) = v_0(x) \geq 0, & x \in D, \end{cases} \tag{1}$$

where $p_1, p_2 \geq 1$ with $p_1 p_2 > 1$. The domain D is a cone in $\mathbf{R}^N$ such as

$$D = \{x \in \mathbf{R}^N; x \neq 0 \text{ and } x/|x| \in \Omega\}, \tag{2}$$

where Ω is some region on S^{N-1} satisfying $\Omega \neq S^{N-1}$ with C^∞-boundary $\partial\Omega$.

The initial data $u_0(x)$ and $v_0(x)$ are bounded and continuous in $\bar{D}$, and $u_0(x) = v_0(x) = 0$ on ∂D. The inhomogeneous terms K_i $(i = 1, 2)$ are nonnegative continuous functions in $D \times (0, \infty)$.

For a given initial value (u_0, v_0), let $T^* = T^*(u_0, v_0)$ be the maximal existence time of the solution of (1). If $T^* = \infty$, the solution is global in time.

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On the other hand, if $T^* < \infty$, then the solution is not global in time in the sense that it blows up at $t = T^*$,

$$\limsup_{t \to T^*} \|u(\cdot, t)\|_\infty + \limsup_{t \to T^*} \|v(\cdot, t)\|_\infty = \infty, \tag{3}$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm with respect to space variable.

Before stating our main results, we recall a history of the studies on the global existence and nonexistence of solutions to the system (1). The initial value problems of the form

$$\begin{cases} u_t = \Delta u + K(x,t)u^p, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^N \end{cases} \tag{4}$$

were studied by many researchers. First, in the case $K(x,t) = 1$, the initial value problems (4) were studied by Fujita [4]. Fujita proved that when $p > 1 + 2/N$ the solution of (4) is global in time if $\|u_0\|_\infty$ is small enough and $u_0(x)$ has an exponential decay. On the other hand, he also proved that if $1 < p < 1 + 2/N$, then the solution of (4) is not global in time for any $u_0 \not\equiv 0$. In the case $p = 1 + 2/N$, the global nonexistence was proved by Hayakawa [9], Kobayashi-Sirao-Tanaka [13] and Weissler [32]. Lee-Ni [16] proved that when $p > 1 + 2/N$ and

$$\limsup_{|x| \to \infty} |x|^a u_0(x) < \infty \qquad \text{with } a \geq \frac{2}{p-1},$$

the solution of (4) is global in time if $\|\langle \cdot \rangle^a u_0\|_\infty$ is small enough, where $\langle x \rangle = (1 + |x|^2)^{1/2}$. On the other hand, they showed that if

$$\liminf_{|x| \to \infty} |x|^a u_0(x) > 0 \qquad \text{with } a < \frac{2}{p-1}$$

or if $\|u_0\|_\infty$ is large enough, then the solution of (4) is not global in time. In the case $K(x,t) \sim |x|^\sigma$ as $|x| \to \infty$ with $\sigma \in \mathbf{R}$, Suzuki [27] proved that if $1 < p \leq 1 + (2 + \sigma)/N$, then all nontrivial solutions of (4) do not exist globally in time, and that if $p > 1 + (2 + \sigma)/N$ and $\|u_0\|$ is sufficiently small, then a global solution of (4) exists (see also [1], [18] and [23]). In the case $K(x,t) = t^q |x|^\sigma$ with $q \geq 0$, $\sigma \geq 0$, Qi [25] proved that if $1 < p \leq 1 + (2 + \sigma + 2q)/N$, then there exists no nontrivial global solution of (4), and that if $p > 1 + (2 + \sigma + 2q)/N$ and $u_0(x)$ is sufficiently small, then there exists a positive global solution of (4).

Some researchers also studied the initial-boundary problem of the form

$$\begin{cases} u_t = \Delta u + K(x,t)u^p, & x \in D,\ t > 0, \\ u(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in D, \end{cases} \tag{5}$$

with cone domain D defined by (2). In the case $K(x,t) = 1$, Levine-Meier [18] obtained that if $1 < p < 1 + 2/(N + \gamma_+)$, then (5) has no nontrivial global solution, where

$$\gamma_+ = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\omega_1}}{2}. \tag{6}$$

Here, $\omega_1 > 0$ denote the first eigenvalue of $-\Delta_\Omega$, where Δ_Ω denote the Laplace-Beltrami operator with homogeneous Dirichlet boundary condition in Ω. On the other hand, they also obtained that if $p > 1 + 2/(N + \gamma_+)$, then nontrivial global solutions of (5) exist. For the case $p = 1 + 2/(N + \gamma_+)$, Levine-Meier [19] proved that the problem (5) possesses no nontrivial global solution. In the case $K(x,t) = |x|^\sigma$ with $\sigma \geq 0$, Levine-Meier [18] obtained that if $1 < p < 1 + (2+\sigma)/(N+\gamma_+)$, then no nontrivial global solution of (5) exists, and that if $p > 1 + (2+\sigma)/(N+\gamma_+)$, then there are nontrivial global solutions of (5). In the case $p = 1 + (2+\sigma)/(N+\gamma_+)$, Hamada [7] proved that if $u_0 \not\equiv 0$ and $0 < \sigma \leq 2(N-2)/(\gamma_+ + 2)$ for $N \geq 3$, there is no global solution. For $p > 1 + (2+\sigma)/(N+\gamma_+)$, Hamada [82] showed that if

$$u_0(x) \leq m\langle x\rangle^{-a}\psi_1\left(\frac{x}{|x|}\right) \qquad \text{with } a > \frac{2+\sigma}{p-1} \text{ and small } m > 0,$$

where $\psi_1(x/|x|)$ denote the eigenfunction corresponding to ω_1, then there exists the unique nontrivial global solution of (5), and that if

$$u_0(x) \geq M\langle x\rangle^{-a}\psi_1\left(\frac{x}{|x|}\right) \qquad \text{with } a < \frac{2+\sigma}{p-1} \text{ and arbitrary } M > 0$$

or

$$u_0(x) \geq M\langle x\rangle^{-a}\psi_1\left(\frac{x}{|x|}\right) \qquad \text{with } a = \frac{2+\sigma}{p-1} \text{ and large } M > 0,$$

then the solution of (5) is not global when $0 \leq \sigma \leq (p-1)(N-2)$ for $N \geq 2$. In the case $K(x,t) \sim t^q$ with $q > -1$ as $t \to \infty$, Levine-Meier [19] asserted that if $p \leq 1 + (2+2q)/(N+\gamma_+)$, there exists no global solution of (5).

The initial value problems for a weakly coupled system

$$\begin{cases} u_t = \Delta u + K_1(x,t)v^{p_1}, & x \in \mathbf{R}^N,\ t > 0, \\ v_t = \Delta v + K_2(x,t)u^{p_2}, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^N, \\ v(x,0) = v_0(x) \geq 0, & x \in \mathbf{R}^N \end{cases} \tag{7}$$

were studied by many researchers. In the case $K_i(x,t) = 1$ $(i = 1,2)$, Escobedo-Herrero [3] and Mochizuki [20] proved that if

$$\frac{\max\{2+2p_1, 2+2p_2\}}{p_1p_2-1} \geq N,$$

$$\liminf_{|x|\to\infty} |x|^{a_1} u_0(x) > 0 \qquad \text{with } a_1 < \frac{2+2p_1}{p_1p_2-1},$$

$$\liminf_{|x|\to\infty} |x|^{a_2} v_0(x) > 0 \qquad \text{with } a_2 < \frac{2+2p_2}{p_1p_2-1},$$

$$u_0(x) \geq M\exp(-v_0|x|^2) \qquad \text{for some } v_0 > 0 \text{ and large } M > 0$$

or

$$v_0(x) \geq M\exp(-v_0|x|^2) \qquad \text{for some } v_0 > 0 \text{ and large } M > 0,$$

then the nontrivial solution of (7) is not global in time, and that if

$$\frac{\max\{2+2p_1, 2+2p_2\}}{p_1p_2-1} < N,$$

$$\limsup_{|x|\to\infty} |x|^{a_1} u_0(x) < \infty \qquad \text{with } a_1 > \frac{2+2p_1}{p_1p_2-1},$$

$$\limsup_{|x|\to\infty} |x|^{a_2} v_0(x) < \infty \qquad \text{with } a_2 > \frac{2+2p_2}{p_1p_2-1}$$

and

$$\|\langle\cdot\rangle^{a_1}u_0\|_\infty + \|\langle\cdot\rangle^{a_2}v_0\|_\infty \text{ is small enough,}$$

then the solution of (7) is global in time (see also [3]). In the case $K_i(x,t) = t^{q_i}$ $(i = 1,2)$, Uda [28] showed that if

$$\frac{\max\{2+2q_1+(2+2q_2)p_1, 2+2q_2+(2+2q_1)p_2\}}{p_1p_2-1} \geq N,$$

then all nontrivial solutions of (7) are nonglobal, and that if

$$\frac{\max\{2+2q_1+(2+2q_2)p_1, 2+2q_2+(2+2q_1)p_2\}}{p_1p_2-1} < N,$$

then there are global nontrivial solutions in (7) with suitable initial data. In the case $K_i(x,t) = |x|^{\sigma_i}$ with $0 \leq \sigma_i \leq N(p_i-1)$ $(i = 1,2)$, Mochizuki-Huang [21] proved that if

$$\frac{\max\{2+\sigma_1+(2+\sigma_2)p_1, 2+\sigma_2+(2+\sigma_1)p_2\}}{p_1p_2-1} \geq N,$$

$$\liminf_{|x|\to\infty} |x|^{a_1}u_0(x) > 0 \qquad \text{with } a_1 < \frac{2+\sigma_1+(2+\sigma_2)p_1}{p_1p_2-1},$$

$$\liminf_{|x|\to\infty} |x|^{a_2}v_0(x) > 0 \qquad \text{with } a_2 < \frac{2+\sigma_2+(2+\sigma_1)p_2}{p_1p_2-1},$$

$$u_0(x) \geq M\exp(-\nu_0|x|^2) \qquad \text{for some } \nu_0 > 0 \text{ and large } M > 0$$

or

$$v_0(x) \geq M\exp(-\nu_0|x|^2) \qquad \text{for some } \nu_0 > 0 \text{ and large } M > 0,$$

then the nontrivial solution of (7) is not global in time, and that if

$$\frac{\max\{2+\sigma_1+(2+\sigma_2)p_1, 2+\sigma_2+(2+\sigma_1)p_2\}}{p_1p_2-1} < N,$$

$$\limsup_{|x|\to\infty} |x|^{a_1}u_0(x) < \infty \qquad \text{with } a_1 > \frac{2+\sigma_1+(2+\sigma_2)p_1}{p_1p_2-1},$$

$$\limsup_{|x|\to\infty} |x|^{a_2}v_0(x) < \infty \qquad \text{with } a_2 > \frac{2+\sigma_2+(2+\sigma_1)p_2}{p_1p_2-1}$$

and

$$\|\langle\cdot\rangle^{a_1}u_0\|_\infty + \|\langle\cdot\rangle^{a_2}v_0\|_\infty \text{ is small enough,}$$

then the solution of (7) is global in time. Thereafter, Igarashi-Umeda [10] extended the results to the case $K_i(x,t)$ $(i=1,2)$ satisfying

$$K_i(x,t) \leq C_U\langle x\rangle^{\sigma_i}(t+1)^{q_i}, \tag{8}$$

and

$$K_i(x,t) \geq C_L|x|^{\sigma_i}t^{q_i} \tag{9}$$

for some $C_U, C_L > 0$, and $\sigma_i, q_i \geq 0$. In this case, we obtained that if

$$\frac{\max\{2+\sigma_1+2q_1+(2+\sigma_2+2q_2)p_1, 2+\sigma_2+2q_2+(2+\sigma_1+2q_1)p_2\}}{p_1p_2-1} \geq N,$$

$$\liminf_{|x|\to\infty} |x|^{a_1}u_0(x) > 0 \qquad \text{with } a_1 < \frac{2+\sigma_1+2q_1+(2+\sigma_2+2q_2)p_1}{p_1p_2-1},$$

$$\liminf_{|x|\to\infty} |x|^{a_2}v_0(x) > 0 \qquad \text{with } a_2 < \frac{2+\sigma_2+2q_2+(2+\sigma_1+2q_1)p_2}{p_1p_2-1},$$

$$u_0(x) \geq M\exp(-\nu_0|x|^2) \qquad \text{for some } \nu_0 > 0 \text{ and large } M > 0$$

or

$$v_0(x) \geq M \exp(-v_0|x|^2) \qquad \text{for some } v_0 > 0 \text{ and large } M > 0,$$

then the nontrivial solution of (7) is not global in time, and that if

$$\frac{\max\{2+\sigma_1+2q_1+(2+\sigma_2+2q_2)p_1, 2+\sigma_2+2q_2+(2+\sigma_1+2q_1)p_2\}}{p_1p_2-1} < N,$$

$$\limsup_{|x|\to\infty} |x|^{a_1}u_0(x) < \infty \qquad \text{with } a_1 > \frac{2+\sigma_1+2q_1+(2+\sigma_2+2q_2)p_1}{p_1p_2-1},$$

$$\limsup_{|x|\to\infty} |x|^{a_2}v_0(x) < \infty \qquad \text{with } a_2 > \frac{2+\sigma_2+2q_2+(2+\sigma_1+2q_1)p_2}{p_1p_2-1}$$

and

$$\|\langle\cdot\rangle^{a_1}u_0\|_\infty + \|\langle\cdot\rangle^{a_2}v_0\|_\infty \text{ is small enough,}$$

then the solution of (7) is global in time.

When D is a cone, that is (1), in the case $K_i(x,t) = 1$ $(i = 1,2)$, Levine [17] proved that if

$$\frac{\max\{2+2p_1, 2+2p_2\}}{p_1p_2-1} \geq N + \gamma_+,$$

then (1) has no nontrivial global solutions, and that if

$$\frac{\max\{2+2p_1, 2+2p_2\}}{p_1p_2-1} < N + \gamma_+,$$

then (1) has both global nontrivial solutions as well as solutions which blow up in a finite time. Thereafter, in the case $K_i(x,t) = t^{q_i}$ $(i = 1,2)$, Uda [28] claimed that if

$$\frac{\max\{2+2q_1+(2+2q_2)p_1, 2+2q_2+(2+2q_1)p_2\}}{p_1p_2-1} \geq N + \gamma_+,$$

then all nontrivial solutions of (7) are global, and that if

$$\frac{\max\{2+2q_1+(2+2q_2)p_1, 2+2q_2+(2+2q_1)p_2\}}{p_1p_2-1} < N + \gamma_+,$$

then there are global nontrivial solutions in (1) with suitable initial data. We shall extend the results to the case $K_i(x,t)$ $(i = 1,2)$ satisfying (8) and (9). Before introducing theorems, we define the constant

$$\alpha_i = \frac{(2+\sigma_i+2q_i)+(2+\sigma_j+2q_j)p_i}{p_1p_2-1} \qquad ((i,j) = (1,2),(2,1)). \tag{10}$$

For $a \geq 0$, we also define the following function space:

$$H_{a,M} = \left\{ \xi \in C(\bar{D}) : \xi(x) \geq M\langle x\rangle^{-a}\psi_1\left(\frac{x}{|x|}\right) \text{ for } x \in D \right\}. \tag{11}$$

Our result of the global nonexistence for (1) treated in [11] is stated as follows.

THEOREM 0 (Theorem 2 of [11]). *Assume that $K_i(x,t)$ $(i = 1,2)$ satisfy* (9). *Suppose that one of the following two conditions holds;*

(i) $\max\{\alpha_1, \alpha_2\} \geq N + \gamma_+$,

(ii) $u_0 \in H_{a_1,M}$ *for* $a_1 < \alpha_1$ *or* $v_0 \in H_{a_2,M}$ *for* $a_2 < \alpha_2$ *with some* $M > 0$, *where* $H_{a,M}$, γ_+ *and* α_i $(i = 1,2)$ *are defined in* (11), (6) *and* (10) *respectively. Then there exists no nontrivial nonnegative global solution of* (1), *that is* $T^* < \infty$.

For $a \geq 0$, we define the following function space:

$$H_m^a = \left\{ \xi \in C(\bar{D}) : \xi(x) \leq m\langle x\rangle^{-a}\psi_1\left(\frac{x}{|x|}\right) \text{ for } x \in D \right\}. \tag{12}$$

On the other hand, the main result of this paper is the following global existence theorem.

THEOREM 1. *Assume that $K_i(x,t)$ $(i = 1,2)$ satisfy* (8). *Suppose that* $\max\{\alpha_1, \alpha_2\} < N + \gamma_+$, *and that*

$$(u_0, v_0) \in H_m^{a_1} \times H_m^{a_2} \qquad \textit{for } a_1 > \alpha_1,\ a_2 > \alpha_2 \textit{ with small } m > 0, \tag{13}$$

where H_m^a, γ_+ *and* α_i $(i = 1,2)$ *are defined by* (12), (6) *and* (10) *respectively. Then the solution* (u,v) *of* (1) *is global in time, that is* $T^* = \infty$. *Moreover, there exists a positive constant* C *such that*

$$u(x,t) \leq C\tilde{u}(x,t+1) \qquad \textit{and} \qquad v(x,t) \leq C\tilde{v}(x,t+1) \qquad \textit{in } D \times (0,\infty),$$

where $\tilde{u}(x,t)$ *and* $\tilde{v}(x,t)$ *are the solutions of the problems*

$$\begin{cases} \tilde{u}_t = \Delta\tilde{u}, & x \in D,\ t > 0, \\ \tilde{u}(x,0) = \langle x\rangle^{-a_1}\psi_1(x/|x|), & x \in D, \\ \tilde{u}(x,t) = 0, & x \in \partial D,\ t \geq 0, \end{cases} \tag{14}$$

and

$$\begin{cases} \tilde{v}_t = \Delta\tilde{v}, & x \in D,\ t > 0, \\ \tilde{v}(x,0) = \langle x\rangle^{-a_2}\psi_1(x/|x|), & x \in D, \\ \tilde{v}(x,t) = 0, & x \in \partial D,\ t \geq 0. \end{cases} \tag{15}$$

Theorems 0 and 1 may be summarized in the following table.

	$\max\{\alpha_1, \alpha_2\} \geq N + \gamma_+$	$\max\{\alpha_1, \alpha_2\} < N + \gamma_+$
$a_1 < \alpha_1$ or $a_2 < \alpha_2$	NG	NG
$a_1 > \alpha_1$ and $a_2 > \alpha_2$	NG	G

NG: There exists no global nontrivial solution in time.
G: There exists a global nontrivial solution in time.

REMARK 1. *Let* $D = \mathbf{R}^N$, *then the above table holds with* $\gamma_+ = 0$. ($\Omega = S^{N-1}$ *and* $\omega_1 = 0$.)

The rest of the paper is organized as follows. Some preliminary lemmata are given in Section 2. In Section 5 we prove Lemma 2.2 in Section 2 of this paper. Theorem 1 is proved in Section 3. In Section 4 we confirm the form of the Green function for the heat equation in the cone domain with the Dirichlet condition. According to the change of variables (16), we express function as follow: $\zeta(x, y, t) = \zeta(r, \theta, \rho, \phi, t)$, $\zeta(x, t) = \zeta(r, \theta, t)$ or $\zeta_0(x) = \zeta_0(r, \theta)$.

2. Preliminaries

In this section we prepare some lemmata for proving Theorem 1.

Let Δ_Ω denote the Laplace-Beltrami operator with homogeneous Dirichlet boundary condition in Ω. Let $\psi_n(\theta)$ $(\theta = x/|x|)$ denote the n-th eigenfunction of $-\Delta_\Omega$ with Dirichlet problem in Ω satisfying $\|\psi_n\|_{L^2(\Omega)} > 0$, where $\|\xi\|_{L^2(\Omega)} = \sqrt{\int_\Omega \xi^2(\phi)d\phi}$. Let $\omega_n > 0$ denote the eigenvalue corresponding to ψ_n. The sequence $\{\psi_n/\|\psi_n\|_{L^2(\Omega)}\}_{n=1}^\infty$ is a complete orthonormal sequence (see [2, p. 53, Chapter III, Theorem 18]).

We introduce the Green's function $G(x, y, t) = G(r, \theta, \rho, \phi, t)$ for the linear heat equation in the cone D, where

$$r = |x|, \qquad \rho = |y|, \qquad \theta = \frac{x}{|x|} \qquad \text{and} \qquad \phi = \frac{y}{|y|} \in \Omega. \tag{16}$$

The heat kernel is explicitly given by

$$G(r, \theta, \rho, \phi, t) = \frac{(r\rho)^{-(N-2)/2}}{2t} \exp\left(-\frac{\rho^2 + r^2}{4t}\right) \sum_{n=1}^{\infty} c_n I_{\nu_n}\left(\frac{r\rho}{2t}\right)\psi_n(\theta)\psi_n(\phi), \tag{17}$$

where

$$\nu_n = [(N-2)^2/4 + \omega_n]^{1/2}, \tag{18}$$

$c_n = 1/\|\psi_n\|^2_{L^2(\Omega)}$ and I_ν is the modified Bessel function:

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\Gamma(\nu+k+1)} \sim \begin{cases} (z/2)^\nu/\Gamma(\nu+1), & \text{as } z \to 0^+ \\ e^z/\sqrt{2\pi z}, & \text{as } z \to +\infty \end{cases} \tag{19}$$

with the Gamma function $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}\,ds$ (see [19] and for detail Section 4).

Remark 2. *The constant* γ_+ *is the positive root of* $\gamma(\gamma+N-2) = \omega_1$ *by* (6) *and satisfies* $\gamma_+ = \nu_1 - (N-2)/2$ *by* (18).

Example 1. *When* $N = 2$ *or* $\Delta_\Omega = \dfrac{\partial^2}{\partial\theta^2}$ *and* $\Omega = (0, a)$ $(a > 0)$, $\omega_1 = \left(\dfrac{\pi}{a}\right)^2$ *namely* $\gamma_+ = \dfrac{\pi}{a}$, *while* $\psi_1(\theta) = A\sin\dfrac{\pi}{a}\theta$ *for any constant* $A(\neq 0)$.

Example 2. *When* $D = \mathbf{R}^{N-1}\times\mathbf{R}_+$, $\omega_1 = N-1$ *namely* $\gamma_+ = 1$.

An operator $e^{t\Delta}$ is defined by

$$e^{t\Delta}\xi(x) = \int_D G(x,y,t)\xi(y)dy = \int_0^\infty\int_\Omega G(r,\theta,\rho,\phi,t)\xi(\rho,\phi)\rho^{N-1}\,d\phi d\rho. \tag{20}$$

Here, letting $w(x,t) = e^{t\Delta}\xi(x)$, $w(x,t)$ is the solution of the initial-boundary value problem

$$\begin{cases} w_t = \Delta w, & x \in D,\ t > 0, \\ w(x,0) = \xi(x), & x \in D, \\ w(x,t) = 0, & x \in \partial D,\ t \geq 0. \end{cases}$$

The solutions of (1) satisfy the following integral equations:

$$\begin{cases} u(x,t) = e^{t\Delta}u_0(x) + \displaystyle\int_0^t e^{(t-s)\Delta}(K_1(x,s)v^{p_1}(x,s))ds, \\ v(x,t) = e^{t\Delta}v_0(x) + \displaystyle\int_0^t e^{(t-s)\Delta}(K_2(x,s)u^{p_2}(x,s))ds. \end{cases}$$

We define for $a > 0$

$$\begin{aligned} \eta_a(x,t) &= e^{(t+1)\Delta}\left(\langle x\rangle^{-a}\psi_1\left(\frac{x}{|x|}\right)\right) = \int_D G(x,y,t+1)\langle y\rangle^{-a}\psi_1\left(\frac{y}{|y|}\right)dy \\ &= \int_0^\infty\int_\Omega G(r,\theta,\rho,\phi,t+1)(1+\rho^2)^{-a/2}\psi_1(\phi)\rho^{N-1}\,d\phi d\rho, \end{aligned} \tag{21}$$

where the heat kernel $G(r,\theta,\rho,\phi,t)$ and the operator $e^{t\Delta}$ are defined by (17) and (20), respectively.

LEMMA 2.1. *Let* η_a *be defined in* (21) *with* $a>0$. *Then we have in* $D\times(0,\infty)$,

$$\eta_a(x,t) \geq C\min\{|x|^{\gamma_+}(t+1)^{-(a+\gamma_+)/2}, |x|^{-a}\}\psi_1\left(\frac{x}{|x|}\right).$$

PROOF. From (19), we may estimate

$$I_\nu(z) \geq \begin{cases} Cz^\nu, & 0<z\leq 1, \\ Cz^{-1/2}e^z, & z>1 \end{cases} \tag{22}$$

with some constant $C>0$. By (17) and (20) we see that

$$\begin{aligned}\eta_a(x,t) &= \int_0^\infty\int_\Omega G(r,\theta,\rho,\phi,t+1)(1+\rho^2)^{-a/2}\psi_1(\phi)\rho^{N-1}\,d\phi d\rho \\ &\geq c_1\|\psi_1\|_{L^2(\Omega)}^2\int_0^\infty \frac{(r\rho)^{-(N-2)/2}}{2(t+1)}\exp\left(-\frac{\rho^2+r^2}{4(t+1)}\right)I_{\nu_1}\left(\frac{r\rho}{2(t+1)}\right) \\ &\quad\times(1+\rho^2)^{-a/2}\psi_1(\theta)\rho^{N-1}\,d\rho.\end{aligned}$$

Note that $c_1 = 1/\|\psi_1\|_{L^2(\Omega)}^2$. By (22) we obtain

$$\begin{aligned}\eta_a(x,t) \geq{}& \frac{C\psi_1(\theta)}{\{2(t+1)\}^{\nu_1+1}} \\ &\times\int_0^{2(t+1)/r} r^{\nu_1-(N-2)/2}\rho^{N/2+\nu_1}(1+\rho^2)^{-a/2}\exp\left(-\frac{\rho^2+r^2}{4(t+1)}\right)d\rho,\end{aligned} \tag{23}$$

and

$$\begin{aligned}\eta_a(x,t) \geq{}& \frac{C\psi_1(\theta)}{\sqrt{2(t+1)}} \\ &\times\int_{2(t+1)/r}^\infty r^{-(N-1)/2}\rho^{(N-1)/2}(1+\rho^2)^{-a/2}\exp\left(-\frac{(\rho-r)^2}{4(t+1)}\right)d\rho.\end{aligned} \tag{24}$$

If $r/\sqrt{t+1}\leq 1$, from (23), by putting $s=\rho/\sqrt{t+1}$, then we obtain

$$\begin{aligned}\eta_a(x,t) \geq{}& C\exp\left(-\frac{r^2}{4(t+1)}\right)r^{\nu_1-(N-2)/2}(t+1)^{-\gamma_+/2}\psi_1(\theta) \\ &\times\int_0^{2\sqrt{t+1}/r} s^{N/2+\nu_1}\exp\left(-\frac{s^2}{4}\right)\{1+s^2(t+1)\}^{-a/2}ds\end{aligned}$$

$$\geq C \exp\left(-\frac{r^2}{4(t+1)}\right) r^{\gamma_+}(t+1)^{-(a+\gamma_+)/2}\psi_1(\theta)$$
$$\times \int_0^{2\sqrt{t+1}/r} s^{N/2+\nu_1} \exp\left(-\frac{s^2}{4}\right)(1+s^2)^{-a/2} ds$$
$$\geq C r^{\gamma_+}(t+1)^{-(a+\gamma_+)/2}\psi_1(\theta) \qquad \text{for } t \geq 0 \text{ and } r/\sqrt{t+1} \leq 1. \tag{25}$$

On the other hand, if $r/\sqrt{t+1} > 1$, from (24), by putting $s = (\rho - r)/\sqrt{t+1}$, we have

$$\eta_a(x,t) \geq C\psi_1(\theta) \int_{\max\{2\sqrt{t+1}/r - r/\sqrt{t+1},0\}}^{\infty} \left(\frac{s\sqrt{t+1}}{r} + 1\right)^{(N-1)/2}$$
$$\times \{1 + (s\sqrt{t+1} + r)^2\}^{-a/2} \exp\left(-\frac{s^2}{4}\right) ds$$
$$\geq C(t+1)^{-a/2}\psi_1(\theta) \int_{\max\{2\sqrt{t+1}/r - r/\sqrt{t+1},0\}}^{\infty} \left(\frac{s\sqrt{t+1}}{r} + 1\right)^{(N-1)/2}$$
$$\times \left\{1 + \left(s + \frac{r}{\sqrt{t+1}}\right)^2\right\}^{-a/2} \exp\left(-\frac{s^2}{4}\right) ds.$$

Letting $\xi = r/\sqrt{t+1}$, then we get

$$r^a \eta_a(x,t) \geq C\psi_1(\theta)\xi^a \int_{\max\{2/\xi - \xi, 0\}}^{\infty} \left(1 + \frac{s}{\xi}\right)^{(N-1)/2} \{1 + (\xi + s)^2\}^{-a/2} \exp\left(-\frac{s^2}{4}\right) ds$$
$$\geq C\psi_1(\theta) \int_1^2 \left(1 + \frac{s}{\xi}\right)^{(N-1)/2} \frac{\xi^a}{\{1 + (\xi + s)^2\}^{a/2}} \exp\left(-\frac{s^2}{4}\right) ds$$
$$\geq C\psi_1(\theta) \qquad \text{for } t \geq 0 \text{ and } \xi = r/\sqrt{t+1} > 1. \tag{26}$$

Summarizing (25) and (26), we obtain the inequality in the lemma. □

Lemma 2.2. *Let* η_a *be defined in* (21) *with* $a > 0$. *Assume* $-\gamma_+ \leq \kappa < \min\{a, N + \gamma_+\}$. *Then*

$$\langle x \rangle^\kappa \eta_a(x,t) \leq C(t+1)^{[\kappa - \min\{N+\gamma_+, a\} + \varepsilon]/2} \psi_1(x/|x|),$$

for any $(x,t) \in D \times (0,\infty)$, *any* $\varepsilon > 0$ *and some positive constant* $C = C(\varepsilon) = C(\varepsilon, \sigma, a, N, \gamma_+)$.

Proof. See Section 5 of this paper. □

LEMMA 2.3. *Let η_a be defined in* (21) *with $a > 0$. Assume $p \geq 1$, $\sigma \geq 0$, $q \geq 0$ and $b > 0$ and*

$$p \min\{a, N + \gamma_+\} - b > 2 + \sigma + 2q. \tag{27}$$

Then there exists a positive constant C such that

$$(t+1)^q \langle x\rangle^\sigma \eta_a^p(x,t) \leq C(t+1)^{[\sigma+2q+b-\min\{a,N+\gamma_+\}p+\varepsilon]/2}\eta_b(x,t), \tag{28}$$

for any $(x,t) \in D \times (0,\infty)$, any $\varepsilon > 0$ and some positive constant $C = C(\varepsilon) = C(\varepsilon, a, b, p, q, \sigma, N, \gamma_+)$.

PROOF. By Lemma 2.1, we obtain

$$\begin{aligned}(t+1)^q \langle x\rangle^\sigma \eta_a^p(x,t) &= (t+1)^q \langle x\rangle^\sigma \eta_a^p(x,t)\eta_b^{-1}(x,t)\eta_b(x,t)\\ &\leq C(t+1)^q \langle x\rangle^\sigma \eta_a^p(x,t)\\ &\qquad \times \max\{|x|^{-\gamma_+}(t+1)^{(b+\gamma_+)/2}, |x|^b\}\psi_1^{-1}\left(\frac{x}{|x|}\right)\eta_b(x,t).\end{aligned}$$

From Lemma 2.2 we have

$$(t+1)^q \langle x\rangle^\sigma \eta_a^p(x,t) \leq C(t+1)^{[\sigma+2q+b-\min\{a,N+\gamma_+\}p+\varepsilon]/2}\eta_b(x,t)\psi_1^{p-1}\left(\frac{x}{|x|}\right)$$

for any $\varepsilon > 0$. If $p \geq 1$, then $\psi_1^{p-1}(x/|x|)$ is bounded. Hence, we obtain (28). □

3. Existence of a global solution

In this section we treat the existence of global in time solutions of (1). Here, we take the same strategy as in [21] and [30].

Since $\max\{\alpha_1, \alpha_2\} < N + \gamma_+$, $a_1 > \alpha_1$ and $a_2 > \alpha_2$ hold, there exist $\tilde{a}_i \in (\alpha_i, a_i]$ $(i = 1, 2)$ such that

$$p_i \min\{\tilde{a}_j, N + \gamma_+\} - \tilde{a}_i > 2 + \sigma_i + 2q_i \qquad ((i,j) = (1,2),(2,1)). \tag{29}$$

Since $H_m^{a_i} \subseteq H_m^{\tilde{a}_i}$ $(i = 1, 2)$, we may let $\tilde{a}_i = a_i$ $(i = 1, 2)$ without loss of generality, where H_m^a are defined in (12).

We define the Banach space X as

$$X = \{v \geq 0 : \|v/\eta_{a_2}\|_\infty < \infty\},$$

where η_a is defined in (21) with $a > 0$ and

$$\|w\|_\infty = \sup_{(x,t) \in D \times (0,\infty)} |w(x,t)|.$$

We consider the associated integral system

$$u(x,t) = e^{t\Delta}u_0(x) + \int_0^t e^{(t-s)\Delta}(K_1(x,s)v^{p_1}(x,s))ds, \tag{30}$$

$$v(x,t) = e^{t\Delta}v_0(x) + \int_0^t e^{(t-s)\Delta}(K_2(x,s)u^{p_2}(x,s))ds \tag{31}$$

with $e^{t\Delta}$ defined in (20). Substituting (30) into (31), we have

$$v(x,t) = V(u_0, v_0, v) \tag{32}$$

with

$$V(u_0,v_0,v) = e^{t\Delta}v_0(x) + \int_0^t e^{(t-s)\Delta} \times \left[K_2(x,s)\left\{e^{s\Delta}u_0(x) + \int_0^s e^{(s-\tau)\Delta}(K_1(x,\tau)v^{p_1}(x,\tau))d\tau\right\}^{p_2}\right]ds.$$

If V is a strict contraction, then its fixed point yields a solution of (1). Moreover, by the fact $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for $a > 0$, $b > 0$ and $p \ge 1$, we obtain

$$V(u_0,v_0,v) \le T(u_0,v_0) + \Gamma(v), \tag{33}$$

where

$$T(u_0,v_0) = e^{t\Delta}v_0(x) + 2^{p_2-1}\int_0^t e^{(t-s)\Delta}\{K_2(x,s)(e^{s\Delta}u_0(x))^{p_2}\}ds,$$

$$\Gamma(v) = 2^{p_2-1}\int_0^t e^{(t-s)\Delta}\left[K_2(x,s)\left\{\int_0^s e^{(s-\tau)\Delta}(K_1(x,\tau)v^{p_1}(x,\tau))d\tau\right\}^{p_2}\right]ds.$$

Lemma 3.1. *Assume the same hypotheses as in Lemma* 2.3. *Then there exists a constant* $C > 0$ *such that*

$$\int_0^t (s+1)^q e^{(t-s)\Delta}(\langle x\rangle^\sigma \eta_a^p(x,s))ds \le C\eta_b(x,t) \tag{34}$$

for any $(x,t) \in D \times (0,\infty)$.

Proof. Put

$$\varepsilon = \frac{p\min\{a, N+\gamma_+\} - b - 2 - \sigma - 2q}{2}.$$

Then from (27) there exists $\varepsilon > 0$ such that

$$\sigma + 2q + b - p\min\{a, N+\gamma_+\} + \varepsilon \equiv \beta < -2. \tag{35}$$

From Lemma 2.3, we have

$$\int_0^t (s+1)^q e^{(t-s)\Delta}(\langle x\rangle^\sigma \eta_a^p(x,s))ds \le C\eta_b(x,t)\int_0^t (s+1)^{\beta/2}ds.$$

From (35) there exists a constant $C' > 0$ such that

$$\int_0^t (s+1)^q e^{(t-s)\Delta}(\langle x\rangle^\sigma \eta_a^p(x,s))ds \le C'\eta_b(x,t). \qquad \square$$

LEMMA 3.2. *Let η_a be defined in* (21) *with $a > 0$.*

(i) *Let (u_0, v_0) satisfy* (13). *Then $T(u_0, v_0) \in X$ and*

$$\|T(u_0, v_0)/\eta_{a_2}\|_\infty \le C_a(m + m^{p_2})$$

with some $C_a > 0$, where m is the constant appeared in (12).

(ii) *Let v be the second element of the solution of* (1). *Then Γ maps X into itself and*

$$\|\Gamma(v)/\eta_{a_2}\|_\infty \le C_b\|v/\eta_{a_2}\|_\infty^{p_1p_2}$$

with some $C_b > 0$.

PROOF. (i) First, it is easily seen that $e^{t\Delta}v_0(x) \le m\eta_{a_2}(x,t)$. Next, from Lemma 3.1 and (29), we obtain

$$\int_0^t e^{(t-s)\Delta}\{K_2(x,s)(e^{s\Delta}u_0(x))^{p_2}\}ds$$

$$\le C_U\int_0^t (s+1)^{q_2}e^{(t-s)\Delta}\{\langle x\rangle^{\sigma_2}(m\eta_{a_1}(x,s))^{p_2}\}ds \le Cm^{p_2}\eta_{a_2}(x,t).$$

Thus, we have

$$|T(u_0, v_0)| \le C\eta_{a_2}(x,t)(m + m^{p_2}).$$

This implies assertion (i).

(ii) Similarly as above, it follows from Lemma 3.1 and (29) that

$$\begin{aligned}\Gamma(v) &\le C\|v/\eta_{a_2}\|_\infty^{p_1p_2}\int_0^t (s+1)^{q_2}e^{(t-s)\Delta} \\ &\quad\times\left[\langle x\rangle^{\sigma_2}\left\{\int_0^s (\tau+1)^{q_1}e^{(s-\tau)\Delta}(\langle x\rangle^{\sigma_1}\eta_{a_2}^{p_1}(x,\tau))d\tau\right\}^{p_2}\right]ds \\ &\le C\|v/\eta_{a_2}\|_\infty^{p_1p_2}\int_0^t (s+1)^{q_2}e^{(t-s)\Delta}(\langle x\rangle^{\sigma_2}\eta_{a_1}^{p_2}(x,s))ds \\ &\le C\|v/\eta_{a_2}\|_\infty^{p_1p_2}\eta_{a_2}(x,t).\end{aligned}$$

Assertion (ii) thus is established. $\square$

Proof of Theorem 1. Let $B_m = \{v \in X; \|v/\eta_{a_2}\|_\infty \le (2C_a + 1)m\}$, where C_a and m are the constants appeared in Lemma 3.2 and (12), respectively. We shall show that $V(u_0, v_0, v)$ is a strict contraction on B_m into itself provided m is small enough.

From (33) and Lemma 3.2 we have

$$\begin{aligned}\|V(u_0, v_0, v)/\eta_{a_2}\|_\infty &\le \|T(u_0, v_0)/\eta_{a_2}\|_\infty + \|\Gamma(v)/\eta_{a_2}\|_\infty \\ &\le C_a(m + m^{p_2}) + C_b\{(2C_a + 1)m\}^{p_1 p_2} \le (2C_a + 1)m.\end{aligned}$$

This proves that V maps B_m into B_m.

Now, we show that $V(u_0, v_0, v)$ is a strict contraction on B_m. By the definition of V we obtain

$$\begin{aligned}&|V(u_0, v_0, v_1) - V(u_0, v_0, v_2)| \\ &\quad \le \int_0^t e^{(t-s)\Delta} K_2(x, s) \left| \left(e^{s\Delta} u_0(x) + \int_0^s e^{(s-\tau)\Delta}(K_1(x, \tau) v_1^{p_1}(x, \tau)) d\tau \right)^{p_2} \right. \\ &\qquad\qquad \left. - \left(e^{s\Delta} u_0(x) + \int_0^s e^{(s-\tau)\Delta}(K_1(x, \tau) v_2^{p_1}(x, \tau)) d\tau \right)^{p_2} \right| ds.\end{aligned}$$

Since $|a^p - b^p| \le p(a + b)^{p-1}|a - b|$ for $a \ge 0$, $b \ge 0$ and $p \ge 1$, we can estimate the difference as follows,

$$|V(u_0, v_0, v_1) - V(u_0, v_0, v_2)| \le p_2 \int_0^t e^{(t-s)\Delta}(K_2(x, s) A(x, s) B(x, s)) ds,$$

where

$$\begin{aligned}A(x, s) &= \left(2e^{s\Delta} u_0(x) + \int_0^s e^{(s-\tau)\Delta}\{K_1(x, \tau)(v_1^{p_1}(x, \tau) + v_2^{p_1}(x, \tau))\} d\tau \right)^{p_2 - 1}, \\ B(x, s) &= \left| \int_0^s e^{(s-\tau)\Delta}\{K_1(x, \tau)(v_1^{p_1}(x, \tau) - v_2^{p_1}(x, \tau))\} d\tau \right|.\end{aligned}$$

Since $(a + b)^p \le 2^{\max\{p-1, 0\}}(a^p + b^p)$ for $a \ge 0$, $b \ge 0$ and $p \ge 0$, we obtain

$$\begin{aligned}A(x, s) \le 2^{\max\{p_2 - 2, 0\}} \Bigg[&(2e^{s\Delta} u_0(x))^{p_2 - 1} \\ &+ \left\{ C_U \int_0^s (\tau + 1)^{q_1} e^{(s-\tau)\Delta}(\langle x \rangle^{\sigma_1} 2\tilde{v}^{p_1}(x, \tau)) d\tau \right\}^{p_2 - 1} \Bigg]\end{aligned}$$

with $\tilde{v} = \max\{v_1, v_2\}$ and

$$B(x,s) \le C_U \int_0^s (\tau+1)^{q_1} e^{(s-\tau)\Delta}(\langle x\rangle^{\sigma_1}|v_1^{p_1}(x,\tau) - v_2^{p_1}(x,\tau)|)d\tau$$
$$\le p_1 C_U \int_0^s (\tau+1)^{q_1} e^{(s-\tau)\Delta}$$
$$\times \{\langle x\rangle^{\sigma_1}(v_1(x,\tau)+v_2(x,\tau))^{p_1-1}|v_1(x,\tau)-v_2(x,\tau)|\}d\tau.$$

From Lemma 3.1 and (29), we have

$$A(x,s) \le 2^{\max\{p_2-2,0\}}\Bigg[(2m\eta_{a_1}(x,s))^{p_2-1}$$
$$+\left\{2C_U\|\tilde{v}/\eta_{a_2}\|_\infty^{p_1}\int_0^s(\tau+1)^{q_1}e^{(s-\tau)\Delta}(\langle x\rangle^{\sigma_1}\eta_{a_2}^{p_1}(x,\tau))d\tau\right\}^{p_2-1}\Bigg]$$
$$\le 2^{\max\{p_2-2,0\}}\{(2m)^{p_2-1}\eta_{a_1}^{p_2-1}(x,s) + (2C(3m)^{p_1})^{p_2-1}\eta_{a_1}^{p_2-1}(x,s)\}$$

and

$$B(x,s) \le p_1 C_U \int_0^s (\tau+1)^{q_1} e^{(s-\tau)\Delta}\{\langle x\rangle^{\sigma_1}(2v(x,\tau))^{p_1-1}|v_1(x,\tau)-v_2(x,\tau)|\}d\tau$$
$$\le 2^{p_1-1}p_1 C_U \int_0^s (\tau+1)^{q_1} e^{(s-\tau)\Delta}$$
$$\times \left\{\langle x\rangle^{\sigma_1}\eta_{a_2}^{p_1}(x,\tau)\left(\frac{\tilde{v}(x,\tau)}{\eta_{a_2}(x,\tau)}\right)^{p_1-1}\left(\frac{|v_1(x,\tau)-v_2(x,\tau)|}{\eta_{a_2}(x,\tau)}\right)\right\}d\tau.$$

We may take m satisfying $(2m)^{p_2-1} + (2C(3m)^{p_1})^{p_2-1} \le 2^{p_2}m^{(p_2-1)/2}$. We then have

$$|V(u_0,v_0,v_1) - V(u_0,v_0,v_2)|$$
$$\le C\int_0^t (s+1)^{q_2}e^{(t-s)\Delta}\{\langle x\rangle^{\sigma_2}(2^{p_2}m^{(p_2-1)/2}\eta_{a_1}^{p_2-1}(x,s))\eta_{a_1}(x,s)\}ds$$
$$\times \|\tilde{v}/\eta_{a_2}\|_\infty^{p_1-1}\|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty$$
$$\le Cm^{p_1+p_2/2-3/2}\int_0^t (s+1)^{q_2}e^{(t-s)\Delta}(\langle x\rangle^{\sigma_2}\eta_{a_1}^{p_2}(x,s))ds\|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty$$
$$\le Cm^{p_1+p_2/2-3/2}\eta_{a_2}(x,t)\|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty.$$

Since $p_1, p_2 \geq 1$ and $p_1 p_2 > 1$, we obtain for some $\rho < 1$

$$\begin{aligned}&\|V(u_0, v_0, v_1)/\eta_{a_2} - V(u_0, v_0, v_2)/\eta_{a_2}\|_\infty \\ &\quad \leq Cm^{p_1+p_2/2-3/2}\|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty \leq \rho\|v_1/\eta_{a_2} - v_2/\eta_{a_2}\|_\infty\end{aligned}$$

with m small enough. Then V is a strict contraction on B_m into itself. Hence, there exists a unique fixed point $v \in X$ which solves (32). Substitute v into (30). Then (u, v) solves (30) and (31). Moreover, since $v \in B_m$, we find

$$v(x,t) \leq Ce^{(t+1)\Delta}\left(\langle x\rangle^{-a_2}\psi_1\left(\frac{x}{|x|}\right)\right) = C\tilde{v}(x, t+1),$$

where $\tilde{v}(x,t)$ is the solution of (15). Substituting this into (30), we have

$$\begin{aligned}u(x,t) &\leq m\eta_{a_1}(x,t) + C\int_0^t (s+1)^{q_1} e^{(t-s)\Delta}(\langle x\rangle^{\sigma_1}\eta_{a_2}^{p_1}(x,s))ds \\ &\leq m\eta_{a_1}(x,t) + C\eta_{a_1}(x,t) \leq C\eta_{a_1}(x,t).\end{aligned}$$

We then have

$$u(x,t) \leq Ce^{(t+1)\Delta}\left(\langle x\rangle^{-a_1}\psi_1\left(\frac{x}{|x|}\right)\right) = C\tilde{u}(x, t+1),$$

where $\tilde{u}(x,t)$ is the solution of (14). The proof of Theorem 1 is completed. □

4. Appendix A: A heat kernel in a cone domain

In this section we confirm the form of the Green function for the heat equation in the cone domain with the Dirichlet condition. In [19] the fact had been shown. In this section, the fact is confirmed.

We consider the initial-boundary value problem for a heat equation

$$\begin{cases} u_t = \Delta u, & x \in D,\ t > 0, \\ u(x,0) = u_0(x), & x \in D, \\ u = 0, & x \in \partial D,\ t \geq 0, \end{cases} \tag{36}$$

where the domain D is a cone in $\mathbf{R}^N$ such as

$$D = \left\{x \in \mathbf{R}^N : x \neq 0 \text{ and } \frac{x}{|x|} \in \Omega\right\},$$

where Ω is some region on S^{N-1} with C^∞-boundary $\partial\Omega$. We introduce the Green's function $G(x, y, t) = G(r, \theta, \rho, \phi, t)$ for the linear heat equation in the cone D. By the variable transformation (16) the problem (36) is expressed as the form

$$\begin{cases} u_t = \Delta u = u_{rr} + \dfrac{N-1}{r} u_r + \dfrac{\Delta_\Omega u}{r^2}, & r > 0,\ \theta \in \Omega,\ t > 0, \\ u(r,\theta,0) = u_0(r,\theta), & r > 0,\ \theta \in \Omega, \\ u = 0, & r > 0,\ \theta \in \partial\Omega, \end{cases} \tag{37}$$

where Δ_Ω is Laplace-Beltrami operator on $\Omega \subset S^{N-1}$.

For the Laplace-Beltrami operator with homogeneous Dirichlet boundary condition on $\Omega \in S^{N-1}$, we denote by $(\omega_n, \psi_n(\theta))$ the corresponding eigenpairs. Then it follows that $\int_\Omega \psi_n^2(\theta) d\theta > 0$ and

$$\int_\Omega \psi_m(\theta)\psi_n(\theta) d\theta = 0$$

for $m \neq n$.

It is known that the Green's function associated with (36) is given by

$$G(r,\theta,\rho,\phi,t) = \frac{(r\rho)^{-(N-2)/2}}{2t} \exp\left(-\frac{\rho^2 + r^2}{4t}\right) \sum_{n=1}^{\infty} c_n I_{\nu_n}\left(\frac{r\rho}{2t}\right) \psi_n(\theta)\psi_n(\phi), \tag{38}$$

where $c_n = 1/\|\psi_n\|_{L^2(\Omega)}^2$ and $\nu_n = [(N-2)^2/4 + \omega_n]^{1/2}$. The function I_ν is the modified Bessel function. The functions satisfy

$$\int_0^\infty e^{-\lambda t} J_\nu(\sqrt{\lambda} r) J_\nu(\sqrt{\lambda}\rho) d\lambda = \frac{1}{t} \exp\left(-\frac{r^2 + \rho^2}{4t}\right) I_\nu\left(\frac{r\rho}{2t}\right) \tag{39}$$

with the Bessel functions J_ν satisfying

$$x^2 J_\nu''(x) + x J_\nu'(x) + (x^2 - \nu^2) J_\nu(x) = 0$$

and

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \Gamma(m + \nu + 1)}$$

(see [31, p. 395]).

From (38) and (39) we see that

$$G(r,\theta,\rho,\phi,t) = \frac{(r\rho)^{-(N-2)/2}}{2} \sum_{n=1}^{\infty} c_n \psi_n(\theta)\psi_n(\phi) \int_0^\infty e^{-\lambda t} J_{\nu_n}(\sqrt{\lambda} r) J_{\nu_n}(\sqrt{\lambda}\rho) d\lambda. \tag{40}$$

The solution of (36) is explicitly given by

$$u(x,t) = u(r,\theta,t) = \int_0^\infty \int_\Omega G(r,\theta,\rho,\phi,t)u_0(\rho,\phi)\rho^{N-1}\, d\phi d\rho. \tag{41}$$

We give the proof of (41) below.

Let $\tilde{u}$ be the inverse Laplace transformed function of u, i.e.

$$u(r,\theta,t) = \int_0^\infty \tilde{u}(r,\theta,s)e^{-st}\, ds.$$

Then this $\tilde{u}$ satisfies the following equation of the form

$$-s\tilde{u} = \tilde{u}_{rr} + \frac{N-1}{r}\tilde{u}_r + \frac{\Delta_\Omega \tilde{u}}{r^2}, \qquad r > 0,\ \theta \in \Omega,\ s > 0. \tag{42}$$

Since $\{\psi_n/\|\psi_n\|_{L^2(\Omega)}\}$ is a complete orthonormal system, we have

$$\tilde{u}(r,\theta,s) = \sum_{n=1}^\infty \tilde{w}_n(r,s)\psi_n(\theta) \tag{43}$$

with

$$\tilde{w}_n(r,s) = c_n \int_\Omega \tilde{u}(r,\phi,s)\psi_n(\phi)d\phi.$$

From (42) and (43) we see that

$$r^2(\tilde{w}_n)_{rr} + (N-1)r(\tilde{w}_n)_r + (r^2 s - \omega_n)\tilde{w}_n = 0. \tag{44}$$

By the Frobenius method we obtain

$$\tilde{w}_n(r,s) = a_n(s)r^{-(N-2)/2}J_{\nu_n}(\sqrt{s}r)$$

with some $a_n(s)$. From (43) we see that

$$\tilde{u}(r,\theta,s) = \sum_{n=1}^\infty \{a_n(s)r^{-(N-2)/2}J_{\nu_n}(\sqrt{s}r)\psi_n(\theta)\}. \tag{45}$$

We thus see that

$$u(x,t) = u(r,\theta,t) = \sum_{n=1}^\infty \int_0^\infty a_n(s)r^{-(N-2)/2}J_{\nu_n}(\sqrt{s}r)e^{-st}\, ds\psi_n(\theta).$$

If we let $t = 0$, we have

$$u_0(x) = u_0(r,\theta) = \sum_{n=1}^{\infty} \int_0^{\infty} a_n(s) r^{-(N-2)/2} J_{\nu_n}(\sqrt{s}r)\,ds\,\psi_n(\theta).$$

Then since

$$\frac{1}{2}\int_0^{\infty}\int_0^{\infty} J_\nu(\sqrt{s}\rho)J_\nu(\sqrt{s}r)f(\rho)\,ds\,d\rho = \int_0^{\infty}\int_0^{\infty} \sigma J_\nu(\sigma\rho)J_\nu(\sigma r)f(\rho)\,d\sigma\,d\rho = \frac{1}{r}f(r)$$

for any $f \in C(0,\infty)$ (see [31, p. 453], see also [6, §2]) and $\{\psi_n/\|\psi_n\|_{L^2(\Omega)}\}$ is a complete orthonormal system (see [2, p. 53, Chapter III, Theorem 18]), we see that

$$a_n(s) = \frac{c_n}{2}\int_0^{\infty}\int_{\Omega} \rho^{N/2} J_{\nu_n}(\sqrt{s}\rho) u_0(\rho,\phi)\psi_n(\phi)\,d\phi\,d\rho.$$

Then we have (41). □

5. Appendix B: Proof of Lemma 2.2

In this section we give a proof of Lemma 2.2. This lemma is equivalent to the following proposition:

PROPOSITION 5.1. *Let η_a be defined in (21) with $a > 0$. Assume $-\gamma_+ \le \kappa < \min\{a, N+\gamma_+\}$. Let $\zeta > 0$ be*

(i) $\zeta = a - \kappa$, *if* $a < N + \gamma_+$,

(ii) $\zeta < N + \gamma_+ - \kappa$, *if* $a = N + \gamma_+$,

(iii) $\zeta = N + \gamma_+ - \kappa$, *if* $a > N + \gamma_+$.

Then there exists a positive constant C such that

$$|x|^\kappa \eta_a(x,t) \le C(t+1)^{-\zeta/2}\psi_1(x/|x|) \quad \text{for } x \in D,\ t > 0, \tag{46}$$

and

$$\langle x\rangle^\kappa \eta_a(x,t) \le C(t+1)^{-\zeta/2}\psi_1(x/|x|) \quad \text{for } x \in D,\ t > 0. \tag{47}$$

PROOF. We follow the argument of Hamada [8, Lemma 3.1].
By (21), we see that

$$r^\kappa \eta_a(x,t) = r^\kappa \int_0^{\infty}\int_{\Omega} G(r,\theta,\rho,\phi,t+1)(1+\rho^2)^{-a/2}\psi_1(\phi)\rho^{N-1}\,d\phi\,d\rho.$$

Since $\{\psi_n\}$ is an orthogonal system, we have

$$|x|^{\kappa}\eta_a(x,t) = r^{\kappa}\left(\int_0^{2(t+1)/r} + \int_{2(t+1)/r}^{\infty}\right)\frac{(r\rho)^{-(N-2)/2}}{2(t+1)}\exp\left(-\frac{\rho^2+r^2}{4(t+1)}\right)$$
$$\times I_{v_1}\left(\frac{r\rho}{2(t+1)}\right)(1+\rho^2)^{-a/2}\rho^{N-1}\,d\rho\psi_1(\theta)$$
$$\equiv (A+B)\psi_1(\theta).$$

First, we estimate A. From (19) we have for some constant $C > 0$

$$I_v(z) \le \begin{cases} Cz^v, & 0 < z \le 1, \\ Cz^{-1/2}e^z, & z > 1. \end{cases} \tag{48}$$

By (48) we obtain

$$A \le Cr^{\kappa}\int_0^{2(t+1)/r}\frac{(r\rho)^{-(N-2)/2}}{2(t+1)}\exp\left(-\frac{\rho^2+r^2}{4(t+1)}\right)\left(\frac{r\rho}{2(t+1)}\right)^{v_1}(1+\rho^2)^{-a/2}\rho^{N-1}\,d\rho.$$

From the definitions of v_1 and γ_+, we have

$$A \le C(2(t+1))^{-N/2-\gamma_+}r^{\kappa+\gamma_+}\exp\left(-\frac{r^2}{4(t+1)}\right)$$
$$\times\int_0^{2(t+1)/r}\exp\left(-\frac{\rho^2}{4(t+1)}\right)\rho^{N-1+\gamma_+}(1+\rho^2)^{-a/2}d\rho.$$

Putting $C_1 = 2^{-N/2-\gamma_+}C$, we get

$$A \le C_1(t+1)^{\{\kappa-(N+\gamma_+)\}/2}\left(\frac{r}{\sqrt{t+1}}\right)^{\kappa+\gamma_+}\exp\left(-\frac{r^2}{4(t+1)}\right)$$
$$\times\int_0^{2(t+1)/r}\exp\left(-\frac{\rho^2}{4(t+1)}\right)\rho^{N-1+\gamma_+}(1+\rho^2)^{-a/2}d\rho.$$

Since $s^{\kappa+\gamma_+}\exp(-s^2)$ is bounded for $s > 0$, there exists a constant $C_2 > 0$ such that

$$A \le C_2(t+1)^{\{\kappa-(N+\gamma_+)\}/2}\int_0^{2(t+1)/r}\exp\left(-\frac{\rho^2}{4(t+1)}\right)\rho^{N-1+\gamma_+}(1+\rho^2)^{-a/2}d\rho$$
$$\equiv C_2(t+1)^{\{\kappa-(N+\gamma_+)\}/2}E(r,t).$$

On the hand, the case $a \le N+\gamma_+$ is considered. Since by the assumption (i) and (ii), $a \ge \zeta+\kappa$, we see that

$$E(r,t) \le 2^{a/2} \int_0^{2(t+1)/r} \exp\left(-\frac{\rho^2}{4(t+1)}\right) \rho^{N-1+\gamma_+}(1+\rho)^{-\zeta-\kappa} d\rho$$
$$\le 2^{a/2} \int_0^{2(t+1)/r} \exp\left(-\frac{\rho^2}{4(t+1)}\right) \rho^{N+\gamma_+-\zeta-\kappa-1}\, d\rho.$$

Put $\xi = \rho/\sqrt{4(t+1)}$. Then we have

$$E(r,t) \le 2^{a/2} \int_0^{\sqrt{t+1}/r} \exp(-\xi^2)(\sqrt{4(t+1)}\xi)^{N+\gamma_+-\zeta-\kappa-1} \sqrt{4(t+1)}\, d\xi$$
$$\le 2^{a/2}(\sqrt{4(t+1)})^{N+\gamma_+-\zeta-\kappa} \int_0^{\infty} \exp(-\xi^2)\xi^{N+\gamma_+-\zeta-\kappa-1}\, d\xi.$$

Since $N+\gamma_+-\zeta-\kappa>0$, there exists a constant $C_3>0$ such that

$$E(r,t) \le C_3(t+1)^{(N+\gamma_+-\zeta-\kappa)/2}.$$

On the other hand, if $a > N+\gamma_+$,

$$E(r,t) \le 2^{a/2} \int_0^{2(t+1)/r} \exp\left(-\frac{\rho^2}{4(t+1)}\right)(1+\rho)^{N+\gamma_+-a-1} d\rho$$
$$\le 2^{a/2} \int_0^{\infty} (1+\rho)^{N+\gamma_+-a-1} d\rho \equiv C_4 < \infty.$$

Since $\zeta \le N+\gamma_+-\kappa$, we obtain for any $t \ge 0$

$$A \le \max\{C_3, C_4\}(t+1)^{-\zeta/2}.$$

Next, B is estimated. From (48) we have

$$B \le C\left\{\int_{[2(t+1)/r,\infty)\cap[2r/3,2r]} + \int_{[2(t+1)/r,\infty)\setminus[2r/3,2r]}\right\}\left(\frac{1}{2(t+1)}\right)^{1/2} \exp\left(-\frac{(\rho-r)^2}{4(t+1)}\right)$$
$$\times r^{-(N-1)/2+\kappa}\rho^{(N-1)/2-a}\, d\rho$$
$$\equiv C(J+K).$$

On one hand, we compute J. If $t+1 \ge r^2$ then $J=0$. When $t+1<r^2$, since $\rho \in [2r/3, 2r]$ we see that

$$J \le \int_{2r/3}^{2r} \left(\frac{1}{2(t+1)}\right)^{1/2} \exp\left(-\frac{(\rho-r)^2}{4(t+1)}\right)\left(\frac{\rho}{r}\right)^{(N-1)/2}\left(\frac{r}{\rho}\right)^a r^{\kappa-a}\, d\rho$$
$$\le 2^{(N-1)/2}\left(\frac{3}{2}\right)^a (t+1)^{(\kappa-a)/2} \int_{-\infty}^{\infty} \left(\frac{1}{2(t+1)}\right)^{1/2} \exp\left(-\frac{(\rho-r)^2}{4(t+1)}\right) d\rho$$
$$\le C_5(t+1)^{-(a-\kappa)/2}$$

with some constant $C_5 > 0$. On the other hand, we estimate K. Since $\rho \in [2(t+1)/r, \infty)/[2r/3, 2r]$, we have $|\rho - r| > \max\{r/3, \rho/2\}$. We thus obtain

$$-\frac{(\rho-r)^2}{4(t+1)} = -\frac{(\rho-r)^2}{8(t+1)} - \frac{(\rho-r)^2}{8(t+1)} \leq -\frac{\rho^2}{32(t+1)} - \frac{r^2}{72(t+1)} \tag{49}$$

and

$$r \geq \frac{2(t+1)}{\rho}. \tag{50}$$

From (49) and (50) we obtain

$$\begin{aligned} K \leq & \int_{[2(t+1)/r,\infty)/[2r/3,2r]} \left(\frac{1}{2(t+1)}\right)^{1/2} \exp\left(-\frac{\rho^2}{32(t+1)} - \frac{r^2}{72(t+1)}\right) \\ & \times \left(\frac{2(t+1)}{\rho}\right)^{-(N-1)/2} r^{\kappa} \rho^{(N-1)/2} \left(\frac{2(t+1)}{r}\right)^{-a} d\rho \\ \leq & \, (2(t+1))^{-(N-1)/2-a} \exp\left(-\frac{r^2}{72(t+1)}\right) \left(\frac{r}{\sqrt{t+1}}\right)^{\kappa+a} (\sqrt{t+1})^{\kappa+a+N-1} \\ & \times \int_0^\infty \left(\frac{1}{2(t+1)}\right)^{1/2} \exp\left(-\frac{\rho^2}{32(t+1)}\right) \left(\frac{\rho}{\sqrt{t+1}}\right)^{N-1} d\rho. \end{aligned}$$

So, there exists a constant $C_6 > 0$ such that

$$K \leq C_6(t+1)^{-(N-1)/2-a+(\kappa+a+N-1)/2} = C_6(t+1)^{-(a-\kappa)/2}.$$

Then we have

$$B \leq \max\{C_5, C_6\}(t+1)^{-(a-\kappa)/2}$$

for any $t \geq 0$. On the other hand from the definition of ζ we have $\zeta \leq a - \kappa$. Then we obtain for any $t \geq 0$

$$B \leq \max\{C_5, C_6\}(t+1)^{-\zeta/2}.$$

We thus have for any $t \geq 0$

$$|x|^{\kappa}\eta_a(x,t) \leq \max\{C_3, C_4, C_5, C_6\}(t+1)^{-\zeta/2}\psi_1(\theta)$$

Hence, we obtain (46) for any $t \geq 0$. Since $c_1\langle x\rangle \leq |x| \leq c_2\langle x\rangle$ for some $c_1, c_2 > 0$, we also have (47). □

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