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Polymorphism and Apartness

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Abstract Using traditional intuitionistic concepts such as apartness and subcountability, we give a relatively simple and direct construction of a natural, set-theoretic model for the second-order polymorphic lambda calculus, a model distinct from that of the modest sets.

1 Introduction The concept of apartness is an intuitionistic "positivization" of the classical notion of inequality between real numbers. In classical mathematics, every set has an apartness defined over it: apartness and inequality coincide. Intuitionistically, things can be *very* different – it is consistent with the full intutionistic set theory IZF to assume that every apartness space is *subcountable*, i.e., a quotient of a set of natural numbers. What follows almost immediately from this is a relatively simple and direct set-theoretic construction of a natural model for the second-order polymorphic lambda calculus $\mathbf{P}\lambda$.

Working within the Kleene realizability universe $\mathcal{V}(Kl)$ for set theory, we construct a small category $\mathcal{C}$ of sets which allow apartness and which, thanks to the presence of local axioms of choice, constitute a natural model of $\mathbf{P}\lambda$. This affords us another clear indication of the mathematical advantages of intuitionistic over classical metamathematics: using classical metamathematics, Reynolds (in [27]) has shown that, on pain of violating Cantor's uncountability theorems, there can be no natural set-theoretic models of $\mathbf{P}\lambda$.

Our construction is one of a number of intuitionistic models for $\mathbf{P}\lambda$ (cf. Pitts [24], Longo and Moggi [16]). The most popular of these is constructed over the category of realizability-valued modest sets $\mathcal{M}$. The model $\mathcal{C}$ of the present paper is distinct from that of modest sets: we prove that $\mathcal{C}$ is a proper subcategory of $\mathcal{M}$ in that the set of objects of the former is a proper subset of that of the latter. Second – and more importantly – our model construction does not leave one with a faulty impression that has been fostered, we think, by the details of the mathematics of $\mathcal{M}$: that the existence of models of the polymorphic lambda cal-

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culus is conceptually (perhaps even logically) tied to Church's Thesis and related computability axioms such as ECT_0. By contrast, our construction is provably independent of such computability axioms over the natural numbers. The construction calls for pure set-theoretic devices only, such as local choice principles, which are reasonably familiar to the constructive mathematician.

2 The polymorphic lambda calculus and its models For purposes of orientation, we begin from a survey of the syntax and elementary theory of the polymorphic λ calculus. For more information, the reader is advised to consult either Bruce and Meyer [6], Fortune et al. [8], or Reynolds [26].

2.1 Polymorphism and impredicativity $\mathbf{P}\lambda$ is the standard second-order polymorphic lambda calculus, elements of which are described in [26] and [6]. (In broad outline, our exposition follows the latter.) In short, $\mathbf{P}\lambda$ extends the ordinary typed lambda calculus: one adds type parameters and a higher-order variable-binder Λ which can be applied to arbitrary lambda expressions. If t is a type variable and M is a term, then

$$\Lambda t.M$$

is a new, higher-order term. Under the intuitive semantics, the term $\Lambda t.M$ denotes a function which will accept arbitrary types as inputs and yield suitably typed functions as outputs. To keep faith with the insight of traditional typed lambda calculus, $\Lambda t.M$ itself has a type. If α is the type symbol assigned to M, then

$$\Delta t.\alpha$$

names the type of $\Lambda t.M$. Intuitively, $\Delta t.\alpha$ ought to be a "large product type": the space of all functions which select, for each type t, an element of $\alpha(t)$.

Along with the extension of the concept of variable abstraction, we also extend its "inverse operation", the notion of application. It makes sense, in $\mathbf{P}\lambda$, to apply the term $\Lambda t.M$ to *any* suitable type expression, since its intuitive semantics is that of a function which has the entire collection of types as its domain. Indeed, we can be fully impredicative in making applications: we apply $\Lambda t.M$ to its own type: the term

$$\Lambda t.M(\Delta t.\alpha)$$

makes perfect sense even though, as a function, $\Lambda t.M$ is a member of its type $\Delta t.\alpha$ and so "ought not" be able to take it as an input value.

For computer scientists, a prime motive for concern with polymorphic terms and the Λ operator is a desire to study, in "clean" formal environments, internal mock-ups of constructs from programming languages such as *Ada* and *Clu*. Such languages seem to permit the creation of *generic* routines—ones allowing types to be passed as parameters. Logicians have been concerned with higher-order lambda terms in their attempts to provide systems of higher-order constructive arithmetic with functional interpretations. Historically, interest in $\mathbf{P}\lambda$ derives, in logic, from results in the thesis of Girard [9]; in computer science, two seminal investigations of polymorphism have been [24] and Milner [22].

2.2 Basic syntax To be more precise about the formal syntax of $\mathbf{P}\lambda$, take $\mathcal{V}_T$ to be an infinite collection of (second-order) type variables. Over these variables, we define the set TE of *type expressions* as follows:

Definition 2.2.1 The set TE of *type expressions* over $\mathcal{V}_T$ is the least collection such that
(1) $\mathcal{V}_T \subseteq TE$ and
(2) if σ and $\tau \in TE$ and $t \in \mathcal{V}_T$, then $\sigma \to \tau$ and $\Delta t.\sigma \in TE$.

We are now in a position to define the set λE of second-order lambda expressions. Let $\mathcal{V}_\lambda$ be an infinite set of first-order variables.

Definition 2.2.2 The set λE of *second-order* lambda *expressions* is the least set such that
(1) $\mathcal{V}_\lambda \subseteq \lambda E$ and
(2) if $M, N \in \lambda E$, $x \in \mathcal{V}_\lambda$, $\sigma \in TE$ and $t \in \mathcal{V}_T$, then all of $(MN), M[\sigma], \lambda x \in \sigma.M$ and $\Lambda t.M$ are members of λE.

The concepts of freedom of a variable occurrence, of freeness of a term for a variable occurrence and of substitution of a term for a variable occurrence are defined as usual.

Following [6], we assume that the variables of $\mathcal{V}_\lambda$ do not "come with type labels attached". Instead, we use syntactic type assignments B to label free variables in lambda expressions for purposes of type checking. Given a syntactic type assignment B—a finite function from a subset of $\mathcal{V}_\lambda$ into the set TE of type expressions—one can readily set down decidable conditions by which correct and incorrect typings of a term can be discriminated. The reader may consult [6] on the precise nature of these conditions.

We move now to the axioms and rules which constitute the pure theory $\mathbf{P}\lambda$.

2.3 Pure theory

2.3.1 Axioms We assume that for each axiom of the form $\sigma = \tau$, σ and τ are both correctly typed with respect to an implicit syntactic type assignment B. The axioms of $\mathbf{P}\lambda$ are all instances of the equalities α_i, β_i, and η_i for $i = 1,2$.

α_1 $\lambda x \in \sigma.M = \lambda y \in \sigma.[y/x]M$ if y is free for x in M and y is not free in M.
α_2 $\Lambda t.M = \Lambda u.[u/t]M$ if u is free for t in M and u is not free in M.
β_1 $(\lambda x \in \sigma.M)N = [N/x]M$ if N is free for x in M.
β_2 $(\Lambda t.M)[\rho] = [\rho/t]M$ if ρ is free for t in M.
η_1 $\lambda x \in \sigma.(Mx) = M$ if x is not free in M.
η_2 $\Lambda t.(M[t]) = M$ if t is not free in M.

2.3.2 Rules of inference Once again the rules are given with respect to an implicit type assignment B. We assume that the terms M, N, P, and Q are from λE while $\gamma \in TE$.

1. $N = M, N = P \vdash M = P$
2. $M = N, P = Q \vdash (MP) = (NQ)$
3. $M = N \vdash M[\gamma] = N[\gamma]$

4. $M = N \vdash \lambda x \in \gamma.M = \lambda x \in \gamma.N$ and
5. $M = N \vdash \Lambda x.M = \Lambda x.N$.

Finally, we can speak intelligibly of λ *theories*:

Definition 2.3.3 A λ *theory* in the second-order polymorphic lambda calculus is a set of equations among terms of λE which is type consistent (in that there is a type assignment B such that all expressions occurring in terms of the set are coherently typed by B), contains all instances of the axioms of $\mathbf{P}\lambda$, and is closed under $\mathbf{P}\lambda$'s inference rules.

2.4 Interpreting polymorphic terms We do not pretend to give a general definition of "model of $\mathbf{P}\lambda$"; for that—and for further discussion— the reader can consult one of [6], [16], or [26]. We have extracted from these articles the concept of *natural structure* for $\mathbf{P}\lambda$.

Definition 2.4.1 Let $\mathcal{C}$ be a small category which is a full subcategory of the category $\mathcal{S}$ of sets and such that there are canonical choices in $\mathcal{C}$ for exponential $\Rightarrow$ and for the product Π taken over the set $\mathcal{C}_0$ of all objects of $\mathcal{C}$. Hence, $\Rightarrow$ is a function from $\mathcal{C}_0 \times \mathcal{C}_0$ into $\mathcal{C}_0$ such that $A \Rightarrow B$ is obtained precisely as in $\mathcal{S}$. Similarly, Π is a function from $\mathcal{C}_0 \Rightarrow \mathcal{C}_0$ into $\mathcal{C}_0$ such that $\Pi(F)$ is obtained precisely as in $\mathcal{S}$. Any such small category $\mathcal{C}$ we call a natural structure for $\mathbf{P}\lambda$.

Once we have said what it is for a category $\mathcal{C}$ to be a structure for $\mathbf{P}\lambda$, we can spell out how the terms of the language of $\mathbf{P}\lambda$ are to be interpreted over such a structure. We must start by assigning an interpretation to each type expression in *TE*. (N.B. That one needs canonical choices for the exponentiation and product operations ought to be obvious from the following.)

Definition 2.4.2 Let $\mathcal{C}$ be a natural structure for $\mathbf{P}\lambda$ and let η be a *type environment*, a function from the set of type variables V_T into the set $\mathcal{C}_0$. To obtain the *type interpretation* $\mathcal{T}_\eta$, we proceed recursively on the structure of a type expression as follows, where $\Rightarrow$ and Π are the canonical exponentiation and product functions, respectively.
(1) for $t \in V_T$, $\mathcal{T}_\eta(t) = \eta(t)$,
(2) $\mathcal{T}_\eta(\gamma \rightarrow \rho) = \mathcal{T}_\eta(\gamma) \Rightarrow \mathcal{T}_\eta(\rho)$.
(3) $\mathcal{T}_\eta(\Delta t.\gamma) = \Pi(\lambda d \in \mathcal{C}_0.\mathcal{T}_{\eta}[d/t](\gamma))$.

As is standard in "referential semantic clauses" for bound variables, $\eta[d/t]$ is the type environment that agrees with η on all type variables except possibly t, in which case

$$\eta[d/t](t) = d.$$

Clearly, the definition of $\mathcal{T}_\eta$ is successful in that, for any type expression γ, $\mathcal{T}_\eta(\gamma) \in \mathcal{C}_0$.

There are certain extra details that must appear as preliminary to the assignment of denotata to terms in λE. We must be very careful so that the denotation assigned to a first-order term τ agrees with the denotation assigned to the second-order type expression that has been associated syntactically to τ by the type assignment B. All that care converges on the definition of a *global B-environment*:

Definition 2.4.3 Let B be a syntactic type assignment. A λ-*environment* for B is a function ρ such that

$$\rho : V_\lambda \to \bigcup_{x \in \mathcal{C}_0} x.$$

(ρ assigns to each λ variable a putative denotation, an element of some type in $\mathcal{C}$.) To ensure that this assignment is compatible with B, with define a *global B-environment* to be a pair

$$\langle \eta, \rho \rangle$$

where

- η is a type environment—as defined above—and
- ρ is a λ-*environment* for B with the property that, for all variables in the domain of B, $\rho(x) \in \mathcal{T}_\eta(B(x))$.

To put it bluntly, if ρ is the second component of a global B-environment with η as first component, then the items assigned to first-order variables by ρ must belong to the types already assigned by the composition of B with the semantic type assignment determined by η.

Next, we extend the assignment ρ to all terms in the expected way:

Definition 2.4.4 Let $e = \langle \eta, \rho \rangle$ be a global B-environment. The denotation map $\mathcal{M}_{B,e}$ is defined by recursion on the λ expressions. We assume that all the terms under consideration are properly typed relative to B.
(1) $\mathcal{M}_{B,e}(x) = \rho(x)$ for $x \in V_\lambda$
(2) $\mathcal{M}_{B,e}((MN)) = \mathcal{M}_{B,e}(M)(\mathcal{M}_{B,e}(N))$
(3) $\mathcal{M}_{B,e}(\lambda x \in \gamma . M) = \lambda d \in \mathcal{T}_\eta(\gamma).\mathcal{M}_{B,e[d/x]}(M)$
(4) $\mathcal{M}_{B,e}(M[\gamma]) = \mathcal{M}_{B,e}(M)(\mathcal{T}_\eta(\gamma))$
(5) $\mathcal{M}_{B,e}(\Lambda t.M) = \lambda d \in \mathcal{C}_0.\mathcal{M}_{B^*,e^*[d/t]}(M)$.
$B^*, e[d/x]$, and $e^*[d/t]$ are variant syntactic type assignments and global B or B^*-environments, respectively. Details on these functions and the conditions on them are to be had from [6].

It is nothing more than a quick inductive check to see that the assignment $\mathcal{M}_{B,e}$ remains compatible with the composition of $\mathcal{T}_\eta$ and B:

Lemma 2.4.5 *If B is a syntactic type assignment and $e = \langle \eta, \rho \rangle$ is a global B-environment and the denotation function $\mathcal{M}_{B,e}$ is defined as above, then for each lambda expression M,*

$$\mathcal{M}_{B,e}(M) \in \mathcal{T}_\eta(B(M))$$

wherein $B(M)$ is the syntactic assignment to M determined by B.

2.5 A soundness theorem At last, we can turn to evaluating over a structure the axioms and rules of $\mathbf{P}\lambda$. As one ought to expect, all the axioms of $\mathbf{P}\lambda$ come out true when interpreted in $\mathcal{C}$, provided that we give = its "naive" interpretation—as straightforward set-theoretic equality in $\mathcal{C}$. Also, all of the inference rules of $\mathbf{P}\lambda$ preserve truth in $\mathcal{C}$.

Definition 2.5.1 Let $\mathcal{C}$ be a natural structure for $\mathbf{P}\lambda$.

(1) We say that

$$\mathcal{C} \models \tau = \sigma[B, e]$$

or that $\mathcal{C}$ *satisfies* $\sigma = \tau$ or that $\sigma = \tau$ is *true* in $\mathcal{C}$ with respect to syntactic type assignment B and global B-environment e, if and only if

$$\mathcal{M}_{B,e}(\tau) = \mathcal{M}_{B,e}(\sigma).$$

2. Then, $\tau = \sigma$ is *valid* in $\mathcal{C}$ whenever $\tau = \sigma$ is true in $\mathcal{C}$ for all (suitable) B and e.
3. Lastly, if $\mathcal{C}$ is a natural structure for $\mathbf{P}\lambda$ and if the collection of all equalities valid in $\mathcal{C}$ constitutes a λ theory, then $\mathcal{C}$ is a *model* of $\mathbf{P}\lambda$.

Theorem 2.5.2 (*Soundness*) *Let $\mathcal{C}$ be a natural structure for* $\mathbf{P}\lambda$. *If $\sigma = \tau$ is an instance of an axiom of* $\mathbf{P}\lambda$, *then $\sigma = \tau$ is valid in $\mathcal{C}$. Also, if B and e are a suitable type assignment and global B-environment, respectively, then if $\Phi \vdash \Psi$ is an inference rule of* $\mathbf{P}\lambda$ *and the equalities of Φ are true in $\mathcal{C}$ with respect to B and e, then so are all the equalities of Ψ.*

Proof: First, since the formal equality sign of $\mathbf{P}\lambda$ is interpreted as "actual equality" in $\mathcal{C}$ and since formal application and abstraction are interpreted as functional application and abstraction, respectively, it is clear that the rules of $\mathbf{P}\lambda$ preserve truth. As for the axioms, first-order axioms such as α_1 are true in $\mathcal{C}$ – their truth conditions coincide with those of the axioms of the original first-order typed λ calculus.

It remains only to consider the essentially second-order axioms: for B a syntactic type assignment and e a global B-environment, the truth of β_2:

$$(\Lambda t.M)[\rho] = \{\rho/t\}M$$

in $\mathcal{C}$ reduces to an instance of the substitution principle:

$$\{\mathcal{T}_\eta(\rho)/d\}M_{B^*,e^*[d/t]}(M) = M_{B,e}(\{\rho/t\}M).$$

Expressions such as '$\{\rho/t\}$' stand for the obvious syntactic substitution operations. This principle of substitution is proved, in turn, via induction on the complexity of term M.

We can verify Axiom η_2 as follows. Assume that t is not free in the λE term M. Then, for suitable B and e,

$$\begin{aligned}
\mathcal{M}_{B,e}(\Lambda t.M[t]) &= \lambda d \in \mathcal{C}_0 \mathcal{M}_{B^*,e^*[d/t]}(M[t]) \\
&= \lambda d \in \mathcal{C}_0 \mathcal{M}_{B^*,e^*[d/t]}(M)[\mathcal{T}_{\eta[d/t]}(t)] \\
&= \lambda d \in \mathcal{C}_0 \mathcal{M}_{B^*,e^*[d/t]}(M[d]) \\
&= \mathcal{M}_{B,e}(M), \text{ since } t \text{ is not free in } M.
\end{aligned}$$

The reader is referred to [6] for further details.

Corollary 2.5.3 *If $\mathcal{C}$ is any natural structure for* $\mathbf{P}\lambda$, *then $\mathcal{C}$ is a model of* $\mathbf{P}\lambda$.

3 Reynold's theorem The highlight of [27] is a proof that there are no natural models of $\mathbf{P}\lambda$. As is readily discerned (cf. [24]), Reynold's proof is not intuitionistically correct as it stands nor can it be easily converted into an intuitionistic proof. Sadly, there is only one nonconstructive step in the argument and it appears in the final sentence. At that stage, Reynolds has just shown that, if $\mathcal{C}$ is a natural structure for $\mathbf{P}\lambda$, then there are nontrivial objects A and $B \in \mathcal{C}_0$ such that $(A \Rightarrow B) \Rightarrow B$ is in one-to-one correspondence with A. In the last sentence, he argues that, as these sets $(A \Rightarrow B) \Rightarrow B$ and A "*are well-known to have different cardinalities, we have a contradiction*" ([27], p. 155). He concludes that there are no natural structures for $\mathbf{P}\lambda$.

All this is classically correct but the statement in italics is not intuitionistically "well known". In fact, it is intuitionistically false. With all manner of techniques, it is possible to construct nontrivial fixed points for functors such as $(X \Rightarrow B) \Rightarrow B$ in models for intuitionistic set theory (cf. McCarty [18] for one type of construction). The category $\mathcal{C}$ which we are about to describe, the category of ω stable subpartitions of ω which allow apartness, will include many such fixed points.

4 Apartness spaces

Apartness is a subrelation of the inequality relation on a set from which "positive" constructive information can be extracted in the course of mathematical reasoning. The concept was first devised by Brouwer [5] for use in analysis as a constructive "positivization" of the relation of inequality between real numbers. Two real numbers, represented by Cauchy sequences, are unequal when it is not the case that the sequences are mutually convergent. As Brouwer realized, "negative facts" such as inequality claims are (relatively speaking) informationally inert. In other words, there is relatively little in the way of constructive information that one can glean from them. By contrast, he said that two real numbers stand in the apartness relation when there is some positive information about their separation, in particular, when there is a rational number which separates the terms of the sequences cofinally. The apartness concept was applied by Brouwer's student and colleague Heyting ([12] and [13]) to algebra and geometry. In each case, apartness played the role of an intuitionistic "positive analogue" to the strictly negative concept of the nonidentity of elements in an algebraic structure.

In truth, there are a number of apartness concepts. The one we define here is known as "strict apartness".

4.1 Definitions

Definition 4.1.1 A pair $\langle A, \# \rangle$ is an *apartness space* whenever it satisfies the four conditions
(1) $\#$ a binary relation on A
(2) $x = y \in A$ iff $\neg x \# y$
(3) if $x \# y$ then, for any $z \in A$, either $x \# z$ or $y \# z$, and
(4) if $x \# y$ then $y \# x$.

Note 4.1.2 The background theory for our discussion of apartness will be one of the standard extensional constructive set theories such as IZF, intuitionistic Zermelo-Fraenkel. We will describe IZF following the definition of the Kleene realizability structure $\mathcal{V}(Kl)$. Because of the treatment of $=$ in these theories, we can assume that each set A has a "built-in" extensional equality $x = y \in A$ defined on it.

Definition 4.1.3 A set A *admits apartness* whenever there is a relation # such that $\langle A, \# \rangle$ is an apartness space. More generally, A is said to *allow apartness* whenever equality on A is stable and it is not impossible ($\neg\neg$) that A admits apartness. The *category of apartness spaces* APS is the full subcategory of the category $\mathcal{S}$ of all sets and set-functions in which each object is a set that allows apartness.

Note 4.1.4 If a set A admits apartness, then equality on A must be stable or closed under $\neg\neg$, i.e., the statement

$$\forall a, b \in A(\neg\neg a = b \rightarrow a = b)$$

is intuitionistically true. That equality on A is stable does not seem to follow from the essentially weaker assumption that it is not impossible that A admit apartness. At most, one can show that equality on A is "almost stable":

$$\neg\neg\forall x, y \in A(\neg\neg x = y \rightarrow x = y).$$

Stability of equality is required for the construction of the natural structure $\mathcal{C}$ which we are about to give. Therefore, stability is explicitly included in the definition of *allowing apartness.*

Classically, apartness collapses into mere negation: every classical set A has the relation $\neg x = y \in A$ as an apartness on it. Consequently, the classical sets which allow apartness can only comprise a proper class; classically, APS is the category of sets itself. A fortiori, there can be no limit on the cardinality of a classical apartness space. Intuitionistically, things are very different. Brouwer was able to give a "weak counterexample" to the claim that every two unequal reals are apart: he showed by intuitionistic means that, if the claim holds, then the general law of the excluded third would be valid. But more is true—as we shall see: the sets which allow apartness are definitely limited in cardinality, so much so that the category of such sets is equivalent to a certain *small* category.

The differences between the classical and the intuitionistic versions are not marked, however, in the next two propositions, each of which holds either classically or constructively.

4.2 Properties of apartness spaces

Proposition 4.2.1 *APS is closed under subsets, function spaces, and arbitrary products—products indexed over any set whatsoever.*

Proof: This is nothing more than a simple check. Every subset of a set which allows apartness inherits an allowed apartness. Next, if $\langle A, \# \rangle$ is an apartness space and B is any set, it is readily checked that the following definition of R specifies an apartness on the collection of functions from B to A:

$$fRg \text{ iff, for some } b \in B, f(b) \ \# \ g(b).$$

Lastly, an analogous definition will specify an apartness suitable to the product of an arbitrary collection of sets, each of which allows apartness.

Any category that is closed under products and equalizers is closed under arbitrary limits and, hence, is complete. Any category that includes suitable products and exponential objects and has a terminal object is said to be cartesian closed (cf. Schubert [29]). We now know that

Corollary 4.2.2 *APS is Cartesian-closed and complete.*

Proof: See [29].

Notably, APS cannot be cocomplete, at least if we cleave to intuitionistic strictures. To use Brouwerian terminology, the statement that APS is cocomplete has a weak counterexample. We can show that, if it is true, then so is a plainly nonintuitionistic principle of mathematical reasoning.

Definition 4.2.3 A *stable quotient* on a set A is an equivalence relation R on A such that R is closed under $\neg\neg$. That is, for any x and y from A,

$$\neg\neg xRy \rightarrow xRy$$

is intuitionistically true.

Theorem 4.2.4 *The statements that APS is closed under arbitrary sums and that APS is closed under stable quotients each imply the general validity of nonconstructure principles of mathematics.*

Proof: Let the principle of *near testability* be the claim

$$\neg\neg\forall\phi(\neg\phi \vee \neg\neg\phi).$$

That near testability fails of constructive correctness is clear; for one thing, the Kleene realizability structure $\mathcal{V}(Kl)$ $(v.i.)$ is a countermodel of it. The realizability of near testability would imply the classical solvability of the general halting problem.

Let $\mathcal{P}$ be the set of propositions ϕ and let P be the set

$$\{\{0:\phi\}:\phi\in\mathcal{P}\}.$$

For purposes of this proof, we will refer to the set $\{0:\phi\}$ as 'ϕ.' Note that, since some propositions are true and others false, $\varnothing \in P$ and $\{0\} \in P$. Assume that P allows apartness #. From the definition of apartness relation, it follows immediately that

$$\neg\neg\varnothing \,\#\, \{0\}$$

and, hence, that

$$\neg\neg\forall\phi\in\mathcal{P}(\phi \,\#\, 0 \vee \phi \,\#\, \varnothing).$$

This is just to say that

$$\neg\neg\forall\phi(\neg\phi \vee \neg\neg\phi),$$

which is the principle of near testability.

Now, the disjoint sum of singletons $\{0\}$,

$$\sum_{x\in P}\{0\},$$

each of which admits the empty apartness relation, is in one-to-one correspondence with *P*. Therefore, if APS were closed under arbitrary sums, then near testability would hold.

To obtain a weak counterexample for the second statement—that sets which allow apartness are closed under stable quotients—we first define a stable quotient on the set of natural numbers and show that the quotient set will allow apartness only if

$$\neg\neg\forall n(\neg\phi(n) \vee \neg\neg\phi(n)).$$

This is a nonconstructive instance of the principle of near testability. The truth of this in $\mathcal{V}(Kl)$ would also imply the solvability of the halting problem.

For each natural number n, let $\phi(n)$ be a proposition and define

$$n \sim m \text{ if and only if } \neg\phi(n) \leftrightarrow \neg\phi(m).$$

This quotient is clearly stable. Without loss of generality, we may assume that $\phi(0)$ holds but $\phi(1)$ does not.

If we now assume that the resultant quotient set allows an apartness #, it then follows that

$$\neg\neg n \# m \leftrightarrow \neg(\neg\phi(n) \leftrightarrow \neg\phi(m)).$$

Also, we know that

$$\neg\neg 0 \# 1.$$

It now follows from the properties of apartness that

$$\neg\neg\forall n(0 \# n \vee 1 \# n).$$

By the definition of the quotient, we then have that

$$\neg\neg\forall n(\neg\phi(n) \vee \neg\neg\phi(n)).$$

Obviously, the set ω of natural numbers allows apartness but the quotient of ω determined by $\sim$ does not. Therefore, allowing apartness is not preserved by taking stable quotients.

We now know that APS, as a category, is not constructively cocomplete: APS is not closed under coproducts even where the index set itself admits apartness.

5 Subcountability in the realizability universe

Definition 5.0.5 A set is *subcountable* when it is a quotient of a collection of natural numbers.

Every collection of natural numbers is trivially subcountable. So, subcountability cannot coincide, at least intutionistically, with countability, even if the set in question has at least one element. For instance, if we look into the Kleene realizability universe $\mathcal{V}(Kl)$ for intuitionistic set theory (which we are about to discuss), we see that the internal truth, in $\mathcal{V}(Kl)$, of the claim that every inhabited subcountable set is countable implies, among other things, that every property of the natural numbers is recursively enumerable.

5.1 A realizability universe

Definition 5.1.1 Let $\mathcal{V}(Kl)$ be the Kleene number realizability universe as defined either in Beeson [3] or in McCarty [17]. Briefly, the universe $\mathcal{V}(Kl)$ itself—as a domain for intuitionistic quantification—is the least fixed-point of the class equation

$$X = \mathcal{P}(\omega \times X)$$

wherein $\mathcal{P}$ is the powerset operator and ω is the standard set of natural numbers. Roughly speaking, if A is a member of $\mathcal{V}(Kl)$ (in other words, if A is a *realizability set*) and $\langle n, a \rangle \in A$, we say that n *realizes* the fact that $a \in A$ or that n is a *realizability witness* for $a \in A$ and write

$$n \models a \in A.$$

Propositional combinations of set-theoretic formulas are interpreted over $\mathcal{V}(Kl)$ just as they were interpreted by Kleene in his original realizability article, [15]. Unbounded set quantifiers receive a "null" or "generic" reading first introduced by Kreisel and Troelstra in their realizability interpretation for second-order arithmetic, cf. [32] and [33]. Finally, if ϕ is a sentence of set theory and n realizes ϕ, we say that $\mathcal{V}(Kl)$ *satisfies* ϕ and write

$$\mathcal{V}(Kl) \models \phi.$$

The set theory which holds sway over the Kleene realizability universe is IZF.

Definition 5.1.2 *Intuitionistic Zermelo-Fraenkel* set theory or *IZF* is a set theory in the language of classical ZF and whose axioms are those of classical Zermelo–Fraenkel formulated so as not to imply, in intuitionistic predicate logic, the law of the excluded third.

Note 5.1.3 A full axiom of choice cannot be added to IZF, if we want to avoid deriving the law of the excluded third, as the Scott–Diaconescu argument shows (cf. Beeson [4]). Also, IZF is equiconsistent with classical ZF (cf. [17], Grayson [10], or [4]).

The Kleene realizability universe is a model for IZF:

Theorem 5.1.4 $\mathcal{V}(Kl) \models$ IZF.

Proof: The argument is standard and is reported in full in [17], [3], and [4].

5.2 Subcountability and apartness In $\mathcal{V}(Kl)$, APS is isomorphic to a small category in virtue of the fact that $\mathcal{V}(Kl)$ validates the principle SCAS ("Sub-Countability of Apartness Spaces"), that *every apartness space is subcountable*.

SCAS is a direct generalization of the more familiar principles SCDS and SCMS (for subcountability of Discrete and of Metric spaces), respectively, that *every set with decidable equality is subcountable* and *every metric space is subcountable*.

Theorem 5.2.1 $\mathcal{V}(Kl) \models$ *Every set which allows apartness is subcountable.* (This result, together with correlative results concerning SCDS and SCMS, first appeared in McCarty [19]).

Proof: We give a brief sketch; details abound in [19].

Assume that A is a realizability set of $\mathcal{V}(Kl)$ which allows apartness. Let $n \vDash a \in A$ be the relation which defines membership for A internally. If we define B so that

$$n \vDash \langle n, a\rangle \in B \leftrightarrow n \vDash a \in A,$$

then B is also a realizability set.

It is readily verified that B is, internally, functional—that

$$\mathcal{V}(Kl) \vDash (\langle n, a\rangle \in B \wedge \langle n, b\rangle \in B) \rightarrow a = b \in A.$$

To see this, first note that there is a partial recursive function Γ such that, for all internal a, b, and natural numbers n, m, if $n \vDash a \in A$, $n \vDash b \in A$, $n \vDash c \in A$, and $m \vDash a \# b$, then $\Gamma(n, m)$ is defined and either

$$\Gamma(n, m) = 0 \wedge \mathcal{V}(Kl) \vDash a \# c$$

or

$$\Gamma(n, m) = 1 \wedge \mathcal{V}(Kl) \vDash b \# c.$$

Working in $\mathcal{V}(Kl)$, if we assume that $\langle n, a\rangle \in B$, that $\langle n, b\rangle \in B$, and that $m \vDash a \# b$, then

$$n \vDash a \in A \wedge n \vDash b \in A.$$

If we replace c in the preceding paragraph by a, we see that $\Gamma(n, m)$ must be 1. But, replacing c by b, we find that $\Gamma(n, m)$ must be 0. Therefore, the assumption that

$$m \vDash a \# b$$

is false, and, since equality is stable on A, $\mathcal{V}(Kl)$ satisfies $a = b$.

With circumspection, we can conclude from this result on subcountability that the category APS isomorphic to a particular small category, the category of ω stable subpartitions of ω which allow apartness. Ordinarily, a subpartition of ω will be a set of pairs of natural numbers which comprise a partial equivalence relation on ω: as a relation-in-extension, the set of pairs is symmetric and transitive. For our purposes, it is best to think of the pairs of the subpartition as coded into individual natural numbers. So, we fix a primitive recursive pairing and take $\lambda x.x^r$ and $\circ$ to be fixed primitive recursive functions on the natural numbers. For each (coded) number pair $\langle x, y\rangle$,

$$\langle x, y\rangle^r = \langle y, x\rangle :$$

x^r hands you back the "reverse" of the pair coded by x. $\circ$ acts on the primitive recursive codes to effect relational composition:

$$\langle x, y\rangle \circ \langle y, z\rangle = \langle x, z\rangle.$$

Definition 5.2.2
(1) A set $A \subseteq \omega$ is *ω-stable* iff $\forall x \in \omega(\neg\neg x \in A \to x \in A)$.
(2) A is an ω-stable subpartition of ω just in case
- $A \subseteq \omega$,
- $\forall x \in \omega(\neg\neg x \in A \to x \in A)$,
- $\forall x \in A\, x^r \in A$ and
- $\forall x, y \in A\, x \circ y \in A$.

(3) The category $\mathcal{C}$ (our candidate for natural structure for $\mathbf{P}\lambda$) is the full subcategory of $\mathcal{S}$ which has the collection of all ω-stable subpartitions of ω which allow apartness as its set $\mathcal{C}_0$ of objects.

Corollary 5.2.3 *In $\mathcal{V}(Kl)$, APS is isomorphic to the small category $\mathcal{C}$.*

Proof: We saw that in $\mathcal{V}(Kl)$ each set in APS is subcounted by a function F_A, which is the obvious "internalization" of its realizability predicate:

$$F_A = \{\langle n, \langle n, a\rangle\rangle : n \vDash a \in A\}.$$

The partitioned domain of F_A provides a member of $\mathcal{C}_0$. Specifically, for $n, m \in \omega$, we set $\langle n, m\rangle \in Q_A$ just in case $F_A(n) = F_A(m) \in A$. Q_A is clearly a partition(ing equivalence relation) on a subset of ω. That it is ω-stable follows directly from the fact that $=$ is stable on A. That it is isomorphic to A and allows apartness is immediate.

6 Axioms of choice

Before we can claim that $\mathcal{C}$ is a natural structure for $\mathbf{P}\lambda$, there is a small matter of the axiom of choice. Were we working within a classical framework in which a global axiom of choice holds, no difficulty would arise. From the above corollary, we might conclude directly that there are *functions* $\Rightarrow$ from $\mathcal{C}_0 \times \mathcal{C}_0$ into $\mathcal{C}_0$ and Π from $\mathcal{C}_0 \Rightarrow \mathcal{C}_0$ into $\mathcal{C}_0$ which select *specific* representatives of the exponential and product constructions—which we know to exist from our earlier work. From the existential quantifiers hidden in the statement that APS is isomorphic to $\mathcal{C}$, we could use the axiom of choice to "choose representatives" to fill the roles of canonical exponential objects and products. As it happens, global choice is unavailable intuitionistically, as the Scott–Diaconescu argument (presented in [4]) shows.

But all is not lost. In the intuitionistic setting, we can achieve the effect of choice within the confines of the realizability universe *for the sorts of sets we are considering*. In other words, we can show that, within $\mathcal{V}(Kl)$, suitable choices can be made locally, over the particular sets $\mathcal{C}_0$ and $\mathcal{C}_0 \Rightarrow \mathcal{C}_0$. To use Peter Aczel's terminology (from [1]), we prove that these sets are *bases*: they both support the realizability truth of local versions of the choice principle. When we work over bases, then we can always choose suitable representatives.

6.1 Bases

Theorem 6.1.1 *Where X is either $\mathcal{C}_0$ or $\mathcal{C}_0 \Rightarrow \mathcal{C}_0$ and Y is any realizability set, $\mathcal{V}(Kl)$ satisfies the statement* *if* $\exists x \in X \forall y \in Y.\Phi(x,y)$ *then there is a function* $f: X \to Y$ *such that* $\forall x \in X \Phi(x, f(x))$.

This is just another way of saying that $\mathcal{C}_0$ *and* $\mathcal{C}_0 \Rightarrow \mathcal{C}_0$ *are both bases.*

Proof: We begin with a technical lemma:

Lemma 6.1.2 *Let* A *be a realizability set. Assume that there is a realizability witness* $j \in \omega$, *a partial recursive function* Γ *and a(n external) function* $f(x)$ *taking internal elements of* A *to internal elements of* A *such that, if* $m \models a \in A$ *and* $\mathcal{V}(Kl)$, *then*

- $\Gamma(m) \models a = f(a)$,
- $j \models f(a) \in A$ *and*
- $\mathcal{V}(Kl) \models a = b$ *only if* $f(a) = f(b)$ *(as external sets).*

Then A *is, in* $\mathcal{V}(Kl)$, *a base.*

Proof: Assume that the antecedent conditions of the lemma hold for A within the realizability universe and assume that the antecedent of the choice principle holds, namely, that

$$n \models \forall x \in A \exists y.\Phi(x,y).$$

If $m \models a \in A$, then—by assumption—$\Gamma(m) \models a = f(a)$ and $j \models f(a) \in A$. Given the realizability interpretation of $\forall\exists$, we know that there is a b such that $\{n\}(j) \models \Phi(f(a), b)$. Let $\Theta(n,j)$ be the partial recursive function $\{n\}(j)$.

At this stage, we can apply the axiom of choice in the metatheory. First, for each internal member a of A, we choose as $F(a)$ a realizability set such that $\Theta(n,j) \models \Phi(f(a), F(a))$. Second, we define the realizability set G to be

$$\{\langle 0, \langle a, F(a)\rangle\rangle : \exists m. m \models a \in A\}.$$

Third, we check that G is an internal function and satisfies the conditions required to show that A is a base:

Assume that a and b are internal elements of A and that $k \models a = b$, $m \models \langle a, c\rangle \in G$, and $n \models \langle b, d\rangle \in G$. It follows that $f(a)$ and $f(b)$ are equal as external sets and hence that $q \models c = d$, where q is effectively and uniformly calculable from m and n. Therefore, G is an internal function. Assume further that $m \models a \in A$ and that

$$n \models \forall x \in A \exists y.\Phi(x,y).$$

As we have just seen, for $b = F(a)$, $0 \models \langle a, b\rangle \in G$ and

$$\Theta(n,j) \models \Phi(f(a), b).$$

Since there is a function Γ such that $\Gamma(m) \models a = f(a)$, there is a partial recursive $\Sigma(n,j,m)$ such that

$$\Sigma(n,j,m) \models \Phi(a,b).$$

We conclude that, under the conditions of the lemma, A is a base.

[Returning to the proof of the main theorem.] It suffices to check that $\mathcal{C}_0$ and $\mathcal{C}_0 \Rightarrow \mathcal{C}_0$ satisfy the antecedent conditions for being a base set down by the lemma.

Assume that $n \models A \in \mathcal{C}_0$. Let $f(A)$ be the realizability set

$$\{\langle k, k\rangle : \exists m . m \models k \in A\}.$$

It is readily seen that there is a partial recursive function Γ such that

$$\Gamma(n) \models A = f(A).$$

Also, there is a fixed $i \in \omega$ — independent of A — such that

$$i \models f(A) \in \mathcal{C}_0.$$

Obviously, for A and $B \in \mathcal{S}$, $\mathcal{V}(Kl) \models A = B$ iff $f(A) = f(B)$ in the external world. The lemma now entails that $\mathcal{C}_0$ is a base.

We turn at last to $\mathcal{C}_0 \Rightarrow \mathcal{C}_0$. Let $g \in \mathcal{C}_0 \Rightarrow \mathcal{C}_0$. $F(g)$ is defined to be the internal function

$$\{\langle m, \langle f(A), f(B)\rangle\rangle : \mathcal{V}(Kl) \models g(A) = B \wedge m \models A \in \mathcal{S}\}.$$

Here, f is the external function carrying $\mathcal{C}_0$ into itself which was defined above. We have already seen that it fulfills the requirements of the lemma. It is straightforward to check that, regardless of g, there is a fixed j such that

$$j \models F(g) \in \mathcal{C}_0 \Rightarrow \mathcal{C}_0.$$

Next, it is clear that, if $\mathcal{V}(Kl) \models g = h$, then $F(g)$ and $F(h)$ are externally the same.

Finally, suppose that $m \models g \in \mathcal{C}_0 \Rightarrow \mathcal{C}_0$ and that $n \models \langle A, B\rangle \in g$. Given the properties of f, there is a partial recursive function $\Theta(m, n)$ such that

$$\Theta(m, n) \models \langle A, B\rangle \in F(g).$$

For the converse, there is a partial recursive Σ such that if $\langle n, \langle f(A), f(B)\rangle\rangle \in F(g)$, then

$$\Sigma(m, n) \models \langle f(A), f(B)\rangle \in g.$$

The conditions of the lemma are again satisfied and $\mathcal{C}_0 \Rightarrow \mathcal{C}_0$ is shown to be a base.

Our proof that $\mathcal{C}_0$ is a natural structure for $\mathbf{P}\lambda$ is complete.

7 Relations with computability Because of the particular details of the modest sets construction (cf. [16]), it has been suggested that there exists a logical or intrinsic connection between the existence of natural structures for $\mathbf{P}\lambda$ and nonclassical computability axioms for the natural numbers. Specifically, crucial properties of the category of modest sets seem to follow from the computability axiom ECT_0 or *extended first-order Church's Thesis.* ECT_0 is the statement that *every function from an ω-stable subset of ω into ω is Turing computable*, or, more precisely (and generally):

> if S is ω-stable and $\forall x \in S \exists y \in \omega \phi(x, y)$, then there is an index e such that, for all $x \in S$, $\{e\}$ is defined on x and $\phi(x, \{e\}(x))$ holds.

It is not difficult to see that the suggested logical connection does not exist. If ECT_0 is involved in the intuitionistic fact that there are natural structures for $\mathbf{P}\lambda$, that involvement does not amount to entailment. There are models $\mathcal{V}(\mathcal{G})$ for intuitionistic set theory in which natural $\mathbf{P}\lambda$ structures exist but in which ECT_0 fails.

7.1 Realizability over the graph model

Definition 7.1.1 $\mathcal{G}$ is the Scott–Plotkin "graph model" (Scott [30] and Plotkin [25]) for the untyped λ calculus, as described, e.g., in Barendegt [2]. In $\mathcal{G}$ each subset X of ω is treated as the set of codes of the graph of a $\mathcal{P}(\omega)$-continuous partial function and application is interpreted as continuous function application. $\mathcal{V}(\mathcal{G})$ is the realizability universe constructed over $\mathcal{G}$ in complete analogy with the method by which the Kleene universe was constructed over the natural numbers.

In $\mathcal{V}(\mathcal{G})$, realizability witnesses are, instead of natural number indices of Turing Machines, sets of natural numbers which "code" into their extensions the concept of continuous function application. Over this model, we can locate natural structures for the second-order polymorphic λ calculus even though ECT_0 fails there.

Theorem 7.1.2 $\mathcal{V}(\mathcal{G}) \models \neg ECT_0$.

Proof: Statements of elementary number theory are absolute with respect to $\mathcal{V}(\mathcal{G})$. If ϕ is a sentence of first-order arithmetic, then

$$\mathcal{V}(\mathcal{G}) \models \phi \text{ if and only if } \phi \text{ is true.}$$

This is proved by induction on the structure of ϕ, making use of the fact that every natural-number-valued function of natural numbers is representable in $\mathcal{G}$ as a continuous function. Therefore, if $\mathcal{V}$ is a model of classical set theory and $\mathcal{V}(\mathcal{G})$ has been constructed over $\mathcal{V}$, then ECT_0 fails in both models.

Note 7.1.3 We need not adopt a classical metatheory in order to construct a universe $\mathcal{V}$ to fulfill the requirements of the preceding theorem. We can construct a model of classical mathematics even over $\mathcal{V}(Kl)$ itself: intuitionistic methods alone show that the universe of Heyting-valued sets with values taken from the regular open subsets of the reals is Boolean and, hence, that classical set theory holds there (cf. Grayson [11]).

7.2 Natural structures over the graph model

Theorem 7.2.1 $\mathcal{V}(\mathcal{G}) \models \mathbf{P}\lambda$ *has a natural model.*

Proof: If we follow the line of argument leading to our earlier conclusion that the internal small category $\mathcal{C}$ is a natural structure for polymorphic lambda calculus, we notice that nothing crucial depends upon the assumption that, in those proofs, symbols such as 'n' and 'm' and such as 'ω' refer to natural numbers and to the set of natural numbers, respectively. All the mathematical tasks performed by these symbols are still performed if we reinterpret them to refer to the members of $\mathcal{G}$ and to $\mathcal{G}$ itself. We have only relied on the set of natural numbers as the domain of a suitable applicative structure over which realizability can be defined. Any of a number of applicative structures would do equally well.

For example, assume that we are working within the universe $\mathcal{V}(\mathcal{G})$. Let Ω be the internal set

$$\{\langle X, X\rangle : X \in \mathcal{G}\}.$$

Ω is our candidate for the $\mathcal{G}$-analogue of the natural numbers. We say that a set A is *majorizable* if there is a surjection from Ω onto A. A is *submajorizable* if there is a partial surjection from (a subset of) Ω onto A.

Now, if we reinterpret reference to natural numbers as references to members of $\mathcal{G}$, then we can show—again, working within $\mathcal{V}(\mathcal{G})$—that every set which allows apartness is submajorizable. The argument is precisely that of our earlier proof that, in $\mathcal{V}(Kl)$, every set which allows apartness is subcountable.

It remains to check that our results on bases and local axioms of choice continue to hold after a move into $\mathcal{V}(\mathcal{G})$. First, we note that the lemma spelling out the conditions under which an internal set is a base remains true with respect to $\mathcal{V}(\mathcal{G})$, provided again that we replace reference to natural numbers and partial recursive functions with reference to members of the graph model and continuous functions, respectively. Second, we can also prove that the category $\mathcal{S}_{\mathcal{G}}$ is a natural structure for $\mathbf{P}\lambda$ where $\mathcal{S}_{\mathcal{G}}$ is the collection of Ω-stable subpartitions of Ω which allow apartness. Just as above, the argument proceeds by applying the lemma to $\mathcal{S}_{\mathcal{G}}$ and showing that each such subpartition A can be identified with a representative set

$$f(A) = \{\langle X, X\rangle : \exists Y. Y \models X \in A\}.$$

Hence, $(\mathcal{S}_{\mathcal{G}})_0$ is seen to be a base. Parallel considerations prove that

$$(\mathcal{S}_{\mathcal{G}})_0 \Rightarrow (\mathcal{S}_{\mathcal{G}})_0$$

is also a base.

8 Apartness and modest sets It remains to show that our construction is really distinct from the modest set construction deriving from the work of Girard and Troelstra ([9], [32], and [33]). Again, we apply—internally within $\mathcal{V}(Kl)$—Brouwer's method of weak counterexamples.

Definition 8.0.2 A set is *modest* if and only if it is an ω-stable subpartition of ω.

Theorem 8.0.3 *The category of modest sets and set functions comprise a natural structure for* $\mathbf{P}\lambda$.

Proof: See [16].

Theorem 8.0.4 *In $\mathcal{V}(Kl)$, the set $\mathcal{C}_0$ of objects of $\mathcal{C}$ is a proper subset of the collection of modest sets.*

Proof: Working within $\mathcal{V}(Kl)$, we exhibit an ω-stable subpartition, a modest set, which demonstrably fails to allow apartness. Note first that, in the Kleene realizability universe, each of the decidability principles

$$\forall n \in \omega[\exists m. T(n, n, m) \vee \neg\exists m. T(n, n, m)]$$

and

$$\neg\neg\forall n \in \omega[\exists m. T(n,n,m) \vee \neg\exists m. T(n,n,m)]$$

fails. Here, T is the unary Kleene "T" computation predicate, as standardly expressed in arithmetic. The failure of these decidability principles is a consequence of the unsolvability of the halting problem. (Thinking of $\mathcal{V}(Kl)$ as a universe for recursive mathematics, we might even say that they are expressions of unsolvability.)

Second, since Markov's principle holds in $\mathcal{V}(Kl)$ (cf. [4]), the "halting predicate" $\exists m. T(n,n,m)$ is ω-stable:

$$\forall n \in \omega[\neg\neg\exists m. T(n,n,m) \rightarrow \exists m. T(n,n,m)].$$

Now, we define the following equivalence relation over ω:

$$n \sim m \Leftrightarrow [\exists p. T(n,n,p) \leftrightarrow \exists p. T(m,m,p)].$$

Thanks to the second fact we just mentioned, this equivalence relation is ω-stable in $\mathcal{V}(Kl)$. Without loss of generality, we can assume that $0 \not\sim 1$ or that $\neg\exists m. T(0,0,m)$ and $\exists m. T(1,1,m)$.

For the sake of argument, assume that the $\sim$-quotient of the natural numbers allows an apartness #. it follows that $\neg\neg 0 \# 1$. Given the definition of apartness, it follows, in turn, that

$$\neg\neg\forall n \in \omega[n \# 0 \vee n \# 1].$$

This entails the weaker decidability principle at once:

$$\neg\neg[\exists m. T(n,n,m) \vee \neg\exists m. T(n,n,m)].$$

Therefore, our assumption that the $\sim$-quotient of the natural numbers allows apartness is false. The set of objects of $\mathcal{C}$ is, consequently, a proper subset of the modest sets.

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