Onoda, N.
Osaka J. Math.
19 (1982), 231-239

REMARKS ON ONE-DIMENSIONAL SEMINORMAL RINGS

NOBUHARU ONODA

(Received June 20, 1980)

Various characterizations of reduced seminormal rings of dimension one are given in Salmon [4], Bombieri [1] and Davis [2]. Among others it is shown that if $(A, \mathfrak{m})$ is a local ring of a closed point on an algebraic curve defined over an algebraically closed field k, then A is seminormal if and only if the completion $\hat{A}$ is k-isomorphic to $k[[X_1, \cdots, X_n]]/(\cdots, X_iX_j, \cdots)$ where $i \neq j$ ([1]) or the associated graded ring $Gr^{\cdot}(A)$ is k-isomorphic to $k[X_1, \cdots, X_n]/(\cdots, X_iX_j, \cdots)$ where $i \neq j$ ([2]). Generalizing these results we prove the following in the first section. Under certain moderate assumptions on A there exist an integer n and an ideal I in $k[X_1, \cdots, X_n]$ such that A is seminormal if and only if $\hat{A} \cong k[[X_1, \cdots, X_n]]/Ik[[X_1, \cdots, X_n]]$ or $Gr^{\cdot}(A) \cong k[X_1, \cdots, X_n]/I$. Moreover the ideal I is generated by quadratic forms and these forms and integer n are determined solely by the k-algebra structure of $\bar{A}/J(\bar{A})$, where $\bar{A}$ is the integral closure of A in the total quotient ring of A and $J(\bar{A})$ is the Jacobson radical of $\bar{A}$. Let C be a plane algebraic curve and let P be a closed point on C. Then it is known that the local ring $O_{P,C}$ is seminormal if and only if P is a simple point or a node (cf. [1], [2], [4]). It is then natural to ask what the seminormalization of $O_{P,C}$ is when P is not a seminormal point. The answer to this question is given in the second section in the case where P is an ordinary multiple point.

The author would like to express his sincere gratitude to Professors Y. Nakai and M. Miyanishi, and Dr. K. Yoshida for their valuable advice and suggestions during the preperation of this article.

0. Notations and conventions

The following notations and conventions are fixed throughout this article. When R is a ring, $J(R)$ stands for the Jacobson radical of R, $Q(R)$ for the total quotient ring of R, $\bar{R}$ for the integral closure of R in $Q(R)$ and ${}^{+}R$ for the seminormalization of R. We denote by $\cong_R$ an R-algebra isomorphism. An R-algebra is always assumed to be commutative, associative and containing 1. The symbols X, Y, Z, T, X_i, etc. are used to denote indeterminates or variables. When we say that $(R, \mathfrak{M})$ is a quasi-local ring, we mean that R is a ring which has the unique maximal ideal $\mathfrak{M}$. A noetherian quasi-local ring is called a local ring.

1. Characterizations of one-dimensional seminormal rings

Lemma 1.1. *Let* $(R, \mathfrak{M})$ *be a reduced one-dimensional quasi-local ring. Then* R *is seminormal if and only if* $\mathfrak{M}=J(\bar{R})$.

Proof. By definition, we have ${}^{+}R=R+J(\bar{R})$. Therefore, $R={}^{+}R$ if and only if $\mathfrak{M}=J(\bar{R})$. Q.E.D.

Lemma 1.2. *Let* k *be a field and let* L *be a reduced noetherian* k*-algebra of dimension* 0. *Then, both* $k+TL[T]$ *and* $k+TL[[T]]$ *are seminormal, where* $k+TL[T]$ *is identified with a subring of* $L[T]$ *and* $k+TL[[T]]$ *is identified with a quasi-local subring of* $L[[T]]$ *with residue field* k.

Proof. Set $R=k+TL[T]$ and $S=k'+TL[T]$, where k' is the integral closure of k in L. First we prove that $\bar{R}=S$. By assumption L is a direct product of fields, hence $L[T]$ is normal. Thus we have $\bar{R}\subset L[T]$. Let $f(T)$ be an element in $L[T]$. Then, it is easy to see that $f(T)\in\bar{R}$ if and only if $f(0)\in k'$, which shows $\bar{R}=S$. Let $\mathfrak{P}$ be the prime ideal $TL[T]$ of R. Then, as is readly seen, we have $\mathfrak{P}R_{\mathfrak{P}}=J(S_{\mathfrak{P}})$, and hence $R_{\mathfrak{P}}$ is seminormal by Lemma 1.1. Let $g(T)$ be an arbitrary element in ${}^{+}R$. Then we have $h(T)g(T)\in R$ for some element $h(T)$ in R but not in $\mathfrak{P}$, because $({}^{+}R)_{\mathfrak{P}}\subset{}^{+}(R_{\mathfrak{P}})=R_{\mathfrak{P}}$. Then we have $h(0)g(0)\in k$, which implies that $g(0)\in k$ because $h(0)\neq 0$. This shows that $g(T)\in R$, and R is seminormal. We can verify that $k+TL[[T]]$ is seminormal by the similar way. Q.E.D.

1.3. In the rest of this section we fix the following notations: Let $(A, \mathfrak{m})$ be a reduced one-dimensional local ring containing the field k isomoiphic to $A/\mathfrak{m}$. We assume that $\bar{A}$ is a finite A-module. Then $\bar{A}$ is a semi-local ring. Let $J=J(\bar{A})$ and let $K=\bar{A}/J$. When M is a finite A-module we denote by $\hat{M}$ the $\mathfrak{m}$-adic completion of M.

Then the following lemma is proved in Davis [2].

Lemma. *The following conditions are equivalent to each other.*

(1) A *is seminormal.*

(2) $Gr^{\cdot}(A)$ *is* k*-isomorphic to* $k+TK[T]$.

(3) $Gr^{\cdot}(A)$ *is reduced and seminormal.*

1.4. Since the $\mathfrak{m}$-adic completion of $\bar{A}$ coincides with the J-adic completion of $\bar{A}$, we have $\hat{\bar{A}}\cong_{k}K[[T]]$ and $\hat{J}=J(\hat{\bar{A}})\cong_{k}TK[[T]]$. Notice that $\hat{\bar{A}}=\bar{\hat{A}}$, because $\hat{\bar{A}}$ is a finite $\hat{A}$-module with $\hat{\bar{A}}\subset Q(\hat{A})$ and $\hat{\bar{A}}\cong_{k}K[[T]]$ is normal.

Lemma 1.5. *The following conditions are equivalent to each other.*

(1) A *is seminormal.*

(2) $\hat{A}$ *is* k*-isomorphic to* $k+TK[[T]]$.

(3) $\hat{A}$ *is seminormal.*

Proof. (1)$\Rightarrow$(2): $\hat{A}$ is a local ring with maximal ideal $\hat{\mathfrak{m}}$ and $k\subset\hat{A}$. Hence we have $\hat{A}\cong_k k+\hat{\mathfrak{m}}=k+\hat{J}\cong_k k+TK[[T]]$, because seminormality of A implies that $\mathfrak{m}=J$.
(2)$\Rightarrow$(3): By Lemma 1.2, if $\hat{A}\cong_k k+TK[[T]]$ then $\hat{A}$ is seminormal.
(3)$\Rightarrow$(1): Since $\hat{A}$ is seminormal we have $\hat{\mathfrak{m}}=\hat{J}$, where $\hat{\mathfrak{m}}=\mathfrak{m}\otimes_A\hat{A}$ and $\hat{J}=J\otimes_A\hat{A}$. Therefore we have $\mathfrak{m}=J$ because $\hat{A}$ is faithfully flat over A. By Lemma 1.1 we see that A is seminormal. Q.E.D.

Lemma 1.6. *Let L be a k-algebra and let $v_1, \cdots, v_n$ be a k-basis of L. Let ρ_{ijk} $(1\leq i, j, k\leq n)$ be the structure constants of L, i.e., ρ_{ijk}'s are elements of k such that*

$$v_iv_j=\sum_{k=1}^{n}\rho_{ijk}v_k\,.$$

Let $\sigma: k[X_1, \cdots, X_n]\to L[T]$ (or $k[[X_1, \cdots, X_n]]\to L[[T]]$) be a k-algebra homomorphism defined by $\sigma(X_i)=v_iT$ $(i=1, \cdots, n)$. Then the kernel I of σ is generated by the quadratic polynomials

$$\psi_{ijk}=(\sum_{m=1}^{n}\rho_{ikm}X_m)X_j-(\sum_{m=1}^{n}\rho_{jkm}X_m)X_i \qquad (1\leq i, j, k\leq n)\,.$$

Proof. First notice that ρ_{ijk}'s satisfy the relations

$$\rho_{ijk}=\rho_{jik}\,, \tag{1}$$

$$\sum_{m=1}^{n}\rho_{ikm}\rho_{mjs}=\sum_{m=1}^{n}\rho_{jkm}\rho_{mis}\,. \tag{2}$$

Let

$$1=\sum_{m=1}^{n}c_mv_m$$

where $c_m\in k$ $(m=1, \cdots, n)$. From $(\sum_{m=1}^{n}c_mv_m)v_i=v_i$ it follows that

$$\sum_{m=1}^{n}c_m\rho_{mis}=\delta_{is}\,. \tag{3}$$

Let I' be the ideal generated by $\{\psi_{ijk}\}$. As is readily seen I is a homogeneous ideal, and $I'\subset I$ from (2). We set $Y=\sum_{m=1}^{n}c_mX_m$. Then, using (3) we get

$$\sum_{i=1}^{n}c_i\psi_{ijk}=X_jX_k-Y\sum_{s=1}^{n}\rho_{jks}X_s$$

i.e.,

$$X_jX_k\equiv Y\sum_{s=1}^{n}\rho_{jks}X_s \qquad (\text{mod } I')\,. \tag{4}$$

Let $F(X_1, \cdots, X_n)$ be a homogeneous polynomial of degree N in I. Clearly $N>1$, and from (4) we easily see that

$$F(X_1, \cdots, X_n) \equiv Y^{N-1}\sum_{s=1}^{n} a_s X_s \qquad (\mathrm{mod}\ I')$$

for some $a_s \in k$. Since $\sigma(F)=0$, $\sigma(Y)=1$ and $\sigma(X_s)=v_s T$ we have $a_s=0$ for $s=1, \cdots, n$ because $v_1, \cdots, v_n$ are linearly independent over k. This proves that $F \in I'$. Q.E.D.

For practical purpose the generators $\{\psi_{ijk}\}$ of I are not easy to handle. For later use we prove the following lemma.

Lemma 1.7. *Let $L=k[X]/(f(X))$, where $f(X)=X^n+a_{n-1}X^{n-1}+\cdots+a_0$ and let α = the residue class of X in L. Define a k-algebra homomorphism σ: $k[X_0, \cdots, X_{n-1}] \to L[T]$ (or $k[[X_0, \cdots, X_{n-1}]] \to L[[T]]$) by $\sigma(X_i)=\alpha^i T$. We set $\xi_m = X_{[m/2]}X_{[(m+1)/2]}$ for $m=0, \cdots, 2n-2$, where [ ] is the Gauss symbol. Let I_f be the ideal generated by quadratic polynomilas*

$$g_{ij} = X_i X_j - \xi_{i+j} \qquad (0 \leqq i,\ j \leqq n-1)\,,$$
$$h_s = \sum_{m=0}^{n} a_m \xi_{m+s} \qquad (0 \leqq s \leqq n-2),\ \textit{where}\ a_n = 1\,.$$

Then the kernel of σ is equal to I_f.

Proof. Let $\alpha^i\alpha^j=\sum_{k=0}^{n-1} \rho_{ijk}\alpha^k$ and set $\psi_{ijk}=(\sum_{m=0}^{n-1} \rho_{ikm}X_m)X_j-(\sum_{m=0}^{n-1}\rho_{jkm}X_m)X_i$. Then, by virtue of Lemma 1.6, Ker σ is generated by ψ_{ijk}'s. It is easy to check that I_f is contained in Ker σ. We shall prove the inverse containment $I_f \supset$ Ker σ. Notice that $\rho_{ijk}=\rho_{stk}$ if $i+j=s+t$. We set $\rho_{i+j,k}=\rho_{ijk}$. Then we have

$$(1) \qquad \rho_{N,m} = \delta_{Nm} \text{ if } N<n \text{ and } \rho_{n,m} = -a_m \qquad \text{for } 0 \leqq m \leqq n-1,$$

and

$$(2) \qquad \rho_{N,0} = -a_0\rho_{N-1,n-1} \text{ and } \rho_{N,k} = \rho_{N-1,k-1}-a_k\rho_{N-1,n-1} \qquad \text{for } 1 \leqq k \leqq n-1.$$

From (2), if $s<n-1$ and $N \geqq 1$ we easily get the relation

$$(3) \qquad \sum_{m=0}^{n-1} \rho_{N,m}\xi_{m+s} \equiv \sum_{m=0}^{n-1} \rho_{N-1,m}\xi_{m+s+1} \qquad (\mathrm{mod}\ I_f)\,.$$

To prove $I_f \supset$ Ker σ it suffices to prove $\psi_{ijk} \in I_f$. Notice that

$$(4) \qquad \psi_{ijk} \equiv \sum_{m=0}^{n-1} \rho_{i+k,m}\xi_{m+j} - \sum_{m=0}^{n-1} \rho_{j+k,m}\xi_{m+i} \qquad (\mathrm{mod}\ I_f)$$

because $X_m X_j \equiv \xi_{m+j}$ and $X_m X_i \equiv \xi_{m+i}$ (mod I_f). We may assume $i \geqq j$ because $\psi_{ijk} = -\psi_{jik}$. If $i+k<n$, from (1) and (4) we have

$$\psi_{ijk} \equiv \xi_{i+j+k} - \xi_{i+j+k} = 0 \qquad (\mathrm{mod}\ I_f)\,.$$

If $i+k \geqq n$ and $j+k<n$, from (1), (3) and (4) we have

$$\psi_{ijk} \equiv \sum_{m=0}^{n-1} \rho_{n,m}\xi_{m+i+j+k-n} - \xi_{i+j+k} \qquad (\text{mod } I_f)$$
$$= -\sum_{m=0}^{n-1} a_m \xi_{m+i+j+k-n} - \xi_{i+j+k} = -h_{i+j+k-n}\,.$$

If $j+k \geqq n$, from (3) and (4) we have

$$\psi_{ijk} \equiv \sum_{m=0}^{n-1} \rho_{i+j+k+1-n,m}\xi_{m+n-1} - \sum_{m=0}^{n-1} \rho_{i+j+k+1-n,m}\xi_{m+n-1} \qquad (\text{mod } I_f)$$
$$=0\,.$$

Thus we have $\psi_{ijk} \equiv 0$ (mod I_f) and we proved the assertion. Q.E.D.

By virtue of Lemmas 1.3, 1.5 and 1.6 we have the following

Theorem 1.8. *Let $v_1, \cdots, v_n$ be a k-basis of a k-algebra $K=\bar{A}/J(\bar{A})$ and let ρ_{ijk} be the structure constants of k-algebra K. Set*

$$\psi_{ijk} = (\sum_{m=1}^{n} \rho_{ikm}X_m)X_j - (\sum_{m=1}^{n} \rho_{jkm}X_m)X_i \qquad (1 \leqq i, j, k \leqq n)\,.$$

Then the following conditions are equivalent to each other.
(1) *A is seminormal.*
(2) *$\hat{A}$ is k-isomorphic to $k[[X_1, \cdots, X_n]]/(\cdots, \psi_{ijk}, \cdots)$.*
(3) *$Gr^{\cdot}(A)$ is k-isomorphic to $k[X_1, \cdots, X_n]/(\cdots, \psi_{ijk}, \cdots)$.*

Lemma 1.9. *We say that a polynomial $f(X)$ in $k[X]$ is reduced in $k[X]$ if $f(X)$ has no multiple factors in $k[X]$, i.e., if the residue ring $k[X]/(f(X))$ is reduced. Let the ideal I_f have the same meaning as in* 1.7. *If $\hat{A}$ is k-isomorphic to $k[[X_0, \cdots, X_{n-1}]]/I_f$ or $Gr^{\cdot}(A)$ is k-isomorphic to $k[X_0, \cdots, X_{n-1}]/I_f$ for some reduced monic polynomial $f(X)$ of degree n in $k[X]$ then A is seminormal.*

Proof. This lemma follows from Lemmas 1.2 and 1.7. Q.E.D.

Theorem 1.10. *Let the ideal I_f have the same meaning as in* 1.7. *Assume that k is a perfect infinite field. Then the following conditions are equivalent to each other.*
(1) *A is seminormal.*
(2) *$\hat{A}$ is k-isomorphic to $k[[X_0, \cdots, X_{n-1}]]/I_f$ for some reduced monic polynomial $f(X)$ of degree n in $k[X]$.*
(3) *$Gr^{\cdot}(A)$ is k-isomorphic to $k[X_0, \cdots, X_{n-1}]/I_f$ for some reduced monic polynomial $f(X)$ of degree n in $k[X]$.*

Proof. If we see that K is k-isomorphic to $k[X]/(f(X))$ for some reduced monic polynomial $f(X)$ in $k[X]$ then the assertion follows from Theorem 1.8, Lemma 1.7 and Lemma 1.9. Let $\mathfrak{M}_1, \cdots, \mathfrak{M}_r$ be all the maximal ideals of $\bar{A}$ and set $K_i=\bar{A}/\mathfrak{M}_i$ for $i=1, \cdots, r$. Then we have $K \cong_k K_1 \times \cdots \times K_r$. Since k is a perfect field we have $K_i \cong_k k[X]/(f_i(X))$ for some irreducible monic polynomial $f_i(X)$ in $k[X]$. We may assume that $f_1(X), \cdots, f_r(X)$ are relatively prime because k is an infinite field. Set $f(X)=f_1(X)\cdots f_r(X)$. Then $f(X)$ is a rdeuced monic polynomial and we have $K \cong_k k[X]/(f(X))$. Q.E.D.

REMARK 1.11. In Theorem 1.10 we assumed that the local ring A is reduced. We may replace this assumption by the condition that depth $A \geqq 1$. In fact the following lemma holds in general.

Lemma. *Let $(R, \mathfrak{M})$ be a seminormal local ring with depth $R \geqq 1$. Then R is reduced.*

Proof. Let $\mathfrak{n}=\mathrm{nil}(R)$, where $\mathrm{nil}(R)$ denotes the nilradical of R, and let S be the set of regular elements of R. Then we have $\mathrm{nil}(\bar{R})=S^{-1}\mathfrak{n}$. From definition, it is easy to check that $R+\mathrm{nil}(\bar{R})\subset {}^+R$, hence we have $S^{-1}\mathfrak{n}\subset R$. Let x be an arbitrary element in $\mathfrak{n}$. Notice that $S\cap\mathfrak{M}\neq\phi$ because depth $R\geqq 1$. Let a be an element in $S\cap\mathfrak{M}$. Then we have $x/a^n\in R$ and $x\in a^nR\subset\mathfrak{M}^n$ for any integer n. Thus we have $x\in\bigcap_{n=0}^{\infty}\mathfrak{M}^n$, and hence we have $x=0$ by Krull's intersection theorem. Q.E.D.

2. Seminormalization of local rings of plane algebraic curves

2.1. Let V be an algebraic variety defined over a field k and let P be a closed point on V. Then P is said to be seminormal if the local ring $O_{P,V}$ is seminormal. Let V be a reduced plane algebraic curve. Then it is known that P is a seminormal point if and only if P is a smooth point or P is a node (cf. [1], [2], [4]). Consequently, if P is a singular point and P is not a node, $O_{P,V}$ is not seminormal. We are interested in how the seminormalization of $O_{P,V}$ can be obtained. For this purpose, we may assume that V is a plane curve defined by a polynomial $F(X, Y)=\sum_{i+j\geqq n} a_{ij}X^iY^j$ in $k[X, Y]$ with $a_{0n}=1$ and P is the origin $(0, 0)$. In this section we shall determine the seminormalization of $O_{P,V}=(k[X, Y]/(F(X, Y)))_{(X,Y)}$ when $f_F(X):=\sum_{i+j=n} a_{ij}X^j$ is a reduced monic polynomial in $k[X]$.

Lemma 2.2. *Let I_0 be the ideal of $k[X_0, \cdots, X_{n-1}]$ $(n\geqq 2)$ generated by $g_{ij}=X_iX_j-\xi_{i+j}$ $(0\leqq i<j\leqq n-1)$, where ξ_m is the same as in 1.7. Then we have $X_{i_1}\cdots X_{i_s}\equiv X_{j_1}\cdots X_{j_s}$ (mod I_0) if $i_1+\cdots+i_s=j_1+\cdots+j_s$. In paticular we have $X_0^{t-1}X_t\equiv X_1^t$ (mod I_0).*

Proof. This is easily seen by simple calculations and we omit the proof.

2.3. Let $F(X, Y)=\sum_{i+j\geqq n} a_{ij}X^iY^j$ be an element in $k[X, Y]$ with $n\geqq 2$ and $a_{0n}=1$. For each ordered pair (i, j) with $i+j\geqq n$, we choose a fixed integer $t=t(i, j)$ satisfying

$$\mathrm{Max}(0, n-i) \leqq t \leqq \mathrm{Min}(n, j)$$

and set

$$\phi_s = \sum_{i+j \geqq n} a_{ij} X_0^{i+t-n} X_1^{j-t} \xi_{t+s} \qquad (0 \leqq s \leqq n-2).$$

Let I be the ideal of $k[X_0, \cdots, X_{n-1}]$ generated by I_0 and ϕ_s's. It should be noticed that I is independent of the choice of t by Lemma 2.2, and $t(n-m, m)=m$ for $m=0, \cdots, n$.

Lemma 2.4. *Let $F(X, Y)$, I_0 and I be the same as in 2.2 and 2.3. Then the ring homomorphism τ: $(k[X, Y]/(F(X, Y)))_{(X,Y)} \to (k[X_0, \cdots, X_{n-1}]/I)_{(X_0, \cdots, X_{n-1})}$ defined by $\tau(X)=X_0$ and $\tau(Y)=X_1$ is injective, birational and integral.*

Proof. The proof is divided into three steps.

Step 1. *The homomorphism τ: $k[X, Y]/(F(X, Y)) \to k[X_0, \cdots, X_{n-1}]/I$ defined by $\tau(X)=X_0$ and $\tau(Y)=X_1$ is well-defined and injective.*

In fact by Lemma 2.2 we have

$$(1) \qquad \begin{cases} X_0^{n-1}\phi_s \equiv \sum_{i+j \geqq n} a_{ij} X_0^i X_1^j X_s = X_s F(X_0, X_1) & (\mathrm{mod}\ I_0) \\ X_0^{n-2}\phi_0 \equiv \sum_{i+j \geqq n} a_{ij} X_0^i X_1^j = F(X_0, X_1) & (\mathrm{mod}\ I_0) \end{cases}$$

because $X_0^{i+t-1}\xi_{t+s} \equiv X_0^i X_1^t X_s$ and $X_0^{i+t-2}\xi_t \equiv X_0^i X_1^t$ (mod I_0) (notice that $i+t \geqq n \geqq 2$). In paticular $F(X_0, X_1) \in I$ and the homomorphism τ is well-defined. To prove that τ is injective let

$$G(X_0, X_1) = \sum_{0 \leqq i < j \leqq n-1} h_{ij} g_{ij} + \sum_{s=0}^{n-2} \lambda_s \phi_s$$

be an element of $I \cap k[X_0, X_1]$ where h_{ij}'s and λ_s's are elements of $k[X_0, \cdots, X_{n-1}]$. Then from (1) we get

$$X_0^{n-1} G(X_0, X_1) \equiv X_0^{n-1} \sum_{0 \leqq i < j \leqq n-1} h_{ij} g_{ij} + F(X_0, X_1) \sum_{s=0}^{n-2} X_s \lambda_s \qquad (\mathrm{mod}\ I_0).$$

Therefore we can write

$$X_0^{n-1} G(X_0, X_1) = \sum l_{ij} g_{ij} + \lambda(X_0, \cdots, X_{n-1}) F(X_0, X_1)$$

for some elements l_{ij} in $k[X_0, \cdots, X_{n-1}]$, where $\lambda(X_0, \cdots, X_{n-1}) = \sum_{s=0}^{n-2} X_s \lambda_s$. As is readly seen, there exist an integer N and a polynomial $\tilde{\lambda}(X, Y)$ in two variables such that

$$\tilde{\lambda}(X, ZX) = X^N \lambda(X, ZX, \cdots, Z^{n-1}X).$$

Since $g_{ij}(X, ZX, \cdots, Z^{n-1}X)=0$, we get, by specializing $X_i \mapsto Z^i X$, the relation

$$X^{N+n-1}G(X, ZX) = \tilde{\lambda}(X, ZX)F(X, ZX).$$

From this we get the identity

$$X_0^{N+n-1}G(X_0, X_1) = \tilde{\lambda}(X_0, X_1)F(X_0, X_1)$$

because X and ZX are independent variables over k. From our assumption $F(X_0, X_1)$ is not divisible by X_0. Hence $\tilde{\lambda}(X_0, X_1)$ is divisible by X_0^{N+n-1}, and we have $G(X_0, X_1) \in F(X_0, X_1)k[X_0, X_1]$.

Step 2. *Let $\mathfrak{Q}$ be a proper prime ideal of $k[X_0, \cdots, X_{n-1}]$ containing I and X_0. Then $\mathfrak{Q}$ is necessarily equal to the maximal ideal $(X_0, \cdots, X_{n-1})$.*

If $X_i \in \mathfrak{Q}$ for some $i \leqq n-3$ we have $X_{i+1}^2 \equiv X_i X_{i+2} \equiv 0 \pmod{\mathfrak{Q}}$ whence $X_{i+1} \in \mathfrak{Q}$. Thus we have $X_0, \cdots, X_{n-2} \in \mathfrak{Q}$. On the other hand, we have $\phi_{n-2} \in I \subset \mathfrak{Q}$ and $\phi_{n-2} = X_{n-1}^2 + \eta$ with $\eta \in \mathfrak{Q}$ because $t(0, n) = n$, $a_{0n} = 1$ and $\xi_s \in \mathfrak{Q}$ for $s < 2n-2$. Thus we have $X_{n-1}^2 \in \mathfrak{Q}$, whence $X_{n-1} \in \mathfrak{Q}$. Obviously, $(X_0, \cdots, X_{n-1})$ is a maximal ideal of $k[X_0, \cdots, X_{n-1}]$, and thence we have $\mathfrak{Q} = (X_0, \cdots, X_{n-1})$.

Step 3. *The ring homomorphism τ: $(k[X, Y]/(F(X, Y)))_{(X,Y)} \to (k[X_0, \cdots, X_{n-1}]/I)_{(X_0, \cdots, X_{n-1})}$ induced naturally by τ is birational and integral.*

Set $B = k[X_0, X_1]/(F(X_0, X_1))$, $C = k[X_0, \cdots, X_{n-1}]/I$, $\mathfrak{p} = (X_0, X_1)B$ and $\mathfrak{M} = (X_0, \cdots, X_{n-1})C$. We shall denote by x_i the residue class of X_i modulo I. Notice that $x_i x_0^{i-1} = x_1^i$ for $1 \leqq i \leqq n-1$ and x_0 is a regular element of B. This implies that $C_\mathfrak{p} \subset Q(B_\mathfrak{p})$. Next we show that $C_\mathfrak{p}$ is integral over $B_\mathfrak{p}$. In fact, we have

$$F(x_0, x_1) = \sum_{i+j \geqq n} a_{ij} x_0^i x_1^j = 0$$

in B. From this it is easy to check that x_1/x_0 is integral over $B_\mathfrak{p}$. Hence $x_i = x_1(x_1/x_0)^{i-1}$ is also integral over $B_\mathfrak{p}$ for $1 \leqq i \leqq n-1$, which shows that $C_\mathfrak{p}$ is integral over $B_\mathfrak{p}$. It remains to prove that $C_\mathfrak{p} = C_\mathfrak{M}$. Let $\mathfrak{Q}C_\mathfrak{p}$ be a maximal ideal of $C_\mathfrak{p}$, where $\mathfrak{Q}$ is a maximal ideal of C. Then we have $\mathfrak{Q}C_\mathfrak{p} \cap B_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p}$ because $C_\mathfrak{p}$ is integral over $B_\mathfrak{p}$, whence $\mathfrak{Q} \cap B = \mathfrak{p}$. Thus we have $\mathfrak{Q} = \mathfrak{M}$ by the step 2, from which we see that $C_\mathfrak{p}$ is a local ring. Therefore, we have $C_\mathfrak{p} = C_\mathfrak{M}$ and the assertion is verified.

Lemma 2.5. *Assume that $f_F(X) = \sum_{i+j=n} a_{ij} X^j$ is a reduced monic polynomial in $k[X]$. Then the local ring $(k[X_0, \cdots, X_{n-1}]/I)_{(X_0, \cdots, X_{n-1})}$ is seminormal.*

Proof. Set $R = (k[X_0, \cdots, X_{n-1}]/I)_{(X_0, \cdots, X_{n-1})}$. Notice that the leading form of ϕ_s is $\sum_{i+j=n} a_{ij} \xi_{j+s}$. Hence, if we set $f = f_F(X)$, we have $Gr^{\cdot}(R) \cong_k k[X_0, \cdots, X_{n-1}]/I_f$, where I_f is the ideal defined in 1.7 (cf. [3; p. 118]). Therefore, by virtue of Lemma 1.9, we see that R is seminormal. Q.E.D.

Summarizing the results stated in Lemmas 2.4 and 2.5, we have the following theorem.

Theorem 2.6. *Let* $F(X, Y)=\sum_{i+j\geqq n} a_{ij}X^iY^j$ *be an element in* $k[X, Y]$ *with* $n\geqq 2$ *and* $a_{0n}=1$. *Assume that* $f_F(X)=\sum_{i+j=n} a_{ij}X^j$ *is a reduced monic polynomial in* $k[X]$. *Then the seminormalization of the local ring* $(k[X, Y]/(F(X, Y)))_{(X,Y)}$ *is isomorphic to* $(k[X_0, \cdots, X_{n-1}]/I)_{(X_0,\cdots,X_{n-1})}$ *where the ideal* I *is generated by* g_{ij} $(0\leqq i<j\leqq n-1)$ *and* ϕ_s $(0\leqq s\leqq n-2)$ *given in* 2.2 *and* 2.3, *respectively.*

REMARK 2.7. When $f_F(X)$ is a reduced monic polynomial in $k[X]$ the seminormalization of $O_{P,V}$ is determined by the leading form of the defining equation even if $f_F(X)$ is not reduced in $\bar{k}[X]$, where $\bar{k}$ denotes the algebraic closure of k. But when $f_F(X)$ is not reduced in $k[X]$ the seminormalization of $O_{P,V}$ is more complicated as is shown by the following examples.

(1) Let $R=(k[X, Y]/(Y^3-XY^2-X^4))_{(X,Y)}$. Then ${}^+R$ is $(k[X_0, X_1]/(X_1^2-X_0X_1-X_0^3)_{(X_0,X_1)}$, and the embedding $R\rightarrow {}^+R$ is given by $X\mapsto X_1$ and $Y\mapsto X_0^2+X_1$.

(2) Let $S=(k[X, Y]/(Y^3-XY^2-X^5))_{(X,Y)}$. Then ${}^+S$ is $(k[X_0, X_1, X_2]/(X_1^2-X_0X_2, X_1X_2+X_0X_1-X_0^3, X_2^2+X_1^2-X_0^2X_1))_{(X_0,X_1,X_2)}$ and the embedding $S\rightarrow {}^+S$ is given by $X\mapsto X_0$ and $Y\mapsto X_2+X_0$.

REMARK 2.8. Let $F(X, Y)$ and I have the same meaning as in 2.6. Assume that the origin P is a unique singular point and $k[X_0, \cdots, X_{n-1}]/I$ is integral over $k[X, Y]/(F(X, Y))$. Then the seminormalization of $k[X, Y]/(F(X, Y))$ is $k[X_0, \cdots, X_{n-1}]/I$.

EXAMPLE. Let $R=k[X, Y]/(Y^3-aX^3-X^6)$. Assume that $\text{char}(k)\neq 3$ or $a^{1/3}\notin k$. Then ${}^+R=k[X, Y, Z]/(Y^2-XZ, YZ-aX^2-X^5, Z^2-aXY-X^4Y)$.

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Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan