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A NOTE ON HAUSDORFF'S SUMMATION METHODS

J. P. BRANNEN

If $\{a_n\}$ is a moment sequence and $(\varDelta a)$ is the difference matrix having base sequence $\{a_n\}$, then $(\varDelta a)$ is symmetric about the main diagonal if and only if the function $\alpha(x)$ such that $a_n = \int_0^1 x^n d\alpha(x)$, $n = 0, 1, 2, \cdots$, is symmetric in the sense that $\alpha(x) + \alpha(1 + x) = \alpha(1) + \alpha(0)$ except for at most countably many x in $[0, 1]$. This property is related to the "fixed points" of the matrix H, where HaH is the Hausdorff matrix determined by the moment sequence $\{a_n\}$.

In each of the papers [2], [3] and [5], there is reference to difference matrices of the form

$$(\varDelta d) = \begin{bmatrix} \varDelta^0 d_0 & \varDelta^0 d_1 & \varDelta^0 d_2 & \\ \varDelta^1 d_0 & \varDelta^1 d_1 & \varDelta^1 d_2 & \vdots \\ \varDelta^2 d_0 & \varDelta^2 d_1 & \varDelta^2 d_2 & \\ & \cdots & & \end{bmatrix}$$

where $\{d_n\}$ is a moment sequence, $\varDelta^0 d_n = d_n$, $n = 0, 1, 2, \cdots$ and $\varDelta^m d_n = \varDelta^{m-1} d_n - \varDelta^{m-1} d_{n+1}$, for $n = 0, 1, 2, \cdots$ and $m = 1, 2, 3, \cdots$. In [2], Garabedian and Wall discussed the importance of $(\varDelta d)$ having the property of being symmetric about the main diagonal, i.e. $\varDelta^m d_n = \varDelta^n d_m$. They also showed that if $\{d_n\}$ is a totally monotone sequence, then $(\varDelta d)$ is symmetric about the main diagonal if and only if the function $f(x)$ which generates $\{d_n\}$ has a certain type continued fraction expansion.

In this paper, the symmetry of $(\varDelta d)$ is investigated with the restriction of total monotonicity removed and a collection of necessary and sufficient conditions are given, Theorem 3, for moment sequences in general. A relation is established between the symmetry of $(\varDelta d)$ and the "fixed points" of the difference matrix

$$(1) \qquad H = \begin{bmatrix} \binom{0}{0} & & \\ \binom{1}{0} & -\binom{1}{1} & \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} \\ & \cdots & \end{bmatrix}.$$

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2. Notation, definitions, and examples. Except for some notation and definitions introduced for convenience, the notation and definitions of this paper will follow [6].

NOTATION. If $\{d_n\}$ is an infinite sequence, d^* and d' denote respectively the diagonal and column matrices determined by $\{d_n\}$.

DEFINITION 1. If $\{d_n\}$ is a number sequence such that for some function $f(x)$ on $[0, 1]$,

$$d_p = \int_0^1 x^p df(x) = \int_0^1 (1-x)^p df(x) \; ; \qquad p = 0, 1, 2, \cdots ,$$

then $\{d_n\}$ is called a symmetric moment sequence.

The Cesàro moment sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots$ provides an example of a moment sequence satisfying Definition 1 since for $p = 0, 1, 2, \cdots$

$$(2) \qquad c_p = \int_0^1 x^p dx = x^{p+1}/p + 1 \Big]_0^1$$
$$= \int_0^1 (1-x)^p dx = -(1-p)^{p+1}/p + 1 \Big]_0^1 = \frac{1}{p+1} .$$

DEFINITION 2. If A is a semi-infinite, lower triangular, matrix having inverse and $\{a_n\}$ and $\{d_n\}$ are sequences such that $A^{-1}d^*Aa' = A^{-1}a^*Ad'$, then $\{a_n\}$ and $\{d_n\}$ are symmetric relative to A.

The Cesàro moment sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, c_p$ of (2), and the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ are symmetric relative to the matrix H of (1).

3. THEOREMS. LEMMA. *Suppose $\{s_n\}$ is a sequence such that $s_p \neq 0$ for $p = 0, 1, 2, \cdots$ and suppose that A is a semi-infinite matrix having inverse such that $As' = s'$; then,*

(i) $A^{-1}s' = s'$,

(ii) *$\{x_n\}$ and $\{s_n\}$ are symmetric with respect to A if and only if $Ax' = x'$, and*

(iii) *if $A^{-1}a^*As' = A^{-1}s^*Aa'$ and $A^{-1}b^*As' = A^{-1}s^*Ab'$, then $A^{-1}b^*Aa' = A^{-1}a^*Ab'$.*

Proof. (i) is obvious. For the proof of (ii), we first suppose $\{x_n\}$ is symmetric with $\{s_n\}$ relative to A so that $A^{-1}x^*As' = A^{-1}s^*Ax'$. Multiplying both sides on the left by A and using $As' = s'$ gives $x^*s' = s^*Ax'$. Under the hypothesis, s^* has inverse s^{*-1} so that

$$(3) \qquad s^{*-1}x^*s' = s^{*-1}s^*Ax' = Ax' .$$

Since $x^*s' = s^*x'$, it follows from (3) that $x' = Ax'$.

On the other hand, if $Ax' = x'$,

$$A^{-1}x^*As' = A^{-1}x^*s' \tag{4}$$

and

$$A^{-1}s^*Ax' = A^{-1}s^*x' \ .$$

Since $s^*x' = x^*s'$, it follows from (4) that x and s are symmetric relative to A.

For the proof of (iii), we suppose that $a' = s^{*-1}a^*s'$ and $b' = s^{*-1}b^*s'$, from which it follows that

$$A^{-1}a^*Ab' = A^{-1}a^*s^{*-1}b^*s' \tag{5}$$

and

$$A^{-1}b^*Aa' = A^{-1}b^*s^{*-1}a^*s' \ . \tag{6}$$

Since diagonal matrices permute, it follows that (5) and (6) are equal establishing (iii).

THEOREM 1. *If* $\{b_n\}$ *is a moment sequence, i.e.,*

$$b_p = \int_0^1 x^p dg(x) \ , \tag{7}$$

$\{b_n\}$ *and the Cesàro sequence* (2) *are symmetric relative to* H *if and only if* $\{b_n\}$ *is a symmetric moment sequence.*

Proof. Let

$$f_n(x) = \begin{cases} \sum_{p=0}^{n-1} \binom{n}{p}(-1)^p x^p & \text{for } n = 2, 4, 6, \cdots \\ \sum_{p=0}^{n-1} \binom{n}{p}(-1)^p x^p - 2x^n & \text{for } n = 1, 3, 5, \cdots . \end{cases}$$

Clearly, if $\{t_n\}$ is any number sequence, $Ht' = t'$ if and only if

$$\sum_{p=0}^{n-1} \binom{n}{p}(-1)^p t_p = 0 \qquad \text{for } n = 2, 4, 6, \cdots$$

and

$$\sum_{p=0}^{n-1} \binom{n}{p}(-1)^p t_p - 2t_n = 0 \qquad \text{for } n = 1, 3, 5, \cdots .$$

Thus if $\{b_n\}$ is defined as in (7), $Hb' = b'$ if and only if

$$\int_0^1 f_n(x)dg(x) = 0 \qquad \text{for } n = 1, 2, 3, \cdots . \tag{8}$$

But, $f_n(x) = (1 - x)^n - x^n$ for $n = 1, 2, 3, \cdots$ so that

$$(9) \qquad \int_0^1 f_n(x)dg(x) = \int_0^1 (1-x)^n dg(x) - \int_0^1 x^n dg(x) ,$$

and consequently (8) holds if and only if $\{b_n\}$ is a symmetric moment sequence. It follows from (9) and (2) that $Hc' = c'$ and from the preceding Lemma that $\{b_n\}$ and $\{c_n\}$ are symmetric relative to H.

Conversely, if $\{b_n\}$ and $\{c_n\}$ are symmetric relative to H, it follows that $Hb' = b'$, and if $\{b_n\}$ is defined as in (7), then $\{b_n\}$ is a symmetric moment sequence.

THEOREM 2. *If $g(x)$ is of bounded variation on $[0, 1]$ and $\{z_n\}$ is the moment sequence determined by $g(x)$, the following two statements are equivalent:*

(i) *$\{z_n\}$ is a symmetric moment sequence, and*

(ii) *there do not exist uncountably many x in $[0, 1]$ for which $g(x) + g(1-x) \neq g(1) + g(0)$.*

Proof. Suppose (i). Then let $u = 1 - x$ so that,

$$z_p = \int_0^1 (1-x)^p dg(x) = \int_0^1 u^p dg(1-x) = -\int_0^1 u^p dg(1-u) .$$

Thus, $\int_0^1 (1-x)^p dg(x) = -\int_0^1 x^p dg(1-x)$ so that for $p = 0, 1, 2, \cdots$,

$$(10) \qquad \int_0^1 x^p d[g(x) + g(1-x)] = 0 .$$

Since $g(x) - g(1-x)$ is of bounded variation on $[0, 1]$, (10) implies that for every $k(x)$ continuous on $[0, 1]$, $\int_0^1 k(x)d[g(x) + g(1-x)] = 0$. This, [4, p. 69], implies (ii). Reversing the steps leading to (10) shows that (ii) implies (i).

An interesting example of a function satisfying (ii) is provided by Evans in [1].

THEOREM 3. *Suppose $g(x)$ is of bounded variation on $[0, 1]$ and suppose $\{a_n\}$ is the moment sequence generated by $g(x)$. The following statements are equivalent:*

(i) *$\{a_n\}$ is a symmetric moment sequence,*

(ii) *$Ha' = a'$,*

(iii) *$\{a_n\}$ and the Cesàro moment sequence $\{c_n\}$ are symmetric relative to H, and*

(iv) *the difference matrix (Δa) having base sequence $\{a_n\}$ is symmetric about the main diagonal.*

Proof. Theorem 1 implies the equivalence of (i), (ii), and (iii).

(i) implies (iv) provided

$$(11)\quad \int_0^1 x^m(1-x)^n dg(x) = \int_0^1 x^n(1-x)^m dg(x) \qquad \text{for } m, n = 0, 1, 2, \cdots .$$

Let $u = 1 - x$ so that $\int_0^1 x^m(1-x)^n dg(x) = \int_1^0 (1-u)^m u^n dg(1-u)$. Thus (11) may be rewritten as

$$(12)\qquad -\int_0^1 (1-x)^m x^n dg(1-x) = \int_0^1 x^n(1-x)^m dg(x)$$
$$= \int_0^1 x^n(1-x)^m d[g(x) + g(1-x)] = 0 .$$

That (12) is the case for $\{a_n\}$ a symmetric moment sequence follows from (ii) of Theorem 2. (iv) implies (ii) since (iv) implies that $a_n = \Delta^n A_0$, which is the same as saying that $Ha' = a'$. Thus the equivalence of the four statements is established.

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