On the exhaustivity and property (p.g.p) of non-additive set functions

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Abstract: *In this paer, a further research on structural characteristics of non-additive set functions is made. Several sufficient and necessary conditions for the exhaustivity and property (p.g.p) of fuzzy measures are given.*

Keywords: *Set functions, exhaustivity, property (p.g.p), null-additivity.*

1 Introduction

The structural characteristics of non-additive set functions are introduced by Dobrakove [1, 2] and Wang [8]. These concepts play very important roles in fuzzy measure theory [3 $\sim$ 9]. In this paper, we further investigate connections between them, and give several sufficient and necessary conditions for the exhaustivity and property (p.g.p) of set functions.

2 Notations and preliminaries

Let X be a non-empty set, $\mathcal{A}$ a σ-ring of subsets of X, and let $\mu : \mathcal{A} \to \overline{R_+} = [0, +\infty]$. Unless stated otherwise, all subsets are supposed to belong to $\mathcal{A}$ and all set functions are asummed to be monotone and equal to zero on the emptyset.

Definition 1 *μ is said to be null-additive if $\mu(F) = 0$ implies $\mu(E \cup F) = \mu(E)$; autocontinuous from above if* $\lim_n \mu(F_n) = 0$ *implies* $\lim_n \mu(E \cup F_n) = \mu(E)$.

Definition 2 *μ is said to be continuous from above at $\emptyset$ (or order continuous) if $E_n \searrow \emptyset$ implies $\lim_n \mu(E_n) = 0$; and exhaustive if $\mu(E_n) \to 0$ for any infinite disjoint sequence $\{E_n\}_n$.*

Definition 3 *μ is called a LSC-fuzzy measure if it is continuous from below, i.e.,* $\lim_n \mu(E_n) = \mu(E)$ *for any $E_n \nearrow E$.*

3 Exhaustivity

In this section, we give three sufficient and necessary conditions for the exhaustivity of non-additive set functions.

Theorem 1 *μ is exhaustive if and only if any monotone sequence $\{E_n\}_n$ is μ-cauchy, i.e.,*

$$\mu(E_n \triangle E_m) \to 0 \ \text{ as } n \wedge m \to +\infty.$$

Proof. Let μ be exhaustive and $\{E_n\}_n$ be a increasing sequence. If $\{E_n\}_n$ is not μ-cauchy, then there exist $\epsilon > 0$ and a subsequence $\{E_{n_k}\}_k$ of $\{E_n\}_n$ such that

$$\mu(E_{n_{k+1}} - E_{n_k}) = \mu(E_{n_{k+1}} \triangle E_{n_k}) \geq \epsilon, \ (\forall k \geq 1).$$

Put $F_k = E_{n_{k+1}} - E_{n_k}$, $k \geq 1$. Then, F_k are pairwise disjoint and $\lim_k \mu(F_k) \geq \epsilon$. This is in contradiction with the exhaustivity of μ. The 'only if' part is true.

Now, we prove the 'if' part. Let $\{E_n\}_n$ be a sequence of pairwise disjoint subsets. Then, $F_n = \bigcup_{k=n}^{+\infty} E_k \searrow \emptyset$ as $n \nearrow +\infty$ and, hence, $\{F_n\}_n$ is μ-cauchy. Therefore, we can get

$$\lim_n \mu(E_n) = \lim_n \mu(F_n - F_{n+1}) = \lim_n \mu(F_n \triangle F_{n+1}) = 0.$$

The theorem is now proved. □

Similarly, we can prove the following theorem.

Theorem 2 *μ is exhaustive if and only if, for any $\epsilon > 0$ and $\{E_n\}_n$, there exists n_0 such that*

$$\mu\left(E_n - \bigcup_{k=1}^{n_0} E_k\right) < \epsilon, \quad (\forall n \geq n_0).$$

Theorem 3 *Let μ be a LSC-fuzzy measure. Then, μ is exhaustive if and only if it is continuous from above at $\emptyset$.*

Proof. The 'if' part is trivial by theorem 2.

Let μ be exhaustive and $\{E_n\}_n$ a decreasing sequence with an empty intersection. Then, it is μ-cauchy and, hence, for any $\epsilon > 0$, there exists n_0 such that

$$\mu(E_n \triangle E_m) < \epsilon \quad (\forall n \wedge m \geq n_0).$$

Thus, we have

$$\mu(E_n) \leq \epsilon, \quad (\forall n \geq n_0).$$

i.e., the 'only if' part is true. □

4 Property (p∘g∘p)

In this section, we further suppose that μ is a LSC-fuzzy measure on $(X, \mathcal{A})$.

Definition 4 *μ is said to have the property (p.g.p) if, for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\mu(A) \vee \mu(B) \leq \delta \Rightarrow \mu(A \cup B) \leq \epsilon \quad [2].$$

Theorem 4 *μ has the property (p.g.p) if and only if, for any $\{E_n\}_n$ and $\{F_n\}_n$ and $\lim_n \mu(E_n) \vee \mu(F_n) = 0$, there exist subsequencs $\{E_{n(k)}\}_k$ and $\{F_{n(k)}\}_k$ such that $\lim_k \mu(E_{n(k)} \cup F_{n(k)}) = 0$.*

Proof. It is obvious by Definition 4. □

Corollary 1 *If μ is autocontinuous from above, then it has the property (p.g.p).*

Proof. It is trivial by Theorem 3 and the autocontinuity from above of μ. □

Theorem 5 *Let μ have the property* (p.g.p.) *and $lim_n \mu(E_n) = 0$. Then, there exist a subsequence $\{E_{n(k)}\}_k$ of $\{E_n\}_n$ and $\delta_k \searrow 0$ such that*

$$\mu\left(\bigcup_{i=k+1}^{+\infty} E_{n(i)}\right) \leq \delta_k, \quad (\forall k \geq 1).$$

Proof. Since μ has the property (p.g.p), there exists $\delta_1 > 0$ such that

$$\mu(E) \vee \mu(F) \leq \delta_1 \Rightarrow \mu(E \cup F) \leq \frac{1}{2}.$$

For above δ_1, there exists $\delta_2 \in (0, \frac{1}{2^2} \wedge \delta_1)$ to satisfy that

$$\mu(E) \vee \mu(F) \leq \delta_2 \Rightarrow \mu(E \cup F) \leq \delta_1,$$

and, similarly, there exists $\delta_3 \in (0, \frac{1}{2^3} \wedge \delta_2)$ to satisfy that

$$\mu(E) \vee \mu(F) \leq \delta_3 \Rightarrow \mu(E \cup F) \leq \delta_2.$$

Repeating this procedure, we can get a sequence $\{\delta_k\}_k$ such that

$$0 < \delta_k < \frac{1}{2^k} \wedge \delta_{k-1}.$$

Since $\lim_n \mu(E_n) = 0$, there exists a subsequence $\{E_{n(k)}\}_k$ to satisfy that

$$\mu(E_{n(k)}) \leq \delta_k, \quad (\forall k \geq 1).$$

Therefore, we have

$$\mu\left(\bigcup_{i=k+1}^{r+1} E_{n(i)}\right) \leq \delta_r, \quad (\forall r, k \geq 2),$$

it follows that the subsequence $\{E_{n(i)}\}_i$ satisfies the requirement of the theorem. □

Definition 5 *μ is said to have the property (s) if* $\lim_n \mu(E_n) = 0$ *implies there exists a subsequence $\{E_{n(k)}\}_k$ such that* $\overline{\lim}_k \mu(E_{n(k)}) = 0$.

As a direct result of Theorem 5, we have the following corollary.

Corollary 2 *If μ has the property* (p.g.p.), *then it has the property* (s).

Theorem 6 *Let μ be exhaustive and null-additive. If μ has the property* (s), *then it has the property* (p.g.p).

Proof. Suppose thst μ has not the property (p.g.p.). Then, there exist $\epsilon_0 > 0$ and sequences $\{E_n\}_n$ and $\{F_n\}_n$ such that

$$\mu(E_n) \vee \mu(F_n) \to 0 \quad \text{while} \quad \mu(E_n \cup F_n) \geq \epsilon_0, \quad (\forall n \geq 1).$$

By using the property (s) of μ, there exist subsequences $\{E_{n(k)}\}_k$ and $\{F_{n(k)}\}_k$ such that $\mu(\overline{\lim}_k E_{n(k)}) = 0$ and $\mu(\overline{\lim}_k F_{n(i)}) = 0$. Thus, we have $\mu(\overline{\lim}_k (E_{n(k)} \cup F_{n(k)})) = 0$ and

$$\overline{\lim}_k \mu(E_{n(k)} \cup F_{n(k)}) \leq \lim_k \mu\left(\bigcup_{i=k}^{+\infty}(E_{n(i)} \cup F_{n(i)})\right) = 0$$

by exhaustivity and null-additivity of μ. This is in contradiction with the fact that

$$\mu(A_{n(k)} \cup B_{n(k)}) \geq \epsilon_0, \quad (\forall k \geq 1).$$

The theorem is now proved. □

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