Laurent Series

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AEM CHAPTER 16

Today's Outline

- Laurent Series
- Residue Integration Method

Wrap up

- A **Sequence** is a set of things (usually numbers) that are in order. $(z_1,z_2,z_3,z_4.....z_n)$
- A **series** is a sum of a sequence of terms. $(s_1=z_1, s_2=z_1+z_2, s_3=z_1+z_2+z_3, \ldots, s_n)$
- A **convergent sequence** is one that has a limit c.

$$\lim_{n\to\infty} z_n = c$$

- A **convergent series** is one whose sequence of partial sums converges.

$$\lim_{n\to\infty} s_n = s.$$

$$s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

Wrap up

- Taylor series:

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

or by (1), Sec 14.4

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^*.$$

Wrap up

- A Maclaurin series is a Taylor series expansion of a function about zero.

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \ldots$$

Wrap up

- Importance special Taylor's series (Sec. 15.4)
  - Geometric series

$$\text{(11)} \qquad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

  - Exponential function

$$\text{(12)} \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}$$

Wrap up

- Importance special Taylor's series (Sec. 15.4)
  - Trigonometric and hyperbolic function

$$\textbf{(14)}\quad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

$$\textbf{(15)}\quad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Wrap up

- Importance special Taylor's series (Sec. 15.4)
  - Logarithmic

$$\text{(16)} \quad \operatorname{Ln}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots$$

$$\text{(17)} \quad -\operatorname{Ln}(1-z) = \operatorname{Ln}\frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

$$\text{(18)} \quad \operatorname{Ln}\frac{1+z}{1-z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right)$$

Lauren Series

- Deret Laurent adalah generalisasi dari deret Taylor.
- Pada deret Laurent terdapat pangkat negatif yang tidak dimiliki pada deret Taylor.

(1)

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

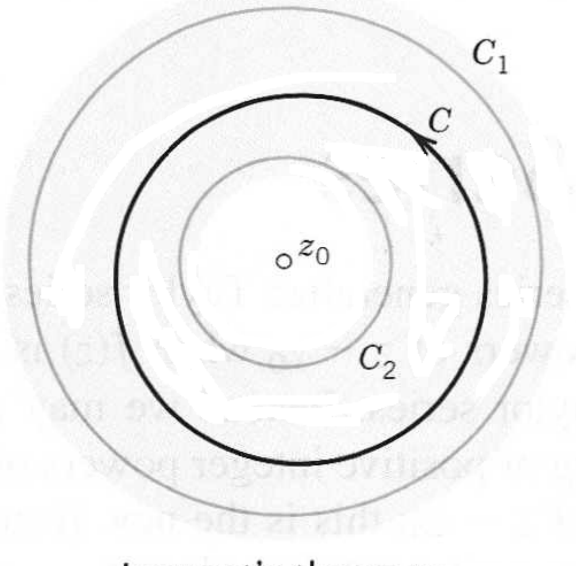
$$\cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

(2)

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}\, dz^*, \quad b_n = \frac{1}{2\pi i} \oint_C (z^*-z_0)^{n-1} f(z^*)\, dz^*,$$

Laurent's theorem

- *Let f(z) be analytic in a domain containing two concentric circles C1 and C2 with center Zo and the annulus between them . Then f(z) can be represented by the Laurent series*



Laurent's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$(1') \qquad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Laurent's theorem

- Semua koefisien dapat disajikan menjadi satu bentuk integral

$$(2') \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^* \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Example 1

Example 1:

Find the Laurent series of $z^{-5}\sin z$ with center 0.

Solution.

menggunakan (14) Sec.15.4 kita dapatkan.

$$z^{-5}\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040} z^2 + - \cdots$$

$$(|z| > 0).$$

Substitution

Example 2

Find the Laurent series of $z^2 e^{1/z}$ with center 0.

Solution

From (12) Sec. 15.4 with z replaced by 1/z we obtain Laurent series whose principal part is an infinite series.

$$z^2 e^{1/z} = z^2 \left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots$$

$(|z| > 0)$.

Development of 1/(1-z)

Example 3

Develop 1/(1-z)

(a) in nonnegative powers of z

(b) in negative powers of z

Solution.

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{(valid if } |z| < 1\text{).}$$

(b) $$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

$$= -\frac{1}{z} - \frac{1}{z^2} - \cdots \qquad \text{(valid if } |z| > 1\text{).}$$

Laurent Expantion in Different Concentric Annuli

Example 4

Find all Laurent series of $1/(z^3 - z^4)$ with center 0.

Solution.

Multiplying by $1/z^3$, we get from Example 3.

$$\text{(I)} \qquad \frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1)$$

$$\text{(II)} \qquad \frac{1}{z^3 - z^4} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots \qquad (|z| > 1).$$

Use of Partial Fraction

Example 5

Find all Taylor and Laurent series of $f(z) = \dfrac{-2z + 3}{z^2 - 3z + 2}$ with center 0.

Solution.

In terms of partial fractions,

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}.$$

- For first fraction

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{(valid if } |z| < 1).$$

(b) $$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

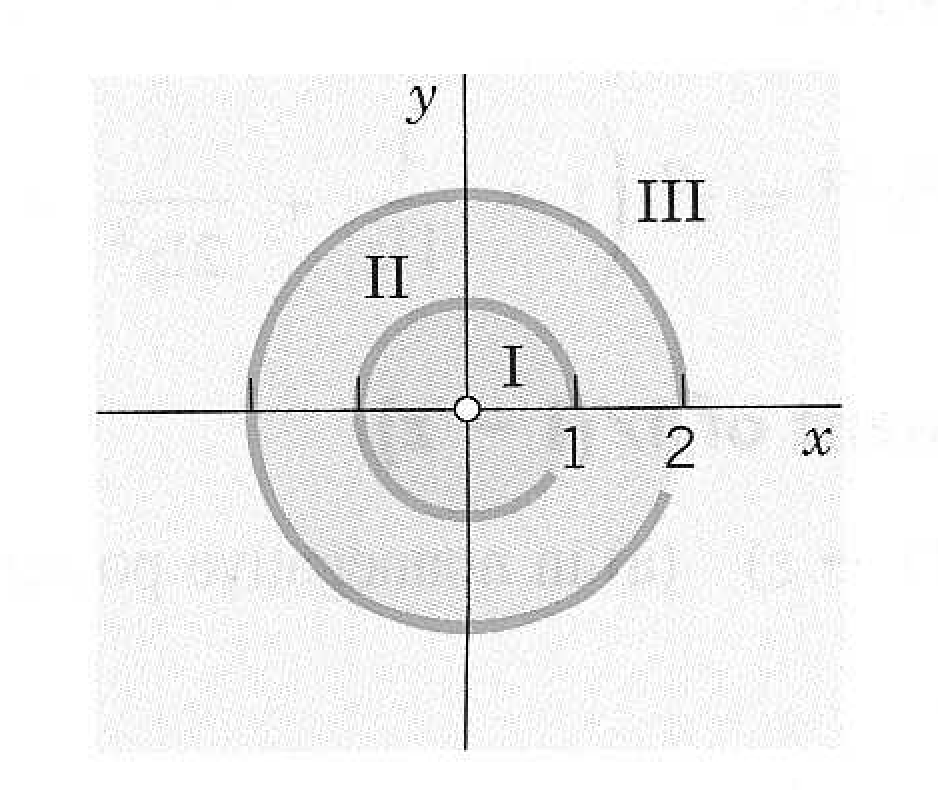
$$= -\frac{1}{z} - \frac{1}{z^2} - \cdots \qquad \text{(valid if } |z| > 1).$$

- For second fraction,

$$\text{(c)} \quad -\frac{1}{z-2} = \frac{1}{2\left(1 - \frac{1}{2}z\right)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \qquad (|z| < 2),$$

$$\text{(d)} \quad -\frac{1}{z-2} = -\frac{1}{z\left(1 - \frac{2}{z}\right)} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \qquad (|z| > 2).$$

- Regions of convergence in Example 5



(I) From (a) and (c), valid for $|z| < 1$

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n = \frac{3}{2} + \frac{5}{4} z + \frac{9}{8} z^2 + \cdots.$$

(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4} z + \frac{1}{8} z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots.$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots.$$

Residue Integration Method

- The purpose of Cauchy's residue integration method is the evaluation of integrals

$$\oint_C f(z)\,dz$$

 taken around a simple close path C.

- If *f (z) is analytic everywhere on C and inside C, such an integral is zero by Cauchy's* integral theorem (Sec. 14.2), and we are done.

- If *f(z) has a singularity at a point z* = z_o *inside* C, *but is otherwise analytic on C and* inside C, then *f(z) has a Laurent series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

that converges for all points near *z* = z_o *(except at z* = z_o *itself) , in some domain of the* form $0<|z - z_o|<R$.

- The coefficient b_1 of the first negative power *$1/(z - z_o)$ of this Laurent series is given by the integral formula (2) in Sec. 16.1* with *$n = 1$, namely,*

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

$$\textbf{(1)} \quad \oint_C f(z)\, dz = 2\pi i b_1.$$

- The coefficient *b1 is called the* ***residue*** *of f(z) at $z=z_o$ and we denote it by*

$$(2) \qquad b_1 = \operatorname*{Res}_{z=z_0} f(z).$$

Evaluation of an Integral by Means of a Residue

Example 1

Integrate the function f(z) = z^{-4} sin *z* counterclockwise around the unit circle C.

Solution.

From (14) in Sec. 15.4 we obtain the Laurent series

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + - \cdots$$

which converges for |z|>0 (that is, for all *z* |=0). *This series shows that f(z) has a pole of third order at z = 0* and the residue b_1= 1/3!. From (l) we thus obtain the answer

$$\oint_C \frac{\sin z}{z^4}\, dz = 2\pi i b_1 = -\frac{\pi i}{3}.$$

Use the Right Laurent series

Example 2

Integrate *f(z)* = $1/(z^3 - z^4)$ *clockwise around the circle C:* $|z| = 1/2$.

Solution.

$z^3 - z^4 = z^3(1 - z)$ *shows that f(z) is singular at* $z = 0$ *and* $z = 1$. *Now* $z = 1$ *lies outside C.* Hence it is of no interest here. So we need the residue of *f(z) at 0. We find it from the Laurent series that* converges for $0<|z|<1$. This is series (I) in Example 4, Sec. 16.1,

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \quad (0 < |z| < 1).$$

we see from it that this residue is 1. Clockwise integration this yields

$$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i.$$

Formula for Residue

- **Simple Poles.** Two formulas for the residue of *f(z)* *at a simple pole at* z_o *are*

$$\text{(3)} \quad \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z)$$

and, assuming that *f(z) = p(z)/q(z),* $p(z_o) \neq 0$*, and q(z) has a simple zero at* z_o *(so that f(z) has at Zo a simple pole, by Theorem 4 in Sec. 16.2),*

$$\text{(4)} \quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

Residue at a Simple pole

Example 3

f(z) = (9z + i)/(z³ + z)

has a simple pole at i because z² + 1 = (z + i)(z - i), and (3) gives the residue

$$\operatorname*{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \lim_{z\to i} (z-i) \frac{9z+i}{z(z+i)(z-i)}$$

$$= \left[\frac{9z+i}{z(z+i)}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

By (4) with $p(i) = 9i + i$ *and* $q'(z) = 3z^2 + 1$ *we confirm the result,*

$$\operatorname*{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \left[\frac{9z+i}{3z^2+1}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

Formula for Residue

- Poles of Any Order. The residue of *f(z) at an mth-order pole at* z_o *is*

$$(5) \quad \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right\}$$

In particular, for a second-order pole *(m = 2)*,

$$(5^*) \quad \operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \{[(z - z_0)^2 f(z)]'\}$$

Residue at a Pole of Higher Order

Example 4

$f(z) = 50z/(z^3 + 2z^2 - 7z + 4)$

has a pole of second order at $z = 1$ *because the denominator equals* $(z+4)(z-1)^2$.

From (5) we obtain the residue*

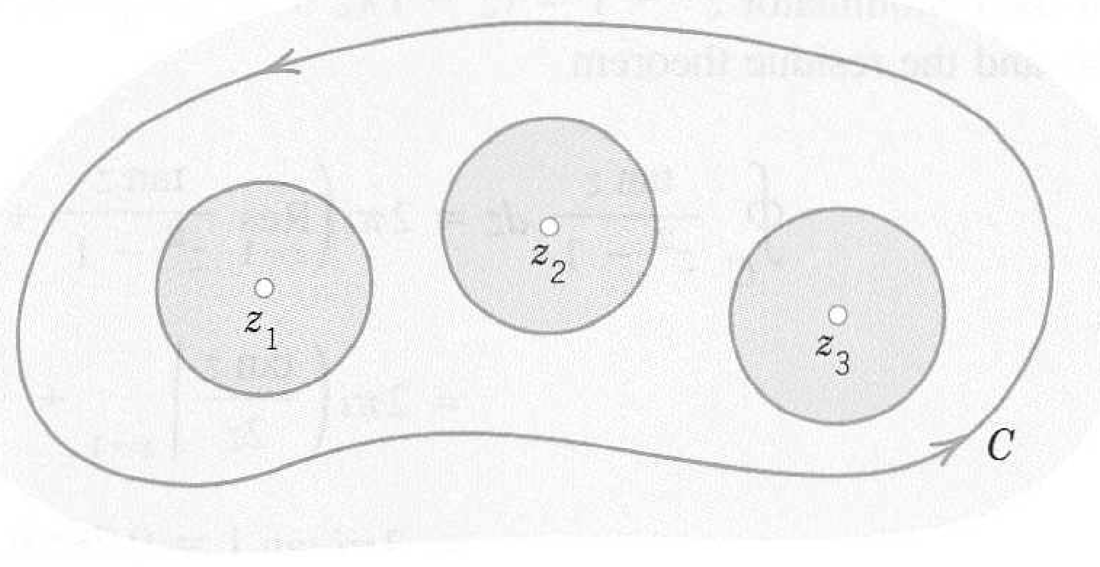
$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left(\frac{50z}{z+4}\right)$$

$$= \frac{200}{5^2} = 8.$$

Several Singularities Inside the Contour

- Residue integration can be extended from the case of a single singularity to the case of several singularities within the contour C.



Residue theorem

- **Residue Theorem**

  Let f(z) be analytic inside a simple closed path C and on C, except for finitely many singular points z_1 , z_2, … , z_k inside C. Then the integral of f(z) taken counterclockwise around C equals 2πi times the sum of the residues of f(z) at z_1, … , z_k:

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z). \tag{6}$$

Example 5

Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside, (d) 0 and I are outside.

$$\oint_C \frac{4 - 3z}{z^2 - z} \, dz$$

Solution

The integrand has simple poles at 0 and 1, with residues [by (3)]

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4,$$

$$\operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

(a) $2\pi i(-4+1) = -6\pi i$, (b) $-8\pi i$, (c) $2\pi i$, (d) 0.

Tugas

PROBLEM SET 16.1

1–6 LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the given function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. (Show the details of your work.)

1. $\dfrac{1}{z^4 - z^5}$ **2.** $z \cos \dfrac{1}{z}$

7–14 LAURENT SERIES NEAR A SINGULARITY AT z_0

Expand the given function in a Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. (Show details.)

7. $\dfrac{e^z}{z - 1}, \quad z_0 = 1$ **8.** $\dfrac{\sin z}{(z - \frac{1}{4}\pi)^3}, \quad z_0 = \frac{1}{4}\pi$

15–23 TAYLOR AND LAURENT SERIES

Find all Taylor and Laurent series with center $z = z_0$ and determine the precise regions of convergence.

15. $\dfrac{1}{1 - z^3}, \quad z_0 = 0$ **16.** $\dfrac{1}{1 - z^2}, \quad z_0 = 1$

17. $\dfrac{z^2}{1 - z^4}, \quad z_0 = 0$ **18.** $\dfrac{1}{z}, \quad z_0 = 1$

Tugas

PROBLEM SET 16.3

3–12 RESIDUES

Find all the singular points and the corresponding residues. (Show the details of your work.)

3. $\dfrac{1}{4 + z^2}$

4. $\dfrac{\cos z}{z^6}$

14–25 RESIDUE INTEGRATION

Evaluate (counterclockwise). (Show the details.)

14. $\oint_C \dfrac{\sin \pi z}{z^4}\, dz, \quad C\colon |z - i| = 2$

15. $\oint_C e^{1/z}\, dz, \quad C\colon |z| = 1$

16. $\oint_C \dfrac{dz}{\sinh \frac{1}{2}\pi z}, \quad C\colon |z - 1| = 1.4$

17. $\oint_C \tan \pi z\, dz, \quad C\colon |z| = 1$

18. $\oint_C \tan \pi z\, dz, \quad C\colon |z| = 2$

Thanks