



## A GENERALIZATION OF NIL-CLEAN RINGS

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*Abstract.* The conditions that allow an element of an associative, unital, not necessarily commutative ring  $R$ , to be represented as a sum of (commuting) idempotents and one nilpotent element are analyzed. Some applications to group rings are also presented.

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### 1. INTRODUCTION

An element  $a$  in an associative unital ring  $R$  is called clean if it can be represented as a sum  $a = e + u$ , where  $e$  is an idempotent element and  $u$  is a unit. This notion was introduced by Nicholson in [8]. If one can find such elements  $e$  and  $u$  such that  $a = e + u$  and  $eu = ue$ , the element  $a$  is called strongly clean. The ring  $R$  itself is called (strongly) clean if every element in  $R$  is (strongly) clean.

Many families of clean rings were investigated in previous decades. In recent years, a particular attention has been paid to the nil-clean rings and its relatives. A nil-clean ring (see [5]) is a ring in which every element is nil-clean, which means that every element can be written as a sum of an idempotent element and a nilpotent one. Analogously, we have a notion of strongly nil-clean elements (and rings). For some of the results concerning this class of rings and some of the related classes of rings, the reader may wish to consult also [1, 3, 7, 10].

A class of strongly 2-nil-clean rings was introduced in [4]. Namely, an element  $a$  in a ring  $R$  is called strongly 2-nil-clean if it can be represented in the form  $a = e + f + n$ , where  $e$  and  $f$  are idempotents,  $n$  is a nilpotent element and they all commute with each other.

In this paper we analyze elements of a ring which can be written as a sum of finitely many idempotents and one nilpotent element which are pairwise commutative. If the number of idempotents which appear in this sum is  $s$ , we call these elements strongly

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$s$ -nil-clean. It turns out that if every element in a ring is strongly  $s$ -nil-clean for some  $s$ , this ring has finite characteristic and every element in this ring is strongly  $(p-1)$ -nil-clean, where  $p$  is the largest prime dividing the characteristic of this ring (see Theorem 1). These rings are naturally called strongly  $(p-1)$ -nil-clean rings and they are all strongly clean (see Corollary 3 and the discussion preceding it). There are many examples of strongly  $(p-1)$ -nil-clean rings. Theorem 2 shows that in every commutative ring  $R$  of finite characteristic  $k$ , elements which are strongly  $s$ -nil-clean for some  $s$ , form a subring which is  $(p-1)$ -nil clean (where  $p$  is the largest prime dividing  $k$ ) and if this ring contains idempotents or nilpotents not belonging to  $\mathbb{Z}_k$  (which is necessarily contained in  $R$ ) we have a non-trivial example of such a ring. Proposition 5 provides examples of finite commutative local rings which are  $(p-1)$ -nil-clean.

The plan of the paper is as follows. In Section 2, we analyze sums of idempotents and one nilpotent element and derive our main criteria for strongly  $(p-1)$ -nil-clean elements in a ring. Section 3 deals with some structure theorems. It is shown that in analyzing strongly  $(p-1)$ -nil-clean rings, we may reduce this analysis to the case when  $p$  is a nilpotent element in a ring under investigation. We also show in this section that strongly  $(p-1)$ -nil-clean rings are strongly  $\pi$ -regular and, consequently, strongly clean.

Section 4 deals with the investigation of group rings  $RG$  where  $R$  is a  $(p-1)$ -nil-clean commutative ring and  $G$  is a group. For example, we show that, for the ring  $RG$  to be strongly  $(p-1)$ -nil-clean, when the characteristic of  $R$  is of the form  $p^s$ , for a prime integer  $p$ , it is necessary that the order of any element of a group  $G$  is of the form  $dp^k$ , for some  $d \mid (p-1)$  and  $k \geq 0$  (see Lemma 3). In the case of a commutative group  $G$  this condition is also sufficient (see Theorem 4).

We emphasize that we work in associative, unital rings which need not be commutative. When a ring is commutative, we drop the adjective “strongly” since it is unnecessary. We use the same symbol  $k$  to denote the integer  $k$  and to denote the ring element  $k1_R$ . It will always be clear what we mean. Finally, we denote the field with  $p$  elements by  $\mathbb{Z}_p$ , the Jacobson radical by  $J(R)$  and the set of nilpotent elements by  $N(R)$ .

## 2. BASIC RESULTS

A ring in which every element is a sum of certain number of idempotents and one nilpotent element, that commute with each other, is a generalization of strongly nil-clean rings and strongly 2-nil-clean rings. In view of this, we introduce the following definition.

**Definition 1.** An element  $a$  of a ring  $R$  is  $s$ -nil-clean if it can be written in the following form:

$$a = e_1 + \cdots + e_s + n, \quad (2.1)$$

where elements  $e_1, \dots, e_s$  are idempotents and  $n$  is nilpotent. If an element  $a$  can be written in the form (2.1) so that elements in this sum are pairwise commutative, we say that this element is strongly  $s$ -nil-clean. If every element in  $R$  is (strongly)  $s$ -nil clean, we say that  $R$  is a (strongly)  $s$ -nil-clean ring.

Of course, if  $a$  is (strongly)  $s$ -nil-clean and  $s < t$ ,  $a$  is also  $t$ -nil-clean — we simply add  $t - s$  zeroes to the presentation of  $a$  as a (strongly)  $s$ -nil-clean element.

*Remark 1.* It is clear that, if  $f: R \rightarrow S$  is a ring homomorphism and an element  $a \in R$  is (strongly)  $s$ -nil-clean, then  $f(a) \in S$  is (strongly)  $s$ -nil-clean. Similarly, an element  $a = (a_1, \dots, a_l) \in R_1 \times \dots \times R_l$  is (strongly)  $s$ -nil-clean iff  $a_i$  is (strongly)  $s$ -nil-clean for all  $i$ . So, homomorphic images and finite direct products of (strongly)  $s$ -nil-clean rings are itself (strongly)  $s$ -nil-clean. Also, a subring of a strongly  $s$ -nil-clean ring has the same property, as we shall see later. However, this does not hold for  $s$ -nil-clean rings. Namely, if we have a ring which is  $s$ -nil-clean, but it is not strongly  $s$ -nil-clean, it is enough to take an element  $a$  which is not strongly  $s$ -nil-clean and look at the subring generated by this element. This subring is commutative, so it cannot be  $s$ -nil-clean – if it were, this element would also be strongly  $s$ -nil-clean.

We begin our analysis with a useful result concerning sum of several idempotents and one nilpotent element.

**Proposition 1.** *Let  $R$  be a ring and suppose that element  $a \in R$  is strongly  $s$ -nil clean. Then  $a(a - 1) \cdots (a - s)$  is nilpotent.*

*Proof.* Let  $a = e_1 + e_2 + \dots + e_s + n$ , where  $e_1, e_2, \dots, e_s$  are idempotents and  $n$  is nilpotent that commute with each other. Observe that

$$1 = ((1 - e_1) + e_1)((1 - e_2) + e_2) \cdots ((1 - e_s) + e_s).$$

After multiplication, we get a sum of products of the form

$$e_{i_1} \cdots e_{i_k} (1 - e_{j_1}) \cdots (1 - e_{j_{s-k}}),$$

where  $1 \leq k \leq s$ ,  $i_1 < \dots < i_k$ ,  $j_1 < \dots < j_{s-k}$  and  $\{i_1, \dots, i_k, j_1, \dots, j_{s-k}\} = \{1, \dots, s\}$ . Next, we get that

$$\begin{aligned} & (k - a)e_{i_1} \cdots e_{i_k} (1 - e_{j_1}) \cdots (1 - e_{j_{s-k}}) \\ &= ((1 - e_{i_1}) + \dots + (1 - e_{i_k}) - e_{j_1} - \dots - e_{j_{s-k}} - n)e_{i_1} \cdots e_{i_k} (1 - e_{j_1}) \cdots (1 - e_{j_{s-k}}) \\ &= -ne_{i_1} \cdots e_{i_k} (1 - e_{j_1}) \cdots (1 - e_{j_{s-k}}). \end{aligned}$$

This follows from the fact that  $(1 - e)e = e - e^2 = 0$ , for an idempotent  $e$ . Since  $n$  is nilpotent, so is this product. Thus, when 1 is multiplied by  $a(a - 1) \cdots (a - s)$ , we get a sum of nilpotent elements that commute with each other. Therefore,  $a(a - 1) \cdots (a - s)$  is nilpotent.  $\square$

The following corollary is simple, but important.

**Corollary 1.** *a) If a ring  $R$  is such that  $-1$  is strongly  $s$ -nil-clean for some  $s \geq 1$ , then this ring has finite characteristic.*

*b) If  $\text{char}(R) = k$ , then  $-1$  is strongly  $(p - 1)$ -nil-clean, where  $p$  is the largest prime dividing  $k$ .*

*Proof.* a) From the previous proposition we conclude that  $(-1)(-2)\cdots(-(s + 1)) = (-1)^{s+1}(s + 1)!$  is nilpotent, so  $((s + 1)!)^m = 0$  for some  $m \geq 1$  and the characteristic of the ring  $R$  is not 0.

b) It is enough to show that  $-1$  is  $(p - 1)$ -nil-clean in the ring  $\mathbb{Z}_k$ , which is contained in  $R$ . If  $k = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$  is the prime factorization of  $k$ , where  $p_1 < \cdots < p_l = p$ , then  $\mathbb{Z}_k \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_l^{\alpha_l}}$ . Since  $-1 \mapsto (-1, \dots, -1)$  under this isomorphism, this reduces the proof to show that  $-1$  is  $(p - 1)$ -nil-clean in  $\mathbb{Z}_{p_i^{\alpha_i}}$ . This is clear since  $-1 = p_i^{\alpha_i} - 1 = \underbrace{1 + \cdots + 1}_{p_i - 1} + p_i(p_i^{\alpha_i - 1} - 1)$ , and  $p_i(p_i^{\alpha_i - 1} - 1)$  is nilpotent in  $\mathbb{Z}_{p_i^{\alpha_i}}$ . □

*Example 1.* The ring  $\mathbb{Z}_k$  is (strongly)  $(p - 1)$ -nil-clean, where  $p$  is the largest prime integer dividing  $k$ . Namely, using the notation from the previous corollary, this reduces to show that  $\mathbb{Z}_{p_i^{\alpha_i}}$  is  $(p - 1)$ -nil-clean. Since any element in  $\mathbb{Z}_{p_i^{\alpha_i}}$  can be written in the form  $\underbrace{1 + \cdots + 1}_s + p_i t$ , for some  $s \in \{0, \dots, p_i - 1\}$  and  $t \in \{0, \dots, p_i^{\alpha_i - 1} - 1\}$ , and since  $p_i$  is nilpotent in  $\mathbb{Z}_{p_i^{\alpha_i}}$ , we are done.

**Lemma 1.** *If  $k = \text{char}(R) = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$  is the prime factorization of the characteristic of the ring  $R$ , then  $R \cong R_1 \times \cdots \times R_l$ , where  $R_i = R/p_i^{\alpha_i} R$ . In particular,  $\text{char}(R_i) = p_i^{\alpha_i}$ .*

*Proof.* This follows easily from the Chinese remainder theorem taking into account the fact that elements  $p_i^{\alpha_i}$  are central. □

Since the products of the form  $a(a - 1)\cdots(a - s)$  are important for our investigation, we introduce the symbol  $(a)_k := a(a - 1)\cdots(a - (k - 1))$  (falling factorial, as is known in combinatorics), where  $k$  is a positive integer.

We have the following corollary.

**Corollary 2.** *Let  $R$  be a ring. Suppose that element  $a \in R$  is strongly  $s$ -nil clean and  $k (< s)$  is nilpotent. Then  $(a)_k$  is a nilpotent element.*

*Proof.* Clearly

$$(a)_s = a(a - 1)\cdots(a - (s - 1)) = a^{t_0}(a - 1)^{t_1}\cdots(a - (k - 1))^{t_{k-1}} + kq(a),$$

for some non-negative integers  $t_i$ , such that  $\sum_{i=0}^{k-1} t_i = s$  and polynomial  $q(X) \in \mathbb{Z}[X]$ . Taking into account that  $k$  is nilpotent, the result follows. □

In order to see what the fact that  $(a)_s$  is nilpotent implies, we start with a simple lemma.

**Lemma 2.** *Let  $p$  be a prime integer and  $r, m \geq 1$  arbitrary positive integers. Then the element  $x$  in the ring  $\mathbb{Z}_{p^r}[X]/\langle X^m(X-1)^m \cdots (X-(p-1))^m \rangle$  is (strongly)  $(p-1)$ -nil-clean, where  $x$  is the class of  $X$  in this quotient ring.*

*Proof.* Ideals  $\langle X-i \rangle$  and  $\langle X-j \rangle$  are coprime in  $\mathbb{Z}_{p^r}[X]$ , for all  $0 \leq i < j < p$ , since  $(X-i) - (X-j) = j-i$  and  $j-i$  is invertible in the ring  $\mathbb{Z}_{p^r}[X]$ . So,  $\langle (X-i)^m \rangle, \langle (X-j)^m \rangle$  are coprime as well. By applying the Chinese remainder theorem we obtain the isomorphism

$$\mathbb{Z}_{p^r}[X]/\langle X^m \cdots (X-(p-1))^m \rangle \cong \mathbb{Z}_{p^r}[X]/\langle X^m \rangle \times \cdots \times \mathbb{Z}_{p^r}[X]/\langle (X-(p-1))^m \rangle,$$

such that  $x \mapsto (x, \dots, x)$ . Thus, it is enough to show that  $x$  has the desired presentation in all factors and since  $x = \underbrace{1 + \cdots + 1}_i + (x-i)$  in the factor  $\mathbb{Z}_{p^r}[X]/\langle (X-i)^m \rangle$ ,

this is true. □

**Proposition 2.** *Let  $R$  be a ring. Suppose that the element  $p$  is nilpotent, where  $p$  is a prime integer, and let  $a \in R$  be such that  $(a)_p$  is nilpotent. Then  $a$  is strongly  $(p-1)$ -nil-clean.*

*Proof.* Consider the homomorphism  $f: \mathbb{Z}[X] \rightarrow R$ , given by  $f(X) = a$ . An immediate consequence of the fact that  $(a(a-1) \cdots (a-(p-1)))^m = 0$  and that  $p^r = 0$  in  $R$ , for some  $m, r \geq 1$ , is the existence of an induced homomorphism

$$\bar{f}: \mathbb{Z}_{p^r}[X]/\langle X^m(X-1)^m \cdots (X-(p-1))^m \rangle \rightarrow R, \text{ such that } x \mapsto a.$$

Since  $x$  is strongly  $(p-1)$ -nil-clean, so is its image  $a$ . □

**Proposition 3.** *Let  $R$  be a ring of characteristic  $k(> 0)$ . If  $p$  is the largest prime dividing  $k$  and  $a \in R$  is such that  $(a)_s$  is nilpotent for some  $s \geq 1$ , then  $a$  is strongly  $(p-1)$ -nil-clean.*

*Proof.* Under isomorphism  $R \cong R_1 \times \cdots \times R_l$ , where  $R_i = R/p_i^{\alpha_i} R$  and  $k = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ , implied by Lemma 1, the element  $a$  goes to  $(a_1, \dots, a_l)$ . Also,  $a$  is strongly  $(p-1)$ -nil-clean iff  $a_i$  is such for all  $i$ . However, from the fact that  $(a)_s$  is nilpotent, it follows that  $(a_i)_s \in R_i$  is nilpotent for all  $i$ . From this, it easily follows that  $(a_i)_{p_i}$  is also nilpotent for all  $i$ . Namely, if  $s < p_i$ , this follows since  $(a_i)_{p_i} = (a_i)_s(a_i-s)_{p_i-s}$  and if  $s > p_i$ , it follows from Corollary 2. Since  $p_i$  is nilpotent in  $R_i$ , Proposition 2 gives that  $a_i \in R_i$  is strongly  $(p_i-1)$ -nil-clean for all  $i$ . From the fact that  $p_i \leq p$  for all  $i$ , it follows that these elements  $a_i$  are all  $(p-1)$ -nil-clean and so is  $a$ . □

**Theorem 1.** *Let  $R$  be a ring such that every element  $a \in R$  is strongly  $s$ -nil-clean for some  $s$ . Then  $R$  has finite characteristic and  $R$  is strongly  $(p-1)$ -nil-clean, where  $p$  is the largest prime dividing  $\text{char}(R)$ .*

*Proof.* Corollary 1 shows that  $R$  has finite characteristic and from Proposition 1 and Proposition 3 it follows that every element is strongly  $(p-1)$ -nil-clean, where  $p$  is the largest prime dividing this characteristic.  $\square$

Theorem 1 shows that one needs only to investigate strongly  $(p-1)$ -nil-clean rings, where  $p$  is a prime integer. For example, the class of all strongly 3-nil-clean rings is the same as the class of strongly 2-nil-clean rings, and the class of strongly 9-nil-clean-rings is the same as the class of strongly 6-nil-clean rings. Namely, if a ring is, say, strongly 9-nil-clean, then  $(a)_{10}$  is nilpotent for all  $a \in R$ . So, this is true for  $a = 10$ . Consequently,  $10! = (10)_{10}$  is nilpotent, so  $\text{char}(R) \mid (10!)^m$ , for some  $m \geq 1$ . We conclude that the largest prime dividing  $\text{char}(R)$  is at most 7 (it may be smaller), so our ring is strongly 6-nil-clean.

**Proposition 4.** *A subring of a strongly  $(p-1)$ -nil-clean ring is also strongly  $(p-1)$ -nil-clean.*

*Proof.* Let  $S$  be a subring of a strongly  $(p-1)$ -nil-clean ring  $R$  and let  $a \in S$ . Since  $a \in R$ ,  $a$  is strongly  $(p-1)$ -nil-clean, and according to Proposition 1 element  $(a)_p$  is nilpotent.  $R$  is of finite characteristic, say  $k$ , which means that  $\text{char}(S) = k$ , with  $p$  being the largest prime dividing  $k$ . When we apply Proposition 3, we get that  $a$  is strongly  $(p-1)$ -nil-clean in  $S$ .  $\square$

**Theorem 2.** *Let  $R$  be a commutative ring of finite characteristic  $k$  and  $p$  the largest prime dividing  $k$ . Let  $S = \{a \in R : a \text{ is } s\text{-nil-clean for some } s\}$ . Then  $S$  is the largest subring of  $R$  which is  $(p-1)$ -nil-clean.*

*Proof.* Since  $k = 0$  in  $R$ , we have:  $-1 = \underbrace{1 + \dots + 1}_{k-1}$ , so  $-1$  is  $(k-1)$ -nil-clean and  $-1 \in S$ . Also, if  $a, b \in S$ , then  $a = e_1 + \dots + e_s + n$ ,  $b = f_1 + \dots + f_t + n'$  and we get  $ab = \sum_{i,j} e_i f_j + N$ , where  $N$  is nilpotent and all  $e_i f_j$  are idempotents ( $R$  is a commutative ring). So,  $ab \in S$ . Similarly,  $a + b = e_1 + \dots + e_s + f_1 + \dots + f_t + n + n' \in S$ . Finally, since  $a - b = a + (-1)b$  we conclude that  $a - b \in S$  as well. So,  $S$  is a ring in which every element is  $s$ -nil-clean for some  $s$ . From Theorem 1, we conclude that  $S$  is actually  $(p-1)$ -nil-clean ring. It is clear that  $S$  is the largest such subring.  $\square$

*Remark 2.* From the Theorem 2 it is clear that, in order to show that a commutative ring of finite characteristic  $k$  is  $(p-1)$ -nil-clean, it is enough to check only its ring generators over  $\mathbb{Z}_k$ . For example, the ring  $\mathbb{Z}_{p^r}[X]/\langle X^m(X-1)^m \dots (X-(p-1))^m \rangle$ , appearing in Lemma 2, is  $(p-1)$ -nil-clean, since it is generated by  $x$  and this element is  $(p-1)$ -nil-clean.

The following proposition provides us with a lot of examples of commutative  $(p-1)$ -nil-clean rings.

**Proposition 5.** *Let  $R$  be a finite commutative local ring,  $M$  its maximal ideal and  $R/M \cong \mathbb{Z}_p$ . Then  $R$  is  $(p - 1)$ -nil-clean.*

*Proof.* We know that every element in  $M$  is nilpotent. If  $x \in R$ , then  $x + M = s + M$ , for some  $s \in \{0, \dots, p - 1\}$ . So,  $x = \underbrace{1 + \dots + 1}_s + m$ , where  $m$  is nilpotent.  $\square$

*Remark 3.* In the case of non-commutative rings, the set of all elements which are strongly  $s$ -nil-clean for some  $s$  do not necessarily form a subring. Such examples will be given in Section 3 and Section 4. However, a simple application of Zorn’s lemma shows that there exist maximal subrings which are strongly  $(p - 1)$ -nil-clean.

The following proposition gives another characterization of strongly  $(p - 1)$ -nil-clean elements.

**Proposition 6.** *Let  $R$  be a ring of finite characteristic  $k$ ,  $p$  the largest prime dividing  $k$  and  $a \in R$ . The following conditions are equivalent.*

- (1)  $(a)_p$  is nilpotent.
- (2)  $a$  is strongly  $(p - 1)$ -nil-clean.
- (3)  $a = b + n$ , where  $b \in R$  is such that  $(b)_p = 0$ ,  $n$  is nilpotent and  $bn = nb$ .

*Proof.* (1)  $\implies$  (2). This is contained in Proposition 3.

(2)  $\implies$  (3). Let  $a = e_1 + \dots + e_{p-1} + n$  be a  $(p - 1)$ -nil-clean decomposition of  $a$ . Take  $b := e_1 + \dots + e_{p-1}$ . The proof of Proposition 1 shows that  $(b)_p = 0$ .

(3)  $\implies$  (1). Assume that  $a = b + n$ . So, we have

$$(a)_p = (b + n)((b - 1) + n) \cdots ((b - p + 1) + n).$$

Therefore, since  $n$  and  $b$  commute,  $(a)_p = (b)_p + nq(n, b) = nq(n, b)$ , for some polynomial  $q(X, Y) \in \mathbb{Z}[X, Y]$ . Since  $n$  is nilpotent, so is  $(a)_p$ .  $\square$

### 3. STRUCTURE THEOREMS

The purpose of this section is to discuss the structure of (strongly)  $(p - 1)$ -nil-clean rings, for prime number  $p$ .

The following proposition sums up the discussion from the previous section.

**Proposition 7.** *Suppose that  $\text{char}(R) = k = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ , where  $p_1 < \dots < p_l = p$ . Then  $R$  is strongly  $(p - 1)$ -nil-clean if and only if  $R_i$  is strongly  $(p_i - 1)$ -nil-clean, where  $R_i = R/p_i^{\alpha_i} R$  and  $1 \leq i \leq l$ .*

This shows that in investigation of strongly  $(p - 1)$ -nil-clean rings, for  $p$  prime, we can reduce our analysis to the case when  $p$  is nilpotent (equivalently, when the characteristic of the ring is a power of a prime).

Let us recall that a ring  $R$  is called strongly  $\pi$ -regular if for every element  $a \in R$  there exists  $n \geq 1$  and  $x \in R$  such that  $a^n = a^{n+1}x$ .

**Theorem 3.** *Every strongly  $(p - 1)$ -nil-clean ring is strongly  $\pi$ -regular.*

*Proof.* It is enough to consider the case when  $p$  is a nilpotent element. Then  $(p-1)!$  is invertible. Let  $a \in R$ . Since  $(a)_p$  is nilpotent, we have that  $((a)_p)^s = 0$  for some  $s$ . But

$$0 = ((a)_p)^s = (a(a-1)\cdots(a-(p-1)))^s = a^s((p-1)!)^s + a^{s+1}y,$$

for some  $y \in R$ . Since  $(p-1)!$  is invertible, we get that  $a^s = a^{s+1}x$ , for some  $x \in R$  and we are done.  $\square$

It is well known that strongly  $\pi$ -regular ring is strongly clean (see [2, Proposition 2.6], [9, Theorem 1], [5, Corollary 2.4]). Also, Jacobson radical of a strongly  $\pi$ -regular ring is nil and commutative strongly  $\pi$ -regular rings have Krull dimension 0 (see [2]). So, we have the following corollary.

**Corollary 3.** *Every strongly  $(p-1)$ -nil-clean ring is strongly clean.*

The following proposition is rather useful.

**Proposition 8.** *Let  $R$  be a ring,  $a \in R$  and let  $p$  be a nilpotent element, where  $p$  is prime. Then  $a^p - a$  is nilpotent if and only if  $(a)_p$  is nilpotent.*

*Proof.* It is well-known that  $X^p - X = (X)_p$  in  $\mathbb{Z}_p[X]$ . So,  $a^p - a - (a)_p = pr(a)$ , for some polynomial  $r(X) \in \mathbb{Z}[X]$ . From this fact, the proof follows immediately.  $\square$

For future reference, we formulate the following corollary which directly follows from Proposition 6 and Proposition 8.

**Corollary 4.** *Let  $R$  be a ring. If  $p$  is nilpotent, then  $R$  is a strongly  $(p-1)$ -nil-clean ring if and only if  $a^p - a$  is nilpotent for every  $a \in R$ .*

Let us proceed with some of the special properties of  $(p-1)$ -nil-clean rings.

**Proposition 9.** *Let  $R$  be a ring and let  $I$  be any nil ideal of  $R$ . Then  $R$  is  $(p-1)$ -nil-clean if and only if  $R/I$  is  $(p-1)$ -nil-clean.*

*Proof.* ( $\implies$ ) As observed before, this is trivial since  $R/I$  is a homomorphic image of  $R$ .

( $\impliedby$ ) Suppose that  $R/I$  is  $(p-1)$ -nil-clean. Let  $x \in R$ . Then  $x + I$  is  $(p-1)$ -nil-clean. Thus,  $x + I = (x_1 + I) + (x_2 + I) + \cdots + (x_{p-1} + I) + (y + I)$ , where  $x_i + I$  are idempotents,  $1 \leq i \leq p-1$ , and  $y + I$  is nilpotent. It is well known that idempotents lift modulo nil ideals (see, e.g. [6, Theorem 21.28]) so there are idempotents  $e_i$  such that  $x_i + I = e_i + I$ . So,  $x - e_1 - e_2 - \cdots - e_{p-1} - y \in I$ , i.e.,  $x = e_1 + e_2 + \cdots + e_{p-1} + y + n$ , for some  $n \in I$ . Element  $y + n$  is nilpotent. Indeed, since  $y + I$  is nilpotent,  $y^k \in I$  for some  $k \in \mathbb{N}$ . Every element different from  $y^k$  in the sum one gets in the expansion of  $(y + n)^k$ , is in  $I$  and we can conclude that  $(y + n)^k \in I$ , so  $y + n$  is nilpotent. Therefore  $R$  is  $(p-1)$ -nil-clean.  $\square$

An analogous result holds for the strongly  $(p-1)$ -nil-clean rings.



**Proposition 10.** *Let  $R$  be a ring and let  $I$  be any nil ideal of  $R$ . Then  $R$  is strongly  $(p - 1)$ -nil-clean if and only if  $R/I$  is strongly  $(p - 1)$ -nil-clean.*

*Proof.* ( $\implies$ ) Again, this is trivial since  $R/I$  is a homomorphic image of  $R$ .  
 ( $\impliedby$ ) Let  $a \in R$ . Since  $R/I$  is strongly  $(p - 1)$ -nil-clean, one has  $((a + I)_p)^k = I$ , for some  $k \in \mathbb{N}$ . Consequently,  $((a)_p)^k \in I$ . As  $I$  is nil ideal,  $((a)_p)^k \in N(R)$ . So,  $(a)_p$  is nilpotent. Since  $(p)_p$  is also nilpotent, the characteristic  $k$  of  $R$  is finite. The characteristic  $l$  of  $R/I$  has the property that  $p$  is the largest prime dividing this characteristic, but this also holds for  $k$ . Namely,  $l \in I$  and therefore  $l$  is nilpotent in  $R$ . So we have that  $k \mid l^s$  for some  $s$ , and also  $l \mid k$ . It follows that the sets of primes dividing  $k$  and  $l$  are the same. Now the result follows from Proposition 6.  $\square$

The following corollary follows directly from the fact that  $J(R)$  is nil for a strongly  $(p - 1)$ -nil-clean ring and Proposition 10.

**Corollary 5.** *A ring  $R$  is strongly  $(p - 1)$ -nil-clean if and only if  $J(R)$  is nil and  $R/J(R)$  is strongly  $(p - 1)$ -nil-clean.*

#### 4. GROUP RINGS

Let us recall the notion of a group ring. Let  $G$  be a group, written multiplicatively, and let  $R$  be a commutative ring. The group ring of  $G$  over  $R$ , denoted by  $RG$ , is a free  $R$ -module with generating set  $G$ , i.e.:

$$RG = \bigoplus_{g \in G} Rg.$$

So, elements of  $RG$  are formal finite sums of the form  $\sum_i r_i g_i$ , with  $r_i \in R$ ,  $g_i \in G$ , while the multiplication is implied by multiplication in  $G$ . The identity of this ring is  $1_R e$ , where  $1_R$  is the identity in  $R$  and  $e$  is the neutral element of  $G$ . We denote the identity simply by 1.

Our main interest here is focused on strongly  $(p - 1)$ -nil-clean group rings  $RG$ . It is obvious that if  $RG$  is strongly  $(p - 1)$ -nil-clean, so is  $R$ . Since we assume that the coefficient ring  $R$  is commutative, we refrain from using adjective “strongly” when referring to  $R$ , we use it only for  $RG$  when appropriate. We begin by discussing rings  $R$ , such that  $\text{char}(R)$  is a power of a prime.

**Lemma 3.** *Let  $R$  be a  $(p - 1)$ -nil-clean ring such that  $\text{char}(R) = p^s$ , for a prime  $p$  and some  $s \geq 1$  and let  $G$  be a group. For the list of conditions:*

- (1)  $RG$  is strongly  $(p - 1)$ -nil-clean;
- (2) For each  $g \in G$ , the element  $g^{p-1} - 1$  is nilpotent;
- (3) For each  $g \in G$ , there exists  $k \geq 0$  and  $d \mid (p - 1)$  such that  $\omega(g) = dp^k$ ,

*the following holds: (1)  $\implies$  (2) and (2)  $\impliedby$  (3). Here,  $\omega(g)$  denotes the order of  $g$  in  $G$ .*

*Proof.* (1)  $\implies$  (2). Let  $g \in G$ . Since  $RG$  is strongly  $(p - 1)$ -nil-clean, from Corollary 4 it follows that  $g^p - g$  is nilpotent. Since  $g$  is invertible, we get that  $g^{p-1} - 1$  is nilpotent.

(3)  $\implies$  (2). Let  $g \in G$ . Then  $\omega(g) = dp^k$ , where  $k \geq 0$  and  $d \mid (p - 1)$ . Let  $p - 1 = ds$ . Since  $\omega(g^d) = p^k$  and  $\gcd(s, p^k) = 1$ , we have that  $\omega(g^{p-1}) = \omega((g^d)^s) = p^k$ . Therefore,

$$(g^{p-1} - 1)^{p^k} = \underbrace{(g^{p-1})^{p^k} - 1}_0 + p \cdot u, \text{ for some } u \in RG.$$

Since  $p$  is nilpotent, the element  $g^{p-1} - 1$  is nilpotent as well.

(2)  $\implies$  (3). Let  $g \in G$ . The order of  $g$  cannot be infinite — in that case, it would not be possible for  $g^{p-1} - 1$  to be nilpotent. Namely, the element  $g^{(p-1)s}$  in the sum  $(g^{p-1} - 1)^s = g^{(p-1)s} + \dots + (-1)^s$  cannot be cancelled out.

So, let us suppose that  $\omega(g) = tp^k$ , for some  $k \geq 0$  and  $t$  such that  $p \nmid t$  and  $\gcd(t, p - 1) = d \neq t$ . Also, let  $t = dt_1$ ,  $p - 1 = dz_1$  and  $h = g^{p-1}$ . Since  $\omega(g^d) = t_1 p^k$  and  $\gcd(z_1, t_1 p^k) = 1$ , it follows that

$$\omega(h) = \omega(g^{p-1}) = \omega((g^d)^{z_1}) = t_1 p^k.$$

Since  $h - 1$  is nilpotent,  $h^{p^k} - 1$  is nilpotent as well. Let  $h_1 = h^{p^k}$ . Then  $\omega(h_1) = t_1$ . Let us focus on the polynomial  $f(X) = (X - 1)^{t_1} - (X^{t_1} - 1)$ , which is clearly divisible by  $X - 1$ :

$$f(X) = (X - 1)((X - 1)^{t_1-1} - (X^{t_1-1} + \dots + X + 1)) = (X - 1)f_1.$$

This follows from the fact that  $t_1 \neq 1$  (which also implies that  $h_1 - 1 \neq 0$ ). We can see that

$$f(X) = (X - 1)(-t_1 + (X - 1)q(X)),$$

for some polynomial  $q \in \mathbb{Z}[X]$ , since  $f_1(1) = -t_1$ . We can conclude that

$$f(h_1) = (h_1 - 1)^{t_1} - \underbrace{(h_1^{t_1} - 1)}_0 = (h_1 - 1)(-t_1 + (h_1 - 1)q(h_1)).$$

We know that  $h_1 - 1$  is nilpotent. As  $p \nmid t_1$  and  $p$  is nilpotent, element  $t_1$  is invertible in  $R$ . So

$$(h_1 - 1)^{t_1} = u(h_1 - 1),$$

for an invertible  $u \in RG$ . So

$$(h_1 - 1)((h_1 - 1)^{t_1-1} - u) = 0,$$

and since  $(h_1 - 1)^{t_1-1} - u$  is invertible, we have that  $h_1 - 1 = 0$ . That is a contradiction.

It is easy to check that the proof of  $2 \iff 3$  is valid, although shorter, even for  $p = 2$ .  $\square$

In the previous lemma,  $G$  was an arbitrary group. If we add commutativity, we actually get equivalence (1)  $\iff$  (2).

**Theorem 4.** *Let  $R$  be a  $(p - 1)$ -nil-clean ring such that  $\text{char}(R) = p^s$ , for a prime  $p$  and some  $s \geq 1$  and let  $G$  be an Abelian group. Then  $RG$  is  $(p - 1)$ -nil-clean iff  $g^{p-1} - 1$  is nilpotent for every  $g \in G$ .*

*Proof.* We only need to prove the “if” part. It follows directly from the Remark following Theorem 2 since elements of the group  $G$  form a generating set for  $RG$  over  $R$ .  $\square$

*Example 2.* The previous theorem does not hold for non-commutative groups. Let us take  $R = \mathbb{Z}_5$  and  $G = \mathbb{D}_4$ , the dihedral group of order 8 generated by elements  $s$  and  $r$  such that  $s^2 = 1 = r^4$ ,  $sr = r^3s$ . In this group,  $g^4 = 1$  for all  $g \in G$ , so the condition that  $g^4 - 1$  is nilpotent is trivially satisfied. However, direct computation shows that

$$\begin{aligned} (s + sr)_5 &= 2s + 2r + 3sr^2 + 3r^3 \\ ((s + sr)_5)^8 &= 3 + 2r^2 \\ (3 + 2r^2)^2 &= 3 + 2r^2, \end{aligned}$$

so  $(s + sr)_5$  is not nilpotent and the group ring  $\mathbb{Z}_5\mathbb{D}_4$  is not strongly 4-nil-clean.

Let us concentrate now on the general case.

**Proposition 11.** *Let  $R$  be a  $(p - 1)$ -nil-clean ring and  $\text{char}(R) = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ , so that  $l > 1$ ,  $p_1 < \cdots < p_l = p$ .*

- (1) *If  $G$  is an elementary Abelian 2-group, then  $RG$  is strongly  $(p - 1)$ -nil-clean.*
- (2) *If  $G$  is an elementary Abelian group in which every element has order  $q$  and  $q \mid \text{gcd}(p_1 - 1, \dots, p_l - 1)$ , then  $RG$  is strongly  $(p - 1)$ -nil-clean.*

*Proof.* (1) As we know,  $R \cong R_1 \times \cdots \times R_l$ , where  $p_i$  is nilpotent in  $R_i$ , and consequently  $RG \cong R_1G \times \cdots \times R_lG$ . So, all the rings  $R_i$  are strongly  $(p - 1)$ -nil-clean, and since  $p_i$  is nilpotent in  $R_i$ ,  $R_i$  is actually strongly  $(p_i - 1)$ -nil-clean. We will use Theorem 4. Since  $2 \mid (p_i - 1)$  for all  $i \geq 2$  and  $g^2 = 1$  for all  $g \in G$ , we have that  $g^{p_i-1} - 1 = 0$  for all  $g \in G$ . If  $p_1 > 2$ , the same holds for  $p_1$ . If  $p_1 = 2$ , then  $(g - 1)^2 = g^2 - 2g + 1 = 2(1 - g)$ . Since 2 is nilpotent in  $R_1G$ ,  $g - 1$  is also nilpotent in  $R_1G$ . In this case also, from  $G$  being Abelian, we can conclude that  $R_iG$  is  $(p_i - 1)$ -nil-clean. Therefore,  $RG$  is  $(p - 1)$ -nil-clean.

(2) Similarly, from  $g^q = 1$ , and  $q \mid \text{gcd}(p_1 - 1, \dots, p_l - 1)$ , we conclude that  $g^{p_i-1} - 1 = 0$  for all  $g \in G$ , and proceed as in (1).  $\square$

**Theorem 5.** *Let  $\text{char}(R) = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ , where  $l > 1$ ,  $p_1 < \cdots < p_l = p$  and let  $G$  be a group. Suppose that  $RG$  is strongly  $(p - 1)$ -nil-clean ring.*

- (1) *For all  $g \in G$  the following holds:  $\omega(g) \mid \text{gcd}(p_2 - 1, \dots, p_l - 1)$  and  $\omega(g) = d_1 p_1^s$ , such that  $d_1 \mid (p_1 - 1)$  and  $s \geq 0$ .*

- (2) If there exists  $h \in G$  such that  $\omega(h) \geq p_1$ , then  $p_1 \mid (p_i - 1)$ , for  $2 \leq i \leq l$ .  
 (3) If for all  $g \in G$ ,  $\omega(g) < p_1$ , then  $\omega(g) \mid \gcd(p_1 - 1, \dots, p_l - 1)$ .  
 (4) If  $p_1 = 2$ , then every element in  $G$  has order  $2^s$  for some  $s \geq 0$ .  
 (5) If  $p_1 = 2$  and  $p_i \equiv 3 \pmod{4}$  for at least one  $i \geq 2$ , then  $G$  is an elementary Abelian 2-group.

*Proof.* (1) As before,  $RG \cong R_1G \times \dots \times R_lG$ , where  $p_i$  is nilpotent in  $R_iG$ . Lemma 3 shows that  $g^{p_i-1} - 1$  is nilpotent in  $R_iG$ . It follows that  $\omega(g) = d_i p_i^{k_i}$  for some  $k_i \geq 0$  and  $d_i \mid (p_i - 1)$ . So,

$$d_1 p_1^{k_1} = d_2 p_2^{k_2} = \dots = d_l p_l^{k_l}.$$

Since  $d_1 < p_1$ , it is clear that  $d_1 p_1^{k_1}$  cannot be divisible by any prime greater than  $p_1$ . So  $k_i = 0$  for  $i \geq 2$ . Therefore,

$$\omega(g) = d_1 p_1^{k_1} = d_2 = \dots = d_l, \quad (4.1)$$

for all  $g \in G$ , where  $d_i \mid (p_i - 1)$ . Hence,  $\omega(g) \mid (p_2 - 1), \dots, \omega(g) \mid (p_l - 1)$ , and we are done.

(2) If  $h \in G$  is such that  $\omega(h) \geq p_1$ , we have  $\omega(h) = d_1 p_1^{k_1} = d_2 = \dots = d_l$ , where  $k_1 \geq 1$ . Since  $d_i \mid (p_i - 1)$ , it follows that  $p_1 \mid (p_i - 1)$ .

(3) Under this assumption, we get that  $k_1 = 0$  in 4.1, hence for all  $g \in G$

$$\omega(g) = d_1 = d_2 = \dots = d_l.$$

We conclude that  $\omega(g) \mid \gcd(p_1 - 1, \dots, p_l - 1)$ .

(4) The fourth assertion follows easily. Namely, in this case  $p_1 = 2$ , so  $\omega(g) = d_1 2^{k_1}$ , where  $d_1 \mid (2 - 1)$ . So,  $\omega(g) = 2^{k_1}$ .

(5) It is enough to show that there are no elements of order 4 in  $G$ . If it were, then for an element  $g \in G$ , we would have equalities

$$4 = d_2 = \dots = d_l,$$

where  $d_i \mid (p_i - 1)$ , for  $2 \leq i \leq l$ . This would imply that  $4 \mid (p_i - 1)$ , that is  $p_i \equiv 1 \pmod{4}$ , for  $2 \leq i \leq l$ , which is a contradiction. Hence, we can conclude that  $G$  is an elementary Abelian 2-group.  $\square$

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