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The Bruhat order on the symmetric group

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Resumo

Neste trabalho, após uma breve introdução sobre conjuntos parcialmente ordenados, cadeias minimais e relação de cobertura, analisamos e caracterizamos os grupos de Coxeter.
Estudamos também aspectos da ordem de Bruhat, em particular quando aplicada ao grupo simétrico, estabelecendo também uma caracterização da relação de cobertura neste caso particular. Como consequência, obtemos uma caracterização combinatória das cadeias minimais no grupo simétrico.
Em último lugar, mostramos como a ordem de Bruhat no grupo simétrico pode assumir um papel importante em aspectos de combinatória, provando a equivalência entre duas ordens parciais definidas no conjuntos dos quadros de Young standard associados a uma dada partição inteira.

Palavras-chave: Cadeias minimais, grupos de Coxeter, ordem de Bruhat, grupo simétrico, quadro de Young.

Resumo Alargado

O grupo simétrico é um dos grupos mais estudados em Álgebra, Combinatória, Teoria da Representação, Geometria e até em Análise. O seu enorme potencial advém em primeiro lugar do facto de qualquer grupo ser isomorfo a um grupo de permutações. A ordem de Bruhat e o grupo simétrico são assim os principais protagonistas deste trabalho. A ordem de Bruhat foi estudada em primeiro lugar por Ehresmann (1934), especialmente no que concerne as suas aplicações às variedades de Schubert. Mais tarde, foi estudada por Verma (1968), que iniciou a abordagem combinatória da ordem de Bruhat nos grupos de Weyl. Recentemente, Anders Bjorner, que é uma das principais referências deste trabalho, analisou e expandiu o estudo desta ordem parcial, especialmente na sua ligação com os grupos de Coxeter. Os grupos de Coxeter devem o seu nome ao matemático britânico, Harold Coxeter (1907-2003) e têm aplicações em diversas áreas da matemática. Exemplos de grupos de Coxeter finitos são os grupos de simetria dos polítopos regulares, e os grupos de Weyl das álgebras de Lie simples. Na família dos grupos de Weyl, encontramos o grupo simétrico. Em geral, os grupos de Coxeter podem ser definidos através de um conjunto de geradores (involuções ou reflexões) que satisfazem uma série de propriedades combinatórias. Devido à versatilidade do grupo simétrico e ao grande potencial da ordem de Bruhat, iremos descrever em detalhe a estrutura do grupo simétrico enquanto conjunto parcialmente ordenado pela ordem de Bruhat, nomeadamente dando uma descrição combinatória das suas cadeias minimais, que se podem indexar com etiquetas em ordem lexicográfica. Seguiremos de perto um artigo de Federico Incitti, [3] onde são descritas as cadeias minimais do grupo simétrico, assim como os casos dos outros grupos de Weyl, em particular o grupo hiperoctaedral e o grupo das permutações pares assinaladas. O conhecimento da estrutura das cadeias minimais está directamente ligado ao conhecimento da relação de cobertura, que por sua vez tem uma descrição combinatória muito curiosa. Em seguida, para dar um exemplo de uma aplicação desta ordem parcial, iremos mostrar que o grupo simétrico, enquanto conjunto parcialmente ordenado atavés da ordem de Bruhat, é a chave para demonstrar um resultado relacionado com a teoria dos quadros de Young. O conjunto dos quadros de Young standard pode ser munido de diversas ordens parciais, sendo a ordem de dominação de longe a mais conhecida e estudada pelas suas inúmeras aplicações. A "ordem de caixa" (box order) é menos conhecida e tem um papel importante na teoria dos subespaços invariantes de operadores lineares nilpotentes. Estas duas ordens têm definições combinatórias aparentemente muito diferentes mas, seguindo o trabalho de Kosakowska, Schmidmeier and Thomas [4], iremos mostrar que são equivalentes.

Este trabalho está organizado em quatro capítulos, cujo conteúdo passamos a descrever.

No primeiro capítulo começamos com uma secção onde fazemos uma breve descrição de noções básicas sobre conjuntos parcialmente ordenados (posets) e terminamos com uma secção onde exploramos a ligação entre os conceitos de cadeia minimal e relação de cobertura. Nesta secção, começamos por definir cadeia saturada, para em seguida chegar ao conceito de poset graduado e função dimensão. Prosseguimos para a definição de sistema de inserção e sistema de cobertura, dois conceitos directamente relacionados com a noção de dimensão. De facto, prova-se que se existir um sistema de cobertura, então o poset é graduado e a relação de cobertura fica completamente caracterizada pelo sistema de cobertura (Proposição 1.10). A secção termina com dois exemplos ilustrativos dos conceitos acima referidos. Estes conceitos serão fundamentais para compreender a estrutura interna do grupo simétrico visto como conjunto parcialmente ordenado pela ordem de Bruhat.

No segundo capítulo seguimos principalmente o livro de Bjorner e Brenti [1]. Na primeira secção, definimos as matrizes de Coxeter, matrizes simétricas de entradas no conjunto $\{1, 2, \dots, \infty\}$ e cujas entradas principais são sempre iguais a 1 e os grafos de Coxeter, que se definem a partir da respectiva matriz de Coxeter. Cada uma destas matrizes fornece um conjunto de relações que permitem definir uma apresentação para um par (W, S), onde S é o conjunto de geradores e W é o grupo em si. Este par é designado sistema de Coxeter associado à matriz de Coxeter e W é designado grupo de Coxeter. Os grupos diedrais e os grupos do tipo $Z_2 \times \ldots \times Z_2$ são exemplos de grupos de Coxeter. Na segunda secção mostramos de que forma é que um grupo de Coxeter pode ser visto como um grupo de permutações através de uma propriedade universal (Teorema 2.18). Na terceira secção, damos uma caracterização combinatória dos grupos de Coxeter (Teorema 2.29), através de duas propriedades: a propriedade de troca (exchange property) e a propriedade de cancelamento (deletion property), que será crucial para concluir que o grupo simétrico é um grupo de Coxeter. O segundo capítulo termina com a quarta secção, onde se define finalmente a ordem de Bruhat para grupos de Coxeter. O principal resultado desta secção é o Teorema 2.32 da propriedade da sub-palavra (sub-word property), que será fundamental no quarto capítulo.

O terceiro capítulo é onde o grupo simétrico assume todo o seu protagonismo. Na primeira secção definimos o grupo simétrico e expomos uma série de propriedades que serão importantes para as secções seguintes, nomeadamente a caracterização do número de inversões de uma permutação. Na segunda secção, provamos que o grupo simétrico é um grupo de Coxeter, considerando como conjunto de geradores as transposições do tipo $(i \;\; i+1)$ mostrando que satisfaz a propriedade da troca. Mostramos ainda que o grupo simétrico de grau n, S_n é um grupo de Weyl do tipo A_{n-1}. Na terceira secção, tratamos o grupo simétrico como conjunto parcialmente ordenado com a ordem de Bruhat. Começamos por mostrar como se constrói o diagrama de Hasse de S_3 de modo a ilustrar o funcionamento da ordem de Bruhat. Em seguida, caracterizamos a relação de cobertura no grupo simétrico, como conjunto parcialmente ordenado pela ordem de Bruhat. A noção de subida livre (free rise) dada na Definição 3.9 é fundamental para caracterizar a relação de cobertura neste poset. Terminamos a secção com o Teorema 3.13, que não demonstramos, onde a relação de ordem de Bruhat no grupo simétrico fica completamente caracterizada através das listas parciais das imagens de cada permutação. Terminamos o capítulo com a quarta secção, onde utilizamos os resultados da terceira secção para descrever as cadeias minimais no grupo simétrico.

No quarto capítulo, começamos por definir diagrama de Ferrers associado a uma partição inteira. O conceito de quadro de Young (Young tableau) standard vai ser crucial neste capítulo e corresponde a um preenchimento de um diagrama de Ferrers com inteiros positivos, de forma crescente nas linhas e colunas. A bem conhecida ordem de dominação para partições será estendida ao conjunto dos quadros de Young standard associados a cada partição. Definimos ainda a ordem de caixa (box order) e mostramos a ordem de caixa implica a ordem de dominação. Associando uma permutação a cada quadro de Young, torna-se possível comparar a ordem de Bruhat com a ordem de dominação e com a ordem de caixa. Terminamos o capítulo mostrando que a ordem de caixa é equivalente à ordem de dominação. Provamos em primeiro lugar que duas permutações estão relacionadas pela ordem de Bruhat se e só se os respectivos quadros de Young estão relacionados pela ordem de dominação. A noção de relação de cobertura na ordem de Bruhat é fundamental para esta conclusão. De facto, provamos que se uma permutação cobre outra, então os respectivos quadro de Young estão relacionados pela ordem de caixa, concluindo assim este trabalho. A última proposição do capítulo fecha o ciclo, mostrando que a relação de cobertura é suficiente para concluir o resultado pretendido.

Abstract

In this thesis, after a brief introduction on ordered sets, minimal chains and covering relations, we analyze and characterize the largely studied Coxeter groups.
Moreover we study some aspects of the Bruhat order, in particular considering it applied to the well known symmetric group, also providing a combinatorial characterization of the covering relation on the latter. Consequently obtaining a direct combinatorial description of the minimal chains of the symmetric group.
Last but not least, we bring an example of how the Bruhat order on the symmetric group can assume a relevant role in several aspects of combinatorics showing the equivalence of two partial orders defined on the set of standard Young tables associated with a certain partition.

Keywords: Minimal chains, Coxeter groups, Bruhat order, symmetric group, Young tableau.

Introduction

The undisputed founders of this work are certainly the symmetric group and a particular type of partial order defined on it, the Bruhat order. The symmetric group, in algebra, as in combinatorics, geometry, representation theory and even sometimes in analysis, is one of the most studied and versatile groups. The enormous potential of this group inserted in many other mathematical problems is evident even from a superficial analysis. Of our interest is the Bruhat order, a particular partial order that was first studied by Ehresmann (1934) applied to the Schubert varieties and subsequently by Verma (1968), who started the combinatorial study of the Bruhat order on the Weyl group. Later Bjorner, a mathematician strongly taken into consideration throughout this thesis, has analyzed and expanded the study of this partial order especially by applying it to the so-called Coxeter Groups. Coxeter groups are named after the British mathematician Harold Coxeter (1907-2003) and find application in many areas of mathematics. Examples of finite Coxeter groups are the symmetry groups of regular polytopes and the Weyl groups of simple Lie algebras. Not least among the Coxeter groups we find the symmetric group. Amazed by the versatility of the latter and by the power of the Bruhat order, we decided to describe in detail the structure of the symmetric group seen as a partially ordered set with the Bruhat order. Moreover we decided to give a direct combinatorial description of the minimal chains of the latter, that is chains with the lexicographically minimal labelling. In making this description we mainly consider the work carried out by Federico Incitti [3], where he not only describes the minimal chains linked to the symmetric group, but also studies in detail the cases linked to different Weyl groups. In particular, the hyperoctahedral group and the even-signed permutation group. Subsequently, to give a concrete example of the vaunted ductility of this partially ordered set, we decided to show how by exploiting the duo, Bruhat order and symmetric group, we are able to solve problems related to different areas. We decided to consider two partial order defined on Young tableaux, the *dominance* order and *box* order which are both defined combinatorially and, among other things, are of importance in the theory of invariant subspaces of nilpotent linear operators. Following the work of Kosakowska, Schmidmeier and Thomas [4], we show how the Bruhat order on the symmetric group is the key to show that those two orders are equivalent.

A more precise description.
We start the first chapter with an introductory section on ordered sets, minimal chains and covering relation. The concepts explained in this very first part will be the basis of the arguments to come and will help us to follow more easily the main content of this work.
Already from the second chapter of we carefully define and describe what are the Coxeter groups. We talk about the latter especially from a combinatorial point of view, which will interest us most throughout the course of this work, and carfully following the huge work that Bjorner and Brenti in [1] developed during years of study of this particular kind of groups. Remarkable is certainly the characterization we give of the Coxeter groups in Theorem 2.29. This is because it is exactly that one that in the next chapter will allow us to show that the symmetric group is indeed a Coxeter group. But, before landing on the third chapter, we finally introduce the much vaunted Bruhat order. We describe it in a general way on Coxeter groups, taking care to reveal the marvelous characteristics of this partial order.
We start the third chapter with a section entirely dedicated to the symmetric group, following one of the most classic manuals on this topic written by Sagan [5]. While we are aware that this is central to many university courses, we certainly could not fail to mention some of its main characteristics.
Finally in section 3.2 we show that the symmetric group is a Coxeter group, and as such, in the next section we describe the Bruhat order in it, focusing on its covering relations. The last section of this third main chapter is entirely dedicated to give a characterization of the minimal chains of the symmetric group.
In the fourth and last chapter we describe the Young tableaux associated with partitions and we define two partial orders on them, the dominance order and the box order. In the second section of this chapter we see how the symmetric group, seen as poset with the Bruhat order, is the key to prove the equivalence of the dominance order and the box order in Theorem 4.12.

Contents

Chapter 1

Partially ordered sets

In this chapter, we give a brief introduction on partially ordered sets and expose some general techniques, concerning gradedness, covering relation and EL-shellability with the aim of characterizing the minimal chains through the covering relations.

We will mainly follow section 3 of the paper [3] from F. Incitti.

1.1 Basic notions on ordered sets

Definition 1.1. Let P be a non empty set endowed with a binary relation R. We say that R is a *partial ordering* of P if it is reflexive, transitive and antisymmetric. That is, if for any $x, y, z \in P$ the following properties hold:

- xRx,
- whenever xRy, yRz, then xRz,
- whenever xRy, yRx, then $x = y$.

Moreover if for any $x, y \in P$,we have xRy or yRx, then we say that R is a *total order* on P.

Normally we use the relation symbol $\leq$ instead of R. We call the pair $(P, \leq)$ *partially ordered set*, or more briefly *poset* whenever $\leq$ is a partial order, or *totally ordered set* when $\leq$ is a total order.

From now on we will say with an abuse of notation that P is a poset, omitting the relation $\leq$.

For $x, y \in P$, we write $x < y$ if $x \leq y$ and $x \neq y$. We will say that x *is less than* y if $x \leq y$ and that x *is greater than* y if $y \leq x$.

Definition 1.2. Let A be a subset of a poset P. An element $x \in P$ is an *upper bound* for A if $a \leq x$ for every $a \in A$. We call *supremum* of A ($\sup A$), to the smallest $x \in P$ upon the upper bounds of A. Dually, we call *lower bound* of A to each element $x \in P$ such that $x \leq a$ for any $a \in A$ and we call *infimum* of A ($\inf A$) to the greater $x \in P$ upon the lower bounds of A.

The poset P is said to be *bounded* if the whole set P has the supremum and the infimum. In this case we call them the top and the bottom elements of the poset P, denoting them respectively as $\hat{1}$ and $\hat{0}$.

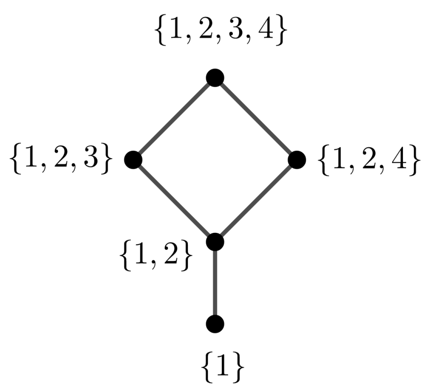
Definition 1.3. Let P be a poset and $x, y \in P$. We say that y *covers* x in P and we write $x \lhd y$, if $x < y$ and whenever $x \leq z \leq y$ for some $z \in P$, then $z = x$ or $z = y$.

Finite posets can be represented in a very intuitive way by the so called *Hasse diagrams.*

Let $P = \{p_1, p_2, \ldots, p_n\}$ be a finite poset. We draw a circle for each element of P and we connect two circles if an only if the corresponding elements are such that one covers the other. For instance suppose that p_2 covers p_1 then we would connect the two small circles associated to p_1 and p_2. Conventionally we draw the element that covers above the one that is covered. In this way we are able to entirely describe the relation on P just by giving the Hasse diagram of it. Let us make an example.

Example 1.4. *Let* $P = \{\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}$ *endowed with the relation "*$\subseteq$*".*

The Hasse diagram of P *is the following*



1.2 Minimal chains and covering relations

In this section, unless otherwise specified, we assume that P is a finite and bounded poset.

Definition 1.5. Let P be a poset. A *chain* of length n in P is a sequence of elements $x_0, x_1, \dots, x_n$ of P, such that

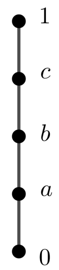
$$x_0 < x_1 < \cdots < x_n.$$

Whenever we have that the relation between the elements is a covering

$$x_0 \lhd x_1 \lhd \cdots \lhd x_n,$$

we say that the chain is *saturated.* A *maximal chain* is a chain that it is not contained in any other chain.

By the definitions we just gave it is clear that any maximal chain is a saturated chain, while the opposite is not true. We can easily see it with a trivial counterexample. Consider the chain,



Then, $0 \lhd a \lhd b$ is a saturated chain, but it is not maximal since it is clearly contained in the chain $0 \lhd a \lhd b \lhd c$.

Definition 1.6. A poset P is *graded* if all its maximal chains have the same length. In particular we say that a graded poset has *rank* n if the length of its maximal chains is equal to n.

Note that, whenever P is a graded poset, it is always possible to define a unique *rank function*,

$$\rho : P \to \mathbb{N} \cup \{0\},$$

such that,

1. $\rho(\hat{0}) = 0$, and

2. $\rho(y) = \rho(x) + 1$, whenever $x \lhd y$.

Whenever the rank of the graded poset P is equal to n then it is also true that the function ρ satisfies the condition,

3. $\rho(\hat{1}) = n$.

Equivalently, we could say that a finite poset P is graded if and only if a function with properties 1. and 2. exists.

Definition 1.7. Let Q be a totally ordered set. An *EL-labelling* of a graded poset P is a function

$$\lambda : \{(x, y) \in P^2 : x \lhd y\} \to Q,$$

such that for any $x, y \in P$ with $x < y$, the following properties hold:

1. There is exactly one saturated chain from x to y with non-decreasing labels,
$$x = x_0 \lhd_{\lambda_1} x_1 \lhd_{\lambda_2} \cdots \lhd_{\lambda_k} x_k = y,$$
with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$, where λ_i is the image through λ of (x_{i-1}, x_i)

2. Whenever we consider another saturated chain from x to y, different from the previous one,
$$x = y_0 \lhd_{\mu_1} y_1 \lhd_{\mu_2} \cdots \lhd_{\mu_k} y_k = y,$$
then
$$(\lambda_1, \ldots, \lambda_k) < (\mu_1, \ldots, \mu_k),$$
in the lexicographic order, that is, suppose i is the first index for which $\lambda_i \neq \mu_i$, then $\lambda_i < \mu_i$.

Whenever a chain has property 2., we say that it has the *lexicographically minimal labelling*.
Moreover, we say that a graded poset P is *EL-shellable* if it has an *EL-labelling*.

In order to arrive to a characterization of the minimal chains, we start by giving the definition of insertion and successor systems on a poset.

Definition 1.8. Let P be a poset. We call a set

$$H \subseteq \{(x, y) \in P^2 : x < y\},$$

a *successor system* of P.
An *insertion system* H of P is a successor system with the following property:
Insertion property: for any $x, y \in P$, with $x < y$, there exists $z \in P$ such that

$$(x, z) \in H \ \ and \ \ z \leq y.$$

A *covering system* of P is a pair (H, ρ) where H is an insertion system of P and $\rho : P \to \mathbb{N} \cup \{0\}$ is a function with the following properties:

ρ-base property: $\rho(\hat{0}) = 0$,

ρ-increasing property: for every $(x, y) \in H$, then $\rho(y) = \rho(x) + 1$.

Next proposition gives a general method to prove that a poset is graded with a given rank function: it suffices to find a covering system of P.

Proposition 1.9. *If there exists a covering system (H, ρ) of P, then P is graded with rank function ρ.*

Proof. Suppose that there exists a covering system (H, ρ). Then by definition ρ has the ρ-base property and therefore $\rho(\hat{0}) = 0$. Moreover if we are considering $x, y \in P$ such that $x \lhd y$, since H is an insertion system we know that there exists $z \in P$ such that

$$(x, z) \in H \ \ and \ \ z \leq y.$$

But, by definition, to be an insertion system H needs to be a successor system, and therefore the fact that $(x, z) \in H$ implies that $x < z$. But since $x \lhd y$ and $z \leq y$, we can conclude that $y = z$. Thus $(x, y) = (x, z) \in H$, and by the ρ-increasing property $\rho(y) = \rho(z) = \rho(x) + 1$. □

Actually a covering system tells us more than what we have just proved. Indeed it characterizes the covering relations in P.

Proposition 1.10. *Let (H, ρ) be a covering system of P. Then for any $x, y \in P$,*

$$x \lhd y \iff (x, y) \in H.$$

Proof. Suppose $x \lhd y$, then it follows directly from the insertion property of H that $(x, y) \in H$.
Suppose $(x, y) \in H$, then by the ρ-increasing property

$$\rho(y) = \rho(x) + 1. \tag{1.1}$$

Note that, since H is an insertion system and ρ has the insertion property we can conclude that ρ is order preserving.
Suppose by contradiction that exists $z \in P$ such that $x < z < y$. Then by the order preserving property of ρ we conclude that $\rho(y) > \rho(z) > \rho(x)$, so that $\rho(y) \geq \rho(x) + 2$, that contradicts (1.1). Thus $x \lhd y$. □

From now on we consider Q to be a totally ordered set, the set of labels. Let S be any set. We call a *labelling* of S to any function

$$\lambda : S \to Q.$$

Definition 1.11. Consider a successor system H of P. We say that a labelling of H,

$$\lambda : H \to Q,$$

is a *good labelling* if it has the injective property.
Injective property: For every $(x, y), (x, z) \in H$, if $\lambda(x, y) = \lambda(x, z)$, then $y = z$.

This injective property gives us some uniqueness results.

Consider a successor system H of P and let λ be a good labelling of it. Fix $x \in P$. We say that a label $i \in Q$ is a *suitable label* for x if there exists $y \in P$ such that $\lambda(x, y) = i$. If such a y exists then by the injective property we know that it is unique. Thanks to this uniqueness we can somehow refer to such element y just using the suitable label i and the element x. Indeed we call it the *transformation* of x respect to i, and we denote it by $t_i^P(x)$.

We define $\Lambda(x)$ to be the set of all suitable labels for a fixed $x \in P$.

Lemma 1.12. *Let H be a successor system of P and $\lambda : H \to Q$ be a good labelling of H. Then the insertion property is equivalent to the following,*

$$\forall x, y \in P,\ x < y,\ \text{there exists}\ \ i \in \Lambda(x)\ \ \text{such that}\ \ t_i^P(x) \leq y. \tag{1.2}$$

Proof. Suppose equation (1.2) holds. The fact that i is a suitable label for x means that there exists a unique $z \in P$ such that $\lambda(x, z) = i$. This means that $(x, z) \in H$. Moreover we know that given i and x, we can write the element z uniquely as $z = t_i^P(x)$. Thus, from $t_i^P(x) \leq y$ we have, $z \leq y$, proving the insertion property. The reciprocal can be proved in a similar way. □

If we consider a covering system (H, ρ), then by the characterization of Proposition 1.10 we have that any good labeling of H is an edge-labelling of P where an edge-labelling of P is just a labelling of the set of edges of the Hasse diagram of P. In this case, each transformation $t_i^P(x)$ is called a covering transformation of x respect to i. Instead of writing $t_i^P(x)$ we write

$$ct_i^P(x),$$

just to underline that we are dealing with a covering system and not just a successor system.

At this point we can conclude that to each pair $(x, i) \in P \times Q$ we associate a unique $ct_i^P(x) \in P$ such that

$$x \lhd ct_i^P(x).$$

Reciprocally, we can always associate a unique $i \in Q$, satisfying $y = ct_i^P(x)$ to a given covering relation $x \lhd y$ in P .

Observe that, given a good labelling λ, the fact that (H, ρ) is a covering system implies that the insertion property holds, and by Lemma 1.12, that the property (1.2) holds. Therefore, for any $x, y \in P$, with $x < y$, we have

$$\{i \in \Lambda(x) : ct_i^P \leq y\} \neq \emptyset.$$

Definition 1.13. Let (H, ρ) be a covering system of P. Let $x, y \in P$, with $x < y$. We denote by $mi_y(x)$, the *minimal label* of x with respect to y and we define it as

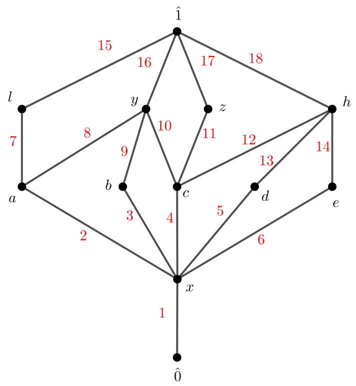
$$mi_y(x) := min\{i \in \Lambda(x) : ct_i^P(x) \leq y\}.$$

We call *minimal covering transformation* of x with respect to y, the covering transformation of x with respect to the minimal label $mi_y(x)$. We denote it by

$$mct_y^P(x) := ct_{mi_y(x)}^P(x).$$

Let us see a concrete example of all these new definitions.

Example 1.14. *Consider the Hasse diagram of the finite and bounded poset P with the edge labelling depicted in red.*



Let (H, ρ) be a covering system of P defined as follows.

- *H is the set of all the edges of the Hasse diagram of P,*
- *ρ is any function that satisfies the base property and the increasing property.*

By Lemma 1.12 *and the fact that λ is an injective edge-labelling we can easily conclude that it is also a good labelling for H.*
For the elements $x, y \in P$, we have that

$$\Lambda(x) = \{2, 3, 4, 5, 6\},$$

the minimal label of x with respect to y is

$$mi_y(x) = 2,$$

and therefore the respective minimal covering transformation is

$$mct_y^P(x) = a.$$

Definition 1.15. Let $x, y \in P$, with $x < y$. We define the *minimal chain* from x to y as the saturated chain

$$x = x_0 \lhd x_1 \lhd \cdots \lhd x_k = y,$$

where $x_i = mct_y^P(x_{i-1})$, for any $i \in [k]$.

So, looking again at Example 1.14 we will try to find the minimal chain from 0 to 1, following the process described by the definition.
Since in our poset the bottom element 0 is covered only by the element x, then

$$\Lambda(x) = \{1\},$$

because $\lambda(0, x) = 1$. Thus of course $mi_1(0) = 1$, and the corresponding covering transformation is $mct_1^P(0) = x$. Therefore, until now we have

$$0 \lhd x \leq 1.$$

At this point we are going to repeat the same exact process for the covering transformation we just found, x.
$\Lambda(x) = \{2, 3, 4, 5, 6\}$ and $mi_1(x) = 2$. The corresponding covering transformation is $mct_1^P(x) = a$, obtaining

$$0 \lhd x \lhd a \leq 1,$$

and again we do the same process to a. $\Lambda(a) = \{7, 8\}$, and $mi_1(a) = 7$. The corresponding covering transformation is $mct_1^P(a) = l$, obtaining

$$0 \lhd x \lhd a \lhd l \leq 1.$$

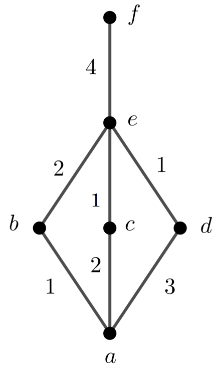
But since $l \lhd 1$, we actually have

$$0 \lhd x \lhd a \lhd l \lhd 1,$$

that is the wanted saturated chain. The associated labelling is $(1, 2, 7, 15)$. It is important to observe that different good labellings lead to different minimal chains but that in any case, having fixed a good labelling λ, the process we have just applied leads always to a saturated chain with the lexicographically minimal labelling. This happens thanks to the minimality choice of the minimal covering transformation. Moreover what we have just seen is that if we know entirely the minimal covering transformations of a poset, then we are able to describe the minimal chains of it.

Looking at Definition 1.7 with the new introduced concepts we can see that we could equivalently define an EL-shellable poset, as a poset in which all the minimal chains have increasing labels and any other saturated chain has at least one decrease in the labels.

On the other hand a good example of EL-labelling on a poset could be the following. Consider



The 2-long minimal chains are

$$a \lhd b \lhd e \text{ and } b \lhd e \lhd f,$$

and we can see that those two have increasing labelling and that all the remaining 2-long saturated chains have a decrease in the labels. We have only one 3-long minimal chain, that is,

$$a \lhd b \lhd e \lhd f.$$

It has the increasing labelling and the other two possible 3-long saturated chains have a decrease in the labels. Since we found an EL-labelling, we can conclude that the poset is $EL - shellable$.

Chapter 2

Coxeter groups

In this chapter we will introduce the Coxeter groups, a particular type of groups that are named after the British mathematician Harold Coxeter (1907-2003) and find application in many areas of mathematics. Coxeter groups can be defined in many ways and we will see the connection between some of these approaches. We will start by defining them from a set of generators with a presentation defined by a particular kind of matrix and we will conclude the chapter by showing that they can be defined by means of generators with certain combinatorial properties.

2.1 Matrices and Coxeter groups

Definition 2.1. Let S be a set. A matrix $m : S \times S \to \{1, 2, \dots, \infty\}$ is called a *Coxeter matrix* if it is a symmetric matrix with $1's$ on the main diagonal,tThat is,

1. $m(s, s') = m(s', s)$,
2. $m(s, s') = 1 \iff s = s'$.

Each Coxeter matrix can be represented by a so called *Coxeter graph* (or *Coxeter diagram*) constructed in the following way;

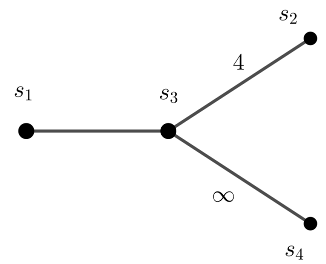
- S is the set of nodes,
- $\{\{s, s'\} : m(s, s') \geq 3\}$ is the set of edges.

For convention, we put a label in the Coxeter graph on each edge such that $m(s, s') \geq 4$ and we label it by the symbol $m(s, s')$ itself.

Example 2.2. *Consider the following Coxeter matrix,*

$$m = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 3 & 4 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{bmatrix}$$

By definition we can see that the Coxeter graph associated to this Coxeter matrix is,



Let us denote

$$S^2_{fin} := \{(s, s') \in S^2 : m(s, s') \neq \infty\}.$$

Each Coxeter matrix determines a group W with the following presentation:

- Generators: S
- Relations: $(ss')^{m(s,s')} = Id, \ \ \forall (s, s') \in S^2_{fin}$.

By definition of Coxeter matrix, we know that for every s in S, $m(s, s) = 1$. Therefore, by the relation we gave,

$$(ss)^1 = s^2 = Id. \tag{2.1}$$

Proposition 2.3. *Let $s \neq s'$. We have that*

$$(ss')^{m(s,s')} = Id \tag{2.2}$$

if and only if

$$\underbrace{ss' \cdot ss' \cdots ss'}_{m(s,s') \ factors} = \underbrace{s's \cdot s's \cdots s's}_{m(s,s') \ factors}. \tag{2.3}$$

Proof. Suppose $(ss')^{m(s,s')} = Id$, then

$$\begin{aligned} \underbrace{ss'ss'\cdots ss'}_{2m(s,s')\ \ factors} = Id &\iff s\ \underbrace{ss'ss'\cdots ss'}_{2m(s,s')\ \ factors} = s \\ &\iff \underbrace{ss}_{=Id}\ \underbrace{s'ss'\cdots ss'}_{2m(s,s')-1\ \ factors} = s. \end{aligned}$$

At this point we can operate the same process, multiplying by s', obtaining,

$$\underbrace{ss'\cdots ss'}_{2m(s,s')-2\ \ factors} = s's,$$

and iterating the all process we have equation (2.3).
Reciprocally, suppose we have equation (2.3),

$$\underbrace{ss'\cdot ss'\cdots ss'}_{m(s,s')\ \ factors} = \underbrace{s's\cdot s's\cdots s's}_{m(s,s')\ \ factors}.$$

Then,

$$\begin{aligned} \underbrace{ss'\cdot ss'\cdots ss'}_{m(s,s')\ \ factors}\underbrace{ss'\cdot ss'\cdots ss'}_{m(s,s')\ \ factors} &= \underbrace{ss'\cdot ss'\cdots ss'}_{m(s,s')\ \ factors}\underbrace{s's\cdot s's\cdots s's}_{m(s,s')\ \ factors} \\ \underbrace{(ss')(ss')\cdots(ss')}_{m(s,s')\ \ parenthesis} &= ss'\cdots s\underbrace{(s's')}_{Id}s\cdots s's. \end{aligned}$$

Therefore, on the right side we would obtain the identity, showing relation (2.2).

□

Observation 2.4. From the previous facts we can show that

$$m(s,s') = 2 \iff s\ \ and\ \ s'\ \ commute.$$

Suppose $m(s,s') = 2$, then from the equivalence just shown of the relations (2.2) and (2.3), we get $ss' = s's$. Similarly in the other direction.

Example 2.5. *Considering the Coxeter matrix of example 2.2 we have,*

- *Set of generators* $S = \{s_1, s_2, s_3, s_4\}$
- $S^2_{fin} = \{(s_1,s_1),(s_2,s_2),(s_3,s_3),(s_4,s_4),(s_1,s_2),(s_1,s_3),(s_1,s_4),\cdots\}$

- *Relations* $\begin{cases} s_1^2 = s_2^2 = s_3^2 = s_4^2 = Id \\ s_1 s_2 = s_2 s_1 \\ s_1 s_3 s_1 = s_3 s_1 s_3 \\ s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2 \\ s_2 s_4 = s_4 s_2 \end{cases}$

Definition 2.6. A Coxeter group W is given by a set S of generators and a presentation associated to a Coxeter matrix m. The pair (W, S) is called a *Coxeter System*.
The cardinality of the set S, is called *rank of* (W, S).

Definition 2.7. We say that a Coxeter system is *irreducible* if its Coxeter graph is connected.

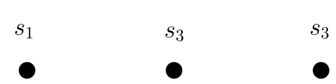
Definition 2.8. A finite Coxeter group for which $m(s, s') \in \{2, 3, 4, 6\}$ for all $(s, s') \in S^2$, with $s \neq s'$ is called *Weyl group*.
If $m(s, s') \in \{2, 3\}$ for all $(s, s') \in S^2$, with $s \neq s'$ then it is called *Simply − laced* group.

In the remainder of this initial section let us just give two exhaustive examples of Coxeter groups.

Example 2.9. *The graph with n isolated vertices is the Coxeter graph of the group*

$$(\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n\ times}, +)$$

of order 2^n.
To make it clear we show it for $n = 3$.
The Coxeter matrix associated with the graph with 3-isolated vertices,



is

$$m = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

We consider as set of generators of the Coxeter group W,

$$S = \{s_1, s_2, s_3\},$$

with the relations $\begin{cases} s_1^2 = s_2^2 = s_3^2 = Id \\ s_1 s_2 = s_2 s_1 \\ s_1 s_3 = s_3 s_1 \\ s_2 s_3 = s_3 s_2 \end{cases}$.

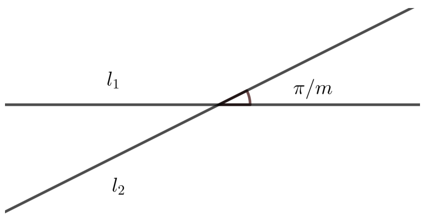
Therefore, $W = \{Id, s_1, s_2, s_3, s_1s_2, s_1s_3, s_2s_3, s_1s_2s_3\}$ *of order* 2^3.
It is now easy to see that the group

$$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2 := \{(1,1,1),(0,0,0),(1,1,0),(1,0,0),(0,1,1),(0,1,0),(0,0,1),(1,0,1)\},$$

is isomorphic to our Coxeter group W via the correspondence

$$\begin{aligned} (0,0,0) &\longleftrightarrow Id_W \\ (1,0,0) &\longleftrightarrow s_1 \\ (0,1,0) &\longleftrightarrow s_2 \\ (0,0,1) &\longleftrightarrow s_3. \end{aligned}$$

Example 2.10 (The Dihedral Group)**.** *Let L_1 and L_2 be the lines in Figure below through the origin of the euclidean plane $\mathbb{E}^2$, where $m \geq 2$.*



Let r_1 be the orthogonal reflection through l_1 and r_2 be the orthogonal reflection through l_2. Then r_1r_2 is a rotation of the plane through the angle $\frac{2\pi}{m}$. Of course doing this rotation exactly m times we would obtain the initial configuration, meaning $(r_1r_2)^m = Id$. Then we can consider the group $G_m = \langle r_1, r_2\rangle$, generated by this two reflections.
We get the following group of order $2m$,

$$G_m := \{(r_1r_2), (r_1r_2)^2, \cdots, (r_1r_2)^m, r_1(r_1r_2), r_1(r_1r_2)^2, \cdots r_1(r_1r_2)^m\}.$$

Note that the element $r_1(r_1r_2)$ is actually r_2 and $r_1(r_1r_2)^m$ is r_1.
For instance, considering $m = 3$ the angle between the two lines in the plain would be $\pi/3$, and $G_3 := \{r_1r_2, (r_1r_2)^2, Id, r_2, r_1(r_1r_2)^2, r_1\}$ the group of the

symmetries of the regular triangle.

We call $I_2(m)$ to the Coxeter group given by the graph

By definition of Coxeter group the elements of $I_2(m)$ are the ones generated by $S = (s_1, s_2)$ with the relation $(s_1 s_2)^m = Id$.
Note that each word of $I_2(m)$ can be represented by a sequence of the type $s_1 s_2 s_1 \dots$ or $s_2 s_1 s_2 \dots$ of length less or equal than m (including the empty word as identity).
So we have two words of each length, plus the identity and the world of length m:

$$\begin{array}{c} Id \\ s_1 \quad s_2 \\ s_1 s_2 \quad s_2 s_1 \\ s_1 s_2 s_1 \quad s_2 s_1 s_2 \\ . \\ . \\ . \\ \underbrace{s_1 s_2 \dots s_2}_{m} = \underbrace{s_2 s_1 \dots s_1}_{m} \end{array}$$

The two words of length m are equal because of relation (2.3), but we don't know if the words of length i, $1 \leq i < m$, coincide, hence

$$|I_2(m)| \leq 2m.$$

Seen this, we can define a surjection

$$f : I_2(m) \to G_m$$

defined by

$$\begin{aligned} s_1 &\to r_1 \\ s_2 &\to r_2, \end{aligned}$$

which is actually an isomorphism because, since $|G_m| = 2m$, we have that $|I_2(m)| = 2m$.
The group $I_2(m)$ is called the Dihedral group of order $2m$.

2.2 Coxeter groups as permutation groups

Let (W, S) be a Coxeter system.

Definition 2.11. We define the set of all the conjugates of the elements of S,

$$T := \{wsw^{-1} : s \in S, w \in W\}.$$

The elements of T are called reflections while the elements of S are called simple reflections.

Since $Id \in W$, we have $S \subseteq T$. Moreover, by an easy computation we obtain that for any $t \in T$,

$$t^2 = wsw^{-1}wsw^{-1} = wssw^{-1} = ww^{-1} = Id. \tag{2.4}$$

Definition 2.12. Let $s_1 s_2 \cdots s_k \in S^*$, where S^* is the free monoid generated by the elements of S. We define the function

$$\hat{T}(s_1 s_2 \cdots s_k) = (t_1, t_2, \ldots, t_k),$$

where

$$t_i := s_1 s_2 \cdots s_{i-1} s_i s_{i-1} \cdots s_2 s_1 = (s_1 \cdots s_{i-1}) s_i (s_1 \cdots s_{i-1})^{-1} \in T,$$

for $1 \leq i \leq k$.

It follows immediately that

$$t_i s_1 \cdots s_k = s_1 \cdots s_{i-1} s_{i+1} \cdots s_1 = s_1 \cdots \hat{s}_i \cdots s_k \tag{2.5}$$

and that

$$s_1 \cdots s_i = t_i t_{i-1} \cdots t_1.$$

In order to state the main result of this section we need a couple of definitions.

Definition 2.13. For $s_1 \cdots s_k \in S^*$ and $t \in T$, we define the function

$$n(s_1 \cdots s_k; t) := \text{the number of times } t \text{ appears in } \hat{T}(s_1 \cdots s_k).$$

Definition 2.14. Let $s \in S$ and $t \in T$, we define the function

$$\eta(s;t) := \begin{cases} -1 & \text{if } s = t, \\ 1 & \text{if } s \neq t. \end{cases}$$

From the previous two definitions we can observe that

$$(-1)^{n(s_1 \cdots s_k; t)} = \prod_{i=1}^{k} \eta(s_i; s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1}).$$

Let us construct an example in order to see all these definitions clearly.

Example 2.15. *Consider the word $s_1 s_2 s_3$, where $s_1 = a$, $s_2 = b$, $s_3 = c$ and the element $aba = aba^{-1} \in T$:*

- $\hat{T}(abc) = (a, aba, abcba)$,
- $n(abc; aba) = 1$ *because the word aba appears one time in* $\hat{T}(abc)$,
- $\displaystyle\prod_{i=1}^{3} \eta(s_i, s_{i-1} \cdots s_1 (aba) s_1 \cdots s_{i-1}) = -1$ *because*

  $\eta(a; aba) \cdot \eta(b; \underbrace{aa}_{Id} b \underbrace{aa}_{Id}) \cdot \eta(c; b \underbrace{aa}_{Id} b \underbrace{aa}_{Id} b = \underbrace{bb}_{Id} b = b) = (1 \cdot -1) \cdot 1 = -1,$

  and $(-1)^{n(abc;aba)} = -1$.

We are finally ready to speak about permutations.

Definition 2.16. Define $S(R)$ as the group of all permutations of the set $R := T \times \{1, -1\}$.
Moreover we define a map $f : S \to S(R)$ which associates to each $s \in S$, a permutation $\pi_s : R \to R$ in $S(R)$ defined as follows,

$$\pi_s(t, \varepsilon) := (sts, \varepsilon\, \eta(s; t)).$$

We should now see that π_s is in fact well defined, but since for every $s \in S$

$$\pi_s^2(t,\varepsilon) = \pi_s(sts, \varepsilon\,\eta(s;t)) = (sstss, \varepsilon\,\eta(s;t)\eta(s;sts)) = (t,\varepsilon), \tag{2.6}$$

we can say that $\pi_s \in S(R)$.

Now consider a word $w = s_1 \cdots s_k \in W$, we say that k is *minimal* if we can not express the word w with less letters.

Lemma 2.17. *Consider the word $w = s_1 \cdots s_k$ with k minimal and let*

$$t_i = s_1 s_2 \cdots s_{i-1} s_i s_{i-1} \cdots s_2 s_1 = (s_1 \cdots s_{i-1}) s_i (s_1 \cdots s_{i-1})^{-1} \in T$$

as before. Then for every $1 \leq i < j \leq k$,

$$t_i \neq t_j.$$

Proof. Suppose by contradiction there exist i and j, $i < j$, such that $t_i = t_j$. Then by equation (2.4) and applying twice equality (2.5) we have,

$$w = t_i t_j s_1 \cdots s_k = t_i s_1 \cdots \hat{s}_j \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k.$$

Which contradicts the minimality of k. □

Theorem 2.18. *Let (W,S) be a Coxeter system, and T the set of reflections.*

i. The map

$$\begin{aligned} f :& S \to S(R) \\ & s \to \pi_s \end{aligned}$$

extends uniquely to an injective homomorphism

$$\begin{aligned} \bar{f} :& W \to S(R) \\ & w \to \pi_w. \end{aligned}$$

ii. For all $t \in T$, $\pi_t(t,\varepsilon) = (t, -\varepsilon)$.

Proof. *i*. To show the first assertion, we want to use the following property,

Universality property: Let (W, S) be a Coxeter system, G a group and $f : S \to G$ a map such that $(f(s)f(s'))^{m(s,s')} = Id, \quad \forall s, s' \in S^2_{fin}$. Then there exists a unique extension of f to a group homomorphism $\bar{f} : W \to G$.

We have already seen with equation (2.6) that

$$\pi_s^2 = Id_R$$

for any $s \in S$.
Now we want to see that, for any $s, s' \in S$ such that $m(s, s') = p \neq \infty$,

$$(\pi_s \pi_{s'})^p = Id_R.$$

Consider the word

$$\bar{s} = s_1 s_2 \cdots s_{2p},$$

where

$$s_i = \begin{cases} s', & \text{if } i \text{ is odd,} \\ s, & \text{if } i \text{ is even,} \end{cases}$$

obtaining

$$\bar{s} = \underbrace{s' s s' s \cdots s' s}_{2p}.$$

By definition $\hat{T}(\bar{s}) = (t_1, \ldots, t_{2p})$, with each

$$t_i = s_1 s_2 \cdots s_i \cdots s_2 s_1 = (s's)^{i-1} s'.$$

For the main relation (2.2) in W, we have $(ss')^p = Id$ and therefore

$$t_{i+1} = t_i, \quad 1 \leq i \leq p.$$

Which means clearly that for any word $t \in T$, the number of times t appears in $\hat{T}(\bar{s})$ (that is by definition $n(\bar{s}; t)$) has to be even.
Call

$$(t', \varepsilon') := (\pi_s \pi_{s'})^p (t, \varepsilon) = \pi_{s_{2p}} \pi_{s_{2p-1}} \cdots \pi_{s_1}(t, \varepsilon) = \underbrace{\pi_s \pi_{s'} \cdots \pi_{s'}}_{2p \ terms}(t, \varepsilon).$$

We want to see that:

- $t' = t$,
- $\varepsilon' = \varepsilon$.

By definition of the operator π_s and by equation (2.2) we have,

$$t' = \underbrace{ss's\cdots s'}_{2p}\,t\,\underbrace{s'\cdots s's}_{2p} = (ss')^p t(s's)^p = t,$$

while

$$\varepsilon' = \varepsilon \prod_{i=1}^{2p} \eta(s_i; s_{i-1}\cdots s_1 t s_1 \cdots s_{i-1}) = \varepsilon(-1)^{n(\bar{s};t)} = \varepsilon.$$

We can therefore apply the Universality property and conclude that we can extend $f : S \to S(R)$ to an homomorphism $\bar{f} : W \to S(R)$.
Now we need to prove that this $\bar{f}$ is in fact injective, to do this we show that its kernel is trivial.
Let $w \in W$ such that $w = s_k s_{k-1}\cdots s_1 \neq Id$, with k minimal and $\hat{T}(s_1 s_2 \cdots s_k) = (t_1, \ldots, t_k)$. By Lemma 2.17, we have that $n(s_1 s_2 \cdots s_k; t_i) = 1$, since the $t_i' s$ are all different.
Hence, by definition,

$$\pi_w(t_i,\, \varepsilon) = (wtw^{-1}, -\varepsilon),$$

and therefore

$$\pi_w \neq Id_R.$$

ii. We show the second assertion by induction on the size of a symmetric expression for t.
Let $t = s_1 s_2 \cdots s_p \cdots s_2 s_1,\ s_i \in S$.
If $t = s_1$, then

$$\pi_t(t, \varepsilon) = \pi_s(s, t) = (sss, \varepsilon\, \eta(s; s)) = (s, \varepsilon \cdot (-1)) = (t, -\varepsilon).$$

Suppose it is true for $p-1$. Then

$$\begin{aligned}
\pi_{s_1\cdots s_p\cdots s_1}(s_1\cdots s_p\cdots s_1, \varepsilon) &= \pi_{s_1}\pi_{s_2\cdots s_p\cdots s_2}(s_2\cdots s_p\cdots s_2, \varepsilon\,\eta(s_1; s_1\cdots s_p\cdots s_1)) \\
&= \pi_{s_1}(s_2\cdots s_p\cdots s_2, -\varepsilon\,\eta(s_1; s_2\cdots s_p\cdots s_2)) \\
&= (s_1\cdots s_p\cdots s_1, -\varepsilon\,\eta^2(s_1; s_2\cdots s_p\cdots s_2)) \\
&= (t, -\varepsilon),
\end{aligned}$$

where we have the second equality by induction hypothesis. □

Thanks to this result we can always extend the function η to the entire group W. Indeed, for $w \in W$ and $t \in T$, let

$$\eta(w; t) = (-1)^{n(s_1 s_2 \cdots s_k; t)}, \tag{2.7}$$

where $w = s_1 s_2 \cdots s_k$, and $s_i \in S$.

Moreover, thanks to this extension, we can write

$$\begin{aligned}
\pi_w(t,\varepsilon) &= \pi_{s_1} \cdots \pi_{s_k}(t,\varepsilon) && (2.8)\\
&= \left(s_1 \cdots s_k t s_k \cdots s_1, \varepsilon \prod_{i=1}^{k} \eta(s_i; s_1 \cdots s_{i-1} t s_{i-1} \cdots s_1) \right) && (2.9)\\
&= \left(wtw^{-1}, \varepsilon (-1)^{n(s_k \cdots s_1, t)} \right) && (2.10)\\
&= \left(wtw^{-1}, \varepsilon\, \eta(w^{-1}; t) \right). && (2.11)
\end{aligned}$$

2.3 A combinatorial characterization of Coxeter groups

In this section we prove a series of combinatorial properties satisfied by the elements of a Coxeter group, namely the Exchange and the Deletion property and, in the end, we give a combinatorial characterization of Coxeter groups.

In particular we are going to show that if a general pair (W, S), where W is a group and S a generating subset of W, with the property $s^2 = Id$, for any $s \in S$, has the so called Exchange or the Deletion property, then (W, S) is a Coxeter system.

In the previous sections we often used the writing

$$w = s_1 s_2 \cdots s_k, \quad for \;\; w \in W,$$

where $s_i \in S$, for any $i \in \{1, \ldots, k\}$. In fact S is a set of generators of the group W, so this should not surprise us.

Definition 2.19. Remembering the definition of k minimal for $w \in W$, we define the *length* of $w \in W$, as this minimal k and we write

$$l(w) = k.$$

In this case

$$w = s_1 \cdots s_k$$

is called a *reduced word* or *reduced expression* for w.

At first sight, the fact that the elements of the free monoid S^* are an ordered sequence of elements of S, while the elements of W are not, could be strange. We therefore adopt the convention that, when we are speaking about elements of S^* we consider the ordered succession of elements of S and when we are referring to elements of W we consider this ordered sequence of elements of S as being actually multiplied.

There are some crucial properties of Coxeter groups that we need to analyze. The main properties we will see are the *Exchange property* and the *Deletion property*. However, to state them, we need some previous work.

Lemma 2.20. *The map*

$$\begin{aligned} \varepsilon : S &\to \{-1, 1\} \\ s &\to -1, \end{aligned}$$

can be extended to a group homomorphism,

$$\bar{\varepsilon} : W \to \{-1, 1\}.$$

Proof. The proof is clear by looking at the Universality property. □

Let $H := \langle w \in W : l(w) \ \ is\ even \rangle$. Since $H = ker(\bar{\varepsilon})$, we can say that H is a subgroup of W and $|W : H| = |W : ker(\bar{\varepsilon})| = |\mathrm{Im}(\bar{\varepsilon})| = 2$.

Looking at the definition of length of a word we can notice that, for any $u, w \in W$,

1. $\varepsilon(w) = (-1)^{l(w)}$,
2. $l(uw) \equiv l(u) + l(w) \ (mod\, n)$,
3. $l(sw) = l(w) \pm 1$ for any $s \in S$,
4. $l(w^{-1}) = l(w)$,
5. $|l(u) - l(w)| \leq l(uw) \leq l(u) + l(w)$,
6. $l(uw^{-1})$ is a metric on W.

At this point, we are ready to state and prove the Exchange property.

Theorem 2.21 (**Exchange property**). *Suppose $w = s_1 \cdots s_k$ where $s_i \in S$ and $t \in T$. If $l(tw) < l(w)$, then $tw = s_1 \cdots \hat{s}_i \cdots s_k$ for some $i \in \{1, \ldots, k\}$.*

Proof. In order to prove it, we are going to prove the equivalence between the following statements

a) $l(tw) < l(w)$,

b) $\eta(w, t) = -1$.

The final result will follow directly from the implication $a) \Rightarrow b)$.

$b) \Rightarrow a)$:
Let $\eta(w,t) = -1$ and consider a reduced expression $w = s'_1 \cdots s'_d$. From the equivalence (2.7), we conclude that

$$n(s'_1 \cdots s'_d; t) \ \textit{is odd}$$

and this means that t appears in $\hat{T}(s'_1 \cdots s'_d)$ at least once. Therefore t is of the form

$$t = s'_1 s'_2 \cdots s'_i \cdots s'_2 s'_1$$

for some $i = 1, \ldots, d$.
Multiplying by w,

$$tw = s'_1 s'_2 \cdots \hat{s'_i} \cdots s'_d,$$

hence,

$$l(tw) = l(s'_1 s'_2 \cdots \hat{s'_i} \cdots s'_d) < d = l(w).$$

$\neg b) \Rightarrow \neg a)$:
Suppose $\eta(w;t) = 1$. Looking at the computation

$$\pi_{(tw)^{-1}}(t, \varepsilon) \overset{1}{=} \pi_{w^{-1}} \pi_t(t, \varepsilon) \overset{2}{=} \pi_{w^{-1}}(t, -\varepsilon) = (w^{-1}tw, -\varepsilon\, \eta(w,t)) = (w^{-1}tw, -\varepsilon),$$

and at the equation (2.11) applied to $\pi_{(tw)^{-1}}$,

$$\pi_{(tw)^{-1}} = ((tw)^{-1} t (tw)^{-1}, \varepsilon\, \eta(tw, t)) = (w^{-1}tw, \eta(tw, t)),$$

we can conclude that

$$\eta(tw, t) = -1.$$

[1] By definition $t = t^{-1}$ for any $t \in T$.

[2] By Theorem 2.18

Hence, from the implication $b) \Rightarrow a)$,

$$l(t \cdot tw) < l(tw),$$

that is, noticing that $t^2 = Id$ for any $t \in T$,

$$l(w) < l(tw).$$

We have finally shown the equivalence $a) \iff b)$.
Now, suppose $l(tw) < l(w)$, we want to show that $tw = s_1 \cdots \hat{s}_i \cdots s_k$, for some $i \in \{1, \dots, k\}$. We have seen that $l(tw) < l(w)$ implies $\eta(w; t) = -1$. Thus by equation (2.7), we know that $n(s_1 \cdots s_k; t)$ *is odd*, which means that t is of the form

$$t = s_1 s_2 \cdots s_i \cdots s_2 s_1$$

for some $i \in \{1, \dots, k\}$.
Hence

$$tw = s_1 \cdots \hat{s}_i \cdots s_k.$$

□

Corollary 2.22. *Let $w = s_1 \cdots s_k$ be a reduced expression and let $t \in T$. The following assertions are equivalent:*

(a) $l(tw) < l(w)$,

(b) $tw = s_1 \cdots \hat{s}_i \cdots s_k$,

(c) $t = s_1 s_2 \cdots s_i \cdots s_2 s_1$.

Furthermore, the index "i" appearing in (b) *and* (c) *is uniquely determined.*

Proof. The equivalence $(b) \iff (c)$ is trivial and uniqueness of the index "i" follows directly from Lemma 2.17.
As regarding $(a) \iff (b)$, we have that the implication from left to write is given by the previous theorem, while from right to left is trivial. □

Now let us see some important definitions, in order to state the Deletion property.

Definition 2.23. Let

$$T_L(w) := \{t \in T : l(tw) < l(w)\}.$$

We call the elements of $T_L(w)$, *left reflections* associated to $w \in W$. Similarly we define the *right reflections* associated to $w \in W$ as the elements of the set

$$T_R(w) := \{t \in T : l(wt) < l(w)\}.$$

Notice that, for any $w \in W$,

$$T_R(w) = T_L(w^{-1}).$$

Indeed,

$$\begin{aligned} t \in T_R(w) &\Rightarrow l(wt) < l(w) \\ &\Rightarrow l((wt)^{-1}) < l(w) \\ &\Rightarrow l(t^{-1}w^{-1}) < l(w^{-1}) \\ &\Rightarrow l(tw^{-1}) < l(w^{-1}) \\ &\Rightarrow t \in T_L(w^{-1}). \end{aligned}$$

Similarly for the other inclusion.
An important consequence of this equality is the following fact.

Corollary 2.24. *For $w \in W$, we have*

$$|T_L(w)| = l(w).$$

Proof. Let $w = s_1 s_2 \dots s_k$, for some $s_1, \dots, s_k \in S$, where k is the length of w. Then, by Corollary 2.22

$$T_L(w) = \{s_1 s_2 \dots s_i \dots s_2 s_1 : 1 \leq i \leq k\}.$$

Moreover, by Lemma 2.17 these elements are all distinct. $\square$

It is possible to restrict these sets of reflections to the set S of generators of W.

Definition 2.25. For $w \in W$ we define the *left descent set* as

$$D_L(w) = T_L(w) \cap S,$$

and the *right descent set* as

$$D_R(w) = T_R(w) \cap S.$$

Corollary 2.26. *For any $s \in S$ and for any $w \in W$,*

i. $s \in D_L(w) \iff$ *Some reduced expression for w begins with the letter s.*

ii. $s \in D_R(w) \iff$ *Some reduced expression for w ends with the letter s.*

Proof. i. If $s \in D_L(w)$ then $s \in \{t \in T : l(tw) < l(w)\} \cap S$ and hence s is such that $l(sw) < l(w)$, which clearly means that w starts with the letter s. Now suppose that w is of the form $w = ss_1 \cdots s_k$, $l(w) = k + 1$. We have seen that

$$l(sw) = l(w) \pm 1,$$

for any $s \in S$ and $w \in W$. In our case we would have

$$l(s \cdot ss_1 \cdots s_k) = l(w) - 1,$$

since $ss = Id$. Then

$$l(sw) < l(w),$$

and $s \in D_L(w)$.
The proof is very similar for *ii*. □

Theorem 2.27 (**Deletion property**). *Let $w = s_1 s_2 \cdots s_k$ and $l(w) < k$, then*

$$w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k,$$

for some $1 \leq i < j \leq k$.

Proof. Choose i maximal such that $s_i \cdots s_k$ is not a reduced expression. For this i we have that

$$l(s_i \cdots s_k) < l(s_{i+1} \cdots s_k),$$

since $s_{i+1} \cdots s_k$ is reduced by the maximality choice of i, while $s_i \cdots s_k$ is not. By the Exchange property we have

$$s_i s_{i+1} \cdots s_k = s_{i+1} \cdots \hat{s}_j \cdots s_k$$

for some $j > i$ and multiplying by $s_1 s_2 \cdots s_{i-1}$ we can conclude,

$$w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k.$$

□

Next result establishes a kind of unicity on the reduced expressions for the elements of a Coxeter group.

Corollary 2.28. *Let (W, S) be a Coxeter system.*

(i) *Any expression $w = s_1 \cdots s_k$ contains a reduced expression for w as a sub-word, obtainable by deleting an even number of letters.*

(ii) *Suppose*

$$w = s_1 \cdots s_k = s'_1 \cdots s'_k$$

are two reduced expressions. Then, the set of letters appearing in the word $s_1 s_2 \cdots s_k$ equals the set of letters appearing in $s'_1 s'_2 \cdots s'_k$.

(iii) *S is a minimal generating set for W.*

Proof. Note that (i) is an immediate consequence of the deletion property, and that (iii) follows directly from (ii).
Let us prove (ii). Consider j as the minimal index such that $s_j \notin \{s'_1, s'_2, \ldots, s'_k\}$, then $t := s_1 s_2 \cdots s_j \cdots s_2 s_1$ that for Corollary 2.22 we can also write as

$$t = s'_1 s'_2 \cdots s'_i \cdots s'_2 s'_1.$$

Hence

$$s_1 s_2 \cdots s_j \cdots s_2 s_1 = s'_1 s'_2 \cdots s'_i \cdots s'_2 s'_1,$$

that is

$$s_j = s_{j-1} \cdots s_1 s'_1 \cdots s'_i \cdots s'_1 s_1 \cdots s_1.$$

By the minimality of j all the terms at the right of the equality are in $\{s'_1, s'_2, \ldots, s'_k\}$, which contradicts the fact that $s_j \notin \{s'_1, s'_2, \ldots, s'_k\}$. □

We are finally ready to prove the main result of this section, that is the combinatorial characterization of a Coxeter system.

Theorem 2.29. *Let W be a group, and S a set of generators of order 2. Then the following are equivalent:*

i. *(W, S) is a Coxeter system.*

ii. *(W, S) has the Exchange property.*

iii. *(W, S) has the Deletion property.*

Proof. Notice that we already proved $i. \Rightarrow ii.$ in the proof of the Exchange property for Coxeter systems and that we showed $ii. \Rightarrow iii.$ in the proof of the Deletion property. It is true that when we proved the deletion property we were referring to a Coxeter system, but looking carefully to the proof we can observe that we never use that hypothesis.
(3. $\Rightarrow$ 2.):
Suppose the Deletion property holds. Let $w = s_1 \cdots s_k$ such that $l(w) = k$. We need to show that if $l(sw) \leq l(w)$ then

$$sw = s_1 \cdots \hat{s}_i \cdots s_k,$$

for some $i \in [k]$.
Consider

$$l(sw) = l(ss_1 \cdots s_k) \leq l(s_1 \cdots s_k) = k,$$

then by the Deletion property, sw can be written with two elements less. Suppose s is not one of those two elements, then

$$sw = ss_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k,$$

for some $i, j \in [k]$.
Then $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$ and

$$l(w) = l(s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k) < k,$$

which is a contradiction.
Therefore we need to have that s is one of those elements, that is

$$sw = ss_1 \cdots s_k = s_1 \cdots \hat{s}_j \cdots s_k.$$

That is exactly what we wanted to prove.

(ii. $\Rightarrow$ i.):
Suppose the Exchange property holds for (W, S), we need to show that it is possible to define a structure of a Coxeter system on it.
Suppose the relation $s_1 \cdots s_r = Id$ holds. We have seen that the Exchange property implies the Deletion property. Hence, since

$$l(s_1 \cdots s_r) = l(Id) = 0,$$

for the Deletion property we could erase the elements of $s_1 \cdots s_r$ pair-wisely until we reach the zero length, and this means that r in $s_1 \cdots s_r$ is even, let us say $r = 2k$ for a certain $k \in \mathbb{N}$. Therefore we have that $s_1 \ldots s_r = Id$ is equivalent to

$$s_1 s_2 \cdots s_k = s'_1 s'_2 \cdots s'_k. \tag{2.12}$$

We build now a structure of a Coxeter system by showing that this relation is a consequence of the relation

$$(ss')^{m(s,s')} = Id,$$

where $m(s, s')$ is defined as the order of the product ss' whenever it is finite. We are going to prove this by induction on k.

Case $k = 1$: trivial.

Induction hypothesis: We assume that any relation of length less then $2k$ can be derived by $(ss')^{m(s,s')} = Id$. For simplicity we say that those relations are "fine".

Case $2k$**:** We are going to consider two particular cases, the case in which $s_1 \cdots s_k$ in relation (2.12) is not reduced, and the case in which it is.

Sub-case 1. $s_1 \cdots s_k$ is not reduced. This means that we can find a maximum $i \in [k]$ such that $s_{i+1} \cdots s_k$ is reduced, but $s_i s_{i+1} \cdots s_k$ is not. Then

$$l(s_i s_{i+1} \cdots s_k) \leq l(s_{i+1} \cdots s_k),$$

which means, by the Exchange property, that

$$s_i s_{i+1} \cdots s_k = s_{i+1} \cdots \hat{s_j} \cdots s_k, \tag{2.13}$$

which is a relation of length less than $2k$ and therefore fine by induction hypothesis.

Now substituting (2.13) in (2.12) we find

$$s_1 \cdots s_{i-1} s_{i+1} \cdots s_k = s'_1 s'_2 \cdots s'_k,$$

that is of length less than $2k$. We conclude by induction hypothesis.

Sub-case 2. $s_1 \cdots s_k$ is reduced.

Without loss of generality we can assume that $s_1 \neq s'_1$, because otherwise, equation (2.12) is trivially equivalent to a shorter relation. Thanks to the Exchange property we have that, for some $1 \leq i \leq k$,

$$s_1 s_2 \ldots s_i = s'_1 s_1 \ldots s_{i-1}. \tag{2.14}$$

From equations (2.12) and (2.14), we conclude that

$$s_1 \ldots \hat{s_i} \ldots s_k = s'_2 s'_3 \ldots s'_k.$$

The latter equation is of length less than $2k$ and therefore is fine. But then also the equation

$$s'_1 s_1 \dots \hat{s}_i \dots s_k = s'_1 s'_2 \dots s'_k \tag{2.15}$$

is fine.

If $i < k$, then also equation (2.14) is fine. But this means that we are done, since equation (2.12) is obtained by substituting equation (2.14) in (2.15).

If $i = k$, then relation (2.15), that is fine, becomes

$$s'_1 s_1 \dots s_{k-1} = s'_1 s'_2 \dots s'_k.$$

So what we need to show is that

$$s'_1 s_1 \dots s_{k-1} = s_1 s_2 \dots s_k \tag{2.16}$$

is fine. Now we want to apply the same exact reasoning of Case 2, using equation (2.16) instead of (2.14). If not settled along the way, at this point we need to understand if

$$s'_1 s_1 \dots s_{k-1} = s_1 s'_1 s_1 \dots s_{k-2}$$

is fine. To understand the latter, we apply another time the reasoning of Case 2, obtaining that the question is whether

$$s_1 s'_1 s_1 \dots s_{k-1} = s'_1 s_1 s'_1 s_1 \dots s_{k-3}$$

is fine, and so on. Eventually we will obtain equation

$$s_1 s'_1 s_1 s'_1 \dots = s'_1 s_1 s'_1 s_1 \dots,$$

which is fine, since it is of course implied by $(s_1 s'_1)^{m(s_1, s'_1)} = Id$.

□

2.4 The Bruhat order

In this section we introduce the Bruhat order, a partial order defined on each Coxeter group that has incredible combinatorial properties. Unless otherwise specified, we will refer to the notions introduced by Anders Bjorner and Francesco Brenti in [1].

Consider (W, S) a Coxeter system with set of reflections

$$T = \{wsw^{-1} : w \in W, s \in S\}.$$

For any $u, w \in W$ we write:

1. $u \xrightarrow{t} w$ meaning that $l(u) < l(w)$ and $u^{-1}w = t \in T$;

2. $u \to w$ meaning that there exists $t \in T$ such that u $\xrightarrow{t}$ w;

3. $u \leq w$ meaning that there exist $u_i \in W$ such that

$$u = u_0 \to u_1 \to \cdots \to u_{k-1} \to u_k = w.$$

The Bruhat order is the partial order defined by property 3. Moreover we define the Bruhat graph as the directed graph whose nodes are the elements of the Coxeter group W and whose edges are given by property 2., meaning that, for $u, w \in W$, there is an arrow from u to w if and only if $u \to w$.

Let us analyze some particular properties of Bruhat order that we will extensively use afterwords.
We want to prove the so called *Subword property*, that is a characterization of the Bruhat order. First of all let us define what a subword is.

Definition 2.30. By *subword* of a word $s_1 \ldots s_q$ we mean a word of the form

$$s_{i_1} s_{i_2} \ldots s_{i_k},$$

where $1 \leq i_1 < \cdots < i_k \leq q$.

Lemma 2.31. *Let $u, w \in W$, with $u \neq w$. Let $s_1 s_2 \ldots s_q$ be a reduced expression for w and suppose that exists a reduced expression of u that is a subword of $s_1 s_2 \ldots s_q$. Then, there exists $v \in W$ such that:*

- *$v > u$;*
- *$l(v) = l(u) + 1$;*
- *Some reduced expression for v is a subword of $s_1 s_2 \ldots s_q$.*

Proof. Suppose that the reduced expression

$$u = s_1 \ldots \hat{s_{i_1}} \ldots \hat{s_{i_k}} \ldots s_q,$$

with $1 \leq i_1 < \cdots < i_k \leq q$, is such that i_k is minimal and let

$$t := s_q s_{q-1} \ldots s_{i_k} \ldots s_{q-1} s_q.$$

So,

$$\begin{aligned} ut &= (s_1 \dots \hat{s_{i_1}} \dots \hat{s_{i_k}} \dots s_q)(s_q s_{q-1} \dots s_{i_k} \dots s_{q-1} s_q) \\ &= s_1 \dots \hat{s_{i_1}} \dots \hat{s_{i_{k-1}}} \dots s_{i_k} \dots s_q \end{aligned}$$

and therefore $l(ut) \le l(u) + 1$.
If we show that $ut > u$ then $v := ut$ satisfies the three required properties.
Suppose by contradiction that $ut < u$, then $l(ut) < l(u)$ and by the strong Exchange property

$$ut = s_1 \dots \hat{s_{i_1}} \dots \hat{s_r} \dots \hat{s_{i_k}} \dots s_q, \tag{2.17}$$

where r is different from each one of the i_j and is such that $1 \le r \le q$. Observe that in equation (2.17) we put just for convenience $\hat{s_r}$ in that specific position but we could have perfectly $r > i_k$, or $r < i_1$. Suppose $r > i_k$, then

$$t = s_q s_{q-1} \dots s_r \dots s_{q-1} s_q,$$

but in this case

$$\begin{aligned} w &= wt^2 \\ &= (s_1 s_2 \dots s_q)(s_q s_{q-1} \dots s_{i_k} \dots s_{q-1} s_q)(s_q s_{q-1} \dots s_r \dots s_{q-1} s_q) \\ &= s_1 s_2 \dots \hat{s_{i_k}} \dots \hat{s_r} \dots s_q, \end{aligned}$$

which contradicts the fact that $l(w) = q$.
So suppose that $r < i_k$, then

$$t = s_q s_{q-1} \dots \hat{s_{i_k}} \dots s_r \dots \hat{s_{i_k}} \dots s_{q-1} s_q,$$

but then

$$\begin{aligned} u &= ut^2 \\ &= (s_1 \dots \hat{s_{i_1}} \dots \hat{s_{i_k}} \dots s_q)(s_q s_{q-1} \dots \hat{s_{i_k}} \dots s_r \dots \hat{s_{i_k}} \dots s_{q-1} s_q)(s_q s_{q-1} \dots s_r \dots s_{q-1} s_q) \\ &= s_1 s_2 \dots \hat{s_{i_1}} \dots \hat{s_r} \dots s_{i_k} \dots s_q, \end{aligned}$$

which contradicts the minimal choice of i_k.
Therefore we should have $ut > u$ concluding the proof. □

Theorem 2.32 (Subword property). *Let $w, u \in W$, $w = s_1 \dots s_q$ a reduced expression. Then*

$$u \le w \iff u = s_{i_1} s_{i_2} \dots s_{i_k}, \;\; 1 \le i_1 < \dots < i_k \le q,$$

where $u = s_{i_1} s_{i_2} \dots s_{i_k}$ is a reduced expression.

Proof. ($\Rightarrow$) Suppose $u \leq w$, then we know that there exist $t_1, \dots, t_m$, $m \in \mathbb{N}$, and $x_0, \dots, x_m \in W$ such that

$$u = x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_m} x_m = w.$$

This implies $w = x_{m-1}t_m$ by definition, that multiplying by t_m is $wt_m = x_{m-1}t_m^2$, that is

$$wt_m = x_{m-1}.$$

Therefore we have $wt_m < w$ and consequently $l(wt_m) < l(w)$, so that we can apply the Exchange property obtaining

$$wt_m = s_1 \dots \hat{s_i} \dots s_q,$$

for some $i \in [q] = \{1, ..., q\}$. Clearly we can apply the same reasoning to x_{m-2}.
$x_{m-2} = x_{m-1}t_{t_{m-1}}$ that by the Exchange property is $s_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_q$.
Going on, we obtain an expression for u by deleting m letters from $s_1 \dots s_q$ of this form

$$s_1 \dots \hat{s_1} \dots \hat{s_m} \dots s_q.$$

If the length of u is $q - m$ we are done and we found the required expression. Otherwise, if $l(u) < q - m$ we can apply the Deletion property and conclude.

($\Leftarrow$) We are going to prove this implication by induction on the quantity $l(w) - l(u)$.
Suppose $l(w) - l(u) = 0$ then $l(w) = l(u)$ and trivially $w = u$. Thus $u \lneq w$.
Suppose $l(w) - l(u) = 1$, then $l(w) = l(u) + 1$. From the previous Lemma we know that there exists $v \in W$ such that $v > u$, $l(v) = l(u) + 1$ and some reduced expression for v is a subword of $w = s_1 \dots s_q$. But then $l(v) = l(w)$ and v is a subword of w, then $v = w$. Thus $w > u$.
Now suppose this result holds for $l(w) - l(u) = n$, and we are showing it for $l(w) - l(u) = n + 1$.
We know that $u = s_1 \dots \hat{s_{i_1}} \dots \hat{s_{i_n}} \dots \hat{s_{i_{n+1}}} \dots s_q$ is a reduced expression for u and that it is a subword of $w = s_1 \dots s_q$. Thus we can apply the previous Lemma and we know that there exists $v \in W$ such that $v > u$, $l(v) = l(u)+1$ and some reduced expression for v is a subword of $w = s_1 \dots s_q$.
Then we need to have

$$v = s_1 \dots \hat{s_{i_1}} \dots s_{i_j} \dots \hat{s_{i_{n+1}}} \dots s_q,$$

where $i_j \in [i_1, \dots, i_{n+1}]$. Hence, by induction hypothesis, $v \leq w$, and by transitivity $u \leq w$. $\square$

Chapter 3

Bruhat order on the symmetric group

The symmetric group is widely studied during the course of university studies, but given the centrality of this group throughout this work, we have decided to make a small introductory section to remember its main definitions and characteristics, taking advantage of it to underline those aspects that we will use most in the next chapters. Initially we will try to follow the introduction that Sagan, in his book 'The symmetric Group' [5], makes of the symmetric group. Then we will also give some small results that are easily found in the book 'Combinatorics of Coxeter Groups' by Bjorner and Brenti [1], widely used in this thesis work.
We will prove that the symmetric group S_n, is a Coxeter group. This means that we can give to each S_n the structure of a Poset, $(S_n, \leq)$, where $\leq$ is the Bruhat order on S_n.

We will characterize the minimal chains on the symmetric group from its covering relation, following the paper of F. Incitti [3].

3.1 The symmetric group

First of all let us define the symmetric group. Let n be a positive integer, the symmetric group denoted by S_n, is the group of all bijections from $[n] := \{1, \ldots, n\}$ to itself, endowed with the composition. The elements of the symmetric group are called permutations. Given two permutations

$\sigma, \tau \in S_n$ we compose σ and τ from the right to the left, meaning

$$\sigma\tau = \sigma \cdot \tau = \sigma \circ \tau,$$

that is, first applying τ and then σ.
Each element of S_n can be written with a synthetic writing in such a way that the permutation is totally described.
Let σ be an element of S_n. Considering $i \in [n]$ we have that for a certain $k \in [n]$, $i, \sigma(i), \sigma^2(i), \dots, \sigma^{k-1}(i)$, are all different, while $\sigma^k(i) = i$. This of course happens because σ is a bijection. So we have the cycle of i, $(i\ \sigma(i)\ \sigma^2(i) \dots \sigma^{k-1}(i))$. Now we take an element of $[n]$ that is not in the cycle of i and we do the same thing. So suppose this element is $j \in [n]$, then we have

$$(j\ \sigma(j)\ \sigma^2(j) \dots \sigma^{p-1}(j)),$$

for a certain $p \in [n]$. Iterating this process until we have all the elements of $[n]$, we obtain a complete description of the permutation σ,

$$(i\ \sigma(i)\ \sigma^2(i) \dots \sigma^{k-1}(i))(j\ \sigma(j)\ \sigma^2(j) \dots \sigma^{p-1}(j)) \cdots .$$

We call this way of writing a permutation *cycle notation*. Moreover we say that the cycle $(i\ \sigma(i) \dots \sigma^{k-1}(i))$, is a $k-$*cycle*, since it has k distinct elements. It can of course happen that a permutation $\sigma \in S_n$ sends an $i \in [n]$ into itself,

$$i = \sigma(i).$$

This means, for what we said before, that we would have a 1-cycle and by convention those cycles are generally omitted in the cycle notation. We call such an element a fixed point of the permutation.

Example 3.1. *Consider the permutation $\sigma \in S_8$ such that*

$$1 \to 7,\ 2 \to 4,\ 3 \to 1,\ 4 \to 8,\ 5 \to 5,\ 6 \to 2,\ 7 \to 3,\ 8 \to 6.$$

Then we can describe it in cycle notation applying the same process we explained before. 1 *is sent in* 7*,* 7 *is sent in* 3 *and* 3 *is sent in* 1*, meaning that we obtain the* 3*-cycle*

$$(173).$$

Than we take another number in $[8]$ *that does not appear in the* 3*-cycle and we apply the same reasoning. At the end we will obtain the cycle notation for* σ*,*

$$(173)(2486),$$

omitting the 1*-cycle* (5)*.*

More in detail, we call *transposition* to each 2-cycle $(i \ \ j)$, with $i, j \in [n]$, $i \neq j$. A particular kind of transpositions are the *adjacent transpositions*, that are permutations of the form $(i \ \ i+1)$, where $i \in \{1, \ldots, n-1\}$. It is well known that S_n can be generated by the set of all adjacent transpositions. We denote the set of all adjacent transpositions by

$$S := \{s_1, \ldots, s_{n-1}\},$$

where $s_i = (i \ \ i+1)$ for any $i = 1, \ldots, n-1$.

Let us introduce another important way of describing a permutation in S_n that we will largely use during this all work, the *complete notation* of a permutation. For any $\sigma \in S_n$, its complete notation consists in orderly listing all the images through σ of the elements $1, .., n$,

$$\sigma(1)\sigma(2) \ldots \sigma(n).$$

For instance, we write the permutation $(173)(2486) \in S_8$ as

$$74185236,$$

where we listed the images of the elements $1, 2, \ldots, 8$ through the permutation.
We observe that with this notation, multiplying an element $\sigma \in S_n$, $\sigma = \sigma(1) \cdots \sigma(n)$, by a transposition $(i \ \ j)$, $i < j$, on the right, means transposing the values in position i and j, that is

$$\sigma(1) \cdots \sigma(i) \cdots \sigma(j) \cdots \sigma(n) \cdot (i \ \ j) = \sigma(1) \cdots \sigma(j) \cdots \sigma(i) \cdots \sigma(n).$$

While multiplying σ on the left by $(i \ \ j)$, means to transpose the elements of σ which are equal to i and j, that is, if $\sigma(k) = i$ and $\sigma(p) = j$, without loss of generality $k < p$, then

$$(i \ \ j) \cdot \sigma(1) \cdots \sigma(k) \cdots \sigma(p) \cdots \sigma(n) = \sigma(1) \cdots \sigma(p) \cdots \sigma(k) \cdots \sigma(n).$$

Thanks to this notation we can we can easily define the length of an element of S_n with respect to the set of generators S.

Definition 3.2. We define the length of $\sigma = \sigma(1) \cdots \sigma(n) \in S_n$, and we denote it by $l_A(\sigma)$, as the minimum number of adjacent transpositions we have to compose to obtain σ from the identity.

For instance consider the element $\sigma = 1342 \in S_4$, written on the complete notation. We have

$$1342 = Id \cdot s_2 \cdot s_3 = 1234 \cdot 1324 \cdot 1243.$$

Since 1342 is not a transposition we have that the length of 1342 is 2.

Definition 3.3. For $\sigma \in S_n$, denote

$$inv(\sigma) = |\{(i, j) \in [n]^2 : i < j, \sigma(i) > \sigma(j)\}|$$

and call it the *inversion number* of σ.

For example let $\sigma = 1342 \in S_4$, then $\sigma(1) = 1, \sigma(2) = 3, \sigma(3) = 4$ and $\sigma(4) = 2$. So $inv(\sigma) = 2$.

Recalling that $_i = (i \quad i+1)$, we observe that

$$inv(\sigma s_i) = \begin{cases} inv(\sigma) + 1, & \text{if } \sigma(i) < \sigma(i+1) \\ inv(\sigma) - 1, & \text{if } \sigma(i) > \sigma(i+1) \end{cases}. \tag{3.1}$$

Thanks to this fact we will be able to prove that the length and the inversion number of a permutation coincide.

Proposition 3.4. *Let $\sigma \in S_n$, then*

$$l_A(\sigma) = inv(\sigma).$$

Proof. 1) $inv(\sigma) \leq l_A(\sigma)$: First of all notice that $inv(Id) = l_A(Id) = 0$. Therefore, by equation (3.1), we can choose to multiply the identity for some elements of S such that the number of inversions increase, and with that also the length increases. But at a certain point, since the maximum number of inversions must be finite we will not be able to increase it anymore, and always by equation (3.1) we obtain $inv(\sigma) \leq l_A(\sigma)$.
2) $inv(\sigma) \geq l_A(\sigma)$: We are going to prove this inequality by induction on $inv(\sigma)$.
Suppose $inv(\sigma) = 0$. Then $inv(\sigma) = 12 \dots n = Id$, since by definition $l_A(Id) = 0$, then of course $inv(\sigma) \geq l_A(\sigma)$.
Suppose that we have the desired result for $inv(\sigma) = k$ for a certain $k \in \mathbb{N}$. Let now $\sigma \in S_n$ such that $inv(\sigma) = k + 1$. If this happens, then $\sigma \neq Id$,

because the number of inversions of the Identity is zero and here is at least one. This means that, by equation (3.1) we could find an $s \in S$ such that

$$inv(\sigma \cdot s) = (k+1) - 1 = k.$$

Note that if it is not possible to find an element starting from σ with less inversions this would mean that σ is the Identity.
By induction hypothesis we have that

$$l_A(\sigma s) \leq inv(\sigma s) = k,$$

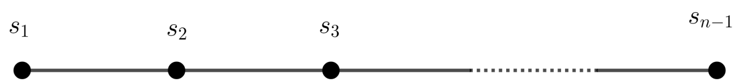
hence

$$l_A(\sigma) \leq k + 1 = inv(\sigma).$$

□

3.2 The symmetric group as Coxeter group

The aim of this section is to prove that the symmetric group S_n together with the set S of the adjacent transpositions is a Coxeter system. We are also going to prove that (S_n, S) is a Coxeter system of a specific type.
Generally we define the type of a Coxeter system by looking at its graph. Whenever the Coxeter graph associated with the Coxeter system is of the form,



we say that the type of the Coxeter system is A_{n-1}.

We will see at first that the length of a permutation is related to the notion of descent set.
Observe that for simplicity we are only going to define the right descent set, because we mostly consider the multiplication on the right. Of course it is possible to give a description like the one in (3.1) by considering the multiplication on the left, but this would be more complicated.

Proposition 3.5. *Let $\sigma \in S_n$ and S be the set of adjacent transpositions. Then*

$$D_R(\sigma) = \{s_i \in S : \sigma(i) > \sigma(i+1)\}.$$

Proof. By Proposition 3.4 and Definition 2.25 of right descent set, we have

$$D_R(\sigma) = \{s \in S : inv(\sigma s) < inv(\sigma)\}.$$

By equation (3.1) we have that the number of inversions in σ is greater than the number of inversions in σs. This means that s is permuting some $\sigma(i)$, $\sigma(i+1)$ that were in decreasing order before multiplying by s. Therefore

$$D_R(\sigma) = \{s_i \in S : \sigma(i) > \sigma(i+1)\}.$$

$\square$

We are finally ready to prove the main result of this section.

Proposition 3.6. *(S_n, S) is a Coxeter system of type A_{n-1}.*

Proof. First of all we observe that, for any $i, j \in [n]$,

$$\begin{cases} s_i s_j = s_j s_i, & if \ \ |i-j| \geq 2 \\ s_i s_j s_i = s_j s_i s_j, & if \ \ |i-j| = 1 \end{cases}.$$

Suppose for a moment that we have already shown that (S_n, S) is a Coxeter system. Then from the observation above we can see that, if $|i-j| = 1$, then $(s_j s_i)^3 = Id$, whereas if $|i-j| \geq 2$, then $(s_j s_i)^2 = Id$. This implies, by definition of Coxeter graph, that the nodes of the graph that are connected are just the ones that have i and j consecutive. Hence the type of the graph would be A_{n-1}.
Now we need to show that (S_n, S) is indeed a Coxeter system.
Thanks to Theorem 2.29, instead of trying to show directly that (S_n, S) is a Coxeter system, we are showing that it has the Exchange Property (in this case the "right" Exchange Property as follows).
Let $i, i_1, \ldots, i_p \in [n-1]$, let $\sigma = s_{i_1} \cdots s_{i_p} \in S_n$ be a reduced expression, and suppose

$$l_A(\sigma s_i) = l_A(s_{i_1} \cdots s_{i_p} s_i) < l_A(s_{i_1} \cdots s_{i_p}) = l_A(\sigma). \tag{3.2}$$

We have to show that there exists $j \in [p]$ such that

$$\sigma s_i = s_{i_1} \cdots \hat{s_{i_j}} \cdots s_{i_p}.$$

By Proposition 3.4 and equation (3.2) we know that

$$inv(\sigma s_i) < inv(\sigma),$$

which means, by (3.1) that

$$\sigma(i) > \sigma(i+1),$$

where $\sigma(i)$ is the i^{th} element of σ in the complete notation $\sigma = \sigma(1)\cdots\sigma(n)$. Therefore, in the complete notation of σ, we have $\sigma(i)$ to the left of $\sigma(i+1)$ even if it is greater. While in the complete notation of the identity we have of course $\sigma(i+1)$ to the left of $\sigma(i)$.
Recall that $\sigma = s_{i_1}\cdots s_{i_p}$ is generated by elements of S starting from the identity, doing successive transpositions. For this reason we have that the elements $\sigma(i)$, $\sigma(i+1)$ have to be in increasing order ($\sigma(i+1)$ an then $\sigma(i)$) until we multiply by a certain transposition, s_{i_j}.
This is, $\sigma(i+1)$ is at the left of $\sigma(i)$ in

$$s_{i_1}\cdots s_{i_{j-1}},$$

while is at the right of $\sigma(i)$ in

$$s_{i_1}\cdots s_{i_j}.$$

So, since $\sigma = s_{i_1}\cdots s_{i_j}\cdots s_{i_p}$, where s_{i_j} makes the change

$$\sigma(i+1) \rightleftarrows \sigma(i),$$

when we apply s_i to σ, we are just changing again

$$\sigma(i+1) \rightleftarrows \sigma(i),$$

which is the same of never applying s_{i_j},

$$\sigma = s_{i_1}\cdots \hat{s_{i_j}}\cdots s_{i_p}.$$

$\square$

Now that it is evident that the symmetric group is indeed a Coxeter group, it should be clear that we can give to each S_n the structure of a Poset, $(S_n, \leq)$, where $\leq$ is the Bruhat order on S_n.

3.3 The covering relation on the symmetric group

In this section we will characterize the covering relation for the Bruhat order on the symmetric group S_n.
Recall that S_n is a Coxeter group with set of generators

$$S = \{(i \;\; i+1) : i \in [n-1]\}.$$

Its set of reflections is

$$T = \{(a\ b) : 1 \leq a < b \leq n\},$$

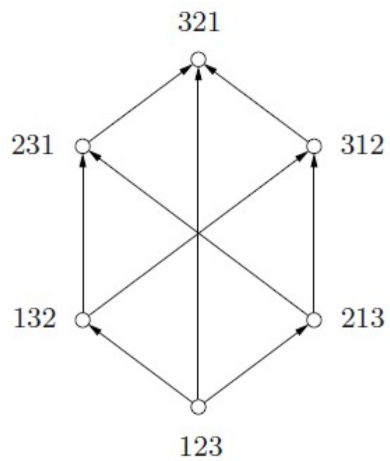
since for every $\sigma \in S_n$ we have

$$\sigma(i \quad i+1)\sigma^{-1} = (\sigma(i)\ \sigma(i+1)).$$

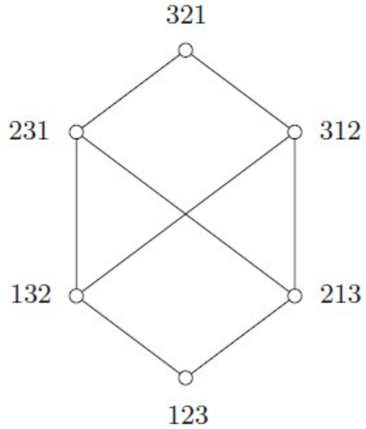
Example 3.7. *Consider the symmetric group*

$$S_3 = \{123, 132, 213, 231, 312, 321\}.$$

The set of generators of S_3 is $S = \{(1\,2), (2\,3)\}$ and the set of reflections is $T = \{(1\,2), (1\,3), (2\,3)\}$.
To draw the Hasse diagram of S_3 we need to understand whether and how two elements of the group are connected. Let us first draw the Bruhat graph from which we can easily obtain the Hasse diagram. Basically the idea with the symmetric group is to start from the identity, going on multiplying by elements of T.
So, consider the identity 123*,* $123 \cdot (1\,2) = 213$*, then* $123 \xrightarrow{(1\,2)} 213$*. Similarly* $123 \xrightarrow{(2\,3)} 132$ *and* $123 \xrightarrow{(1\,3)} 321$*.*
Now $213 \cdot (1\,3) = 312$ *and* $213 \cdot (2\,3) = 231$*, so that* $213 \xrightarrow{(1\,3)} 312$ *and* $213 \xrightarrow{(2\,3)} 231$*. Going on with this process we will obtain the following directed graph.*



So now to draw the Hasse diagram we just need to delete the edges of the Coxeter graph that do not correspond to the covering relation, since by convention if u is connected to v that is connected to h, even if of course by *transitivity u is connected to h, we don't draw a direct arrow from u to h. We therefore obtain the following Hasse diagram*



Of course the method we just used in the example is not a good way to understand if two elements are related in the Bruhat order. The reason is that when we have a bigger group it is way more difficult to do all those computations.
Luckily we have a criterion to decide whether an element σ is covered by another element τ in the Bruhat order of the symmetric group.

Lemma 3.8. *Let $\sigma, \tau \in S_n$. Then σ is covered by τ in Bruhat order (and we write $\sigma \lhd \tau$) if and only if $\tau = \sigma \cdot (a\, b)$ for some $a < b$ such that $\sigma(a) < \sigma(b)$ and there does not exist any c such that $a < c < b$ and $\sigma(a) < \sigma(c) < \sigma(b)$.*

Proof. Suppose that there exists $(a\, b)$ with $a < b$ such that $\tau = \sigma \cdot (a\, b)$ and there is no c, $a < c < b$, such that $\sigma(a) < \sigma(c) < \sigma(b)$. This means that $inv(\tau) = inv(\sigma) + 1$. Thus we have a covering. On the other hand, suppose $\sigma \lhd \tau$. Then $\sigma \cdot (a\, b)$ for some $(a\, b)$. By contradiction, suppose there exists c, $a < c < b$ such that $\sigma(a) < \sigma(c) < \sigma(b)$. This means that $\sigma < \sigma(a\, c) < \tau$ (because we can see τ as $\sigma(a\, c)(a\, b)(c\, b)$). But this is a contradiction because we would have that τ does not cover σ. $\square$

In 2003, Federico Incitti [3] states this Lemma in a more elegant way by introducing the following definition.

Definition 3.9. Let σ be an element of S_n. We say that a pair $(a, b) \in [n]^2$ such that $a < b$ and $\sigma(a) < \sigma(b)$ is a *rise* of σ. Moreover, if there is no $c \in [n]$ such that $a < c < b$ and $\sigma(a) < \sigma(c) < \sigma(b)$ then we say that (a, b) is a *free rise* of σ.

Thanks to this definition the covering relation in S_n can be characterized in terms of free rises. This characterization will be very useful in the next chapter.

Lemma 3.10. *Let $\sigma, \tau \in S_n$ such that $\sigma < \tau$. Then $\sigma \lhd \tau$ if and only if $\tau = \sigma(a\,b)$ where (a, b) is a free rise of σ.*

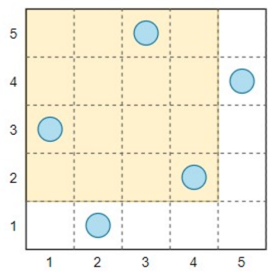
Unfortunately, already for $n > 5$ also this criteria is not very effective if we want to know whether two elements of S_n are comparable.

Following the paper of F. Incitti [3], we can define for any permutation $\sigma \in S_n$, the quantity

$$\sigma[i, j] := |\{a \in [i] : \sigma(a) \geq j\}|, \quad for\ i, j = 1, \ldots, n. \tag{3.3}$$

This apparently very abstract definition has a clear graphical display, called dot-counting interpretation. In a general way, given a permutation $\sigma \in S_n$ we draw a square and we place a dot in each point with coordinates $(a, \sigma(a))$, for $a \in [n]$. Then $\sigma[i, j]$ defined in (3.3) counts exactly the number of dots in the northwest corner above the point (i, j).

As an example, consider the permutation $\sigma = 31524 \in S_5$. Drawing the 5×5 square and placing the dots like in figure below we can easily see for instance that $\sigma[4, 2] = 3$.



Thanks to the previous definition, it is possible to obtain a useful criteria that gives a better characterization of comparable permutations in the Bruhat order.

Theorem 3.11. *Let $\sigma, \tau \in S_n$. Then, the following are equivalent:*

(i) $\sigma \leq \tau$.

(ii) $\sigma[i, j] \leq \tau[i, j]$, *for all* $i, j \in [n]$.

Proof. $(i) \Rightarrow (ii)$: Suppose $\sigma \leq \tau$. Without loss of generality we can assume that there exists $t \in T$, of the form (a, b), with $a < b$, such that $\tau = \sigma(a, b)$. Moreover since $\sigma \leq \tau$, we need to have $\sigma(a) < \sigma(b)$. Thanks to (3.3), this means

$$\tau[i,j] = \begin{cases} \sigma[i,j] + 1 & \text{if } a \leq i < b \text{ and } \sigma(a) < j \leq \sigma(b), \\ \sigma[i,j] & \text{otherwise.} \end{cases} \tag{3.4}$$

Indeed, write σ as $\sigma(1) \cdots \sigma(a) \cdots \sigma(b) \cdots \sigma(n)$. Then τ is obtained from σ by swapping $\sigma(a)$ and $\sigma(b)$, which is $\sigma(1) \cdots \sigma(b) \cdots \sigma(a) \cdots \sigma(n)$. This means that the entries $\sigma(1) \ldots \sigma(a-1)$ and $\sigma(b+1) \ldots \sigma(n)$ are not modified. Writing σ and τ in this way it is not difficult to see that the only way $\tau[i,j]$ can differ from $\sigma[i,j]$ is when $a \leq i < b$ and $\sigma(a) < j \leq \sigma(b)$. Thus, for any $i, j \in [n]$, we have $\sigma[i,j] \leq \tau[i,j]$.

(ii. $\Rightarrow$i.:) Let $M(i,j) := \tau[i,j] - \sigma[i,j]$ for all $i, j \in [n]$.
Clearly if $M(i,j) = 0$ for any $i, j \in [n]$ we have to conclude that σ and τ are the same permutation.
Now let us choose $(a_1, b_1) \in [n]^2$ in such a way that $M(a_1, b_1) > 0$ and $M(i,j) = 0$ for any $(i,j) \in [1, a_1] \times [b_1, n] - \{(a_1, b_1)\}$.
Looking at the dot-counting interpretation we explained just before this theorem, we can notice that this choice of a_1 and b_1 implies that $\tau(a_1) = b_1$ and $\sigma(a_1) < b_1$. At this point, let $(a_2, b_2) \in [n]^2$ be the bottom right corner of a maximal positive connected submatrix of M having (a_1, b_1) as the upper left corner. Observing that for any permutation $x \in S_n$ the following holds,

$$x[n,i] = n + 1 - i \text{ and } x[i,1] = i, \text{ for any } i \in [n],$$

we have that $a_2 < n$ and $b_2 > 1$. Because of the maximality there exist $c \in [a_1, a_2]$ and $d \in [b_2, b_1]$ such that

$$M(c, b_2 - 1) = 0 \text{ and } M(a_2 + 1, d) = 0,$$

and therefore

$$M(a_2 + 1, b_2 - 1) - M(c, b_2 - 1) - M(a_2 + 1, d) + M(c, d) > 0.$$

We can notice that for any permutation $x \in S_n$ and for all $1 \leq k \leq i \leq n$ and $1 \leq j \leq l \leq n$, we have

$$x[i,j] - x[k,j] - x[i,l] + x[k,l] = |\{a \in [k+1, i] : j \leq x(a) < l\}|,$$

which is clear by the dot-counting interpretation.
Hence in our situation we have

$$|\{e \in [c+1, a_2+1] : \tau(e) \in [b_2 - 1, d - 1]\}| > 0.$$

Now, choose $(a_0, b_0) \in [c+1, a_2+1] \times [b_2-1, d-1]$ such that $\tau(a_0) = b_0$, then we must have $a_1 < a_0$ and $\tau(a_1) = b_1 > b_0 = \tau(a_0)$. At this point, considering $\nu := \tau(a_1, a_0)$, we must have $\nu \to \tau$ and so that $\nu \leq \tau$. But looking at equation (3.4) and the wise choice of (a_2, b_2) we made, we have that $\sigma[i,j] \leq \nu[i,j]$ for any $i, j \in [n]$. This means, by induction, that $\sigma \leq \nu$ and therefore that $\sigma \leq \tau$. $\square$

Example 3.12. *Consider the permutations $\sigma, \tau \in S_8$*

$$\sigma = 368475912 \quad \textit{and} \quad \tau = 694287531.$$

Observing that

$$\sigma[1,6] < \tau[1,6] \quad \textit{and} \quad \sigma[4,3] > \tau[4,3],$$

we can conclude by the previous theorem that σ and τ are not comparable.

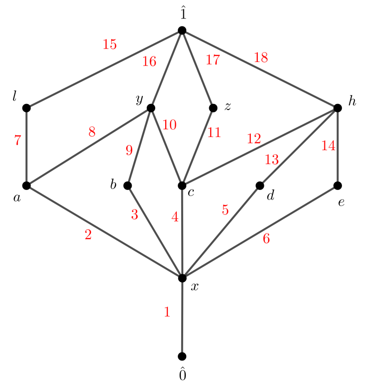
We end this chapter with a technical result that is going to be useful later on. A proof of this theorem can be found in [1], Theorem 2.6.3

Given a sequence of numbers $(n_1, n_2, .., n_k)$ we write $sort(n_1, n_2, \dots, n_k)$ for the sequence sorted into an increasing order.

Theorem 3.13. *Let $\sigma, \tau \in S_n$. Then $\sigma \leq \tau$ if and only if for every $i \in \{1, \dots, n\}$, $sort(\sigma(1), \dots, \sigma(i))$ has each entry less then or equal to the corresponding entry in $sort(\tau(1), \dots, \tau(i))$.*

3.4 Minimal chains of the symmetric group

The main purpose of this section is to give a characterization for the minimal chains of S_n as a poset with the Bruhat order.
First of all, we need to find a covering system of S_n. The easiest way to define one is considering as successor system H the set of edges of the Hasse diagram of S_n and as ρ any function satisfying the base and the increasing property. Finding such a ρ is always possible when we have a finite bounded poset because intuitively it is just assigning the 'level' to each element on the Hasse diagram.
For instance, consider the Hasse diagram of the poset in Example 1.14.



We would consider the bottom element 0 to be in the '0-level' assigning to it $\rho(0) = 0$, the elements x to be in the first level assigning to it $\rho(x) = 1$, the elements $\{a, b, c, d, e\}$ to be in the second level, etc.

Define the edge-labelling

$$\lambda : \{(\sigma, \tau) \in S_n^2 : \sigma \lhd \tau\} \to \{(i, j) \in [n]^2 : i < j\},$$

that associates to each $\sigma, \tau \in S_n$, with $\sigma \lhd \tau$, the pair $(i, j) = \lambda(\sigma, \tau)$, where (i, j) is the free rise of σ such that $\tau = \sigma(i, j)$.

This particular edge-labelling is called *standard labelling* of S_n. The standard labelling is well defined because we can consider the lexicographic order on the set $\{(i, j) \in [n]^2 : i < j\}$:

$$(i, j) < (k, h) \iff i < k \text{ or if } \ i = k \text{ then } j < h.$$

This standard labelling is also a labelling of the successor system H above defined, since the set $\{(\sigma, \tau) \in S_n^2 : \sigma \lhd \tau\}$ is the set of all the edges of the Hasse diagram of S_n.

Lemma 3.14. *The standard labelling of S_n is a good labelling of*

$$H = \{(\sigma, \tau) \in S_n^2 : \sigma \lhd \tau\}.$$

Proof. Let $(\sigma, \tau), (\sigma, \mu) \in H$, then, by definition of H, we have $\sigma \lhd \tau$ and $\sigma \lhd \mu$. We have also that, by definition of standard labelling, $\lambda(\sigma, \tau)$ and $\lambda(\sigma, \mu)$ are free rises of σ. This means that there exist (i, j) and (h, k) in $\{(i, j) \in [n]^2 : i < j\}$ such that

$$\lambda(\sigma, \tau) = (i, j) \text{ and } \lambda(\sigma, \mu) = (h, k),$$

with $\tau = \sigma(i, j)$ and $\mu = \sigma(h, k)$.
Suppose that $\lambda(\sigma, \tau) = \lambda(\sigma, \mu)$, then

$$(i, j) = (h, k).$$

Therefore, $\sigma(i, j) = \sigma(h, k)$, which implies that

$$\tau = \sigma,$$

showing the injectivity property. $\square$

Now we have settled up a convenient situation to obtain a characterization of the minimal chains of S_n by defining a covering system (H, ρ) and a good labelling λ for H.

Observe that for $\sigma \in S_n$, its suitable labels are the free rises of σ.
We already know that each free rise (i, j) of σ uniquely identifies a permutation $\tau = \sigma(i, j)$. This permutation is exactly the covering transformation of σ with respect to the label (i, j),

$$ct^{S_n}_{(i,j)} = \sigma(i, j) = \tau.$$

Moreover, let $\sigma, \tau \in S_n$ such that $\sigma < \tau$. Then we denote by $di_\tau(\sigma)$ the minimal position where the two permutations differ,

$$di_\tau(\sigma) = \min\{i \in [n] : \sigma(i) \neq \tau(i)\}.$$

Lemma 3.15. *Let $\sigma, \tau \in S_n$ such that $\sigma < \tau$. Then*

$$\sigma(di) < \tau(di),$$

where $di := di_\tau(\sigma)$.

Proof. Recall that $\sigma < \tau$ implies by Theorem 3.11 that, for any $h, k \in [n]$, we have $\sigma[h, k] \leq \tau[h, k]$.
Suppose by contradiction that $\sigma(di) > \tau(di)$.
By definition, $\sigma[di, \sigma(di)] = \{j \in [di] : \sigma(j) \in [\sigma(di), n]\}$. Since we know that $\sigma(di)$ is the first entry of σ that differs from the entries of τ and that $\sigma(di) > \tau(di)$, then, when we count how many entries between $\tau(1), \ldots, \tau(di)$ are greater then $\sigma(di)$, we clearly have that the number is the same that for σ except for $\sigma(di)$. Therefore $\sigma[di, \sigma(di)] - 1 = \tau[di, \sigma(di)]$. Which means that there exists $h = di$ and $k = \sigma(di)$ such that $\tau[h, k] \leq \sigma[h, k]$. That is a contradiction.

$\square$

Lemma 3.16. *Let $\sigma, \tau \in S_n$ such that $\sigma < \tau$ and $di := di_\tau(\sigma)$. Then the set*

$$\{j \in [di+1,\, n] : \sigma(j) \in [\sigma(di)+1, \tau(di)]\},$$

is not empty.

Proof. Let $k := \sigma^{-1}(\tau(di))$.
Suppose $k \in [di-1]$, then, since σ and τ are equal until position di, we have $\sigma(k) = \tau(k)$. Thus

$$\tau(k) = \sigma(k) = \sigma(\sigma^{-1}(\tau(di))) = \tau(di),$$

that means

$$di = k,$$

which is a contradiction.
Suppose $k = di$, then

$$\sigma(di) = \sigma(\sigma^{-1}(\tau(di))) = \tau(di),$$

that is a contradiction by definition of di.
Therefore $k \in [di+1, n]$ and the desired set is non-empty. □

Consider $\sigma, \tau \in S_n$, with $\sigma < \tau$. We define the *covering index* of σ with respect to τ as,

$$ci_\tau(\sigma) := \min\{j \in [di_\tau(\sigma)+1,\, n] : \sigma(j) \in [\sigma(di_\tau(\sigma))+1, \tau(di_\tau(\sigma))]\}.$$

Note that thanks to Lemma 3.16 the covering index always exists.

By now, we defined two particular indexes,

- $di_\tau(\sigma) =: di$, that is the minimum index for which σ and τ differ;
- $ci_\tau(\sigma) =: ci$, that is the minimum position in $[di+1, n]$ such that σ in that specific position has a value in $[\sigma(di)+1, \tau(di)]$.

Since $ci \in [di+1, n]$ then we have that $di < ci$.
Moreover, since $\sigma(ci) \in [\sigma(di)+1, \tau(di)]$ then $\sigma(di) < \sigma(ci)$.
This two observations allow us to say that the pair (di, ci) is a rise of σ by definition. Moreover by minimality of ci, the rise (di, ci) is a free rise and thus a suitable label of σ.

Actually what we are going to see with the next two results is that this suitable label (di, ci) is indeed the minimal label of σ with respect to τ, where we are always considering (H, ρ) and λ as we defined them at the beginning of this section.

Proposition 3.17. *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then*

$$ct^{S_n}_{(di,ci)}(\sigma) = \sigma(di, ci) \leq \tau.$$

Proof. Let $\chi := \sigma(di, ci)$. We want to show that for every $(h, k) \in [n]^2$, $\chi[h, k] \leq \tau[h, k]$, so that we can conclude by Theorem 3.11, that

$$\chi = \sigma(di, ci) \leq \tau.$$

Define $R := [di, ci - di] \times [\sigma(di) + 1, \sigma(ci)]$.
At first we show that for any $(h, k) \in R^c$, $\chi[h, k] \leq \tau[h, k]$ happens, and then that the same happens also for any $(h, k) \in R$.
Without loss of generality suppose $di = 1$. Recall that

$$\chi[h, k] = |\{j \in [h] : \chi(j) \in [k, n]\}|,$$

and note that

$$\chi[h, k] = \begin{cases} \sigma[h, k] + 1, & \text{if } (h, k) \in R \\ \sigma[h, k] & \text{if } (h, k) \notin R \end{cases}.$$

This happens because if we are considering $(h, k) \in R$, since $\chi = \sigma(1, ci)$, in the positions from 1 to h of χ we will always have $\sigma(ci)$, while we won't have it in the first $[h]$ positions of σ.
By hypothesis $\sigma < \tau$, that always by Theorem 3.11, implies that $\sigma[h, k] \leq \tau[h, k]$ for any $(h, k) \in [n]^2$. So, since $\chi[h, k] = \sigma[h, k]$ for any $(h, k) \in R^C$, we conclude that whenever $(h, k) \in R^C$ then

$$\chi[h, k] \leq \tau[h, k].$$

Now suppose $(h, k) \in R$, then

$$\sigma[h, k] = \sigma[h, \tau(1) + 1] \leq \tau[h, \tau(1) + 1] \leq \tau[h, k] - 1.$$

Which means that $\tau[h, k] \geq \sigma[h, k] + 1 = \chi[h, k]$, for any $(h, k) \in R$, concluding the proof. □

Proposition 3.18. *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then*

$$mi_\tau(\sigma) = (di, ci).$$

Proof. We already observed that the suitable labels of a permutation $\sigma \in S_n$ are its free rises. What we want to show is that $(di, ci) \leq (i, j)$, for any free rise (i, j) of σ such that

$$ct_{(i,j)^{S_n}(\sigma)} = \sigma(i, j) \leq \tau.$$

So, let (i, j) be such a free rise and suppose by contradiction that
$(i, j) < (di, ci)$. Thus we need to have $i < di$ or $i = di$ and $j < ci$.
Suppose $i < ci$:
trivially $\sigma(i, j) \leq \tau$ implies that $\sigma(i) \neq \tau(i)$, which contradicts the minimality of di.
Now suppose $i = di$ and $j < ci$:
Call $\xi := \sigma(i, j) = \sigma(di, j)$. Note that (i, j) is a free rise of σ, which implies by definition of free rise that $i < j$ and $\sigma(i) < \sigma(j)$ that in our case is $\sigma(di) < \sigma(j)$.
Since $\xi = \sigma(di, j)$, then $\xi(di) = \sigma(j)$. Moreover, by the choice of (i, j) we know that $\xi \leq \tau$ which implies by Lemma 3.15 that $\xi(di) \leq \tau(di)$. But then $\sigma(j) \leq \tau(di)$. Therefore what we have is,

- $j < ci$,
- $\sigma(di) < \sigma(j)$,
- $\sigma(j) < \tau(di)$.

But this contradicts the minimality of ci. $\square$

Therefore, by Definition 1.13, we have that the minimal covering transformation of σ with respect to τ is

$$mct_\tau^{S_n} = ct_{(di,ci)}^{S^n}(\sigma) = \sigma(di, ci).$$

By Definition 1.15, since we described the minimal covering transformation between any two related elements of S_n, we have also a description of the minimal chains of S_n.

Chapter 4

Ordering standard Young tableaux

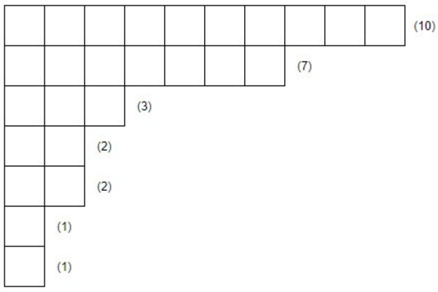
A Young tableau is a combinatorial configuration that provides useful coding for describing the representations of the symmetric group and for studying their properties. Young tableaux were introduced in 1900 by Alfred Young, at that time a professor of mathematics at the University of Cambridge. They were then applied to the study of symmetric group by Georg Frobenius in 1903. The theory was later developed by Alfred Young and many other mathematicians. Young tableaux in fact constitute a central theme also for the developments of today's algebraic combinatorics.

In this chapter, we introduce some basic concepts regarding integer partitions, like diagrams of Ferrers and Young tableaux and we show how powerful and intensely connected to many aspects of combinatorics is the Bruhat order defined on the symmetric group.

Definition 4.1. Let n be a positive integer. A *partition* of n is a non-increasing sequence of non negative integers $p := (p_1, p_2, \dots, p_k)$ such that $\sum_{i=1}^{k} p_i = n =: |p|$. Each $p_i > 0$ is called a *part* of the partition.

An useful way to visualize integer partitions is the so called *diagram of Ferrers* that is a left justified diagram that encodes the parts of the partitions in rows of cells, where the number of cells in each row corresponds to the size of a part. The first row corresponds to the largest part, the second to the second largest part and so on. Here we propose an example.

Example 4.2. *Consider the partition* $(10, 7, 3, 2, 2, 1, 1)$ *of* $26 = 10 + 7 + 3 + 2 + 2 + 1 + 1$*. The Ferrers diagram for this partition is*



Definition 4.3. Let n be a positive integer and let $p = (p_1, p_2, \ldots, p_n)$ be a partition of n. We call *Young tableau of shape* p, to any filling of the cells of the diagram of Ferrers of p by distinct integers such that each row and each column is strictly increasing when read left to right and top to bottom. This integers are called *entries* of the tableau. Moreover a Young tableau is called *standard Young tableau* whenever its entries are exactly $1, 2, \ldots, n$.

If p is a partition, we denote by T_p the set of all possible standard Young tableaux of shape p.

In the sequel we will define two partial orders on the set T_p, the set of all standard Young tableaux of shape p, $\leq_{dom}$ and $\leq_{box}$, and we will illustrate the power of the Bruhat order by using the tools we studied before to show that these two partial orders are indeed equivalent.

Definition 4.4. Let $p = (p_1, p_2, \ldots, p_k)$, $\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m)$ be partitions of $n \in \mathbb{N}$. We say that p and $\hat{p}$ are in the dominance partial order, $p \leq_{dom} \hat{p}$, if

$$p_1 + p_2 + \ldots + p_j \leq \hat{p}_1 + \hat{p}_2 + \ldots + \hat{p}_j$$

for any $j \in \{1, \ldots, max\{k, m\}\}$, where, without loss of generality, if $k < m$, we set $p_{k+1} = p_{k+2} = \cdots = p_m = 0$.

It is possible to describe a standard Young tableau X of shape p as a sequence of partitions

$$X = [\gamma^{(1)}, \ldots, \gamma^{|p|}],$$

where each $\gamma^{(i)}$, is the partition that corresponds to the sub-tableau of X containing the entries from 1 to i.

For example, consider the partition $p = (2, 2, 1)$ and the following standard Young tableau of shape p

1	4
2	5
3	

This Young tableau can be represented by the sequence of partitions

$$[\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}, \gamma^{(5)}] = [(1), (1, 1), (1, 1, 1), (2, 1, 1), (2, 2, 1)].$$

Definition 4.5. Let p be a partition. Given two standard Young tableaux $X = [\gamma^{(1)}, \dots, \gamma^{(|p|)}]$ and $Y = [\delta^{(1)}, \dots, \delta^{(|p|)}]$ in T_p, we say that $X \leq_{dom} Y$ if

$$\gamma^{(i)} \leq_{dom} \delta^{(i)},$$

for each $i \in \{1, \dots, |p|\}$.

Let us make an example always considering the partition $p = (2, 2, 1)$.

Example 4.6. *All the possible standard Young tableaux of shape p are*

W:

1	2
3	4
5	

U:

1	2
3	5
4	

Z:

1	3
2	4
5	

Y:

1	3
2	5
4	

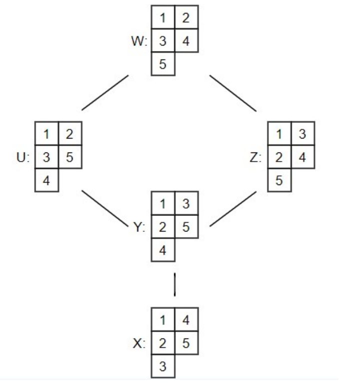
X:

1	4
2	5
3	

and we can describe these tableaux as $X = [(1), (1, 1), (1, 1, 1), (2, 1, 1), (2, 2, 1)]$, $Y = [(1), (1, 1), (2, 1), (2, 1, 1), (2, 2, 1)]$, $Z = [(1), (1, 1), (2, 1), (2, 2), (2, 2, 1)]$, $U = [(1), (2), (2, 1), (2, 1, 1), (2, 2, 1)]$, $W = [(1), (2), (2, 1), (2, 2), (2, 2, 1)]$.
For instance we can observe that $X \leq_{dom} Y$, since

- $(1) \leq_{dom} (1)$;
- $(1, 1) \leq_{dom} (1, 1)$;
- $(1, 1, 1) \leq_{dom} (2, 1)$;
- $(2, 1, 1) \leq_{dom} (2, 1, 1)$;
- $(2, 2, 1) \leq_{dom} (2, 2, 1)$.

Whereas the tableaux U and Z are not comparable since $(1,1) \leq_{dom} (2)$ but $(2,1,1) \leq_{dom} (2,2)$. The dominance order on $T_{(2,2,1)}$ has the following Hasse diagram



In the sequel we define the *box order.*

Definition 4.7. Let p be a partition. Given standard Young tableaux $X = [\gamma^{(1)}, \ldots, \gamma^{(|p|)}]$ and $Y = [\delta^{(1)}, \ldots, \delta^{(|p|)}]$ in T_p, we say that $X \leq_{box} Y$ if X is obtained from Y by *decreasing box moves*. Where we call a decreasing box move any move that swap two entries in such a way that the smaller entry winds down to an higher-numbered row.

Consider the previous example on $p = (2,2,1)$. We have that X is obtained from Y by the decreasing box move $3 \rightleftarrows 4$, Y is obtained by Z by the decreasing box move $4 \rightleftarrows 5$, and for instance X is obtained by W from the sequence of decreasing box moves $2 \rightleftarrows 3$, $4 \rightleftarrows 5$ and $3 \rightleftarrows 4$.

It seems reasonable to ask if the partial orders, dominance and box order, are indeed equivalent. Let us show one of the implications, the other one is much more complex and we will be able to prove it shortly thanks to the power of the Bruhat order.

Proposition 4.8. *Given standard Young tableaux X and Y of the same shape p, if $X \leq_{box} Y$ then $X \leq_{dom} Y$. That is, the box order implies the dominance order.*

Proof. First of all we call a box move based on the couple $a - b$, $a < b$, to any sequence of box moves which takes into account a as minimum number and b as maximum. Observe that whenever we do a box move based on $a - b$ on a standard Young tableau the only numbers we are moving are the numbers a, b and the ones in between. Therefore, if $X = [\gamma^{(1)}, \dots, \gamma^{(|p|)}]$ and $Y = [\delta^{(1)}, \dots, \delta^{(|p|)}]$ and we obtain X from Y by doing a decreasing box move based on $a - b$, $a < b$, we have that $\delta^{(1)}, \dots, \delta^{(a-1)}$, as well as , $\delta^{(b)}, \dots, \delta^{(|p|)}$, remain unchanged. Regarding $\gamma^{(l)}, \delta^{(l)}$ for $a - 1 < l < b$, we will always have that

$$\gamma^{(l)} \leq_{dom} \delta^{(l)},$$

since what we do by a decreasing box move is moving down a smaller entry, so that the partial sum we defined can only decrease. This allows us to conclude that $X \leq_{dom} Y$. $\square$

Now we are going to relate permutations with standard Young tableaux with the aim of showing the vice-versa of the previous proposition.

Let $p = (p_1, p_2, \dots)$ be a partition of n. We fill the boxes of the diagram of Ferrers of p, increasingly with the integers $1, \dots, n$, starting from the first row, from left to right, then going into the second row from left to right and so on.

Let now X be a standard Young tableau of shape p, that is a filling of the boxes of the Ferrers diagram of p, as we previously explained. We associate to X a permutation (written in the complete notation)

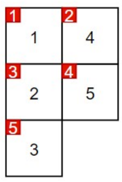
$$\pi(X) = \pi(X)(1)\pi(X)(2) \dots \pi(X)(n) \in S_n,$$

in such a way that each entry $\pi(X)(i)$ is the filling of the i^{th} box in the standard Young tableau X.

Actually we could extend the concept of associated permutation to any Young tableau in which the fillings are from 1 to n.

Let us make an example.

Example 4.9. *As we previously explained, we number the boxes of the standard Young tableau X in the following way (red),*



The filling of the first box is 1*, the filling of the second box is* 4*, the one of the third is* 2*, of the fourth is* 5 *and of the fifth is* 3*, for this reason we have that*

$$\pi(X) = 14253.$$

Now that we have defined a correspondence between permutations and standard Young tableaux, we can get to the heart of this chapter and try to understand how and how much, what we have studied on the Bruhat partial order helps us in proving the converse of Proposition 4.8.

Proposition 4.10. *Let p be a partition and X and Y be standard Young tableaux of shape p. Then $\pi(Y) \leq \pi(X)$ in the Bruhat order if and only if $X \leq_{dom} Y$.*

Proof. Let $X = [\gamma^{(1)}, \dots, \gamma^{(|p|)}]$ and $Y = [\delta^{(1)}, \dots, \delta^{(|p|)}]$. Observe that whenever $\pi(Y) \leq \pi(X)$ happens, then also $\pi(Y)^{-1} \leq \pi(X)^{-1}$ holds. Indeed, without loss of generality, we can assume that there exists a reflexion $t \in T$ such that $\pi(X) = \pi(Y)\, t$, so that

$$\pi(X)^{-1} = t^{-1}\pi(Y)^{-1} = t\, \pi(Y)^{-1},$$

where the second equality is justified by the fact that a reflection is an involution. At this point we can multiply the right part of the previous equation by $\pi(Y)^{-1}\pi(Y)$, obtaining

$$\pi(X)^{-1} = \pi(Y)^{-1}(\pi(Y)\, t\, \pi(Y)^{-1}),$$

where $(\pi(Y)\, t\, \pi(Y)^{-1})$ is an element of T, and thus

$$\pi(Y)^{-1} \leq \pi(X)^{-1}.$$

Now that we have $\pi(Y)^{-1} \leq \pi(X)^{-1}$, we can apply Theorem 3.13 to $\pi(Y)^{-1}$ and $\pi(X)^{-1}$, obtaining that for any $i \in \{1, \dots, |n|\}$,

$$sort(\pi(Y)(1), \dots, \pi(Y)(i))$$

has each entry less or equal than the corresponding entry in

$$sort(\pi(X)(1), \dots, \pi(X)(i)).$$

But it is clear by the way we constructed the permutation associated to a standard Young tableau that this means that for each $i \in \{1, \dots, |n|\}$,

$$\delta^{(i)} \geq_{dom} \gamma^{(i)},$$

which means

$$X \leq_{dom} Y.$$

□

Now suppose that $\pi(X)$ and $\pi(Y)$ are permutations associated respectively to the standard Young tableaux X and Y of the same shape. Then the following result holds.

Proposition 4.11. *Whenever $\pi(X)$ covers $\pi(Y)$ in the Bruhat order ($\pi(Y) \lhd \pi(X)$), there is a decreasing box move from Y to X. Which means that $X \leq_{box} Y$.*

Proof. The proof of this proposition is really immediate. It suffices to think about what means that a permutation covers another one in the Bruhat order. Indeed the fact that $\pi(X)$ covers $\pi(Y)$ means that $\pi(X)$ is obtained by $\pi(Y)$ by swapping two entries in such a way that we move to the right the smaller entry and to the left the greater. This means by construction that we are moving down to a greater numbered row the smaller entry in the standard Young diagram, so that in this way we have exactly what we have called a decreasing box move from Y to X, which implies $X \leq_{box} Y$. □

Until now we proved that two associated permutations, let us say $\pi(X)$ and $\pi(Y)$, are related in the Bruhat order, $\pi(Y) \leq \pi(X)$, if and only if the corresponding standard Young tableaux X and Y are related in the dominance order, $X \leq_{dom} Y$. Moreover, whenever the relation in the Bruhat order is a covering relation, $\pi(Y) \lhd \pi(X)$, then X and Y are related in the box order, $X \leq_{box} Y$. Also, we already know that the box order implies the dominance order. Therefore, if we show that whenever we have two associated permutations $\pi(X), \pi(Y)$, related in the Bruhat order $\pi(Y) \leq \pi(X)$, then there exists another associated permutation $\pi(Z)$ such that $\pi(Y) \lhd \pi(Z)$ and $\pi(Z) \leq \pi(X)$, we can conclude that also the dominance order implies the box order. The reason it clear and it is thanks to the fact that S_n is a finite group. Let us try to explain it in detail.

Suppose that $\pi(Y) \leq \pi(X)$, which we know by Proposition 4.10 that is equivalent to $X \leq_{dom} Y$. Suppose there exists $\pi(Z)$ such that $\pi(Y) \lhd \pi(Z)$ and $\pi(Z) \leq \pi(X)$. Then by Proposition 4.11 and again Proposition 4.10 we know that $Z \leq_{box} Y$ and $X \leq_{dom} Z$. We can of course apply the same reasoning to X and Z, since $X \leq_{dom} Z$ implies by Proposition 4.10 that $\pi(Z) \leq \pi(X)$. We then obtain, for some standard Young tableau W, a chain

$$W \leq_{box} Z \leq_{box} Y.$$

But at a certain point, due to the fact that S_n is finite, we need to find $\pi(X)$, obtaining

$$X \leq_{box} \cdots W \leq_{box} Z \leq_{box} Y.$$

Therefore we only have to show what has just been stated.

For simplicity we call L the set of all permutations of S_n which are associated to a standard Young tableau of shape p, where p is a partition of n.

Proposition 4.12. *Let $x, z \in L$ such that $x < z$. Then there exists $y \in L$ such that $x \lhd y$ and $y \leq z$.*

Proof. Let $s_{i_1} \cdots s_{i_p}$ be a reduced expression for z. Since by hypothesis $x < z$, then by Theorem 2.32 there exists a reduced expression for x that is a subword of $s_{i_1} \cdots s_{i_p}$,

$$x = s_{i_1} \cdots s_{\hat{i_{j_1}}} \cdots s_{\hat{i_{j_q}}} \cdots s_{i_p},$$

where we choose j_q the minimum for which this happens. Define

$$y = s_{i_1} \cdots s_{\hat{i_{j_1}}} \cdots s_{\hat{i_{j_{q-1}}}} \cdots s_{i_{j_q}} \cdots s_{i_p},$$

and exactly as we did in the proof of Theorem 2.32, a permutation constructed in such a way is such that $x \lhd y$ and $y \leq z$. Thus we just need to prove that y is in L.

Since $x \lhd y$ then we know that there exists $t \in T$ such that $y = xt$. But knowing the expressions of y and x, we easily get the expression for t,

$$t = s_{i_p} s_{i_{p-1}} \cdots s_{i_{j_q}} \cdots s_{i_{p-1}} s_{i_p}.$$

But then, $l(zt) < l(z)$, which by Corollary 2.22 implies $zt < z$.
Moreover, since all the transpositions of the symmetric group are of the form $(a\, b)$ with $a < b$, let us fix $t = (a\, b)$.
Now, we know that $x, z \in L$, this means that there exist two standard Young tableaux associated to them. Let us call them respectively X and Z. Since we observed that $z\, t = z(a\, b) < z$, we need to have that the entry in position a of z is larger than the entry in position b of z. Moreover, since $z \in L$, we need to have that the box numbered by a is not in the same row neither in the same column of the one numbered by b in Z, otherwise Z would not be a standard Young tableau.

Now we know that $y = x(a\, b)$ and this implies that $x(a) < x(b)$ and moreover, by Lemma (3.8), we know also that it does not exist any c such that $a < c < b$ such that $x(a) < x(c) < x(b)$. Thus when we swap the entries in $x(a)$ and $x(b)$ the resulting tableau is a standard Young tableau, because we don't have any value between $x(a)$ and $x(b)$ in between the boxes a and b. Therefore $y \in L$. $\square$

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