

Divisor function

In mathematics, and specifically in number theory, a **divisor function** is an arithmetical function related to the divisors of an integer. When referred to as *the* divisor function, it counts the *number of divisors of an integer*. It appears in a number of remarkable identities, including relationships on the Riemann zeta function and the Eisenstein series of modular forms. Divisor functions were studied by Ramanujan, who gave a number of important congruences and identities.

A related function is the divisor summatory function, which, as the name implies, is a sum over the divisor function.

Definition

The **sum of positive divisors function** $\sigma_x(n)$, for a real or complex number x , is defined as the sum of the x th powers of the positive divisors of n , or

$$\sigma_x(n) = \sum_{d|n} d^x.$$

The notations $d(n)$, $\nu(n)$ and $\tau(n)$ (for the German *Teiler* = divisors) are also used to denote $\sigma_0(n)$, or the **number-of-divisors function**^{[1][2]} (sequence A000005 in OEIS). When x is 1, the function is called the **sigma function** or **sum-of-divisors function**,^{[3][4]} and the subscript is often omitted, so $\sigma(n)$ is equivalent to $\sigma_1(n)$ (☞ A000203).

The **aliquot sum** $s(n)$ of n is the sum of the proper divisors (that is, the divisors excluding n itself, ☞ A001065), and equals $\sigma_1(n) - n$; the aliquot sequence of n is formed by repeatedly applying the aliquot sum function.

Example

For example, $\sigma_0(12)$ is the number of the divisors of 12:

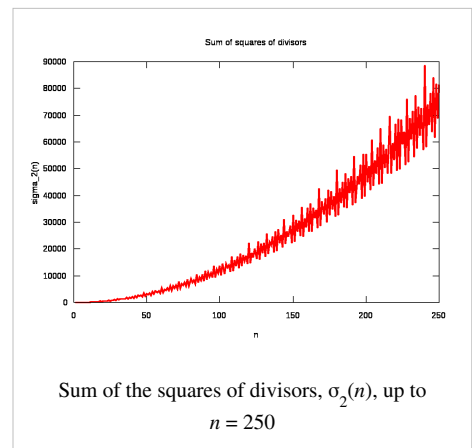
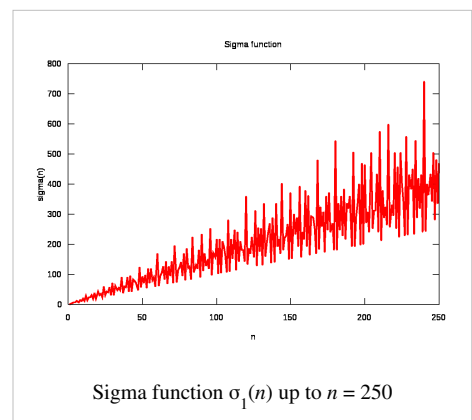
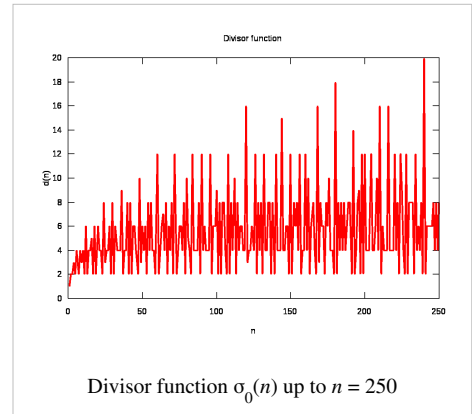
$$\begin{aligned} \sigma_0(12) &= 1^0 + 2^0 + 3^0 + 4^0 + 6^0 + 12^0 \\ &= 1 + 1 + 1 + 1 + 1 + 1 = 6, \end{aligned}$$

while $\sigma_1(12)$ is the sum of all the divisors:

$$\begin{aligned} \sigma_1(12) &= 1^1 + 2^1 + 3^1 + 4^1 + 6^1 + 12^1 \\ &= 1 + 2 + 3 + 4 + 6 + 12 = 28, \end{aligned}$$

and the aliquot sum $s(12)$ of proper divisors is:

$$\begin{aligned} s(12) &= 1^1 + 2^1 + 3^1 + 4^1 + 6^1 \\ &= 1 + 2 + 3 + 4 + 6 = 16. \end{aligned}$$



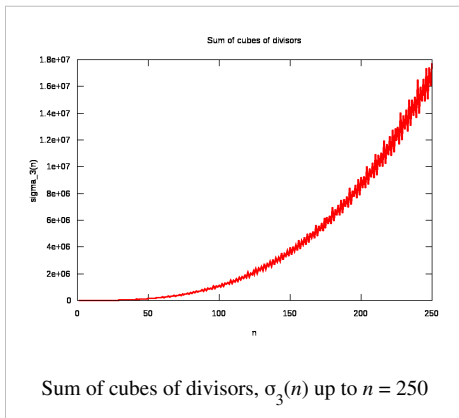


Table of values

n	Divisors	$\sigma_0(n)$	$\sigma_1(n)$	$s(n) = \sigma_1(n) - n$	Comment
1	1	1	1	0	square number: $\sigma_0(n)$ is odd; power of 2: $s(n) = n - 1$ (almost-perfect)
2	1,2	2	3	1	Prime: $\sigma_1(n) = 1+n$ so $s(n) = 1$
3	1,3	2	4	1	Prime: $\sigma_1(n) = 1+n$ so $s(n) = 1$
4	1,2,4	3	7	3	square number: $\sigma_0(n)$ is odd; power of 2: $s(n) = n - 1$ (almost-perfect)
5	1,5	2	6	1	Prime: $\sigma_1(n) = 1+n$ so $s(n) = 1$
6	1,2,3,6	4	12	6	first perfect number: $s(n) = n$
7	1,7	2	8	1	Prime: $\sigma_1(n) = 1+n$ so $s(n) = 1$
8	1,2,4,8	4	15	7	power of 2: $s(n) = n - 1$ (almost-perfect)
9	1,3,9	3	13	4	square number: $\sigma_0(n)$ is odd
10	1,2,5,10	4	18	8	
11	1,11	2	12	1	Prime: $\sigma_1(n) = 1+n$ so $s(n) = 1$
12	1,2,3,4,6,12	6	28	16	first abundant number: $s(n) > n$
13	1,13	2	14	1	Prime: $\sigma_1(n) = 1+n$ so $s(n) = 1$
14	1,2,7,14	4	24	10	
15	1,3,5,15	4	24	9	
16	1,2,4,8,16	5	31	15	square number: $\sigma_0(n)$ is odd; power of 2: $s(n) = n - 1$ (almost-perfect)

The cases $x=2$, $x=3$ and so on are tabulated in [OEIS A001157](#), [OEIS A001158](#), [OEIS A001159](#), [OEIS A001160](#), [OEIS A013954](#), [OEIS A013955](#) ...

Properties

For a non-square integer every divisor d of n is paired with divisor n/d of n and $\sigma_0(n)$ is then even; for a square integer one divisor (namely \sqrt{n}) is not paired with a distinct divisor and $\sigma_0(n)$ is then odd.

For a prime number p ,

$$\begin{aligned} d(p) &= 2 \\ d(p^n) &= n + 1 \\ \sigma(p) &= p + 1 \end{aligned}$$

because by definition, the factors of a prime number are 1 and itself. Also, where $p_n\#$ denotes the primorial,

$$d(p_n\#) = 2^n$$

since n prime factors allow a sequence of binary selection (p_i or 1) from n terms for each proper divisor formed.

Clearly, $1 < d(n) < n$ and $\sigma(n) > n$ for all $n > 2$.

The divisor function is multiplicative, but not completely multiplicative. The consequence of this is that, if we write

$$n = \prod_{i=1}^r p_i^{a_i}$$

where $r = \omega(n)$ is the number of distinct prime factors of n , p_i is the i th prime factor, and a_i is the maximum power of p_i by which n is divisible, then we have

$$\sigma_x(n) = \prod_{i=1}^r \frac{p_i^{(a_i+1)x} - 1}{p_i^x - 1}$$

which is equivalent to the useful formula:

$$\sigma_x(n) = \prod_{i=1}^r \sum_{j=0}^{a_i} p_i^{jx} = \prod_{i=1}^r (1 + p_i^x + p_i^{2x} + \dots + p_i^{a_i x}).$$

It follows (by setting $x = 0$) that $d(n)$ is:

$$d(n) = \prod_{i=1}^r (a_i + 1).$$

For example, if n is 24, there are two prime factors (p_1 is 2; p_2 is 3); noting that 24 is the product of $2^3 \times 3^1$, a_1 is 3 and a_2 is 1. Thus we can calculate $d(24)$ as so:

$$\begin{aligned} d(24) &= \prod_{i=1}^2 (a_i + 1) \\ &= (3 + 1)(1 + 1) = 4 \times 2 = 8. \end{aligned}$$

The eight divisors counted by this formula are 1, 2, 4, 8, 3, 6, 12, and 24.

We also note $s(n) = \sigma(n) - n$. Here $s(n)$ denotes the sum of the proper divisors of n , i.e. the divisors of n excluding n itself. This function is the one used to recognize perfect numbers which are the n for which $s(n) = n$. If $s(n) > n$ then n is an abundant number and if $s(n) < n$ then n is a deficient number.

If n is a power of 2, e.g. $n = 2^k$, then $\sigma(n) = 2 \times 2^k - 1 = 2n - 1$, and $s(n) = n - 1$, which makes n almost-perfect.

As an example, for two distinct primes p and q with $p < q$, let

$$n = pq.$$

Then

$$\begin{aligned} \sigma(n) &= (p + 1)(q + 1) = n + 1 + (p + q), \\ \phi(n) &= (p - 1)(q - 1) = n + 1 - (p + q), \end{aligned}$$

and

$$n + 1 = (\sigma(n) + \phi(n))/2,$$

$$p + q = (\sigma(n) - \phi(n))/2,$$

where $\varphi(n)$ is Euler's totient function.

Then, the roots of:

$$(x-p)(x-q) = x^2 - (p+q)x + n = x^2 - [(\sigma(n) - \phi(n))/2]x + [(\sigma(n) + \phi(n))/2 - 1] = 0$$

allows us to express p and q in terms of $\sigma(n)$ and $\varphi(n)$ only, without even knowing n or $p+q$, as:

$$p = (\sigma(n) - \phi(n))/4 - \sqrt{[(\sigma(n) - \phi(n))/4]^2 - [(\sigma(n) + \phi(n))/2 - 1]},$$

$$q = (\sigma(n) - \phi(n))/4 + \sqrt{[(\sigma(n) - \phi(n))/4]^2 - [(\sigma(n) + \phi(n))/2 - 1]}.$$

Also, knowing n and either $\sigma(n)$ or $\varphi(n)$ (or knowing $p+q$ and either $\sigma(n)$ or $\varphi(n)$) allows us to easily find p and q .

In 1984, Roger Heath-Brown proved that

$$d(n) = d(n + 1)$$

will occur infinitely often.

Series relations

Two Dirichlet series involving the divisor function are:

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \zeta(s)\zeta(s - a),$$

which for $d(n) = \sigma_0(n)$ gives

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s),$$

and

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s - a)\zeta(s - b)\zeta(s - a - b)}{\zeta(2s - a - b)}.$$

A Lambert series involving the divisor function is:

$$\sum_{n=1}^{\infty} q^n \sigma_a(n) = \sum_{n=1}^{\infty} \frac{n^a q^n}{1 - q^n}$$

for arbitrary complex $|q| \leq 1$ and a . This summation also appears as the Fourier series of the Eisenstein series and the invariants of the Weierstrass elliptic functions.

Approximate growth rate

In little-o notation, the divisor function satisfies the inequality (see page 296 of Apostol's book^[5])

$$\text{for all } \epsilon > 0, \quad d(n) = o(n^\epsilon).$$

More precisely, Severin Wigert showed that

$$\limsup_{n \rightarrow \infty} \frac{\log d(n)}{\log n / \log \log n} = \log 2.$$

On the other hand, since there are infinitely many prime numbers,

$$\liminf_{n \rightarrow \infty} d(n) = 2.$$

In Big-O notation, Dirichlet showed that the average order of the divisor function satisfies the following inequality (see Theorem 3.3 of Apostol's book^[5])

$$\text{for all } x \geq 1, \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}),$$

where γ is Euler's constant. Improving the bound $O(\sqrt{x})$ in this formula is known as Dirichlet's divisor problem

The behaviour of the sigma function is irregular. The asymptotic growth rate of the sigma function can be expressed by:

$$\limsup_{n \rightarrow \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma,$$

where \limsup is the limit superior. This result is **Grönwall's theorem**, published in 1913 (Grönwall 1913). His proof uses Mertens' 3rd theorem, which says that

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \prod_{p \leq n} \frac{p}{p-1} = e^\gamma,$$

where p denotes a prime.

In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality:

$$\sigma(n) < e^\gamma n \log \log n \text{ (Robin's inequality)}$$

holds for all sufficiently large n . In 1984 Guy Robin proved that the inequality is true for all $n \geq 5,041$ if and only if the Riemann hypothesis is true (Robin 1984). This is **Robin's theorem** and the inequality became known after him. The largest known value that violates the inequality is $n=5,040$. If the Riemann hypothesis is true, there are no greater exceptions. If the hypothesis is false, then Robin showed there are an infinite number of values of n that violate the inequality, and it is known that the smallest such $n \geq 5,041$ must be superabundant (Akbariy & Friggstad 2009). It has been shown that the inequality holds for large odd and square-free integers, and that the Riemann hypothesis is equivalent to the inequality just for n divisible by the fifth power of a prime (Choie et al. 2007).

A related bound was given by Jeffrey Lagarias in 2002, who proved that the Riemann hypothesis is equivalent to the statement that

$$\sigma(n) \leq H_n + \ln(H_n) e^{H_n}$$

for every natural number n , where H_n is the n th harmonic number, (Lagarias 2002).

Robin also proved, unconditionally, that the inequality

$$\sigma(n) < e^\gamma n \log \log n + \frac{0.6483 n}{\log \log n}$$

holds for all $n \geq 3$.

Notes

- [1] Long (1972, p. 46)
- [2] Petofozeo & Byrkit (1970, p. 63)
- [3] Long (1972, p. 46)
- [4] Petofozeo & Byrkit (1970, p. 58)
- [5] Apostol, Tom M. (1976), *Introduction to analytic number theory*, Undergraduate Texts in Mathematics, New York-Heidelberg: Springer-Verlag, ISBN 978-0-387-90163-3, MR0434929

References

- Akbariy, Amir; Friggstad, Zachary (2009), "Superabundant numbers and the Riemann hypothesis" (<http://webdocs.cs.ualberta.ca/~zacharyf/Papers/superabundant.pdf>), *American Mathematical Monthly* **116** (3): 273–275, doi:10.4169/193009709X470128.
- Bach, Eric; Shallit, Jeffrey, *Algorithmic Number Theory*, volume 1, 1996, MIT Press. ISBN 0-262-02405-5, see page 234 in section 8.8.

- Caveney, Geoffrey; Nicolas, Jean-Louis; Sondow, Jonathan (2011), "Robin's theorem, primes, and a new elementary reformulation of the Riemann Hypothesis" (<http://www.integers-ejcnt.org/133/133.pdf>), *INTEGERS: the Electronic Journal of Combinatorial Number Theory* **11**: A33
- Choie, YoungJu; Lichiardopol, Nicolas; Moree, Pieter; Solé, Patrick (2007), "On Robin's criterion for the Riemann hypothesis" (http://jtnb.cedram.org/item?id=JTNB_2007__19_2_357_0), *Journal de théorie des nombres de Bordeaux* **19** (2): 357–372, arXiv:math.NT/0604314, doi:10.5802/jtnb.591, ISSN 1246-7405, MR2394891
- Grönwall, Thomas Hakon (1913), "Some asymptotic expressions in the theory of numbers", *Transactions of the American Mathematical Society* **14**: 113–122, doi:10.1090/S0002-9947-1913-1500940-6
- Lagarias, Jeffrey C. (2002), "An elementary problem equivalent to the Riemann hypothesis", *The American Mathematical Monthly* **109** (6): 534–543, doi:10.2307/2695443, ISSN 0002-9890, JSTOR 2695443, MR1908008
- Long, Calvin T. (1972), *Elementary Introduction to Number Theory* (2nd ed.), Lexington: D. C. Heath and Company
- Nicolas, Jean-Louis; Ramanujan, Srinivasa; Robin, Guy (1997), "Highly composite numbers", *The Ramanujan Journal* **1** (2): 119–153, doi:10.1023/A:1009764017495, ISSN 1382-4090, MR1606180
- Pettofrezzo, Anthony J.; Byrkit, Donald R. (1970), *Elements of Number Theory*, Englewood Cliffs: Prentice Hall
- Robin, Guy (1984), "Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann", *Journal de Mathématiques Pures et Appliquées, Neuvième Série* **63** (2): 187–213, ISSN 0021-7824, MR774171
- Weisstein, Eric W., "Divisor Function (<http://mathworld.wolfram.com/DivisorFunction.html>)" from MathWorld.
- Weisstein, Eric W., "Robin's Theorem (<http://mathworld.wolfram.com/RobinsTheorem.html>)" from MathWorld.
- Elementary Evaluation of Certain Convolution Sums Involving Divisor Functions (<http://mathstat.carleton.ca/~williams/papers/pdf/249.pdf>) PDF of a paper by Huard, Ou, Spearman, and Williams. Contains elementary (i.e. not relying on the theory of modular forms) proofs of divisor sum convolutions, formulas for the number of ways of representing a number as a sum of triangular numbers, and related results.

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