SEVERAL INEQUALITIES OF FRECHET SPACES

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Abstract. Several inequalities of Frechet spaces are given in this paper, which can be regarded as the Frechet spaces versions of the well-known polarization identity occurring in Hilbert spaces. Our results generalize many inequalities in Banach spaces. The inequalities developed here have various applications in a number of fields. By using these inequalities, many recent results can be easily generalized from Banach spaces to Frechet spaces.

1. Introduction

Among all Frechet spaces, the Hilbert spaces are generalized regarded as ones with the simplest and perhaps most immediately and clearly discernible geometric structure. This observation is supported and indeed characterized by the availability of parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \tag{*}$$

or equivalently the polarization identity

$$\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2. \tag{**}$$

With this understanding we shall say that Hilbert spaces are spaces with the best structure. However, in applications many problems do not fall naturally into spaces with this structure. Therefore, Chang S. S. [1] generalize (*) and (**) to Banach spaces. The main results can be summarized as follows:

$$\|x+y\|^p \le \|x\|^p + p\langle y, j_p\rangle, \quad \forall j_p \in J_p(x+y),$$

where $J_p : X \to X^*$ is an appropriate duality mapping.

In this paper, we shall present a version of (**) in Frechet spaces. Our results generalize the inequalities in Chang S. S. [1, 2]. The inequalities developed here have applications in a number of different fields. By using these inequalities, many recent results in [2] – [6] can be easily generalized from Banach spaces to the Frechet spaces.

1.1. **Preliminary introduction to Frechet spaces.**

Definition 1 ([7])**.** *The paranorm of x, written $|x|$, is a real number defined for all x in a vector space X and satisfying, for all $x, y \in X$,*

(a) $|0| = 0$,
(b) $|x| \ge 0$,
(c) $|-x| = |x|$,
(d) $|x+y| \le |x| + |y|$, *(triangle inequality)*

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(e) *if* $\{t_n\}$*is a sequence of scalars with* $t_n \to t$ *and* $\{x_n\} \subset X$ *with* $|x_n - x| \to 0$*, then* $|t_n x_n - tx| \to 0$ *(continuity of multiplication).*

Remark 1. *The concept of paranorm is a generalization of absolute value.*

Definition 2 ([7])**.** *A Frechet space is a separated complete paranormed space.*

Remark 2. *Every Banach space is a Frechet space, but not conversely.*

Example 1 ([7])**.** ω *is a Frechet space , but it is not a Banach space, where* ω *denotes all sequences , with*$|x| = \sum \frac{1}{2^n} \frac{|x_n|}{|x_n|+1}$.

Let F be a Frechet space with paranorm $|\cdot|$ and topological dual F^* . Let $\langle \cdot, \cdot \rangle$ be the duality pair between F and F^*.

2. Main Results

Proposition 1. F^* *is a paranorm space.*

Proof. Let

$$|f| = \sup_{x \neq 0} \frac{|\langle x, f\rangle|}{|x|}, \forall f \in F^*. \tag{2.1}$$

Now we verify the conditions in Definition 1. Conditions (a), (b), (c) are obviously satisfied. Since

$$|f + g| = \sup_{x\neq 0} \frac{|\langle x, f\rangle + \langle x, g\rangle|}{|x|} \leq \sup_{x\neq 0} \frac{|\langle x, f\rangle|}{|x|} + \sup_{x\neq 0} \frac{|\langle x, g\rangle|}{|x|} = |f| + |g|,$$

condition (d) is satisfied. If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{f_n\} \subset F^*$ with $|f_n - f| \to 0$. Then

$$\begin{aligned} |t_n f_n - tf| &= \sup_{x\neq 0} \frac{|\langle x, t_n f_n\rangle - \langle x, tf\rangle|}{|x|} \\ &\leq |t_n - t| \sup_{x\neq 0} \frac{|\langle x, f_n\rangle|}{|x|} + t \sup_{x\neq 0} \frac{|\langle x, f_n - f\rangle|}{|x|} \\ &= |t_n - t||f_n| + t|f_n - f|. \end{aligned}$$

Let $n \to \infty$, $|t_n f_n - tf| \to 0$. From Definition 1, we obtain that $|\cdot|$ defined by (2.1) is a paranorm. Let $J_p : F \to 2^{F^*}$ be paranormalized duality mapping of F defined by

$$J_p(x) = \{x^* \in F^* : \langle x, x^*\rangle = |x||x^*|, |x^*| = |x|^{p-1}, x \in F\}.$$

∎

Lemma 1. $J_p(x) \subset \partial\psi(x)$ *, where* $\psi(x) = p^{-1}|x|^P, x \in F$.

Proof. If $f \in J_p(x)$, then for any $y \in F$

$$\langle f, y - x\rangle = \langle f, y\rangle - \langle f, x\rangle \leq |f||y| - |x||f|. \tag{2.2}$$

Now we show that the inequality

$$p|x|^{p-1}|y| \leq |y|^p + (p - 1)|x|^p \tag{2.3}$$

holds for any $p \in N$.

(i) $p = 2$, it is obvious that $2|x||y| \leq |x|^2 + |y|^2$,so inequality (2.3) holds.

(ii) Assume $p = k$,

$$k|x|^{k-1}|y| \leq |y|^k + (k-1)|x|^k, \tag{2.4}$$

holds.

(iii) We consider $p = k + 1$.
It can easily draw a conclusion that

$$(|x| - |y|)(|x|^k - |y|^k) \geq 0, \tag{2.5}$$

which implies $|y|^{k+1} + |x|^{k+1} \geq |x|^k|y| + |x||y|^k$.

From (2.4) and (2.5), we have

$$\begin{aligned}(k+1)|x|^k|y| &= |x|^k|y| + |x||y|^k + k|x|^k|y| \\ &\leq |x|^k|y| + |x||y|^k + (k-1)|x|^{k+1} \\ &\leq |y|^{k+1} + k|x|^{k+1}.\end{aligned}$$

So by induction, for any $p \in N$, inequality (2.3) is satisfied. Since $|f| = |x|^{p-1}$, from (2.3), we have

$$p|f||y| - p|x|^p \leq |y|^p - |x|^p.$$

i.e.

$$|f||y| - |x||f| \leq p^{-1}|y|^p - p^{-1}|x|^p. \tag{2.6}$$

From (2.2) and (2.6), we have

$$\langle f, y - x\rangle \leq p^{-1}|y|^p - p^{-1}|x|^p = \psi(y) - \psi(x),$$

i.e. $f \in \partial\psi(x)$.
So, the proof is completed. ∎

Theorem 1. *Let F be a Frechet space and $J_p : F \to 2^{F^*}$ be paranormalized duality mapping of F, then, for any $x, y \in F, |x + y|^p \leq |x|^p + p < y, j_p(x + y) >$, for all $j_p(x + y) \in J_p(x + y)$.*

Proof. By Lemma 1, we have $J_p(x) \subset \partial\psi(x)$, where $\psi(x) = p^{-1}|x|^P, x \in F$. It follows from the definition of subdifferential of ψ that

$$\psi(x) - \psi(x + y) \geq \langle x - (x + y), j_p\rangle\,, \quad \forall j_p \in J_p(x + y).$$

Substituting $\psi(x)$ by $p^{-1}|x|^p$ and simplifying we have

$$|x + y|^p \leq |x|^p + p\,\langle y, j_p\rangle\,, \quad \forall j_p \in J_p(x + y).$$

∎

Remark 3. *Theorem 1 generalizes the corresponding results in Zhang shi-sheng* [1].

Corollary 1. *Let F be a Frechet space and $J : F \to 2^{F^*}$ be paranormalized duality mapping of F , then, for any $x, y \in F$, $|x + y|^2 \leq |x|^2 + 2\,\langle y, j(x + y)\rangle$, for all $j(x + y) \in J(x + y)$.*

Remark 4. *Corollary 1 generalizes the corresponding results in S. S. Chang* [2].

Remark 5. *Corollary 1 has many applications in the variational inequality, variational inclusion and iterative process for fixed point problems. And many recent results in* [3] – [8] *can be easily generalized by using the inequality in Corollary 1.*

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