

ever, bring out so prominently the simplicity and directness of the relation existing between the quantities involved in the equation and those given by observation.

As an illustration of the utility of equations (2) and (2'), let us take the determination of a scalenohedron on a mineral (such as calcite) whose elements are known. Measurement of two of the angles between adjacent faces suffices for the determination. If PP_1 and PP_2 are the two angles measured, we know the three sides of the triangle aPa_1 ; and the angle $Pab = bQ = \frac{\pi}{2} - OQ$ is the first quantity deduced from the measurements. Equation (2) then gives a simple equation in terms of the indices h, k, l . If PP_1 or PP_2 be given with the angle of the middle edge of the scalenohedron, we know the sides of the triangles aPa_1 or a_2Pa_2 . In the first case OQ is determined as before, in the second OR ; and we must employ (2) or (2') accordingly.

To complete the analysis, I need only point out that the relations connecting the indices of dirhomboidal forms can be most simply obtained by aid of the equations connecting the indices of a face with those of the zone in which it lies. Thus E , the face of the dirhomboidal form corresponding to P , lies in the zones $[OP]$ and $[b_1P_1]$, whence its indices can be at once obtained, and all the geometrical relations connecting it with P can be proved. Professor Maskelyne has, I believe, already given this method of deducing the indices of the dirhomboidal form in his lectures at Oxford.

XVI. *The Dilatation of Crystals on Change of Temperature.*

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DURING the last fifty years a considerable amount of energy has been devoted by mathematicians and crystallographers to the investigation of the nature of the alteration of a crystal due to change of temperature, and it cannot even yet be said that the state of our knowledge is very satisfactory. It must be hoped that this short paper may serve to once more direct attention to this important and interesting question.

Mitscherlich was the first to observe that the angles of crystals may vary with the temperature (Pogg. *Ann.* vol. i. 1824, vol. x. 1827); and he concluded from his experiments that the alteration is probably related to the axes of optical elasticity.

F. E. Neumann was the next to take up the question (Pogg. *Ann.* vol. xxvii. 1833), and assuming, as a deduction from experiment, that the luminiferous æther is in every crystal symmetrical to three perpendicular planes, and that the symmetry of the æther is superinduced by the symmetry of the arrangement of the crystal-molecules, decided that the matter of the crystal must also be symmetrical to these three planes, and that on change of temperature the perpendiculars to these planes must remain permanently fixed in direction in space: to these lines he applied the term *thermic axes*, and gave formulæ by the help of which their position could be calculated in the case of an oblique crystal from goniometrical observations at any two temperatures. Employing for gypsum the data supplied by Mitscherlich, he found that, within the limits of errors of experiment, these thermic axes did actually coincide with the axes of optical elasticity; and he therefore concluded that there really exist for all crystals what may be called rectangular "morphological axes." If this be true, it follows that the oblique and anorthic systems are probably mere hemihedral developments of the rhombic. About the time of the publication of this paper the crossed dispersion of the optic axes of borax was discovered by Herschel and Nörrenberg; and thus considerable doubt may be felt as to the truth of Neumann's assumption that the luminiferous æther and therefore also the matter of every crystal are symmetrical to three rectangular planes. Ångström (Pogg. *Ann.* vol. lxxxvi. 1852) next made a series of experiments on gypsum and felspar, and enunciated as result that the axes of optical elasticity, of cohesion, of ordinary elasticity, and of heat expansion did not all coincide, and that therefore the idea of rectangular morphological axes must be given up. Grailich and Lang (*Sitzungsber. d. Akad. in Wien*, vol. xxxiii. 1859), in their celebrated memoir on the physical relations of crystals, treat of this subject. They come to the conclusion that, for all changes of temperature, the parametral ratios are permanently either rational or irrational, and that in an oblique or anor-

thic crystal rectangular thermic axes do not necessarily, and in fact in gypsum do not, exist—thermic axes being defined to be “such directions in crystals of the non-tesseral systems as at every temperature are equally inclined to each other and present the same unaltered relation to the crystallographic elements.” It was, however, soon pointed out that an expression which they had found on substitution of the data for gypsum to become imaginary, could in the most general case be expressed as the square root of the sum of the squares of two real quantities, and therefore was necessarily real. It is clear that some numerical error had been made in their calculation, and that their formulæ really prove that in the symmetry-plane of an oblique crystal two lines can be found, at any one temperature, which are at right angles at a second. It must be remarked that it is not proved that the lines are at right angles at the intermediate temperatures. It may be here stated that the relations given by Grailich and Lang, as connecting the angles between any five planes, are only proved for the case where two of the planes are crystalloid-planes and the other three are zone-planes, and so can scarcely be called general relations between five crystalloid-planes. If, however, we remark, as will be shown later, that the constancy of the indices, on which the proof depends, still holds even if they be irrational, it follows that the relations mentioned above are true in any case, it being a matter of no moment whether the planes be crystalloid-planes, zone-planes, or neither. C. Neumann (*Pogg. Ann.* vol. cxiv. 1861) shows that, in an anorthic crystal, for any two temperatures three lines can be found which are at right angles at both, and shows how their positions may be calculated from the observed angles of the crystal at the two temperatures. In 1868 (*Pogg. Ann.* vol. cxxxv.) appeared a paper by C. Pape, on the “thermic and chemical axes” of gypsum and chalcantite, in which he (somewhat inaccurately) states that, in the preceding article, Neumann has proved the existence of axes which remain at right angles at all temperatures. From his experiments he infers the coincidence of the thermic and chemical axes in both gypsum and chalcantite, whence he rapidly passes to the belief that he has established the existence of rectangular morphological axes for all crystals.

It is difficult to see how the dispersion of the planes of the optic axes, met with in many crystals, can be simply explained by the theory of the existence of these rectangular morphological axes. In the same volume of the *Annalen* is published one of Fizeau's papers on the dilatation of crystals as determined by experiment; and he finds, by his admirable method, that the coefficients of dilatation of felspar along two directions equally inclined to an optic mean line, and on opposite sides of it, in the plane of symmetry, are almost in the ratio of 5 : 1; and thus the belief in the existence of any simple relation between the positions of the axes of dilatation and the axes of optical elasticity ought to be finally discarded. In his treatment of the theoretical part of the question, Fizeau assumes that the directions of maximum, mean, and minimum dilatation are permanent for all temperatures. Summing up, we may say that up to the present it has merely been proved, and that in a laborious manner, that at any two temperatures there are three crystal-lines at right angles at both: it has not been proved that these lines are at right angles at other temperatures or are fixed in space; but still Pape and Fizeau seem to think they may legitimately assume it.

We shall assume that the physical and geometrical properties along all parallel lines of a crystal are the same. Hence it immediately follows that any set of particles of the crystal in a right line at any one temperature will be so at any other, although the position in space and the length of the line may vary; equal parallel lines will remain equal and parallel; parallel planes will remain parallel planes; parallelograms will remain parallelograms; and parallelepipeds will remain parallelepipeds. We may remark, in the first place, that since all the edges of tautozonal planes are parallel, and lines parallel at one temperature are parallel at every other, the property of tautozonality is a permanent one; and, further, since the condition for the tautozonality of planes, whether crystalloid or not, depends only on their indices, the latter, whether rational or irrational, must also be unaltered by change of temperature. This may also be proved as follows:—Imagine the crystal freely suspended in space: the forces due to change of temperature being all internal and in the nature of actions and reactions, the centre of mass must remain unaltered in

position, and may be taken as a fixed origin. At the initial temperature let $OX \cdot OY \cdot OZ$ be any three crystal-lines passing through the centre of mass O , and let $ABC \cdot HKL$ be any two planes cutting them in A, B, C, H, K, L respectively; if ABC be the parametral plane, the indices of HKL will be

$$\frac{OA}{OH'} \quad \frac{OB}{OK'} \quad \frac{OC}{OL'}$$

At any other temperature let A', B', C', H', K', L' be the new positions of the crystal-points A, B, C, H, K, L ; the indices of H', K', L' , if $A' B' C'$ be the parametral plane, will be

$$\frac{OA'}{OH'} \quad \frac{OB'}{OK'} \quad \frac{OC'}{OL'}$$

But points which are in a right line and equidistant from each other at one temperature will possess these properties at any other, and thus parts of a crystal-line which have any ratio at one temperature will have the same ratio permanently; we therefore must have

$$\frac{OA'}{OH'} = \frac{OA}{OH}, \quad \frac{OB'}{OK'} = \frac{OB}{OK}, \quad \frac{OC'}{OL'} = \frac{OC}{OL};$$

and the indices, whether rational or irrational, are invariable. Since parallelograms remain parallelograms, it follows, from the known properties of homographic figures, that a circle will in general become an ellipse, and a sphere an ellipsoid. This may, however, be shown very simply as follows:— O being, as before, the centre of mass, let $OA \cdot OB \cdot OC$ be three perpendicular crystal-lines at the first temperature; construct a parallelepiped having $OA \cdot OB \cdot OC$ for adjacent edges, and let P be the other extremity of the diagonal which passes through O ; at a second temperature the lines $OA \cdot OB \cdot OC$ will in general not only have taken new directions in space, but have varied in mutual inclination and in length. Let OA', OB', OC' be these new directions and lengths; on these lines as adjacent edges construct a parallelepiped, and let P' be the other extremity of the diagonal which passes through O : since parallelepipeds at one temperature remain so at any other, it follows that P' is the new position of P , and its coordinates $x' y' z'$ referred to the oblique axes will be $OA' \cdot OB' \cdot OC'$ respectively.

Let the unit lengths in the directions $OA \cdot OB \cdot OC$ at the first temperature have increased to the lengths $\alpha \cdot \beta \cdot \gamma$ in the directions $OA' \cdot OB' \cdot OC'$ at the second; we must have

$$OA' = \alpha \cdot OA, \quad OB' = \beta \cdot OB, \quad OC' = \gamma \cdot OC,$$

and

$$\left(\frac{x'}{\alpha}\right)^2 + \left(\frac{y'}{\beta}\right)^2 + \left(\frac{z'}{\gamma}\right)^2 = OA^2 + OB^2 + OC^2 = OP^2.$$

If the locus of P at the first temperature be a sphere of unit radius, the equation to the surface at the second temperature and referred to oblique axes becomes

$$\frac{x'^2}{\alpha^2} + \frac{y'^2}{\beta^2} + \frac{z'^2}{\gamma^2} = 1.$$

But this is the well-known equation of an ellipsoid referred to three conjugate diameters as axes: thus a sphere at any temperature will in general become an ellipsoid at any other, and any triad of perpendicular lines in the sphere will become a triad of conjugate diameters of the ellipsoid; conversely, any triad of conjugate diameters of the ellipsoid must have been at right angles in the sphere. But one and only one triad of conjugate diameters of an ellipsoid, namely the principal axes, is rectangular; and thus there must always exist one and only one triad of lines which is rectangular at two temperatures. If, however, the sphere remains a sphere or becomes a spheroid, it is clear that it will be possible to find an infinity of triads which are rectangular at two temperatures. The axes of the ellipsoid will evidently be the triad of lines which have experienced respectively a maximum, mean, and minimum dilatation. It may also be remarked that in general two, and only two, great circles of the sphere will remain circles—those, in fact, which become at the second temperature the circular sections of the ellipsoid. The above is true whatever the magnitude of the change, and whatever the cause which produces it; the only requisite is that all parallel equal lines should remain parallel and equal. The state of the crystal in the interval has not yet been considered.

It will be convenient now to enter into a short digression as to the permanency of a plane of symmetry at all temperatures, and thus to consider the question of the permanency of the crystallographic systems. It has been shown above that

the indices of planes remain constant, and that for this reason the change of position of any plane of the system of given indices will be known if we are given the changes of the elements $\xi, \eta, \zeta; a : b : c$. Now it may be proved, by help of the rationality of the anharmonic ratios of any four of its planes, that any crystalloid system must present one or other of six types of symmetry, according to the values of ξ, η, ζ and the rationality or irrationality of the ratios $a : b : c$. For instance, if $\xi = \eta = \zeta = \frac{\pi}{2}$ and $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ be all irrational, the crystalloid system of planes will present symmetry to three and only three perpendicular planes; while if one and only one of these ratios, say $\frac{a}{b}$, is irrational and the other elements are unaltered, the crystalloid system will present symmetry to the above three planes and also to two additional planes perpendicular to each other and bisecting the angles between two of the former. It becomes an interesting question as to whether or not, in the case of a crystal presenting symmetry to only three perpendicular planes, $\frac{a}{b}$ can ever be so altered by change of temperature as to become rational. In the above memoir Grailich and Lang answer this question in the negative; for, they argue, if such an event were possible, a crystalloid system of planes might pass, on change of temperature, from rhombic to tetragonal symmetry: and this they think is disproved by the statement that no crystal has been observed to pass on change of temperature from one type of symmetry to the other, as shown by the fact that the optical properties of a crystal at various temperatures point to a permanent type of symmetry of the luminiferous æther, and therefore of the mass of the crystal. In the first place, we may remark that the difference between an irrational number and the nearest rational one is not very large, being somewhere near the infinitieth decimal figure: if, then, $\frac{a}{b}$ experience any finite change, it must necessarily pass through a large number of rational values, for every one of which the crystalloid system of planes must possess tetragonal symmetry. The fallacy of the argument is this: it assumes that the symmetry of the crystal, as shown in all its physical properties, is the same as the symmetry of disposition of the

crystalloid planes. But it is easy to see that this need not be always the case.

Suppose the centres of mass of the molecules to be arranged in rectangular-parallelepipedal order, the edges of an elementary parallelepiped having absolute lengths a, b, c respectively; and, as usual, suppose that all the planes through molecular mass-centres are crystalloid planes: it follows from the above that such a set of *planes* will possess the symmetry characteristic of the rhombic system so long as the ratios $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ are all irrational, but the symmetry of the tetragonal if one and only one of them (say $\frac{a}{b}$) become rational; but it is clear that unless a is not only in a rational ratio to b , but actually equal to it, the symmetry of disposition of the *mass-centres* themselves, and therefore the symmetry of the physical properties, will still be no higher than rhombic. But we may go still further, and assert that even if a actually become equal to b , the physical properties of the crystal need only be symmetrical to three perpendicular planes. For if the molecules themselves "have sides," or different properties in different directions (and no one has ventured to assume that they have not), it is possible to imagine that each molecule has an internal symmetry, upon which, as well as upon the arrangement itself of the molecules, the symmetry of the physical properties must ultimately depend. If, then, in the case we are considering, the internal symmetry of the molecule be only symmetry to the above three planes, even though the symmetry of disposition both of the molecular mass-centres and of the molecular planes be tetragonal, those physical properties (among which we must include dilatation on change of temperature) which depend on the internal symmetry of the molecules as well as upon their grouping will still present only that symmetry which characterizes the rhombic system. And we may further observe that, as the dilatations along the directions perpendicular to these planes are independent, $\frac{a}{b}$ will again become irrational, and the crystalloid system itself once more pass to rhombic symmetry.

It might appear at first sight that, from the measurement of the angles of a crystal at only one temperature, no safe con-

clusion could be arrived at as to the crystal-system to which it must be referred, and that either the crystal must be measured at a second temperature or some physical tests must be applied. We must, however, remember that the faces of a crystal will be in general called into existence in such a way as to accord in their symmetry of disposition with that which is common to the internal arrangement of the molecules and to their grouping, even if the crystal is formed at some temperature at which the latter possesses a symmetry corresponding to a higher type.

In what follows, it will be assumed that in a crystal there is an extremely large number of points arranged in parallelepipedal order, and that at each of these points there is a centre of mass of a molecule or molecular group—all these groups being equal and parallel in orientation. For brevity, such a molecular group will be spoken of as a molecule.

We infer from the above, that the only symmetry-planes which are permanent on change of temperature are those which are at once planes of symmetry to the molecules themselves and to the molecular grouping, and also that it would be possible for a single crystal to determine at each of six different temperatures a crystalloid system presenting symmetry corresponding to a different crystal-system.

Now nothing in the above would conflict with the possibility of new planes of symmetry common to the molecules and their arrangement starting into existence. All we can say is that, if they do come into existence, they must do so symmetrically to the preexisting common symmetry-planes; but once in existence, they cannot be made to disappear again by any cause which would produce internal reactions symmetrical to these common symmetry-planes. Even if the molecules explode, they must do so simultaneously and symmetrically to these planes, and the products of these explosions must therefore also be symmetrical to them. As a crude illustration of the way in which such planes might come into existence, take the previous example of a rhombic arrangement of molecules themselves possessing rhombic symmetry, and let the temperature change to that at which the arrangement of the molecules is tetragonal. It is quite conceivable that at this temperature the internal equilibrium of the molecules

might become unstable, and that each molecule might rearrange itself in such a way as to be still symmetrical to the old planes of symmetry, but also at the same time to the two additional planes requisite for tetragonal symmetry. From this instant the whole system, including the arrangement and the internal constitution of the molecules, would present a symmetry corresponding to the tetragonal type; in fact, the crystal would henceforth be tetragonal.

But this symmetrical state could only be so permanent in a space where no disturbing force could enter. We know from experiment that the cohesion of a body diminishes as the temperature increases, and that in general the body may come to find itself in a fluid condition, a state in which the slightest exertion of force will disturb the general arrangement of the molecules. If the cohesion has diminished to this extent in any single direction in the æolotropic body we are considering, the body has reached a condition at any rate analogous to the fluid state; and although, under the sole action of change of temperature, it would for ever retain its symmetrical crystalline arrangement, still, if any attempt were made to determine its hardness while in this state, the result must inevitably be the total collapse of the parallelepipedal nature of the system. If the crystal is within reach of the force of gravity, it will not be necessary even to try the hardness; the parallelepipedal system on arriving at the above state will instantly be destroyed.

It would thus appear that, although by the action of change of temperature the introduction of new planes of symmetry common to the molecules and their grouping may be possible, the reverse operation cannot be achieved until after a serious interference with the parallelepipedal grouping, and not at all in regions far away from disturbing forces. This seems to point to the greater stability of the systems of higher symmetry, and to the fact that we may entertain the hope of being able to transfer most crystals under favourable circumstances into the cubic system—hexagonal crystals, however, being denied this privilege by virtue of their possessing an equatorial plane of symmetry. The reverse operation might, however, be effected by the action of some force, such as magnetism, which would not necessarily induce internal reactions symmetrical to the common planes of symmetry. As remarkable

instances of this kind of transformation from a lower to a higher type of symmetry, we may mention the transition of:— (1) amorphous bodies to the crystalline state, as in wrought iron, barley sugar, and the fusible alloys of Rose and Newton; (2) an oblique to a rhombic crystal, as in the case of artificial sulphur; (3) a rhombic to tetragonal, as in that of iodide of mercury; (4) a rhombic to rhombohedral, as in the change of aragonite to calcite. And although the red tetragonal iodide of mercury does become yellow, and therefore presumably rhombic, on heating, still the general rule seems to be that if a crystal is to pass, under the action of the above kind of forces, from a type of symmetry in which there is any number of symmetry-planes common to the molecules and their grouping, to a type in which any of these planes has disappeared, the crystal must first pass through the fluid condition; and in all cases we may say that the parallelepipedal nature of the system will first be destroyed.

Neglecting cases where this destruction of the parallelepipedal nature of the system takes place, we shall henceforth assume the permanency for all temperatures of any of these common symmetry-planes.

Returning to the so-called thermic axes, we have seen that their rectangularity and permanency has been generally assumed for all temperatures. But on consideration we must decide that this is an unfair assumption; for just as Neumann was led by his theory of the symmetrical nature of every crystal with respect to three perpendicular planes to conclude that these perpendicular planes would remain fixed in space for all temperatures, so we might in the same way almost conclude the converse of this proposition, and say that if these planes are permanent in position in space for all temperatures, they are planes of symmetry to the molecules and their grouping. It is true that if these planes only retain their mutual inclination whilst altering their directions in space, we cannot come to the above conclusion; but in that case we should probably say that thermic axes which revolve with the temperature do not bear any very intimate relation to the structure of the crystal—that, in fact, they are mere geometrical inventions, and something akin to the axes of that hypothetical ellipse which the moon describes round a certain point as focus.

For simplicity we shall first consider the alteration of a crystal presenting at least one plane of symmetry common to the molecules and their grouping, and this at all temperatures.

Imagine a circular cylinder cut at any temperature from such a crystal, with its axis perpendicular to the plane of symmetry. From the above, if the cylinder be freely suspended in space and its temperature changed, the axis will remain permanent in direction. In general, the planes perpendicular to the axis will be translated parallel to it; but one plane, that passing through the centre of mass of the crystal or the middle point of the axis, will remain permanently fixed in space; the crystal-lines, however, which lie in this plane will in general change their directions in space and likewise their mutual inclination. To investigate the nature of these changes, we shall take in this plane rectangular axes OX. OZ fixed in space. At any instant determined by the time t , let the crystal-line of unit length coinciding with OX be rotating in space with the angular velocity $\frac{d\theta}{dt}$ or $\dot{\theta}$, and be increasing in length at the rate $\frac{d\alpha}{dt}$ or $\dot{\alpha}$; and let $\frac{d\phi}{dt}$ or $\dot{\phi}$, $\frac{d\gamma}{dt}$ or $\dot{\gamma}$, be the corresponding values for the crystal-line of unit length coinciding at the same instant with the direction OZ. If $x z$ be the coordinates of any point P at this instant, its resolved velocities in the directions OX. OZ will be $x\dot{\alpha} - z\dot{\phi}$, $x\dot{\theta} + z\dot{\gamma}$ respectively. If the change of OP at this instant be one of simple elongation, we must have

$$\frac{x\dot{\alpha} - z\dot{\phi}}{x} = \frac{x\dot{\theta} + z\dot{\gamma}}{z},$$

or

$$\dot{\theta}x^2 - (\dot{\alpha} - \dot{\gamma})xz + \dot{\phi}z^2 = 0. \quad \dots \quad (1)$$

We may first remark that there is either a single pair of such lines, whether real or imaginary, or an infinity of them; for so long as the coefficients do not all vanish, this equation represents only two lines; while if all the coefficients do vanish, and thus the equation reduce to any identity, we must have $\dot{\theta} = \dot{\phi} = 0$ and $\dot{\alpha} = \dot{\gamma}$, in which case the change is one of simple linear dilatation in all directions.

If the crystal present the common symmetry-planes characteristic of the cubic, tetragonal, or hexagonal systems, it follows from the permanency of direction of those planes that there will be more than two lines in certain of these symmetry-planes likewise permanent in direction in space: in these cases, therefore, the change of every line of the system lying in the common symmetry-plane containing these lines of fixed direction will be one of simple elongation without rotation. In the case of the orthorhombic system, the two lines in a symmetry-plane which remain unchanged in direction in space will be the same for all temperatures, and, moreover, will be at right angles.

We shall now show that these lines, in the case of an oblique crystal, are real, though it is clear that in general they are not at right angles. From dynamical considerations it is known that, so long as the forces acting on a material system are internal, and therefore of the nature of actions and reactions, there can be no change of the moment of momentum of the system about any line. As the molecular system under consideration starts from rest, the moment of momentum of the system about any line must therefore be zero throughout the motion. Further, the moment of momentum of a material system about any straight line is known to be equal to the moment of momentum of the system, supposed collected at its centre of mass and moving with it, plus the moment of momentum of the system relative to its centre of mass about a straight line drawn parallel to the given straight line through the centre of mass. In the case of a molecular group, the latter of these terms will be either absolute zero, or, at any rate, of an order of magnitude which may safely be neglected in comparison with the former term. If, then, r be the distance of the centre of mass of a molecular group of mass m from the above axis, ω the angular velocity at this instant of the radius vector perpendicular to the axis and passing through the centre of mass of the group, the moment of momentum of the molecule about the axis is $mr^2\omega$, and for the whole system $\Sigma mr^2\omega$. From the above this must be zero; and thus at any instant the values of ω for some molecules must be positive and for others negative, and there must be at least one intermediate position for which ω must be zero. From the above

equation, or from more simple considerations, it follows that there is a second real line at this instant fixed in space. For these lines Professor Maskelyne (to whom I may here express my hearty thanks for his many valuable suggestions) proposes the convenient term *atropic*. The change of configuration of the system at this instant may be treated as a simple linear dilatation along these two directions. Expressed in rectangular coordinates, the above condition $\Sigma mr^2\omega = 0$ may be written as

$$\dot{\theta}\Sigma mx^2 - (\dot{\alpha} - \dot{\gamma})\Sigma maxz - \dot{\phi}\Sigma mz^2 = 0. \quad \dots \quad (2)$$

As the atropic lines are in general not at right angles, and yet one pair of crystal-lines rectangular at one temperature has been shown to be also rectangular at a second, it is seen that these latter lines will in general have changed their direction in space. The position of this pair of lines may be found in the following way:—Let $r \psi$ be the polar coordinates of any point P, OX being the initial line, and let ω be the angular velocity of OP at any instant, while $\frac{1}{r} \frac{dr}{dt}$, the rate of elongation of the unit crystal-line in the direction OP, may be denoted by α . We have immediately, by resolution perpendicularly to the radius vector OP,

$$\omega = [\dot{\theta} \cos \psi + \dot{\gamma} \sin \psi] \cos \psi - [\dot{\alpha} \cos \psi - \dot{\phi} \sin \psi] \sin \psi,$$

or

$$2\omega = \dot{\theta} + \dot{\phi} + [\dot{\theta} - \dot{\phi}] \cos 2\psi - [\dot{\alpha} - \dot{\gamma}] \sin 2\psi. \quad \dots \quad (3)$$

Similarly for the line perpendicular to OP, and by hypothesis having the same angular velocity,

$$2\omega = \dot{\theta} + \dot{\phi} - [\dot{\theta} - \dot{\phi}] \cos 2\psi + [\dot{\alpha} - \dot{\gamma}] \sin 2\psi.$$

Equating these values of ω , we deduce

$$\omega = \frac{1}{2} [\dot{\theta} + \dot{\phi}] \quad \dots \quad (4)$$

$$\tan 2\psi = \frac{\dot{\theta} - \dot{\phi}}{\alpha - \gamma} \quad \dots \quad (5)$$

So long as the atropic lines are fixed in space, the ratios of the coefficients of x^2 , xz , and z^2 in equation (1) must be constant, and just so long will ψ remain constant. If, then, the lines given by equation (1) are atropic for more than one instant, the directions of the crystal-lines which are at right angles and

are rotating in the same direction with equal velocities will also be fixed in space for more than one instant. But it must be noticed that, by reason of the angular velocity of such lines, different pairs of crystal-lines will coincide at different instants with these fixed directions in space: in other words, if two lines are permanently atropic and not at right angles, not only will the lines which at any instant are retaining their perpendicularity have an angular velocity, but at different instants different crystal-lines will possess the property. We shall now proceed a step further, and show that this retention of the mutual inclination for any instant is possessed by an infinity of pairs of lines. If χ be the angle any crystal-line makes with the axis OX, it follows from (3) that, if ω be the angular velocity of this line,

$$2\omega = \dot{\theta} + \dot{\phi} + (\dot{\theta} - \dot{\phi}) \cos 2\chi - (\dot{\alpha} - \dot{\gamma}) \sin 2\chi. \quad (6)$$

If $\psi, \psi + \frac{\pi}{2}$ be the angular coordinates of those lines which at this instant are retaining their perpendicularity, we may write from (5),

$$\dot{\theta} - \dot{\phi} = a \sin 2\psi, \quad \dot{\alpha} - \dot{\gamma} = a \cos 2\psi,$$

and thus

$$2\omega = \dot{\theta} + \dot{\phi} + a \sin 2(\psi - \chi).$$

Similarly, if ω be also the angular velocity of a line whose angular coordinate is χ' ,

$$2\omega = \dot{\theta} + \dot{\phi} + a \sin 2(\psi - \chi').$$

Equating these values of ω , we deduce

$$(\chi' - \psi) + (\chi - \psi) = \frac{\pi}{2}.$$

Thus, at any instant, any two lines will be retaining their mutual inclination if the sum of the angles which they make with one of the lines of that pair which is retaining its perpendicularity be equal to $\frac{\pi}{2}$ —in other words, if they are equally inclined to and on opposite sides of a bisector of the angle between the lines of this latter pair.

The following relations between the angular velocities and between the rates of elongation of perpendicular lines are interesting.

If ω' be the angular velocity of the crystal-line whose

angular coordinate is $\chi + \frac{\pi}{2}$, it follows, as above, that

$$2\omega' = \dot{\theta} + \dot{\phi} - (\dot{\theta} - \dot{\phi}) \cos 2\chi + (\dot{\alpha} - \dot{\gamma}) \sin 2\chi; \quad \dots \quad (7)$$

and by addition of (6) and (7),

$$\omega + \omega' = \dot{\theta} + \dot{\phi}.$$

Thus at any instant the sum of the angular velocities of any two perpendicular lines in the plane of symmetry is constant. In the case of an orthorhombic crystal, two lines in a plane of symmetry have been shown to be perpendicular and without angular velocity; thus at any instant, in such a crystal any two perpendicular lines in a symmetry-plane will be rotating with equal velocities in opposite directions. Again, resolving the velocities $x\dot{\alpha} - z\dot{\phi}$, $x\dot{\theta} + z\dot{\gamma}$ along the radius vector OP, we get

$$\begin{aligned} \kappa &= \frac{1}{r} \frac{dr}{dt} = (\dot{\theta} \cos \chi + \dot{\gamma} \sin \chi) \sin \chi + (\dot{\alpha} \cos \chi - \dot{\phi} \sin \chi) \cos \chi \\ &= \dot{\alpha} \cos^2 \chi - (\dot{\phi} - \dot{\theta}) \sin \chi \cos \chi + \dot{\gamma} \sin^2 \chi, \end{aligned}$$

and similarly, for the perpendicular line,

$$\kappa' = \dot{\alpha} \sin^2 \chi + (\dot{\phi} - \dot{\theta}) \cos \chi \sin \chi + \dot{\gamma} \cos^2 \chi$$

and

$$\kappa + \kappa' = \dot{\alpha} + \dot{\gamma};$$

or at any instant the sum of the rates of elongation along perpendicular lines in the plane of symmetry of a crystal is constant for all pairs.

To sum up, we may say that at any instant the change of configuration of the molecular distribution in a plane of symmetry may be treated as (1) a simple linear dilatation along two straight lines, in general not at right angles, or (2) a linear dilatation along any pair of an infinity of pairs of lines, accompanied by a rotation of the system as a rigid body: the lines of one of these pairs are at right angles; and the dilatation along them is in one case a maximum and in the other a minimum.

It may be shown in a somewhat similar way that, whatever be the magnitude of the changes, if the crystal-lines of unit length which at a temperature τ coincide with the fixed rectangular axes OX . OZ have become at a second temperature τ' of lengths α and γ respectively, and have revolved through angles of θ and ϕ , the coordinates $\xi \zeta$ of the point P' (the

position at τ' of the point P, of which the coordinates at τ were $x z$),

$$\xi = \alpha x \cos \theta - \gamma z \sin \phi,$$

$$\zeta = \alpha x \sin \theta + \gamma z \cos \phi.$$

Whence it follows that one pair of lines, whether real or imaginary, will have the same directions at the two temperatures, their equation being

$$\alpha \sin \theta x^2 - (\alpha \cos \theta - \gamma \cos \phi) x z + \gamma \sin \phi z^2 = 0. \quad (8)$$

We have seen above that in the limiting case these lines are real. Also one pair of crystal-lines, inclined at angles of ψ and $\psi + \frac{\pi}{2}$ to the axis OX at the initial temperature τ , will be at right angles at the temperature τ' , the angle ψ being given by the very simple relation

$$\tan 2\psi = \frac{2 \sin (\theta - \phi)}{\frac{\alpha}{\gamma} - \frac{\gamma}{\alpha}}.$$

If the lines given by (8) have been atropic *during* the interval from τ to τ' , the same crystal-lines cannot have been at right angles during the whole of this interval; for if the ratios $\alpha \sin \theta : \alpha \cos \theta - \gamma \cos \phi : \gamma \sin \phi$ be constant during the interval, ψ can only be constant when θ and ϕ are zero and $\alpha = \gamma$. Further, it may be shown that if any crystal-line OP at the temperature τ makes an angle ψ with the axis OX, a second crystal-line OQ making an angle $\psi + \epsilon$ with the axis OX can be found which has the same inclination to the crystal-line OP at the two temperatures. If $\alpha \sin \theta = a$, $\gamma \cos \phi = b$, $\alpha \cos \theta = c$, $\gamma \sin \phi = d$, and ψ_1 be the inclination of the crystal-line OP to the axis OX at the temperature τ' ,

$$\tan \psi_1 = \frac{a + b \tan \psi}{c - d \tan \psi},$$

$$\tan (\psi_1 + \epsilon) = \frac{a + b \tan (\psi + \epsilon)}{c - d \tan (\psi + \epsilon)}.$$

Eliminating ψ_1 between these two equations and dividing through by $\tan \epsilon$, we obtain a linear relation connecting $\tan \psi$ and $\tan \epsilon$, and thus for any value of ψ we get a real value of ϵ . Thus, considering only the molecular distribution in a common plane of symmetry, the system may be brought from the con-

figuration at one temperature to the configuration at any other by (1) a simple linear dilatation along two directions in general not at right angles, or (2) a linear dilatation parallel to any one of an infinity of pairs of lines followed by a rotation of the system through some angle round the perpendicular to the symmetry-plane: the lines of one of these pairs are rectangular; and in this case they are the crystal-lines which experience the maximum and minimum elongation.

In exactly the same way it may be proved that, in the case of an anorthic crystal, if the crystal-lines of unit length coinciding at a temperature τ with fixed rectangular axes in space have at a second temperature τ' increased to α , β , and γ respectively, and have taken new directions defined respectively by the direction-cosines

$$\lambda_1\mu_1\nu_1, \quad \lambda_2\mu_2\nu_2, \quad \lambda_3\mu_3\nu_3, \text{---}$$

(1) One triad of lines not at right angles will have the same directions in space at both temperatures, each line being determined by the three equations

$$\begin{aligned} (\lambda_1\alpha - \delta)x + \lambda_2\beta y + \lambda_3\gamma z &= 0, \\ \mu_1\alpha x + (\mu_2\beta - \delta)y + \mu_3\gamma z &= 0, \\ \nu_1\alpha x + \nu_2\beta y + (\nu_3\gamma - \delta)z &= 0, \end{aligned}$$

where δ is any one of the roots of the cubic

$$\begin{vmatrix} \lambda_1\alpha - \delta & \lambda_2\beta & \lambda_3\gamma \\ \mu_1\alpha & \mu_2\beta - \delta & \mu_3\gamma \\ \nu_1\alpha & \nu_2\beta & \nu_3\gamma - \delta \end{vmatrix} = 0.$$

In any case one of these lines must be real; and by the principle of the "conservation of areas" it may be shown, as above, that in the limit all three must be real.

(2) An infinity of triads are equiangular at the two temperatures; one only of these triads is rectangular. This property of two homographic point-systems is virtually proved in a paper by Prof. H. J. S. Smith, "On the Focal Properties of Homographic Figures," published by the London Mathematical Society. The dilatations along the lines which are at right angles at the two temperatures will be maximum, mean, and minimum respectively.

Thus we may pass from the configuration at the first to that at the second temperature by (1) simple linear dilatations

along three lines in general not at right angles, or (2) linear dilatations along equiangular triads followed by a rotation of the system as a rigid body.

It would appear that the term thermic axes, if applied at all, ought really to be devoted to those which have been here called atropic.

The following problems may be suggested for solution:—

(1) What is the relation between these atropic lines and the directions of the edges of an elementary parallelepiped?

(2) Are the same lines atropic for all temperatures?

XVII. *Examination of two new Amalgams, and a Specimen of Native Gold.* By WALTER FLIGHT, D.Sc., of the Mineral Department of the British Museum.

A SPECIMEN of “native silver” from Kongsberg proved, on analysis, to be an amalgam, and, as it appears to me, to be an amalgam new to science. The composition of the specimen in question was found to be:—

Silver . .	75·900	$\frac{75\cdot900}{108} = 0\cdot70.$
Mercury . .	23·065	$\frac{23\cdot065}{100} = 0\cdot23.$
Insol. part .	0·490	
	99·455	

These numbers indicate the formula $\text{Ag}_3 \text{Hg}$ as that of the amalgam, or a composition the converse of that of the amalgam from Moschellandsberg, Ag Hg_3 . It is worth recording how firmly and tenaciously mercury is retained by silver in the molten state. In one experiment, a fragment of the above amalgam was fused in a glass tube and kept at a bright red heat for more than ten minutes. The bead of silver, when cold, was beaten flat, cut into strips, and again heated, when what proved to be almost an additional one per cent. of mercury came off.

Another specimen of supposed “native silver” from Kongsberg was found to possess a somewhat different composition. The analysis gave the following numbers:—