Semiclassical analysis and Berezin–Toeplitz quantization

Alix Deleporte

Laboratoire de Mathématiques d'Orsay
Université Paris-Saclay

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What is semiclassical/microlocal analysis

Observed link between (linear) PDEs and classical mechanics.

Example: light propagates the same way as billiard balls... But WiFi doesn't!

What's the link between the d'Alembert wave equation and the shortest path (geodesic) equation?

$$\partial_t^2 u = \Delta_x u \qquad\qquad \ddot{\gamma}(t) = 0$$

XXth century revival:

- build quantum mechanics from classical mechanics (quantization)
- recover classical mechanics from scale limit of quantum mechanics (semiclassical limit)

Semiclassical correspondence

Objects	Classical	Quantum

Semiclassical correspondence

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Quantization: associate to $f \in C^\infty(\mathbb{R}^{2d},\mathbb{R})$ a family $Op_\hbar(f) \in L(H)$, such that, as $\hbar \to 0$,

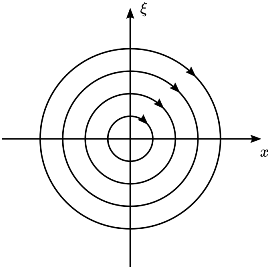
$$[Op_\hbar(f), Op_\hbar(h)] = i\hbar Op_\hbar(\{f,g\}) + O(\hbar^2).$$

One practical recipe: pseudodifferential operators

$$H = L^2(\mathbb{R}^d_x) \qquad\qquad Op_\hbar(\xi) = -i\hbar\nabla$$

Example: harmonic oscillator (mass on a spring)

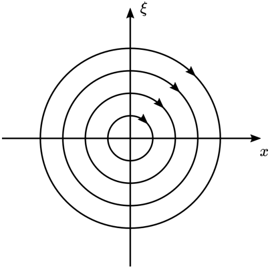
classical	$\rightsquigarrow$	quantum
$(\dot{x}, \dot{\xi}) = (\xi, -x)$		



$$x(t) = \sqrt{x_0^2 + \xi_0^2} \cos\left[t - \arctan\left(\frac{\xi_0}{x_0}\right)\right]$$

Example: harmonic oscillator (mass on a spring)

classical	$\rightsquigarrow$	quantum
$(\dot{x}, \dot{\xi}) = (\xi, -x)$		$i\hbar\partial_t\psi = \frac{1}{2}(-\hbar^2\Delta + x^2)\psi$



???

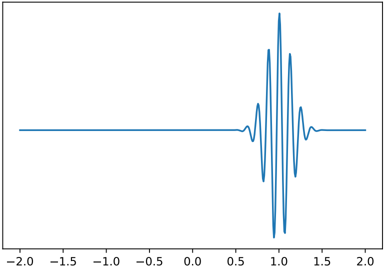
$$x(t) = \sqrt{x_0^2 + \xi_0^2}\cos\left[t - \arctan\left(\frac{\xi_0}{x_0}\right)\right]$$

$$H_{n;\hbar}(x)e^{-\frac{x^2}{2\hbar}}$$

Gaussian wavepackets

Introduce family of quantum states “corresponding to” a physical state:

$$\psi_{(x,\xi)}(y) = \frac{1}{(2\pi\hbar)^{\frac{d}{4}}} \exp\left[-\frac{1}{\hbar}\left(\frac{|x-y|^2}{2} + i(x-y)\cdot\xi\right)\right].$$



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- $\mathrm{Op}_\hbar(a)\psi_{(x,\xi)} = a(x,\xi)\psi_{(x,\xi)} + O(\hbar)$ $\qquad a \in C_c^\infty.$
- $\langle \psi_{(x,\xi)}, \mathrm{Op}_\hbar(a)\psi_{(x',\xi')}\rangle = O(\hbar^\infty)$ $\qquad (x,\xi) \neq (x',\xi').$
- Exact harmonic oscillator formula:

$$e^{-\frac{it}{2\hbar}(-\hbar^2\Delta + x^2)}\psi_{(x_0,\xi_0)} = e^{\frac{i}{\hbar}\int_0^t \xi(s)dx(s)}\psi_{(x(t),\xi(t))}.$$

In this example the Gaussian wavepackets behave classically!

The Bargmann transform

Can one “reconstruct” any function from Gaussian wavepackets? Introduce the Bargmann (FBI) transform

$$\mathcal{B}_\hbar u : (x,\xi) \mapsto (2\pi\hbar)^{-d}\langle u, \psi_{(x,\xi)}\rangle$$

Proposition

$\mathcal{B}_\hbar$ sends $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$. It is an isometry on its image.

$$\mathcal{B}_\hbar(L^2(\mathbb{R}^d)) = \left\{ u \in L^2(\mathbb{R}^{2d}), e^{\frac{|\xi|^2}{2\hbar}} u \text{ is holomorphic} \right\}.$$

Bargmann space: $\mathcal{B}_\hbar(L^2(\mathbb{R}^d)) = B_\hbar$
Bergman projector: $\mathcal{B}_\hbar\mathcal{B}_\hbar^* = \Pi_\hbar : L^2(\mathbb{R}^{2d}) \to B_\hbar$

Bargmann space I - a better gauge

From now on $\frac{x+i\xi}{\sqrt{2}} = z$.

Gauge change: the weight $e^{\frac{|\mathrm{im}(z)|^2}{\hbar}}$ is not very symmetric. Tool:

$$|\mathrm{im}(z)|^2 = \frac{|z|^2}{2} - \underbrace{\frac{z\cdot z}{2}}_{\text{holomorphic}} + \underbrace{i\,\mathrm{re}(z)\cdot\mathrm{im}(z)}_{\text{purely imaginary}}.$$

Hence the L^2 isometry $u \mapsto e^{-i\frac{\mathrm{re}(z)\cdot\mathrm{im}(z)}{\hbar}} u$ changes $B_\hbar$ into

$$\left\{ f \in L^2(\mathbb{C}^d), e^{\frac{|z|^2}{2\hbar}} f \text{ is holomorphic} \right\}.$$

This corresponds to the (apparently weird) convention

$$\psi_{(x,\xi)} : y \mapsto (2\pi\hbar)^{-\frac{d}{4}} \exp\left[-\frac{1}{\hbar}\left(\frac{|x-y|^2}{2} + i(\frac{x}{2} - y)\cdot\xi\right)\right]$$

Bargmann space II - Hilbert basis and projector

Hilbert basis of monomials (normalised, orthogonal, span $B_\hbar$):

$$\mathbb{N}^d \ni \nu \rightsquigarrow e_\nu = \frac{\pi^d \hbar^{d+|\nu|}}{\sqrt{\nu!}} z^\nu e^{-\frac{|z|^2}{2\hbar}}$$

Consequence: the projector has an integral kernel

$$\Pi_\hbar(z,w) = (\pi\hbar)^{-d} \exp\left[-\frac{1}{2\hbar}(|z-w|^2 + 2i\mathrm{im}(z\cdot\overline{w}))\right].$$

- For fixed $z \neq w$, $\Pi_\hbar(z,w) \to 0$ very fast.
- $z \mapsto \Pi_\hbar(z,w) = C_\hbar \mathcal{B}_\hbar(\psi_{\sqrt{2}w})$ is a Gaussian with center w.

Berezin–Toeplitz operators

We want to manipulate operators while staying in $B_\hbar$.
Natural object associated with $a : \mathbb{C}^d \to \mathbb{R}$: quadratic form:

$$\forall u \in B_\hbar, Q_a(u) = \int a|u|^2$$

Bounded if $a \in L^\infty$, positive if $a \geqslant 0$, dense domain if a has polynomial growth.

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To Q_a corresponds a Toeplitz operator

$$T_\hbar(a) = \Pi_\hbar a \Pi_\hbar$$

Goal: show that $a \rightsquigarrow T_\hbar(a)$ is a quantization, linked with pseudodifferential operators.

Quantizing polynomials

First computations

$$T_\hbar(1) = I \qquad T_\hbar(z) = z \qquad T_\hbar(\bar{z}) = (T_\hbar(z))^* = \hbar\mathfrak{d}$$

Here $\mathfrak{d} = e^{-\frac{|z|^2}{2\hbar}} \partial e^{\frac{|z|^2}{2\hbar}}$.

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Here $\mathfrak{d} = e^{-\frac{|z|^2}{2\hbar}}\partial e^{\frac{|z|^2}{2\hbar}}$.
More generally

$$\langle u, T_\hbar(z^\alpha \overline{z}^\beta)v\rangle = \int u z^\beta \overline{z}^\alpha \overline{v} = \langle z^\beta u, z^\alpha v\rangle$$

so that

$$T_\hbar(z^\alpha\overline{z}^\beta) = (\hbar\mathfrak{d})^\beta z^\alpha.$$

z is like x and $\overline{z}$ is like ξ... With anti-Wick ordering.

The harmonic oscillator revisited

Creation and annihilation operators (proof postponed):

$$\mathcal{B}_\hbar^* T_\hbar(z)\mathcal{B}_\hbar = \frac{x+\hbar\nabla}{\sqrt{2}} \qquad \mathcal{B}_\hbar^* T_\hbar(\bar{z})\mathcal{B}_\hbar = \frac{x-\hbar\nabla}{\sqrt{2}},$$

so that

$$\mathcal{B}_\hbar^* T_\hbar(|z|^2)\mathcal{B}_\hbar = \frac{1}{2}(-\hbar^2\Delta + x^2) + \frac{\hbar}{2}.$$

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Eigenfunctions on the Bargmann side: the monomials $z^\nu e^{-\frac{|z|^2}{4\hbar}}$.

Propagator $e^{itT_\hbar(|z|^2)/\hbar}$=rotation around the origin.

Calculus of Berezin–Toeplitz operators

From the anti-Wick property, we obtain

$$T_\hbar(a)T_\hbar(b) \approx T_\hbar\left[\sum_{\alpha\in\mathbb{N}^d}\frac{\hbar^{|\alpha|}}{\alpha!}\partial^\alpha a\overline{\partial}^\alpha b\right]$$

Exact formula (finite sum) when a or b is a polynomial; valid mod. $O_{L^2\to L^2}(\hbar^\infty)$ when all high enough derivatives of a and b are bounded.

Remark: it is not true that the product of two Berezin–Toeplitz operator is again a Berezin–Toeplitz operator.

Wick/covariant/lower symbol

Since $\Pi_\hbar$ has an integral kernel, then (as long as a has polynomial growth) $T_\hbar(a) = \Pi_\hbar a \Pi_\hbar$ has an integral kernel.

Wick symbol of $T_\hbar(a)$=restriction to the diagonal of the integral kernel.

- Obtained by forward heat:

$$T_\hbar(a)(z,z) = e^{i\frac{\Delta}{\hbar}} a(z)$$

- The operator with Wick symbol $z^\alpha \overline{z}^\beta$ is $z^\alpha (\hbar\partial)^\beta$.
- $T_\hbar(a)(z,z)$ determines $T_\hbar(a)(z,w)$ by holomorphy.

Wick symbols of pseudodifferential operators

Pairing between Gaussian states: define the Wick symbol of $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ as

$$(x, \xi) \mapsto \langle \psi_{(x,\xi)}, A\psi_{(x,\xi)} \rangle.$$

- Consistent with last slide (via $\mathcal{B}_\hbar$).
- $\langle \psi_{(x,\xi)}, Op_\hbar(a)\psi_{(x,\xi)} \rangle = e^{\frac{\Delta}{2\hbar}} a(x, \xi)$ (forward heat at half time).
- Conclusion: $\mathcal{B}_\hbar T_\hbar(a)\mathcal{B}_\hbar^* = Op_\hbar(e^{\frac{\Delta}{2\hbar}} a)$ (for every $a \in L^\infty$).

Comments on regularity

If $a \in C_c^\infty(\mathbb{R}^{2d}, \mathbb{C})$, one can approximately invert the heat evolution

$$\mathcal{B}_\hbar^* \mathrm{Op}_\hbar(a) \mathcal{B}_\hbar = T_\hbar \left[\sum_{k=0}^{+\infty} \frac{(-1)^k \Delta^k a}{2^k k!} \right] + O_{L^2 \to L^2}(\hbar^\infty).$$

Remark: $T_\hbar(a)$ is well-defined and bounded whenever $a \in L^\infty$ but $\mathrm{Op}_\hbar(a)$ is not.

Application: Gårding inequality:

$$C_c^\infty \ni a \geqslant 0 \Rightarrow T_\hbar(a) \geqslant 0 \Rightarrow \mathrm{Op}_\hbar(a) \geqslant -C\hbar.$$

Why you should like Berezin–Toeplitz operators

- Positivity, works in lower regularity.
- The harmonic oscillator is simpler.
- Microlocalisation is localisation.
- No caustics.

You may already be close to Berezin–Toeplitz operators if you're studying

- FBI or other wavepackets transforms.
- The physicists' second quantization (anti-Wick ordering).
- Strong magnetic fields (B_k is the first Landau level for a constant magnetic field)

Caveat emptor

- Beware of factors $\sqrt{2}$ when changing quantizations.
- Beware of gauge changes.
- Sometimes the natural FBI transform yields anti-holomorphic function spaces.
- Beware of symbol spaces.

Some serious references (with locally constant conventions):

- G. B. Folland, Harmonic analysis in phase space.
- M. Zworski, Semiclassical analysis.
- A. Martinez, Introduction to semiclassical analysis.
- B. Hall, Holomorphic methods in analysis and mathematical physics.

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From Euclidian space to the torus

Goal: build a space of functions invariant by translations along a discrete lattice, and study how Berezin–Toeplitz operators with periodic symbol act on them.

First: what are translations? If $f \in B_\hbar$ then $f(v + \cdot) \in L^2$ but $f(v + \cdot)e^{\frac{|\cdot|^2}{2\hbar}}$ is not holomorphic.

Magnetic translations

Idea: in the classical world, translation on the x axis means following the Hamilton flow of ξ.
Set

$$U_x(t) = \exp(i\tfrac{t}{\hbar}T_\hbar(\mathrm{im}(z))) \qquad U_\xi(t) = \exp(-i\tfrac{t}{\hbar}T_\hbar(\mathrm{re}(z))).$$

Now $T_\hbar(\mathrm{im}(z)) = \frac{T_\hbar(z)-T_\hbar(\overline{z})}{2i} = \frac{z-\hbar\partial}{2i}$.

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Now $T_\hbar(\mathrm{im}(z)) = \frac{T_\hbar(z)-T_\hbar(\overline{z})}{2i} = \frac{z-\hbar\partial}{2i}$.
Transport equations! Solutions

$$U_x(t)f : z \mapsto f(z-t)e^{\frac{it}{\hbar}\mathrm{im}(z)}$$

$$U_\xi(t)f : z \mapsto f(z-it)e^{-\frac{it}{\hbar}\mathrm{re}(z)}$$

Noncommutative geometry

Beware that $T_\hbar(\mathrm{im}(z))$ and $T_\hbar(\mathrm{re}(z))$ do not commute! Neither do $U_x(t)$ and $U_\xi(s)$ in general.
A computation yields

$$U_x(t)U_\xi(s)U_x(-t)U_\xi(-s) = e^{2i\frac{ts}{\hbar}}.$$

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Requirement for commutativity: Area of fund. domain is a multiple of $\pi\hbar$.

From now on, $\hbar = \frac{1}{k}$, with $k \to +\infty$, and the area of the fundamental domain is a multiple of π.

Floquet theory

For a lattice Λ as above, it makes sense to consider functions f such that

- $z \mapsto f(z)e^{\frac{k|z|^2}{2}}$ is holomorphic
- f is invariant under translations by elements of Λ.

How large is this space?

Proposition

At least when k is large, the dimension of this space is $\frac{k\mathrm{Area}(\mathbb{C}/\Lambda)}{\pi}$.

Dimension of quantum space

Quick proof: the projector on this space is $\sum_{\lambda \in \Lambda} U_k(\lambda)\Pi_k$ and Π_k decays rapidly away from the diagonal so that

$$\begin{aligned}
\dim &= \mathrm{tr}(\text{projector}) \\
&= \int_{\mathbb{C}/\Lambda} \sum_{\lambda \in \Lambda} \Pi_k(z-\lambda, z) e^{ik\mathrm{im}(\overline{\lambda} z)} dz \\
&= \int_{\mathbb{C}/\Lambda} \Pi_k(z,z) dz + O(e^{-ck}) \\
&= \frac{k}{\pi} \mathrm{Area}(\mathbb{C}/\Lambda) + O(e^{-ck}).
\end{aligned}$$

Coherent states

Let $P_k = \sum_{\lambda\in\Lambda} U_k(\lambda)\Pi_k = \sum_{j=1}^{k} e_j^* e_j$ be the reproducing projector of our space. It has an integral kernel, and freezing the second variable we obtain the coherent states

$$\psi_w : z \mapsto \sum_{\lambda\in\Lambda} U_k(\lambda)\Pi_k(z,w).$$

Remember that $U_k(\lambda)$ acts by translation and multiplication by the exponential of a linear term and Π_k is a Gaussian.

Bottom line: ψ_w is a Jacobi theta function.

Computing Toeplitz operators

Set $\Lambda = \mathbb{Z} + i\pi\mathbb{Z}$. Given $f : \mathbb{C}/\Lambda \to \mathbb{R}$, we can quantize into $T_k(f) = P_k f P_k$. But theta functions are not very practical...

Proposition

- *The spectrum of* $T_k(\cos(2\pi x))$ *is* $\{\cos(2\pi j/k), 1 \leqslant j \leqslant k\}$.
- *The map between the eigenbases of* $T_k(\cos(2\pi x))$ *and* $T_k(\cos(2\xi))$ *is the discrete Fourier transform.*
- *The composition rule for these Toeplitz operators is the same as on* $\mathbb{C}$*:*

$$T_k(a)T_k(b) = T_\hbar\left[\sum_{\alpha\in\mathbb{N}^n} \frac{1}{k^{|\alpha|}\alpha!}\partial^\alpha a \overline{\partial}^\alpha b\right] + O(\hbar^\infty)$$

Example: finite differences for ODEs on the circle.

The quantum space

What exactly is the space here? It is NOT a weighted space of holomorphic functions over the torus (there are none except constants).

As one translates along the horizontal or vertical direction, one has to update the gauge.

Right setting: Sections of a $\mathbb{C}$-bundle L over the torus.

Topological invariant of $\mathbb{C}$-bundles over oriented surfaces: first Chern class. For L it is exactly $\frac{\mathrm{Area}(\mathbb{C}\backslash\Lambda)}{\pi}$.

The projector P_k acts on $L^2(M, L^{\otimes k})$ and projects on those sections that are holomorphic.

Stereographic coordinates

stereographic coordinates. Consider the weighted holomorphic space

$$\{f \in L^2(\mathbb{C}, \mathbb{C}), s \mapsto f(s)(1 + |s|^2)^{\frac{k+2}{2}} \text{ is holomorphic}\}.$$

Spanned by the orthonormal basis

$$e_k : s \mapsto \sqrt{\frac{\pi(k+1)!}{m!(k-m)!}} s^m (1 + |s|^2)^{-\frac{k+2}{2}} \qquad 0 \leqslant m \leqslant k,$$

projector kernel

$$(s, s') \mapsto \frac{\pi(k+1)}{(1 + |s|^2)(1 + |s'|^2)} \left(\frac{1 + s\overline{s'}}{\sqrt{(1 + |s|^2)(1 + |s'|^2)}} \right)^k$$

Computing Toeplitz operators

Given $f : S^2 \to \mathbb{R}$, first apply the stereographic coordinates, then compute its matrix elements in the monomial basis above $\langle e_k, f e_{k'} \rangle$.
E.g. (x, y, z) coordinate functions of $S^2 \to \mathbb{R}^3$.

$$x = \frac{2\mathrm{re}(s)}{1+|s|^2} \qquad y = \frac{2\mathrm{im}(s)}{1+|s|^2} \qquad z = \frac{1-|s|^2}{1+|s|^2}.$$

Result: $T_\hbar(x), T_\hbar(y), T_\hbar(z)$ are the spin matrices S_x, S_y, S_z, at spin $S = \frac{k}{2}$.

Example at $k = 1$, Pauli matrices

$$T_1(x) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T_1(y) = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad T_1(z) = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Representation of $SU(2)$

In general, one finds the exact commutation relations

$$[T_k(x), T_k(y)] = \frac{i}{k} T_k(z) \text{ and cyclic permutations.}$$

That's because the Hamiltonian dynamics associated with (x, y, z) are the rotations of the sphere along the axes, which preserve the structure (in general one would only have approximative commutation relations).

Interesting for physicists: many spins (tensor product of these spaces and operators).

Understanding the quantum space - I

What's the relationship between the geometry of S^2 or $\mathbb{C}^n$ and the weights $(1+|s|^2)^{\frac{k}{2}+1}$ or $e^{k\frac{|z|^2}{2}}$?

Fact 1: the volume form in stereographic coordinates is $(1+|s|^2)^{-2}ds$ (explains the offset by 1).

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Fact 1: the volume form in stereographic coordinates is $(1+|s|^2)^{-2}ds$ (explains the offset by 1).

Remember gauge changes? The only thing which matters seems to be $\partial\overline{\partial}\log((1+|s|^2))$ or $\partial\overline{\partial}|z|^2$.

Fact 2: we obtain the natural Riemannian structure in both cases.

Understanding the quantum space - II

The sphere is covered by the domains of the two stereographic maps (from the North and South pole), and the map between the charts is $s \mapsto \frac{1}{s}$.

Understanding the quantum space - II

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Gauge change: an element of this space is of the form $(a_0 + a_1 s + ... + a_k s^k)(1+|s|^2)^{-\frac{k}{2}-1}$, and after this transformation we obtain

$$\begin{aligned}
&(a_0 + \frac{a_1}{s} + \ldots + \frac{a_k}{s^k})(1 + \frac{1}{|s|^2})^{-\frac{k}{2}-1} \\
&= (a_0 s^k + a_1 s^{k-1} + \ldots + a_k)(1+|s|^2)^{-\frac{k}{2}-1} \underbrace{\left(\frac{s}{|s|}\right)^{\frac{k}{2}}}_{\text{modulus } 1} \frac{\text{dvol}}{\text{dvol}}
\end{aligned}$$

Understanding the quantum space - III

Again, the clearest geometric setting is that of a $\mathbb{C}$-bundle over S^2 (natural bundle $\mathcal{O}(k)$ in complex geometry). The quantum states are holomorphic sections of this bundle.

Notations

- $L^2(S^2, L^{\otimes k})$ space of all square-integrable sections.
- $H^0(S^2, L^{\otimes k})$ subspace of holomorphic sections.
- $\Pi_k : L^2(S^2, L^{\otimes k}) \to H^0(S^2, L^{\otimes k})$ orthogonal Bergman projector.
- $T_k(f) = \Pi_k f \Pi_k$ Berezin–Toeplitz quantization of $f : \mathbb{S}^2 \to \mathbb{R}$.

Local weighted holomorphic spaces

Local picture: an open set $\Omega \subset \mathbb{C}^n$ and a weight $\phi : \Omega \to \mathbb{R}$ such that

$$\left[\frac{\partial^2 \phi}{\partial z_j \partial \overline{z_k}}\right]_{j,k} \gg 0. \qquad \text{(PSH)}$$

Among $L^2(\Omega, \det(\partial\overline{\partial}\phi))$, consider the functions f such that $z \mapsto f(z)e^{k\frac{\phi(z)}{2}}$ is holomorphic $\rightsquigarrow$ subspace $H_k^{\phi}(\Omega)$.

Gauge change: change ϕ while keeping $\partial\overline{\partial}\phi$ constant (keep track of coordinate changes!).

Kähler geometry

The complex structure and the weight ϕ gives

- A Riemannian metric g
- A symplectic form ω (eats two tangent vectors, antisymmetric, $d\omega = 0$; think of it as a magnetic field)

In these coordinates, both g and ω are deduced from $\frac{\partial^2 \phi}{\partial z_j \partial \overline{z_k}}$.
Compatibility: $g(u, Jv) = \omega(u, v)$.

Globally, the data (M, J, ω, g) is called a Kähler manifold.

Gluing the spaces

Can one always successfully glue the spaces $H_k^{\phi}(\Omega)$? No, there is a compatibility condition (remember what happened on tori?).

Integrability condition: for every closed surface $\Sigma \subset M$,

$$\int_{\Sigma} \omega \in 2\pi\mathbb{Z}.$$

If it is satisfied, one can glue the different $H_k^{\phi}(\Omega)$ into $H^0(M, L^{\otimes k})$ for some $\mathbb{C}$-bundle L. It is finite-dimensional when M is compact.

Berezin–Toeplitz operators

- Bergman projector $\Pi_k : L^2(M, L^{\otimes k}) \to H^0(M, L^{\otimes k})$
- Berezin–Toeplitz operators $T_k(f) = \Pi_k f \Pi_k$ for $f : M \to \mathbb{R}$.

What happens as $k \to +\infty$?
As soon as $f \in L^\infty$, $(T_k(f))_{k\in\mathbb{N}}$ is a sequence of self-adjoint matrices whose size tends to infinity.

Some references

- O. Debarre, Complex tori and abelian varieties.
- E. Lieb, The classical limit of quantum spin systems.
- D. Borthwick, Introduction to Kähler quantization.
  (hard to find; email me!)
- Y. Le Floch, A short introduction to Berezin–Toeplitz quantization.

Table of Contents

Spectral gap

The first order of business is the Bergman projector Π_k.

Operator $\overline{\partial}_k^*\overline{\partial}_k$ on $L^2(M, L^{\otimes k})$ such that Π_k projects on its kernel.

Spectral gap for this magnetic Laplacian: [Kohn 63, Hörmander 68] As soon as $\phi \in C^{1,1}$,

$$\exists c > 0, C,\ \sigma(\overline{\partial}_k^*\overline{\partial}_k) \subset \{0\} \cup [ck - C, +\infty).$$

Off-diagonal kernel decay

Recall that

$$|\Pi_k(z,w)|^2 = C_n k^d \exp(-k|z-w|^2) \text{ on } \mathbb{C}^d$$

$$|\Pi_k(s,s')|^2 = C(k+1)\left(\frac{|1+s\overline{s'}|}{\sqrt{(1+|s|^2)(1+|s'|^2)}}\right)^k \text{ on } S^2.$$

[Christ 91, Delin 98]: in general,

$$|\Pi_k(z,w)|^2 \leqslant Ck^{\dim(M)} \exp(-\sqrt{k}c\,\mathrm{dist}(z,w)).$$

This decay rate is sharp among C^∞ metrics [Christ 13].

Idea of the proof

Witten deformation: given $\rho \in C^{1,1}(M,\mathbb{R})$, deform $\overline{\partial}_k^*\overline{\partial}_k$ into

$$e^{\alpha\sqrt{k}\rho}\overline{\partial}_k^*\overline{\partial}_k e^{-\alpha\sqrt{k}\rho} = \overline{\partial}_k^*\overline{\partial}_k + \alpha\sqrt{k}P_1 + \alpha^2 k P_0.$$

In particular, for $|\lambda|$ small and fixed, by the resolvent formula,

$$\|(\lambda - k^{-1}e^{\alpha\sqrt{k}\rho}\overline{\partial}_k^*\overline{\partial}_k e^{-\alpha\sqrt{k}\rho})^{-1}\|_{L^2\to L^2} = O(1);$$

hence the contour integral, which has integral kernel

$$(z,w) \mapsto e^{\alpha\sqrt{k}(\rho(z)-\rho(w))}\Pi_k(z,w)$$

is uniformly bounded $L^2 \to L^2$.

Application: concentration of eigenfunctions

Proposition

If the Kähler potentials are $C^{1,1}$ *and* $f : M \to \mathbb{R}$ *is* $C^{1,1}$*, then solutions of* $T_\hbar(f)u_k = \lambda_k u_k$ *are* $O(e^{-c\sqrt{k}})$ *at positive distance from* $\{f = \lambda_k\}$*.*

First remark: very different from the pseudodifferential case!

Observed first in [Kordyukov 20] where it is stated in the C^∞ setting. More explicit, low-regularity version in [Deleporte 21], also with uniformity in the dimension.

Method of proof: Agmon-type estimates.

The Bergman kernel in smooth regularity

Proposition

For every small $\Omega \subset M$ *and Kähler potential* ϕ*, there exists smooth functions* $\psi, a_0, a_1, \ldots$ *on* $\Omega \times \Omega$ *such that*

$$\left\| \Pi_k(z,w) - k^{\dim} e^{k\psi(z,w)} \sum_{j=0}^{J} k^{-j} a_j(z,w) \right\|_{C^{\ell}(\Omega\times\Omega)} \leqslant O_{J,\ell}(k^{\dim + \ell - J}).$$

Global version: "$e^{k\psi(z,w)}$" is a section $\Psi^{\otimes k}$ of $(L \boxtimes L^*)^{\otimes k}$.

Decay:

$$\mathrm{re}(\psi)(z,w) \leqslant -c|z-w|^2.$$

Almost holomorphic extensions

Since the kernel Π_k projects onto the weighted holomorphic space, we seem to need

$$(\overline{\partial}_z, \partial_w)[\underbrace{\psi(z,w) + \phi(z) + \overline{\phi(w)}}\,] = 0 \qquad \psi(z,z) = 0.$$

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Not doable unless ϕ is real-analytic...

Actually we do it modulo $O(\mathrm{dist}(z,w)^\infty)$, and it's enough because of the off-diagonal decay. Same strategy for the amplitudes a_j.

Usual difficulties of Fourier Integral Operators with complex-valued phases; see [Melin-Sjöstrand 75].

Proofs of the Bergman kernel asymptotics

There are many different methods... But they all rely on the spectral gap.

- [Boutet de Monvel-Sjöstrand 74]: microlocal version (no exponential weight but boundary)
- [Zelditch 98, Shiffman-Zelditch 02, Charles 03, ...]: Translation of the above in our setting.
- [Tian 90, ...]: "Peak section" method: construct by hand the right candidate for the element $\psi_{x_0} \in H^0$ such that $\langle \psi_{x_0}, u \rangle = u(x_0)$ for all $u \in H^0$.
- [results by Bismut and Demailly, Ma-Marinescu 06, ...] Heat-type asymptotics and/or resolvent estimates.

Big application: Kodaira almost isometry

- Choosing a basis $(s_0, \ldots, s_{d_k})$ of $H^0(M, L^{\otimes k})$ gives a map $M \to \mathbb{C}^{d_k}/\mathbb{C}^*$.
- Since $\Pi_k(x,x) = \sum_j |s_j(x)|^2$ is non-zero for k large, they never vanish together, and we obtain $M \to \mathbb{CP}^{d_k - 1}$.
- From the C^2 convergence of Π_k on the diagonal, we know more: The pulled-back metric on $\mathbb{CP}^{d_k-1}$ is close to the original metric.

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Conclusion: all compact Kähler manifolds such that $\forall \Sigma, \int_\Sigma \omega \in 2\pi\mathbb{Z}$ can be realised as projective submanifolds, in an almost isometric way as the dimension of the ambiant space increases.

True in the $C^{1,1}$ case: [Coman-Ma-Marinescu 14].

Calculus of Toeplitz operators

Proposition

[Bordemann-Meinrenken-Schlichenmaier 94, Charles 03]There exists a sequence $(C_j)_{j\in\mathbb{N}}$ of degree 2j differential operators such that, for every $a, b \in C^\infty(M,\mathbb{R})$,

$$T_k(a)T_k(b) = T_k\left(\sum_{j=0}^{J} k^{-j} C_j(a,b)\right) + O_J(\hbar^{-J-1}).$$

[Charles 03]: generalisation of the Wick symbol into "covariant Toeplitz operators", with integral kernels of the form

$$(z,w) \mapsto k^{\dim} \Psi^{\otimes k}(x,y) \sum_{j=0}^{+\infty} k^{-j} a_j(z,w)$$

for general sequences $(a_j)_{j\in\mathbb{N}}$; they also form an algebra and one can pass from one to the other.

Geometric quantization

Bottom line of the previous results: whenever $a, b \in C^\infty$,

$$[T_k(a), T_k(b)] = ik^{-1}T_k(\{a,b\}) + O(k^{-2})$$

so we are really implementing the "classical mechanics to quantum mechanics" program.

Largely unknown: behaviour in low-regularity.

$$a, b \in C^1 \Rightarrow [T_k(a), T_k(b)] = ik^{-1}T_k(\{a,b\}) + o(k^{-1})?$$

Partial results: [Charles-Polterovich 15] $a, b \in C^4$.

Prequantum dynamics I - coherent states

Evolution of (micro)localised wavepackets? Useful for constructing quasimodes, etc.
First we must say what is a wavepacket.

Coherent state: to $(x, v) \in L$ (where x is the base point on M), apply Riesz representation theorem and obtain

$$\forall u \in H^0(M, L^{\otimes k}), \langle u(x), v \rangle_{L_x} =: \langle u, \psi_{x,v} \rangle_{H^0}.$$

The dependence on $v \in L$ is only via a multiplicative constant.

Prequantum dynamics II - parallel transport

Given $f : M \to \mathbb{R}$, can one solve approximately

$$ik^{-1}\partial_t u = T_k(f)u \qquad\qquad u(0) = \psi_{x,v}?$$

Answer: yes. More or less

$$e^{-itkT_k(f)}\psi_{x,v} \approx \psi_{x(t),v(t)}$$

where $t \mapsto x(t)$ follows the Hamiltonian dynamics of f and $v(t)$ is the parallel transport of $v(0)$.

Application: trace formula

Does a closed orbit lead to a quasimode? Yes, if and only if the parallel transport preserves the phase.

Trace formula: already in [Boutet-Guillemin 81].
Bohr-Sommerfeld rules for integrable systems: work by Y. Le Floch.

This is also true for pseudodifferential operators on cotangent spaces... But here the line bundle L is topologically trivial!

Structure of the propagator

[Charles-Le Floch 21]: Kernel of the propagator.
There exists a section $\Psi(t)$ of $L \boxtimes \overline{L}$ and a sequence of functions $(a_j(t))_{j\in\mathbb{N}}$ on $M \times M$ such that

$$e^{itkT_k(f)}(z,w) = k^{\dim}\Psi(t)(z,w)^{\otimes k}\sum_{j\geqslant 0} k^{-j}a_j(t,z,w) + O(k^{-\infty});$$

moreover $|\Psi(t)| \leqslant \exp(-c\,\mathrm{dist}(z,\phi_t(w)))$.

Also in this article: a geometric interpretation for Ψ (via parallel transport along L) and a_0 (via the linearised dynamics).

More advantages of Toeplitz quantization

- Bohr-Sommerfeld rules are hard-coded into the formalism.
- The quantum propagator is a Fourier Integral Operator, without phase variables, for all times.

Motivations for real-analytic regularity

Why are we interested in this?

- Many objects are more natural in analytic regularity (e.g. the holomorphic extension of the weights).
- $O(e^{-ck})$ estimates in spectral theory, tunneling.
- Non-self-adjoint evolution.
- Applications to Kähler geometry.

Analytic stationary phase - I

What's the link between real-analytic regularity and $O(e^{-ck})$ estimates in the calculus of oscillatory integrals?

The first step is to understand asymptotic properties of integrals of the form

$$\int e^{k\varphi(x)}a(x)dx$$

when φ and a are real-analytic.

Analytic stationary phase - II

Standard 1D example: $\varphi(x) = -\frac{x^2}{4}$. Usual stationary phase/saddle-point theorem tells us that

$$\int e^{-\frac{kx^2}{4}} a(x)dx = \exp(k^{-1}\Delta)a(0)$$
$$= \frac{1}{\sqrt{\pi k}} \sum_{j=0}^{N} \frac{\Delta^j a(0)}{k^j j!} + O(k^{-N-\frac{3}{2}}).$$

Size of the j-th term if a is real-analytic, with convergence radius ρ:

$$\left|\frac{\Delta^j a(0)}{k^j j!}\right| \leqslant \frac{1}{k^j j!}\|a\|_{C^{2j}} \leqslant \frac{(2j)!}{\rho^{2j}k^j j!} \leqslant \frac{j!}{(2k\rho^2)^j}.$$

Analytic stationary phase - III

Size of the j-th term if a is real-analytic, with convergence radius ρ:

$$\left|\frac{\Delta^j a(0)}{k^j j!}\right| \leqslant \frac{1}{k^j j!}\|a\|_{C^{2j}} \leqslant \frac{(2j)!}{\rho^{2j} k^j j!} \leqslant \frac{j!}{(2k\rho^2)^j}.$$

The rhs series does not converge!! The smallest term is at

$$j \approx 2k\rho^2,$$

and the size of this term, by Stirling formula, is:

$$\frac{(2k\rho^2)!}{(2k\rho^2)^{2k\rho^2}} \sim \sqrt{4\pi k\rho^2} e^{-2k\rho^2}.$$

Analytic stationary phase - IIII

We obtain analytic stationary phase by optimisation of the term of the expansion: for some $\alpha > 0, \beta > 0$, one has

$$\int e^{-\frac{kx^2}{4}} a(x)dx = \frac{1}{\sqrt{\pi k}} \sum_{j=0}^{\alpha k} \frac{\Delta^j a(0)}{k^j j!} + O(e^{-\beta k}).$$

This result can be generalised: the output of the stationary phase involves an analytic symbol, up to an exponentially small error.

Analytic symbols

Definition

A function $a \in \mathbb{R}^n_x \times [0,1]_h \to \mathbb{C}$ is an analytic symbol when a is smooth and its Borel transform

$$\mathcal{B}a : (x; h) \mapsto \sum_{j=0}^{+\infty} \frac{\partial_h^j a(x;0) h^j}{j!^2}$$

sums into a real-analytic function near $\{h = 0\}$.

In practice, $a = \sum h^j a_j(x)$ is a formal series satisfying

$$\|a_j\|_{C^n} \leqslant C \frac{j! n!}{\rho^j R^n} \forall j, n.$$

Banach spaces of analytic functions $\rightsquigarrow$ Banach spaces of symbols.

Practical example: pseudodifferential operators

Theorem ([Boutet-Krée 67, Sjöstrand 82])

There are Banach norms of analytic symbols $\|\cdot\|$ in which the Moyal product is continuous: if a, b, c satisfy

$$\mathrm{Op}_h(a)\mathrm{Op}_h(b) = \mathrm{Op}_h(c),$$

then

$$\|c\| \leqslant C\|a\|\|b\|.$$

Proof: by hand, counting derivatives in the formula

$$c(x, \xi; h) = \sum_j \frac{(ih)^j}{j!}(\nabla_x^j a \cdot \nabla_\xi^j b - \nabla_\xi^j a \cdot \nabla_x^j b).$$

Remark: almost the same formula for Berezin–Toeplitz operators on $\mathbb{C}^n$.

My first proof in analytic microlocal analysis

Problem: prove that, if a is bounded away from 0, then $\mathrm{Op}(a)$ has an inverse mod $O(e^{-ch^{-1}})$.

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Usual proof for $O(h^N)$: prove that

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then correct a^{-1} by induction. Are the coefficients in this induction bounded as analytic symbols? Very hard to prove.

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New idea: Since $1 + hr$ is close to 1, use the Banach algebra norm to invert it with the convergent series

$$\mathrm{Op}_h(1 - hr)^{-1} = 1 + \mathrm{Op}_h(hr) + (\mathrm{Op}_h(hr))^2 + (\mathrm{Op}_h(hr))^3 + \ldots$$

The Bergman kernel

Theorem ([Rouby-Sjöstrand-Vũ Ngọc 18, Deleporte 18])

Suppose the Kähler potentials ϕ are real-analytic. Let $\psi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the holomorphic extension (polarisation) of ϕ.
Then there exists an analytic symbol s such that

$$\Pi_k(x,y) = k^d e^{k(2\psi(x,y)-\phi(x)-\phi(y))} s(x,y;k^{-1}) + O(e^{-\beta k}).$$

Covariant Toeplitz operators

Strategy of proof: look at operators of the form

$$T_k(a)(x,y) = k^d \Psi^{\otimes k}(x,y) a(x,y;k^{-1}),$$

where Ψ is the candidate for the phase (in charts, holomorphic extension of the weight) and a is any analytic symbol.

Theorem

These "analytic covariant Toeplitz operators" form a unit algebra modulo exponentially small remainders.

The unit will exactly be the Bergman projector.

Remark: not clear that it is true for operators of the form $\Pi_k f \Pi_k$.

Unit algebra?

Path in [Rouby-Sjöstrand-Vũ Ngọc,Deleporte-Hitrik-Sjöstrand 21]: conjugate to pseudo-differential operators.

Path used in [Deleporte 18, Charles 20, ...]: we know from the smooth case that

$$T_k(a)T_k(b) = T_k\left(\sum_{j\geqslant 0} k^{-j}C_j(a,b)\right).$$

So we can imitate the proof of the Banach algebra property (with a similar count of derivatives).

Non-self-adjoint complex evolution and spectral theory

- [Deleporte-Zelditch 22]: FIO formula for purely imaginary propagation $e^{tkT_k(f)}$, link with change of Kähler structure.
- [Alphonse, Bernier, White...]: Precise study of quantum evolution in the quadratic case.
- [Rouby 19, Duraffour...]: Bohr-Sommerfeld rules for non-self-adjoint integrable systems.

Thanks for your attention!