Lecture Notes on PDEs, part II: Laplace's equation, the wave equation and more

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Contents

1 The wave equation (introduction)

The wave equation is the third of the essential linear PDEs in applied mathematics. In one dimension, it has the form

$$u_{tt} = c^2 u_{xx}$$

for $u(x,t)$. As the name suggests, the wave equation describes the propagation of waves, so it is of fundamental importance to many fields. It describes electromagnetic waves, some surface waves in water, vibrating strings, sound waves and much more.

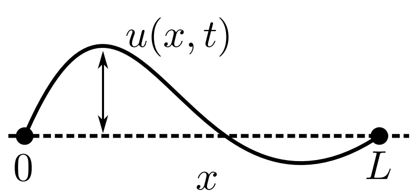
Consider, as an illustrative example, a string that is fixed at ends $x = 0$ and $x = L$. It has a constant tension T and linear mass density (i.e. mass per length) of λ_m. Assuming gravity is negligible, the vertical displacement $u(x,t)$, if it is not too large, can be described by the wave equation:

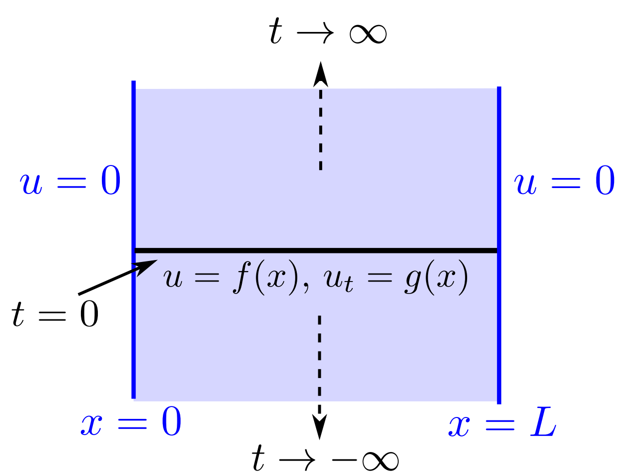
$$u_{tt} = c^2 u_{xx}, \qquad x \in (0, L)$$

with $c = \sqrt{T/\lambda_m}$. The derivation is standard (see e.g. the book). Suppose that the string has, at $t = 0$, an initial displacement $f(x)$ and initial speed of $g(x)$. We'll leave f and g arbitrary for now. The initial boundary value problem for $u(x,t)$ is

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad x \in (0, L), t\ in\mathbb{R} \\ u(0,t) &= 0, \quad u(L,t) = 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x). \end{aligned} \tag{1}$$

Note that we have two initial conditions because there are two time derivatives (unlike the heat equation). A sketch and the domain (in the (x,t) plane) is shown below. Note that we **do not** restrict $t > 0$ as in the heat equation.





How many boundary conditions are needed? Typically, for a PDE, to get a unique solution we need one condition (boundary or initial) for each derivative in each variable. For instance:

$$u_t = u_{xx} \implies \text{one } t\text{-deriv, two } x \text{ derivs} \implies \text{one IC, two BCs}$$

and

$$u_{tt} = u_{xx} \implies \text{two } t\text{-derivs, two } x \text{ derivs} \implies \text{two ICs, two BCs}$$

In the next section, we consider Laplace's equation $u_{xx} + u_{yy} = 0$:

$$u_{xx} + u_{yy} = 0 \implies \text{two } x \text{ and } y \text{ derivs} \implies \text{four BCs.}$$

1.1 Solution (separation of variables)

We can easily solve this equation using separation of variables. We look for a separated solution

$$u = h(t)\phi(x).$$

Substitute into the PDE and rearrange terms to get

$$\frac{1}{c^2}\frac{h''(t)}{h(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda.$$

This, along with the boundary conditions at the ends, yields the BVP for ϕ:

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = \phi(L) = 0,$$

which has solutions

$$\phi_n = \sin\frac{n\pi x}{L}, \quad \lambda_n = n^2\pi^2/L^2.$$

Now, for each λ_n, we solve for the solution $g_n(t)$ to the other equation:

$$\phi_n'' + c^2\lambda_n\phi_n = 0.$$

There are no initial conditions here because neither initial condition is separable; the initial conditions will be applied after constructing the full series (note that if, say, $g = 0$ or $f = 0$ then we would have a condition to apply to each ϕ_n).

The solution is then

$$\phi_n = a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}.$$

Thus the separated solutions are

$$u_n(x,t) = (a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L})\sin\frac{n\pi x}{L}.$$

The full solution to the PDE with the boundary conditions $u = 0$ at $x = 0, L$ is a superposition of these waves:

$$u(x,t) = \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L})\sin\frac{n\pi x}{L}. \tag{2}$$

To find the coefficients, the first initial condition $u(x,0)=f(x)$ gives

$$f(x)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}.$$

and the second initial condition $u_t(x,0)=g(x)$ gives

$$g(x)=\frac{c\pi}{L}\sum_{n=1}^{\infty} nb_n \sin\frac{n\pi x}{L}.$$

Both are Fourier sine series, so we easily solve for the coefficients and find

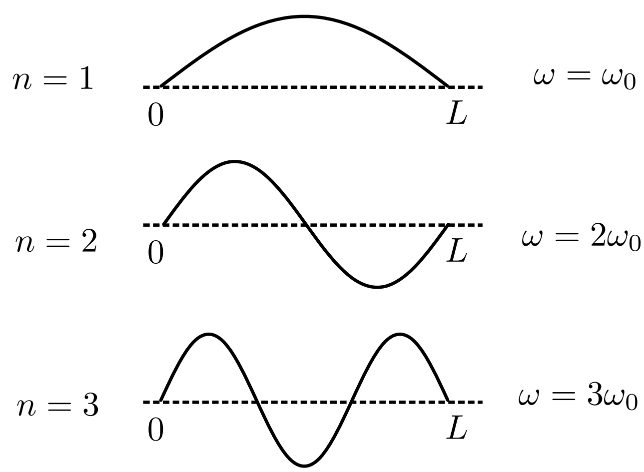
$$a_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad b_n=\frac{2}{nc\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}.\,dx. \tag{3}$$

1.2 Standing waves

Consider a string of length π (for simplicity) and fixed ends. The separated solutions have the form

$$u_n(t)=(a_n\cos nct+b_n\sin nct)\sin nx.$$

These solutions are called **standing waves**, because they have points at which the string 'stands' still. Plotting at some time t look like



The solutions oscillate over time, but observe that

$$u_n(t)=0 \text{ where } x=\frac{k\pi}{n}, \quad k=0,1,2\cdots,n.$$

The points $k\pi/n$ are called the **nodes**. Note also that the amplitude is largest at the points where $\sin nx=\pm 1$, namely at

$$nx=\pi/2+k\pi.$$

These are called the **anti-nodes**.

In fact, with some effort we can show that the solution to the wave equation is really a superposition of **two** waves travelling in opposite directions, reflecting off the boundaries and interfering with each other. At the nodes, they cancel out exactly (destructive interference) and at the anti-nodes, they add together exactly (constructive interference).

For the separated solutions, this is easy to show:

$$\cos nct \sin nx = \frac{1}{2}(\sin n(x+ct) + \sin n(x-ct)) = \frac{1}{2}h_n(x+ct) + \frac{1}{2}h_n(x-ct).$$

It is straightforward to check that both parts of the sum are solutions to the wave equation ('travelling waves') although they do not individually satisfy the boundary conditions.

We can be much more general about this (it is not just true for standing waves); see later.

Standing waves: Defining the **fundamental frequency** in radians per time,

$$\omega_0 = \pi c/L,$$

we can also write h_n in terms of this frequency and its multiples (the *harmonics*):

$$h_n = a_n \cos \omega_n t + b_n \sin \omega_n t, \qquad \omega_n = n\omega_0.$$

Physically, these correspond to *standing waves*, which oscillate at the frequency ω_n and have fixed nodes (with $u = 0$) at equally spaced points $x = Lm/n\pi$ for $m = 0, \cdots n$. Note that the 'fundamental frequency' is typically written in cycles per time, so

$$f_0 = \frac{c}{2L}, \quad f_n = nf_0.$$

Example: plucking a string

Suppose a string from a guitar or harp is plucked. The initial displacement will be something like a triangular shape, such as

$$f = \begin{cases} 2Ax/L & 0 \le x < L/2 \\ 2A(L-x)/L & L/2 < x < L \end{cases}$$

where A is the initial displacement at $x = L/2$. The initial speed is $g = 0$. In that case, it is straightforward to show that $b_n = 0$ and

$$a_n = \frac{8A}{\pi^2 n^2} \sin \frac{n\pi}{2}.$$

In terms of the frequencies ω_n, the response of the string is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(2\pi\omega_n t) \sin \frac{n\pi x}{L}.$$

Since $\sin n\pi/2 = 0$ for n even, the string, when plucked exactly at the center, vibrates with only the odd harmonics, and the amplitude of the harmonics decay quadratically with n. Note that because the initial displacement is not an eigenfunction, there are an infinite number of harmonics present. For a musical instrument, this is ideal, since the sound is much better when it is a mix of frequencies (a pure tone of one frequency is not pleasant).

Superposition of initial conditions: The principle of superposition gives some insight into the parts of the solution (2). We can split the IBVP (1) into one part with zero initial velocity ($u_t = 0$) and one with zero initial displacement ($u = 0$). To be precise, let v solve

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \quad t > 0, \quad x \in (0, L), \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0 \\ v(x,0) &= f(x), \quad v_t(x,0) = 0 \end{aligned} \tag{4}$$

and let w solve

$$\begin{aligned} w_{tt} &= c^2 w_{xx}, \quad t > 0, \quad x \in (0, L), \\ w(0,t) &= 0, \quad w(L,t) = 0, \quad t > 0 \\ w(x,0) &= 0, \quad w_t(x,0) = g(x). \end{aligned} \tag{5}$$

Then the solution $u(x,t)$ to (1) is

$$u = v + w.$$

Notice that the two pieces correspond to the sine/cosine terms in the full solution:

$$u(x,t) = \underbrace{\sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{L} \sin \frac{n\pi x}{L} \right)}_{v(x,t)} + \underbrace{\sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi ct}{L} \sin \frac{n\pi x}{L} \right)}_{w(x,t)}.$$

To see this, note that from (3), if $g(x) = 0$ then the b_n's are zero; if $f(x) = 0$ then the a_n's are zero. It follows that the two terms above are really v and w.

1.3 Solution (eigenfunction expansion)

The derivation using an eigenfunction expansion follows the same pattern as the heat equation. Again, let us consider the Dirichlet problem (1). Write the PDE as

$$u_{tt} = -L[u], \qquad Lu = u_{xx}. \tag{6}$$

The eigenfunction problem for $\phi(x)$ is

$$L\phi = \lambda\phi, \qquad \phi \text{ satisfies the BCs}$$

which is just the familiar eigenvalue problem from the heat equation,

$$-\phi'' = \lambda\phi, \qquad \phi(0) = \phi(L) = 0.$$

The eigenvalues/functions are

$$\lambda_n = n^2\pi^2/L^2, \qquad \phi_n = \sin\frac{n\pi x}{L}, \qquad n \geq 1.$$

The solution therefore has an eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} h_n(t)\phi_n(x).$$

Plug into the PDE:

$$\begin{aligned}\sum_{n=1}^{\infty} h_n''(t)\phi_n(x) &= \sum_{n=1}^{\infty} h_n(t)\phi_n''(x)\\ &= -\sum_{n=1}^{\infty} \lambda_n h_n(t)\phi_n(x).\end{aligned}$$

Rearrange to get

$$\sum_{n=1}^{\infty}(h_n''(t) + \lambda_n h_n(t))\phi_n(x) = 0$$

so

$$h_n''(t) + \lambda_n h_n(t) = 0 \text{ for } n \geq 1.$$

From here, the solution is the same as for separation of variables - solve for ϕ_n, then apply the initial conditions.

2 Laplace's equation

In two dimensions the heat equation[1] is

$$u_t = \alpha(u_{xx} + u_{yy}) = \alpha\Delta u$$

where $\Delta u = u_{xx} + u_{yy}$ is the **Laplacian** of u (the operator Δ is the 'Laplacian'). If the solution reaches an equilibrium, the resulting steady state will satisfy

$$u_{xx} + u_{yy} = 0. \tag{7}$$

This equation is **Laplace's equation** in two dimensions, one of the essential equations in applied mathematics (and the most important for time-independent problems). Note that in general, the Laplacian for a function $u(x_1, \cdots, x_n)$ in $\mathbb{R}^n \to \mathbb{R}$ is defined to be the sum of the second partial derivatives:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

Laplace's equation is then compactly written as

$$\Delta u = 0.$$

The inhomogeneous case, i.e.

$$\Delta u = f$$

the equation is called **Poisson's equation**. Innumerable physical systems are described by Laplace's equation or Poisson's equation, beyond steady states for the heat equation: inviscid fluid flow (e.g. flow past an airfoil), stress in a solid, electric fields, wavefunctions (time independence) in quantum mechanics, and more.

The two differences with the wave equation

$$u_{tt} = c^2 u_{xx}$$

are:

- We specify **boundary conditions** in both directions, not initial conditions in t.
- There is an opposite sign; we have $u_{xx} = -u_{yy}$ rather than $u_{tt} = c^2 u_{xx}$.

The first point changes the way the problem is solved slightly; the second point changes the answer. Note that there is also no coefficient, but this is not really important (we can just as easily solve $u_{xx} + k^2 u_{yy} = 0$).

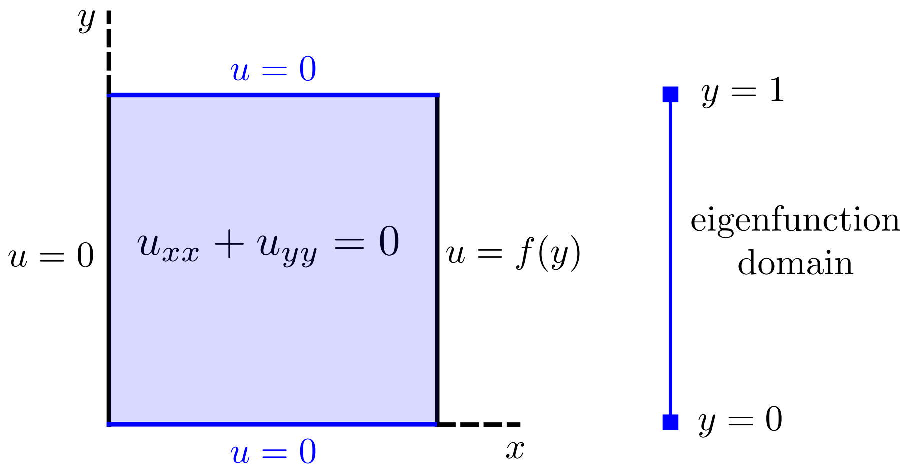
[1] The derivation follows the same argument as what we did in one dimension.

2.1 Solution in a rectangle

We can solve Laplace's equation in a bounded domain by the same techniques used for the heat and wave equation.

Consider the following boundary value problem in a square of side length 1:

$$\begin{aligned} 0 = u_{xx} + u_{yy}, \quad x \in (0,1), \quad y \in (0,1) \\ u(x,0) = 0, \quad u(x,1) = 0, \quad x \in (0,1) \\ u(0,y) = 0, \quad u(1,y) = f(y), \quad y \in (0,1). \end{aligned}$$



The boundary conditions are all homogeneous (shown in **blue** above) except on the right edge ($y = 0$)). Motivated by this, we will try to get eigenfunctions $\phi(y)$, since the eigenvalue problem requires us to impose **homogeneous boundary conditions**.

Look for a separated solution

$$u = g(x)\phi(y).$$

Substitute into the PDE to get

$$0 = g''(x)\phi(y) + g(x)h''(y)$$

and then separate:

$$-\frac{h''(y)}{\phi(y)} = \frac{g''(x)}{g(x)} = \lambda.$$

This leads to the pair of ODEs

$$\phi''(y) + \lambda\phi(y) = 0, \qquad g''(x) = \lambda g(x).$$

Applying the boundary conditions on the sides $x = 0$ and $x = 1$, we get the BVP

$$\phi''(y) + \lambda\phi(y) = 0, \quad \phi(0) = \phi(1) = 0.$$

We know the solutions to the above; they are

$$\phi_n(y) = \sin n\pi y, \quad \lambda_n = n^2\pi^2, \qquad n \geq 1.$$

Now we solve for g for each λ_n. Note that there is only one boundary condition (at $x = 1$); we leave the $f(x)$ condition for later (it will require using the full series). We solve

$$g'' - n^2\pi^2 g = 0, \qquad g(0) = 0$$

to get

$$g_n(x) = a_n \sinh n\pi x.$$

The solution

$$u_n = g_n(x)\phi_n(y)$$

satisfies the PDE and all the boundary conditions except $u(x, 0) = f(x)$. To satisfy this, we need to write u as a sum all of the separated solutions $g_n\phi_n$:

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh n\pi x \; \sin n\pi y.$$

Now apply $u(1, y) = f(y)$ (the boundary condition at $x = 1$) to get

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh n\pi \; \sin n\pi y.$$

Note that the functions $\phi_n = \sin \pi y$ are orthogonal in $L^2[0, 1]$ (we have shown this several times at this point!). As always, take inner products of both sides with $h_m = \sin m\pi y$ to get the coefficients:

$$\langle f, h_m\rangle = (a_m \sinh m\pi)\,\langle h_m, h_m\rangle$$

so

$$a_n = \frac{1}{\sinh n\pi}\frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}, \qquad n \geq 1.$$

Remark (convergence): In terms of the eigenfunction basis (set $\phi_n = \sin n\pi y$; the basis is $\{\phi_n(y)\}$), we have

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh n\pi x \phi_n(y).$$

Unlike the heat equation, the coefficients have an $\sinh \pi x$, which contains a **positive** exponential ($\sinh n\pi x \sim e^{n\pi|x|}/2$ as $n \to \infty$). However, the a_n's have a $\sinh n\pi$ in the denominator, which makes sure the coefficients are small enough that the series convergences. it can be shown that

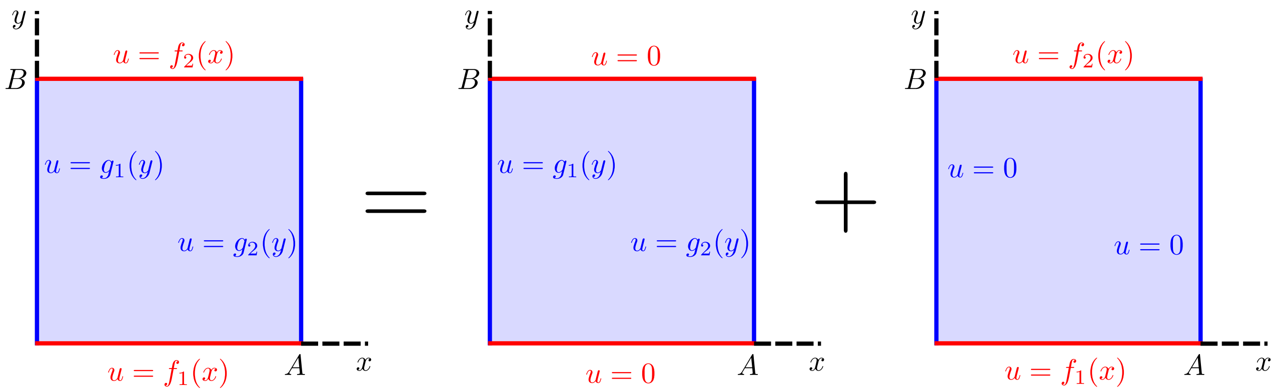
$$a_n = \frac{\sinh n\pi x}{\sinh n\pi} \text{ decays exponentially as } n \to \infty$$

for x in the domain. This is true since as $n \to \infty$, the numerator grows like $e^{n\pi|x|}/2$ while the denominator grows like $e^{n\pi}/2$, so if $|x| < 1$ the rate is faster for the denominator.

2.2 Rectangle, with more boundary conditions

Let's return to the rectangle example and consider how to solve the problem when there are inhomogeneous boundary conditions applied at all the sides for Laplace's equation in a rectangle of width A and height B:

$$\begin{aligned} 0 &= u_{xx} + u_{yy}, \quad x \in (0,a), \quad y \in (0,b) \\ u(x,0) = f_1(x), \quad u(x,1) &= f_2(x), \quad x \in (0,A) \\ u(0,y) = g_1(y), \quad u(1,y) &= g_2(y), \quad y \in (0,B). \end{aligned} \tag{8}$$



Both pairs of opposite sides (in blue and red above) could have non-homogeneous BCs. Our method only works if one of those pairs is homogeneous.

To solve (8), we use superposition and break the problem up into parts. Each part will take care of one (or two) of the boundaries and leave all the others zero. When added together, the sum of the parts will satisfy all the boundary conditions.

We find v, w solving

$$\begin{aligned} 0 &= v_{xx} + v_{yy}, \quad x \in (0,A), \quad y \in (0,B) \\ v(x,0) = 0, \quad v(x,B) &= 0, \quad x \in (0,A) \\ v(0,y) = g_1(y), \quad v(A,y) &= g_2(y), \quad y \in (0,B). \end{aligned} \tag{9}$$

$$\begin{aligned} 0 &= w_{xx} + w_{yy}, \quad x \in (0,A), \quad y \in (0,B) \\ w(x,0) = f_1(x), \quad w(x,B) &= f_2(x), \quad x \in (0,A) \\ w(0,y) = 0, \quad w(A,y) &= 0, \quad y \in (0,B). \end{aligned} \tag{10}$$

The sum $u = v + w$ is then the solution to (8). The solutions u along with v, w for a specific choice of initial condition are shown in Figure 1.

Solving for v: To solve (9), look for a separated solution

$$v = h(x)\phi(y).$$

This leads to the boundary value problem

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = \phi(b) = 0.$$

The solutions are

$$\phi_n(y) = \sin\frac{n\pi y}{B}, \quad \lambda_n = n^2\pi^2/B^2.$$

There are no boundary conditions we can apply for h (both boundaries have inhomogeneous terms), which satisfies

$$h_n'' - \lambda_n h_n = 0,$$

so we take the general solution. Set $\mu_n = n\pi/a$. The right choice of basis for solutions to the ODE has one basis function vanish at $x = 0$ and the other at $x = A$:

$$h_n(x) = a_n \sinh(\mu_n(A - x)) + b_n \sinh \mu_n x.$$

See subsection 7.2 for details. Adding up all the separated solutions, the solution for v is

$$v(x, y) = \sum_{n=1}^{\infty} \Big[a_n \sinh(\mu_n(A - x)) + b_n \sinh(\mu_n x)\Big]\phi_n(y).$$

Now the choice of basis becomes useful because it makes only one set of coeffiients appear at each boundary. At $x = 0$:

$$g_1(y) = v(0, y) = \sum_{n=1}^{\infty} a_n \sinh(\mu_n)\phi_n(y)$$

and so

$$a_n \sinh \mu_n = \frac{\langle g_1, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{2}{B}\int_0^B g_1(y) \sin\frac{n\pi y}{B}\, dy.$$

where $\langle f, g\rangle$ is the inner product on $L^2[0, B]$. At the $x = A$ boundary:

$$g_2(y) = v(A, y) = \sum_{n=1}^{\infty} b_n \sinh(\mu_n A)\phi_n(y)$$

and so

$$b_n \sinh(\mu_n A) = \frac{\langle g_2, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{2}{B}\int_0^B g_2(y) \sin\frac{n\pi y}{B}\, dy.$$

Finding w that solves (10) is the same process, and one gets a similar expression (left as an exercise). Finally, the solution to the original problem (8) is

$$u = v + w.$$

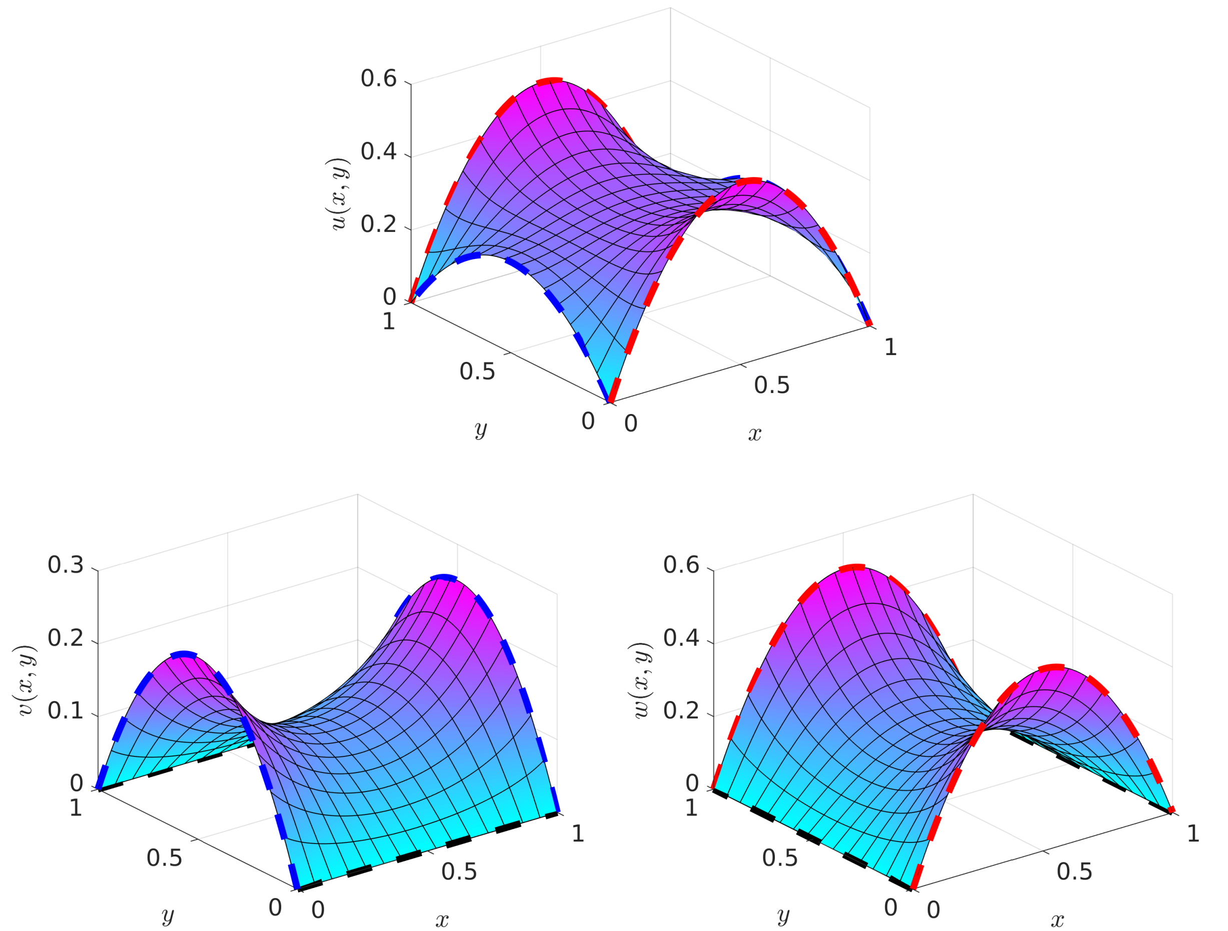


Figure 1: Solution u to (8) and the solutions v and w to (9) and (10) for $f_1 = f_2 = x(1-x)$ and $g_1 = g_2 = y(1-y)$.

2.2.1 Remarks on separation of variables for Laplace's equation

Just as with the heat equation, if there are more complicated inhomogeneous terms, e.g.

$$0 = u_{xx} + u_{yy} + f(x,y)$$

then the eigenfunction method is required unless you are lucky and there is a 'particular' solution you can subtract out to remove the inhomogeneous terms.

When applying the eigenfunction method, one must pick a direction for the eigenfunctions, either

$$u = \sum c_n(x)\phi_n(y) \quad \text{or} \quad u = \sum c_n(y)\phi_n(x).$$

The correct choice is one where the boundary conditions are homogeneous (if both work, then it does not matter which you choose). The details are somewhat involved but straightforward in concept.

Non-separable boundary conditions: There is a more fundamental concern: the geometry of the problem. Notice that in order to 'separate' the problem, we needed each boundary condition to involve only one coordinate. This was true, for instance, for the rectangle (one edge is $x = \text{const}$, the other is $y = \text{const}$) and for a circle, annulus or wedge (which have edges $r = \text{const}$ or $\theta = \text{const}$).

Dealing with boundary conditions that are not separable - for instance, flow of water past a fish (at least a non-spherical fish) - is challenging and requires more sophisticated techniques.

3 More solution techniques

The following techniques are all ways of reducing more complicated problems to simpler ones. The 'simple' problems are

a) a **homogeneous** PDE and **homogeneous** BCs

b) an **inhomogeneous** PDE (i.e. with a source term) and **homogeneous** BCs

Both (a) and (b) can be solved using eigenfunction expansions; (a) is even simpler and can be solved using separation of variables.[2] Note that 'homogeneous BCs' here means that there are enough homogeneous BCs to get eigenfunctions; the other boundaries may be allowed to stay inhomogeneous.

3.1 Steady states

We convert an **inhomogeneous** heat equation to a homogeneous problem when the inhomogeneous terms are all **time-independent.** In doing so, we obtain an easy method for determining the limit of the solution as $t \to \infty$.

Consider the IBVP

$$\begin{gathered} u_t = u_{xx} + h(x), \quad x \in [0, \ell], \ \ t > 0 \\ u(0,t) = A, \quad u(\ell, t) = B, \quad t > 0 \\ u(x, 0) = f(x) \end{gathered} \tag{11}$$

which represents heat flow with a time-independent source and/or ends fixed at some temperature. The expectation is that over time, the heat will approach a steady state (equilibrium):

$$\overline{u}(x) = \lim_{t \to \infty} u(x,t).$$

[2] There is a method for reducing (b) to (a) that we will not cover here since we can solve (a); this method is called **Duhamel's principle**.

Formally, we can obtain this equilibrium shape as follows: if $\overline{u}(x)$ is a steady-state, then it solves the PDE but does not depend on time. Thus it must satisfy

$$0 = \overline{u}_{xx} + h(x), \qquad \overline{u}(0) = A, \quad \overline{u}(\ell) = B.$$

This we can then solve. The important point is that the difference betwee the PDE solution and steady state,

$$v = u - \overline{u}$$

solves the homogeneous IBVP

$$\begin{aligned} v_t &= v_{xx}, \quad x \in [0,\ell], \;\; t > 0 \\ v(0,t) &= 0, \quad v(\ell,t) = 0, \quad t > 0 \\ v(x,0) &= f(x) - \overline{u}(x) \end{aligned} \tag{12}$$

So to solve (11) we can find the steady state (formally), subtract it out and then solve (12) for the 'homogeneous' part.

Note that the inhomogeneous term could appear in the source or in the boundaries.

Example: Consider the inhomogeneous problem

$$\begin{aligned} u_t &= u_{xx} + 1, \quad x \in (0,1), \;\; t > 0 \\ u(0,t) &= 0, \quad u(1,t) = 0, \quad t > 0 \\ u(x,0) &= f(x). \end{aligned}$$

There is a constant input source of heat, the heat is kept fixed at both ends and there some initial distribution $f(x)$ of heat. We want to show that the heat distribution converges to some equilibrium shape as $t \to \infty$. This is done in two steps:

1. Compute the steady state, assuming it exists: From knowledge of the heat equation, we expect a steady state to exist (diffusion wants to spread things out). Let us *assume that it exists*, i.e. there is a function $\overline{u}(x)$ such that

$$\overline{u}(x) = \lim_{t\to\infty} u(x,t). \tag{13}$$

Plug $\overline{u}(x)$ into the PDE and BCs to get

$$\overline{u}_{xx} = -1, \quad \overline{u}(0) = \overline{u}(1) = 0$$

which is easily solved (directly) to obtain

$$\overline{u}(x) = \frac{1}{2}x(1-x). \tag{14}$$

Key point: At this point, we have shown that *if a steady state exists*, it must be (14). It does not yet follow that the limit (13) exists!

2. Show directly that it is really a steady state: Now let

$$v(x,t) = u(x,t) - \overline{u}(x).$$

The function v is the difference between the solution and the steady state, so we want to show that $v \to 0$ as $t \to \infty$. Note that $\overline{u}$ is the solution to the IBVP

$$\begin{aligned} u_t = u_{xx} + 1, \quad & x \in (0,1), \;\; t > 0 \\ u(0,t) = 0, \quad u(1,t) = 0, \quad & t > 0 \\ u(x,0) = \overline{u}(x) & \end{aligned}$$

By linearity/superposition v satisfies

$$\begin{aligned} v_t = v_{xx}, \quad & x \in (0,1), \;\; t > 0 \\ v(0,t) = 0, \quad v(1,t) = 0, \quad & t > 0 \\ v(x,0) = f(x) - \overline{u}(x) & \end{aligned}$$

which is the difference of the IBVP satisfied by u and the one satisfied by $\overline{u}$.

But this equation is just the heat equation (homogeneous) with Dirichlet boundary conditions; the solution is

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin \lambda_n x \tag{15}$$

where $\lambda_n = n^2\pi^2$. and

$$b_n = 2\int_0^1 v(x,0) \sin n\pi x \, dx = 2\int_0^1 (f(x) - \overline{u}(x)) \sin \pi x \, dx.$$

The solution to the inhomogeneous problem (14) is then

$$u(x,t) = \overline{u} + v = \frac{1}{2}x(1-x) + \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin \lambda_n x.$$

To verify that $\overline{u}$ is actually the steady state, note that all the eigenvalues are positive, so

$$\lim_{t\to\infty} u(x,t) = \overline{u}(x) + \lim_{t\to\infty} v(x,t) = \overline{u}(x).$$

When does the method fail? This trick works when there is a steady state but *only when the source term and boundary conditions do not depend on time*. For instance,

$$u_t = tu_{xx} + \sin x$$

cannot be solved using this method. Assuming $u_t = 0$ is not enough since we also need to take $t \to \infty$ and we cannot find a $u = \overline{u}(x)$ that solves

$$tu_{xx} + \sin x = 0.$$

We can, however, solve the full problem using the eigenfunction method and find the steady state directly.

3.2 Moving inhomogeneous BCs into a source

Suggesiton: We will need to use the eigenfunction expansion method here; a review example is included in the appendix (subsection 7.1). We can use a trick similar to the steady state to remove inhomgeneous **boundary** conditions. It is too much to try to find a simple function that satisfies the PDE and the BCs simultaneously. However, we can instead compromise and find a function w that satisfies the boundary conditions but not the PDE. Then

$$v = u - w$$

will satisfy the **homogeneous** boundary conditions but will have both a different IC **and an extra source term**. For example, consider the heat equation with time-dependent boundary conditions:

$$\begin{aligned} u_t &= u_{xx} + f(x,t), \quad x \in [0,1], \quad t > 0, \\ u(0,t) = g(t), \quad u(1,t) &= h(t), \quad t > 0, \\ u(x,0) &= u_0(x). \end{aligned}$$

There are many choices for a function w that satisfies the boundary conditions. For Dirichlet BCs, the easiest is just to construct a line that goes from $(0, g(t))$ to $(1, h(t))$:

$$w(x,t) = (1-x)g(t) + xh(t).$$

Now define $v(x,t) = u(x,t) - w(x,t)$. Then

$$v(0,t) = u(0,t) - w(0,t) = g(t) - g(t) = 0,$$

and similarly for $v(1,t)$, so

$$v(0,t) = v(1,t) = 0.$$

Now we find the PDE for v by linearity/superposition. First, we need to find the 'source' term for w. To do so, plug w into the PDE $w_t = w_{xx}$; what is left-over is the source term. Explicitly, we have

$$w_t = w_{xx} + g(x,t)$$

where

$$g = w_t - w_{xx} = (1-x)g'(t) + xh'(t).$$

It follows that

$$v_t = u_t - w_t = u_{xx} + f - (w_{xx} + g) = v_{xx} + \tilde{f}$$

where the new sourcce term for v is

$$\tilde{f}(x,t) = f - g = f - (1-x)g'(t) - xh'(t).$$

Finally, for the initial condition,

$$v(x,0) = u(x,0) - w(x,0) = u_0(x) - (1-x)g(0) - xh(0).$$

The problem to solve for v is then

$$\begin{aligned} v_t &= v_{xx} + \tilde{f}(x,t), \quad x \in [0,1], \quad t > 0, \\ v(0,t) = 0, \quad v(1,t) &= 0, \quad t > 0, \\ v(x,0) &= u_0(x) - (1-x)g(0) - xh(0). \end{aligned}$$

We obtain v using the eigenfunction expansion method and then add w back in to get

$$u = v + w.$$

Illustrative example

We solve

$$\begin{aligned} u_t &= u_{xx}, \quad x \in (0,\pi), \quad t > 0, \\ u(0,t) &= 0, \quad u(\pi,t) = At, \quad t > 0, \\ u(x,0) &= 0. \end{aligned}$$

Set

$$w = \frac{Axt}{\pi} \tag{16}$$

and

$$v = u - w.$$

Then $w_t = w_{xx} + Ax/\pi$, so the PDE for v is

$$\begin{aligned} v_t &= u_t - w_t \\ &= u_{xx} - (w_{xx} + Ax/\pi) \\ &= v_{xx} - \frac{Ax}{\pi}. \end{aligned}$$

Since $w(x,0) = 0$, the initial condition for v is the same for u (i.e. $v(x,0) = 0$). Thus the IBVP for v we need to solve is

$$\begin{aligned} v_t &= v_{xx} - \frac{Ax}{\pi}, \quad x \in (0,\pi),\ t > 0, \\ v(0,t) &= 0, \quad v(\pi,t) = 0, \quad t > 0, \\ v(x,0) &= 0. \end{aligned}$$

For the source term, expand x in terms of the eigenfunctions:

$$x = \sum_{n=1}^{\infty} b_n \phi_n, \qquad b_n = -2\cos(n\pi)/n.$$

Now solve for $v = \sum c_n(t)\phi_n(x)$ using an eigenfunction expansion to get

$$c_n'(t) + \lambda_n c_n(t) = -\frac{Ab_n}{\pi}, \qquad c_n(0) = 0.$$

The solution for this ODE is

$$c_n(t) = \frac{Ab_n}{\lambda_n \pi}(e^{-\lambda_n t} - 1), \tag{17}$$

so the solution to the IBVP is

$$u(x,t) = \frac{At}{\pi}x + \sum_{n=1}^{\infty} c_n(t)\phi_n(x)$$

with c_n's given by (17) and b_n's by (16). Note that the Atx/π term is **not** a 'particular solution' to the PDE, since it **only satisfies the boundary conditions** and not the PDE.

In the next section, we will use another method that can be used to solve the original problem directly, which gives us u as an actual sum of a homogeneous and particular solution.

4 Inhomogeneous boundary conditions

When boundary conditions are inhomogeneous, the method above will not work. The issue is that the eigenfunctions must satisfy the *homogeneous* boundary conditions, so it seems

$$u(x,t) = \sum_n c_n(t)\phi_n(x) \tag{18}$$

should also satisfy the homogeneous BCs. Thus we cannot just substitute u into the PDE/BCs and solve for c_n. For instance, if

$$u(0,t) = 0, \qquad u(\pi,t) = At$$

and $\phi_n = \sin nx$ then

$$u(\pi,t) = \sum_n c_n(t)\phi_n(\pi) = 0$$

which gets us nowhere. However, the series (18) is still a good starting point.

4.1 A useful lemma

Recall that the inner product on $L^2[0,\ell]$ is $\langle f,g\rangle = \int_0^\ell f(x)g(x)\,dx$.

Lagrange's identity [simple version]: Consider the operator

$$Lf = -f''.$$

If f and g are functions in $L^2[0,\ell]$ then

$$\langle Lf, g\rangle = (fg' - f'g)\Big|_0^\ell + \langle f, Lg\rangle. \tag{19}$$

Moreover, if we have standard **homogeneous** boundary conditions then

$$\langle Lf, g\rangle = \langle f, Lg\rangle \text{ for all } f,g \text{ satisfying the BCs.}$$

In this case we say the the operator L with the boundary conditions[3] is **self-adjoint**.

The identity generalizes the fact that for a real symmetric matrix A and vectors $x,y\in\mathbb{R}^n$,

$$(Ax)\cdot y = (Ax)^T y = x^T A^t y = x^T(Ay) = x\cdot(Ay).$$

The proof of (19) is simple; just integrate by parts twice (left as an exercise).

[3]Dirichlet, Neumann or Robin, i.e. $\alpha_1 f(0) + \alpha_2 f'(0) = 0$ and $\beta_1 f(\ell) + \beta_2 f'(\ell) = 0$.

4.2 PDE with Inhomogeneous BCs: example

An example will serve to illustrate the idea. We solve the heat equation in $[0,\pi]$ with a time-dependent boundary condition:

$$\begin{aligned} u_t &= u_{xx}, \quad x \in (0,\pi), \quad t > 0, \\ u(0,t) = 0, \quad u(\pi,t) &= At, \quad t > 0, \\ u(x,0) &= f(x). \end{aligned} \tag{20}$$

The eigenfunctions/values are

$$\phi_n = \sin nx, \quad \lambda_n = n^2, \qquad n \geq 1.$$

The solution has an eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x).$$

The idea is to **use the formula for the coefficients**, the **PDE** and some manipulations (including the identity (19)) to get equations for $c_n(t)$.

4.2.1 The new part: equations for the coefficients

Since the ϕ_n's form an orthogonal basis, we have that

$$c_n(t) = \frac{\langle u, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{2}{\pi}\langle u, \phi_n \rangle. \tag{21}$$

Of course, u is unknown, so we need to get ODEs for $c_n(t)$. To do so, take the inner product of the PDE with ϕ_n:

$$\langle u_t, \phi_n \rangle = \langle u_{xx}, \phi_n \rangle.$$

Our goal is to write everything in the form $\langle \cdots, \phi_n \rangle$, since this will give back $c_n(t)$. The first term is $c_n'(t)$ since the $\frac{\partial}{\partial t}$ can be factored out:

$$\langle u_t, \phi_n \rangle = \frac{\partial}{\partial t}\langle u, \phi_n \rangle = c_n'(t).$$

Letting $Lu = -u_{xx}$, the second term is $-\langle Lu, \phi_n \rangle$. It follows that

$$\begin{aligned} \frac{\pi}{2}c_n'(t) &= \langle u_t, \phi_n \rangle \\ &= -\langle Lu, \phi_n \rangle \\ &= (u\phi_n' - u_x\phi_n)\Big|_0^\pi - \langle u, L\phi_n \rangle && \text{(using the identity (19))} \\ &= -At\cos n\pi - \langle u, L\phi_n \rangle && \text{(using the BCs)} \\ &= -At\cos n\pi - \langle u, \lambda_n\phi_n \rangle && \text{(since } \phi_n \text{ is an eigenfunction)} \end{aligned}$$

This gives

$$\frac{\pi}{2}c_n'(t) = -At\cos n\pi - \lambda_n\langle u, \phi_n \rangle = -At\cos n\pi - \frac{\pi}{2}c_n(t). \tag{22}$$

using the formula for the coefficients (21) again.

Boundary term details: We know that u has boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = At$$

and ϕ_n (by definition) has

$$\phi_n(0) = \phi_n(\pi) = 0, \qquad \phi_n'(0) = n,\ \phi_n'(\pi) = \cos n\pi$$

which lets us simplify the boundary term:

$$(u\phi_n' - u_x\phi_n)\Big|_0^\pi = u(\pi)\phi_n'(\pi) - u(0)\phi_n'(0) = At\cos n\pi.$$

Because of the inhomogeneous BC, the boundary term is **not zero**!

4.2.2 The rest of the solution

Now that we have equations for the c_n's, the rest of the solution work the same way as before. For brevity, set

$$\gamma_n = -\frac{2An\cos(n\pi)}{\pi}. \tag{23}$$

The ODE for c_n is then

$$c_n'(t) + \lambda_n c_n(t) = \gamma_n t$$

As before, write the initial condition in terms of the eigenfunction basis:

$$f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x), \qquad a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx. \tag{24}$$

Then $u(x,0) = f(x)$ gives the initial condition for c_n:

$$c_n(0) = a_n.$$

To solve, use an integrating factor:

$$(e^{\lambda_n t}c_n)' = \gamma_n e^{\lambda_n t}t$$

to obtain

$$c_n = a_n e^{-\lambda_n t} + \gamma_n e^{-\lambda_n t}\int_0^t e^{\lambda_n s}s\,ds.$$

Evaluating the integral we get

$$c_n(t) = a_n e^{-\lambda_n t} + \frac{\gamma_n}{\lambda_n^2}\left(\lambda_n t - 1 + e^{-\lambda_n t}\right).$$

While not required, we can plug in λ_n^2 and γ_n from (23) we get

$$c_n(t) = a_n e^{-n^2 t} - \frac{2A\cos(n\pi)}{\pi n^3}\left(n^2 t - 1 + e^{-n^2 t}\right). \tag{25}$$

The solution is then given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x)$$

with $c_n(t)$ given by (25) and the a_n's by (24). Note that the first term in the expression (25) for $c_n(t)$ gives the homogeneous solution; the second term is the response to the inhomogeneous boundary conditions:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t}\phi_n(x) - \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^3}\left(n^2 t - 1 + e^{-n^2 t}\right)\phi_n(x).$$

In fact, this solution is the same as the one we derived using the superposition trick in section 3.2.

4.3 Theoretical note: What does it mean to be a solution?

Consider the solution we just found for the IBVP (20) with inhomogeneous boundary conditions. It has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x). \tag{26}$$

However, the eigenfunctions satisfy the *homogeneous* boundary conditions:

$$\phi_n(0) = \phi_n(\pi) = 0.$$

It follows that $u(0,t) = 0$ and $u(\pi,t) = 0$. But we solved for u using the inhomogeneous boundary conditions! The apparent contradiction is due to the fact that the equality in the representation (26) is *not* pointwise; it is equal in the same sense we had for Fourier series.

Thus when we say that (26) is a solution to the IBVP, it does not mean that the series satisfies the boundary conditions **at the endpoints.** However, the right/left limits satisfy the left/right BCs For example,, for (26) solving (20) we have

$$\lim_{x\to 0^+} u(x,t) = 0, \qquad \lim_{x\to \pi^-} u(x,t) = At$$

even though $u(\pi,t) = 0$. This means that the series for u, close to the boundary, is correct; it just might be incorrect at the boundary exactly). Thus the fact that the series is wrong at the boundaries is not much of a worry (if handled carefully).

5 Robin boundary conditions

Returning to homogeneous problems, we now solve the heat equation with a boundary condition involving u and u_x (Robin). Two new issues arise:

- We will be unable to get explicit solutions for the eigenvalues λ_n, so a graphical method must be used to find and estimate them.
- There can be eigenvalues in all of the cases (negative and positive!).

Consider a metal bar of length 1 with temperature fixed at one end, with the other end being heated. Suppose the temperature $u(x,t)$ satisfies

$$u_t = u_{xx}, \qquad \text{for } x \in (0,1),\ t > 0 \tag{27}$$

with boundary conditions

$$u(0,t) = 0, \quad u_x(1,t) - \beta u(1,t) = 0 \tag{28}$$

and initial condition

$$u(x,0) = f(x).$$

Here $\beta > 0$ describes the strength of the heating; the flux at the boundary is

$$-u_x = -\beta u.$$

Heat is added at a rate βu if $u > 0$ (and lost if $u < 0$). The value of β determines the balance between inflow and outflow of heat, so the solution behavior as $t \to \infty$ will depend on β; we will be able to determine it by looking at the eigenvalues.

5.1 Eigenvalues in nasty cases, graphically

The eigenvalue problem is

$$-\phi'' = \lambda\phi, \quad \phi(0) = 0, \quad \phi'(1) - \beta\phi(1) = 0 \tag{29}$$

For this problems, the eigenvalues cannot be obtained exactly. Instead, we need to use a graphical argument to locate them.

Negative eigenvalues: First, suppose $\lambda < 0$. Then the general solution is

$$\phi = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

where $\mu = \sqrt{-\lambda}$. Since $\phi(0) = 0$, we need $c_1 = -c_2$, so

$$\phi = e^{\mu x} - e^{-\mu x}.$$

Now impose the Robin boundary condition:

$$0 = \mu(e^{\mu} + e^{-\mu}) - \beta(e^{\mu} - e^{-\mu}).$$

We cannot solve for μ here. Instead, rearrange to get

$$\beta = \mu \frac{e^{2\mu}+1}{e^{2\mu}-1} := g(\mu).$$

A solution exists for each $\mu > 0$ such that $g(\mu) = \beta$ (draw a horizontal line at β across the graph of $g(\mu)$ for $\mu > 0$. From a plot of $g(\mu)$ (check that $g(\mu) > 1$ for $\mu > 0$), it is clear that there are no positive solutions for $\beta \leq 1$ and exactly one when $\beta > 1$.

No zero eigenvalue: If $\lambda = 0$ the general solution is

$$\phi = c_1 x + c_2.$$

The boundary condition $\phi(0) = 0$ forces $c_2 = 0$ so $\phi = c_1 x$. Imposing the other boundary condition,

$$c_1 - \beta c_1 = 0$$

which has no solutions (except zero) when $\beta \neq 1$ but has a solution when $\beta = 1$; the eigenfunction is then $\phi = x$.

Positive eigenvalues: Finally, If $\lambda > 0$ then (with $\mu = \sqrt{\lambda}$)

$$\phi = c_1 \sin \mu x + c_2 \cos \mu x.$$

The boundary condition $\phi(0) = 0$ requires $c_2 = 0$. The other condition gives

$$\mu \cos \mu - \beta \sin \mu = 0.$$

Rearranging, we need μ that satisfies

$$\frac{\mu}{\beta} = \tan \mu. \tag{30}$$

There is no exact solution, but the solutions are the intersections of the line μ/β and $\tan \mu$ at positive values μ.

Note that $\tan \mu$ has asymptotes at $\pi/2 + n\pi$. Let $I_0 = (0, \pi/2)$ and let $I_n = (\pi/2 + (n-1)\pi, \pi/2 + n\pi$ be the intervals for each of the branches of $\tan \mu$ (for $\mu > 0$.

First note that in I_0, $\tan \mu$ starts at 0 and increases to ∞. Since $(\tan \mu)' = 1$ at $\mu = 0$ and $(\mu/\beta)' = 1/\beta > 1$, the line starts above $\tan \mu$ then intersects it once in I_0 (see plot).

For the other intervals, note that $\tan \mu$ is one-to-one and goes from $-\infty$ to ∞ in each I_n, so the line μ/β clearly intersects the branch at a unique point in I_n.

It follows that there is a sequence of positive solutions μ_n to (30) with

$$0 < \mu_0 < \pi/2, \quad \pi/2 + (n-1)\pi < \mu_n < \pi/2 + n\pi \text{ for } n \geq 1.$$

The corresponding eigenvalues and eigenfunctions are

$$\lambda_n = \mu_n^2, \quad \phi_n = \sin \mu_n x \text{ for } n = 1, 2, \cdots.$$

5.2 Solving the PDE

In all cases, there are eigenvalues $\lambda_n > 0$ for $n \geq 1$ with eigenfunctions

$$\phi_n = \sin\sqrt{\lambda_n}x, \quad n = 1, 2, \cdots.$$

However, the first eigenvalue is different for each of the three cases, which determines the behavior of the solution. The details of the eigenvalue problem are in subsection 5.1.

Summary: solution to the eigenvalue problem for (27)-(28) The eigenvalues depend on β. For all $\beta \neq 0$, there is a sequence

$$0 < \lambda_1 < \lambda_2 < \cdots \to \infty$$

of eigenvalues with eigenfunctions

$$\phi_n = \sin\sqrt{\lambda_n}x, \text{ quad } n = 1, 2, \cdots.$$

- If $\beta < 1$ there are no other eigenvalues.
- If $\beta = 1$ there is a zero eigenvalue:

$$\lambda_0 = 0, \quad \phi_0 = x.$$

- If $\beta > 1$ there is no zero eigenvalue but there is a single negative eigenvalue:

$$\lambda_0 < 0, \quad \psi_0 = \sinh(\sqrt{-\lambda_0}x).$$

Case 1 ($\beta < 1$; decay):

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

Following the same steps as **??** (note that the eigenfunctions are different, but the steps are exactly the same here), we get

$$a_n'(t) + \lambda_n a_n(t) = 0 \implies a_n(t) = b_n e^{-\lambda_n t}.$$

Thus the solution to the PDE (27) with the BCs (28) is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \sin\sqrt{\lambda_n}x. \tag{31}$$

Finally, to satisfy the initial condition, evaluate the series at $t = 0$ and set it equal to $f(x)$:

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n, \qquad \text{with } \phi_n = \sin\sqrt{\lambda_n}x. \tag{32}$$

The ϕ_n's are an orthogonal basis (a result of the theorem stated in our original discussion of eigenvalue problems), so we simply take the inner product with ϕ_m on both sides to get

$$\langle f, \phi_m \rangle = \sum_{n=1}^{\infty} a_n \langle \phi_n, \phi_m \rangle = a_m \langle \phi_m, \phi_m \rangle.$$

Thus the constants a_n are given by

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_0^1 f(x)\phi_n(x)\,dx}{\int_0^1 \phi_n^2\,dx}. \tag{33}$$

For completeness, we could evaluate the denominator by direction integration. Define

$$\kappa_n = \int_0^1 \phi_n^2\,dx = \int_0^1 \sin^2 \sqrt{\lambda_n} x\,dx = \int_0^1 \frac{1 - \cos\sqrt{\lambda_n} x}{2}\,dx = \frac{1}{2} - \frac{1}{2\sqrt{\lambda_n}} \sin\sqrt{\lambda_n}.$$

Then, explicitly, the coefficients a_n are given by

$$a_n = \frac{1}{\kappa_n} \int_0^1 f(x) \sin\sqrt{\lambda_n} x\,dx.$$

Case 2 ($\beta = 1$; steady state):

There is an extra eigenvalue $\lambda_0 = 0$ and eigenfunction $\phi_0 = x$. The solution has the form

$$u(x,t) = a_0(t)\phi_0(x) + \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

Solving for the coefficients $a_n(t)$ is exactly the same; we get

$$a_n(t) = b_n e^{-\lambda_n t} \text{ for } n = 1, 2, \cdots.$$

Since $L[\phi_0] = 0$, the $n = 0$ term yields

$$a_0'(t) = 0.$$

It follows that the solution is

$$u(x,t) = b_0 x + \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin\sqrt{\lambda_n} x.$$

To solve for b_0, use the initial condition. We require

$$f(x) = b_0 x + (\text{terms orthogonal to } x).$$

Taking the inner product with x, we get

$$b_0 = \frac{\int_0^1 x f(x)\,dx}{\int_0^1 x^2\,dx} = 3 \int_0^1 x f(x)\,dx.$$

Since $\lambda_n > 0$ except for $n = 0$, all the terms past $n = 0$ vanish (quickly) as $t \to \infty$. This suggests that

$$\lim_{t\to\infty} u(x,t) = a_0 x$$

i.e. the solution converges to the 'steady state' $a_0 x$ Note that if it happens to be true that

$$\int_0^1 x f(x)\,dx = 0$$

then $a_0 = 0$ and the solution converges to zero. That is, the zero mode is responsible for the convergence to a non-zero steady state; without it, solutions will just go to zero.

Case 3 (**$\beta > 1$; growth):**

The solution is

$$u(x,t) = b_0 e^{-\lambda_0 t} \sinh \mu_0 x + \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin \sqrt{\lambda_n} x.$$

By the theorem, $\phi_0 = \sinh \mu_0 x$ is orthogonal to the other basis functions, so

$$b_0 = \frac{\langle f, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{\int_0^1 f(x) \sinh \mu_0 x\,dx}{\int_0^1 \sinh^2 \mu_0 x\,dx}.$$

Since $\lambda_0 < 0$, the first term **grows exponentially**. We call this term an **unstable mode** of the system. Thus the solution $u(x,t)$ will have exponential growth unless it happens to be true that the initial condition has no component in the unstable mode, i.e.

$$\int_0^1 f(x) \sinh \mu_0 x\,dx = 0$$

which makes $b_0 = 0$.

Physical interpretation:

$\beta < 1$**:** More heat leaves through $x = 0$ than enters the system, so the heat decays to zero.

$\beta = 1$**:** There is a balance between heat entering/leaving the system, so there is a steady state as $t \to \infty$. Typically, this steady state will be non-zero, which occurs precisely when

$$b_0 = 3 \int_0^1 x f(x)\,dx \neq 0.$$

$\beta > 1$ Enough heat enters that it collects in the metal bar and the temperature grows exponentially due to the **unstable mode**. For most initial conditions, $u(x,t) \to \infty$ as $t \to \infty$ for $x \in (0,1)$. More precisely, the exponential growth rate is $-\lambda_0$. The exception is if f does not have a ϕ_0 component ($b_0 = 0$)

So long as f has a ϕ_0 component ($b_0 \neq 0$), no matter how small the coefficient is to start, it will grow exponentially with time and eventually dominate the solution (since all the other terms decay).

6 Equations in other geometries

6.1 Laplace's equation in a disk

We now solve Laplace's equation inside a disk of radius R. In polar coordinates, Laplace's equation $\Delta u = 0$ for $u(r, \theta)$ in the disk becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r \in (0, R), \quad \theta \in [0, 2\pi]$$

Let us assume that u is specified on the boundary of the circle:

$$u(R, \theta) = f(\theta).$$

We start by looking for a separated solution

$$u = g(r)h(\theta).$$

Substituting into the PDE we get

$$g''(r)h(\theta) + \frac{1}{r}g'(r)h(\theta) + \frac{1}{r^2}g(r)h''(\theta) = 0.$$

Divide by $h(\theta)$ and $g(r)$ and move all the θ terms to the right:

$$\frac{r^2 g''(r) + r g'(r)}{g(r)} = -\lambda \frac{h''(\theta)}{h(\theta)}.$$

We arrive at the equations

$$r^2 g'' + r g' - \lambda g = 0, \qquad h''(\theta) + \lambda h(\theta) = 0.$$

Since the inhomogeneous part $f(r)$ is at a boundary $r = R$, we want to look in the other direction first (θ) where the boundary conditions are homogeneous.

There are no explicit boundary conditions in θ; however, because θ is an angle there are implied periodic boundary conditions

$$u(r, 0) = u(r, 2\pi), \quad u_\theta(r, 0) = u_\theta(r, 2\pi).$$

Thus we need to solve

$$h''(\theta) + \lambda h(\theta) = 0, \qquad h(0) = h(2\pi), \quad h'(0) = h'(2\pi).$$

This is an eigenvalue problem with **periodic boundary conditions**. The solutions are $h_0(\theta) = a_0$ for $\lambda = 0$ and

$$h_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad \lambda_n = n^2, \quad n \geq 1$$

where a_n, b_n are arbitrary. To obtain g_n, we need to solve

$$r^2 g_n''(r) + r g_n'(r) - n^2 g_n(r) = 0.$$

Again, there are no explicit boundary conditions, but we want the solution in the disk to be finite, so implicitly we have the condition

$$g_n(r) \text{ is bounded }, \quad r \in [0, R].$$

The ODE is an Euler equation, so guess $g = r^p$ and obtain

$$p(p-1) + p - n^2 = 0 \implies p = \pm n.$$

The solution is then

$$g_n = c_n r^n + d_n r^{-n}.$$

By the boundedness condition, $d_n = 0$. The separated solutions for u are then

$$u_0 = \frac{a_0}{2}, \quad u_n = r^n(a_n \cos n\theta + b_n \sin n\theta), \quad n \geq 1$$

for arbitrary constants a_n and b_n. Note that the c_n's are not necessary (since a_n, b_n are already arbitrary) and that the constant for u_0 was chosen to be $a_0/2$ for convenience.

The full solution is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta). \tag{34}$$

We now find the constants so that u satisfies the condition at $r = R$:

$$f(\theta) = u(R,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n(a_n \cos n\theta + b_n \sin n\theta).$$

This is just the Fourier series (with coefficients $R^n a_n$ and $R^n b_n$) for $f(\theta)$, so

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \quad b_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta. \tag{35}$$

Now we are done. In summary, the series (34) with coefficients (35) is the solution to

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad r \in (0,R), \quad \theta \in [0, 2\pi]$$

with boundary condition

$$u(R, \theta) = f(\theta)$$

and implied boundary conditions

$$u \text{ bounded for } r \in [0, R], \quad u \text{ periodic in } \theta.$$

Integral formula: This particular problem is a remarkable case where the infinite series can be simplified. After some calculations, one ends up with *Poisson's integral formula*

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(\phi)}{R^2 + r^2 - 2rR\cos(\theta - \phi)}\, d\phi.$$

The formula, however, is not of any use to us here besides looking nice.

7 Appendix

7.1 PDEs with source terms; example

Consider the following IBVP for the heat equation with a time-dependent source:

$$\begin{aligned} u_t &= u_{xx} + e^{-t}, \quad x \in (0, \pi), \quad t > 0, \\ u(0, t) &= 0, \quad u(\pi, t) = 0, \quad t > 0, \\ u(x, 0) &= f(x). \end{aligned} \tag{36}$$

Refer to the previous notes for the outline of the method.

First, we find the appropriate eigenvalues/eigenfunctions for the **homogeneous problem** (as if the source term were zero). For (36), the eigenfunctions/values are

$$\phi_n = \sin nx, \quad \lambda_n = n^2, \qquad n \geq 1.$$

The eigenfunctions form a basis for $L^2[0, \pi]$. Now write all the functions in the PDE in terms of the eigenfunction basis. The solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) \phi_n(x)$$

for unknown $c_n(t)'$. The source term is

$$e^{-t} = \sum_{n=1}^{\infty} \gamma_n(t) \phi_n(x).$$

The γ_n's are easily found through the usual formula:

$$\gamma_n(t) = \frac{\langle e^{-t}, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{2}{\pi} \int_0^{\pi} e^{-t} \sin nx\, dx = e^{-t} \frac{2}{n\pi}(1 - \cos(n\pi)).$$

The inner product is an integral in x, so the dependence on t causes no trouble. For convenience let us write $\gamma_n(t) = a_n e^{-t}$ where

$$a_n = \frac{2}{n\pi}(1 - \cos(n\pi)). \tag{37}$$

Note that in this case we also could have just written

$$1 = \sum_{n=1}^{\infty} a_n \phi_n$$

and then multiplied by e^{-t} instead of directly computing the expansion of e^{-t}.

Finally, for the initial condition,

$$f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x), \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx. \tag{38}$$

Now plug in these series into the PDE to obtain

$$\sum_{n=1}^{\infty} c_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} c_n(t)\phi_n''(x) + \sum_{n=1}^{\infty} \gamma_n(t)\phi_n(x) \tag{39}$$

$$= \sum_{n=1}^{\infty} (-c_n(t)\lambda_n + \gamma_n(t))\phi_n(x) \tag{40}$$

$$\tag{41}$$

using that $\phi_n'' = -\lambda_n \phi_n$. Since $\{\phi_n\}$ is a basis, the coefficients of ϕ_n on either side must be equal for all n, so

$$c_n'(t) + \lambda_n c_n(t) = \gamma_n(t), \qquad n \geq 1.$$

From the initial condition:

$$u(x,0) = f(x) \implies \sum_{n=1}^{\infty} c_n(0)\phi_n(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

We then solve the IVP for the coefficients,

$$c_n'(t) + \lambda_n c_n(t) = \gamma_n(t), \qquad c_n(0) = b_n, \tag{42}$$

to obtain

$$c_n(t) = b_n e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t e^{\lambda_n s}\gamma_n(s)\, ds.$$

We solved for γ_n before, so the term on the right is

$$e^{-\lambda_n t}\int_0^t e^{\lambda_n s}\gamma_n(s)\, ds = a_n e^{-\lambda_n t}\int_0^t e^{\lambda_n s} e^{-s}\, ds = \begin{cases} \frac{a_n}{\lambda_n - 1}(e^{-t} - e^{-\lambda_n t}) & n > 1 \\ a_1 t e^{-t} & n = 1 \end{cases}.$$

Highlight (word of caution): Some casework may be required in finding the c_n's, depending on λ_n and the inhomogeneous term γ_n. Here, we could get the general solution to (42) for all n at once, but then had to do some casework to compute because $\gamma_1(t)$ is a homogeneous solution, but γ_n for $n > 1$ is not.

Finally, combining everything, the solution $u(x,t)$ is

$$u(x,t) = (b_1 e^{-t} + a_1 t e^{-t}) \sin x + \sum_{n=2}^{\infty} \left[b_n e^{-n^2 t} + \frac{a_n}{n^2-1}(e^{-t} - e^{-n^2 t}) \right] \sin nx \, dx.$$

with a_n and b_n from (37) and (38). Note that

$$u(x,t) = \left[\sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx \right] + \left[a_1 t e^{-t} \sin x + \sum_{n=2}^{\infty} \left(\frac{a_n}{n^2-1}(e^{-t} - e^{-n^2 t}) \right) \sin nx \, dx \right]$$

The first term in square brackets depends only on the initial condition; it is the homogeneous solution. The second term depends only on the source term (a 'particular' solution with a zero initial condition).

7.2 Good basis functions for Laplace's equation

It will be useful tto recall

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

In solving Laplace's equation we end up needing to deal with

$$g'' - \lambda g = 0$$

for positive eigenvalues $\lambda > 0$. The solutions, with $\mu = \sqrt{\lambda}$, are

$$g = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

Instead, we could choose any linear combination of the two as basis functions. Given some boundary conditions, we want to choose the basis functions so that one of them vanishes at each boundary condition. For instance, if the boundary conditions are

$$u(0,y) = f_1(y), \quad u(1,y) = f_2(y)$$

Then for each n we seek basis solutions g_1, g_2 such that

$$g_1(0) = 0, \quad g_2(0) = 0.$$

For g_1, we get $c_2 = -c_1$ so

$$g_1 = c_1(e^{\mu x} - e^{-\mu x}) = 2c_1 \sinh x \implies g_1 = \sinh x.$$

In a similar way, we find [see homework] that

$$g_2 = \sinh(\mu(x-1)).$$

As another example, if the boundary conditions at $x = 0$ and $x = 1$ are

$$u(0, y) = 0, \quad u_x(1, y) = f(y)$$

then we would choose

$$g_n(x) = a_n \sinh(\mu_n x) + b_n \cosh(\mu_n(x-1)).$$

For each boundary condition, one of the basis functions vanishes:

$$g_n(0) = b_n \cosh(\mu_n), \quad g_n'(1) = a_n \cosh(\mu_n).$$

7.3 Compatibility conditions

Consider the Neumann problem for Laplace's equation in a disk of radius R,

$$0 = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad u_r(R, \theta) = f(\theta). \tag{43}$$

Recall that the general solution to the PDE in the disk (without the condition at R) is

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta).$$

In order to be able to satisfy the boundary condition $u_r(r, \theta) = 0$, we must have $\int_0^{2\pi} f(\theta)\, d\theta = 0$ (why?). This is called the **compatibility condition**, which says that the Neumann problem only has a solution for certain constraints at the boundary.

For physical context: Laplace's equation is a steady-state for heat conduction. If the system is in equilibrium, nothing is entering or leaving, so the total flux through the boundary must be zero, which is exactly the compatibility condition.

The compatibility condition can be obtained without solving the PDE. Integrating both sides of the PDE (43) over the disk and using the boundary condition, we find that

$$0 = \int_0^{2\pi} f(\theta)\, d\theta.$$

Note that the integral over the disk is $\int_0^{2\pi} \int_0^R \cdots r\, dr d\theta$.

The compatibility condition for a general Neumann problem is that the integral of the flux through the boundary must be zero. Notice that if this condition is satisfied, **the solution is only unique up to a constant**. To get a unique solution, we would need to also specify some other constraint (that would depend on the problem), e.g. the amount of stuff in the domain:

$$\int u\, dA = M_0.$$

Contrast this with other steady state problems with Dirichlet ($u = A$) or radiation boundary conditions, where there is only one solution. For the heat equation with a steady state, we always have an initial condition that was not a Neumann condition, $u(x, 0) = f(x)$, so the solution (with boundary conditions) is unique and there is no compatibility condition needed.