Novi Sad J. Math.
Vol. 43, No. 2, 2013, 51-57

SOME CRITERIA FOR UNIVALENCE OF A CERTAIN INTEGRAL OPERATOR

Laura F. Stanciu[1], Daniel Breaz[2] and Hari M. Srivastava[3]

Abstract. The main objective of this paper is to obtain new conditions for the integral operator $F_{\alpha,\beta}(z)$ to be univalent in the open unit disk $\mathbb{U}$. This integral operator $F_{\alpha,\beta}(z)$ was considered in a recent work [4]. A number of known or new univalence conditions are shown to follow upon specializing the parameters involved in our main results.

AMS Mathematics Subject Classification (2010): 30C45, 30C75

Key words and phrases: Analytic functions; Open unit disk; Univalent functions; Starlike functions; Integral operators; Univalence conditions; General Schwarz Lemma.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\}$$

and satisfy the following usual normalization condition:

$$f(0) = f'(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers. We denote by $\mathcal{P}$ the class of the functions $p(z)$ which are analytic in $\mathbb{U}$ and satisfy the following conditions:

$$p(0) = 1 \qquad \text{and} \qquad \Re\{p(z)\} > 0, \quad z \in \mathbb{U}.$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$. Suppose also that $\mathcal{S}^*$ denotes the subclass of $\mathcal{S}$ consisting of all functions $f(z)$ in $\mathcal{S}$ which are starlike in $\mathbb{U}$.
The following univalence condition was derived by Ozaki and Nunokawa [2].

[1]Department of Mathematics, University of Piteşti, Strada Târgul din Vale 1, R-110040 Argeş, România, e-mail: laura_stanciu_30@yahoo.com

[2]Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, Strada Nicolae Iorga 11-13, R-510000 Alba Iulia, România, e-mail: dbreaz@uab.ro

[3]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada, e-mail: harimsri@math.uvic.ca

Theorem 1.1 (see [2]). *Let the function $f \in \mathcal{A}$ satisfy the following inequality:*

$$\left|\frac{z^2 f'(z)}{[f(z)]^2} - 1\right| \leqq |z|^2, \qquad z \in \mathbb{U}. \tag{1.1}$$

Then $f(z)$ is in the univalent function class $\mathcal{S}$ in $\mathbb{U}$.

The problem of finding sufficient conditions for univalence of various integral operators has been investigated in many recent works (see, for example, [6] and the references cited therein). In our present investigation we study the univalence conditions for the following integral operator:

$$F_{\alpha,\beta}(z) := \left(\beta \int_0^z t^{\beta-\alpha-1}[f(t)]^{\alpha} g(t)dt\right)^{\frac{1}{\beta}} \tag{1.2}$$

$$(\alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \{0\}, f \in \mathcal{A}, g \in \mathcal{P}).$$

In the proof of our main result (Theorem 2.1 below), we need each of the following univalence criteria. The first univalence criterion, which is asserted by Theorem 1.2 below, is a generalization of the Ozaki-Nunokawa criterion (1.1); it was obtained by Răducanu et al. [5]. The second univalence criterion, which is asserted by Theorem 1.3 below, is a generalization of Ahlfors's and Becker's univalence criterion; it was proven by Pescar [3].

Theorem 1.2 (see [5]). *Let $f \in \mathcal{A}$ and $m > 0$ be so constrained that*

$$\left|\left(\frac{z^2 f'(z)}{[f(z)]^2} - 1\right) - \frac{m-1}{2}|z|^{m+1}\right| \leqq \frac{m+1}{2}|z|^{m+1}, \qquad z \in \mathbb{U}. \tag{1.3}$$

Then the function $f(z)$ is analytic and univalent in $\mathbb{U}$.

Theorem 1.3 (see [3]). *Let the parameters $\beta \in \mathbb{C}$ and $c \in \mathbb{C}$ be so constrained that*

$$\Re(\beta) > 0 \qquad \textit{and} \qquad |c| \leqq 1, \qquad c \neq -1.$$

If $f \in \mathcal{A}$ satisfies the following inequality:

$$\left|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{zf''(z)}{\beta f'(z)}\right| \leqq 1, \qquad z \in \mathbb{U}, \tag{1.4}$$

then the function $F_\beta(z)$ given by

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)dt\right)^{\frac{1}{\beta}} = z + \cdots \tag{1.5}$$

is analytic and univalent in $\mathbb{U}$.

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma (see, for details, [1]).

Lemma 1.4 (General Schwarz Lemma (see [1])). *Let the function $f(z)$ be regular in the disk*

$$\mathbb{U}_R = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < R, \quad R > 0\}$$

with

$$|f(z)| < M, \qquad z \in \mathbb{C},\ M > 0$$

for a fixed number $M > 0$. If the function $f(z)$ has one zero with multiplicity order bigger than a positive integer m for $z = 0$, then

$$|f(z)| \leqq \frac{M}{R^m} |z|^m, \qquad z \in \mathbb{U}_R. \tag{1.6}$$

The equality in (1.6) *holds true only if*

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is a real constant.

2. The Main Univalence Criterion

Our main univalence criterion for the integral operator $F_{\alpha,\beta}(z)$ defined by (1.2) is asserted by Theorem 2.1 below.

Theorem 2.1. *Let the function $f \in \mathcal{A}$ satisfy the hypothesis* (1.3) *of Theorem* 1.2. *Suppose that M, N are real positive numbers, $m > 0$ and $g \in \mathcal{P}$. Also let*

$$\Re(\beta) \geqq \big[|\alpha| \left((m+1) M + 1\right) + N\big], \qquad \alpha, \beta \in \mathbb{C}.$$

If

$$|f(z)| < M, \qquad z \in \mathbb{U}, \qquad \left|\frac{zg'(z)}{g(z)}\right| \leqq N, \qquad z \in \mathbb{U} \tag{2.1}$$

and

$$|c| \leqq 1 - \frac{1}{\Re(\beta)} |\alpha| \left[(m+1) M + 1\right] - \frac{1}{\Re(\beta)} N, \qquad c \in \mathbb{C}, \tag{2.2}$$

then the function $F_{\alpha,\beta}(z)$ defined by (1.2) *is analytic and univalent in $\mathbb{U}$.*

Proof. We begin by observing that the integral operator $F_{\alpha,\beta}(z)$ in (1.2) can be rewritten as follows:

$$F_{\alpha,\beta}(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)^\alpha g(t) dt\right)^{\frac{1}{\beta}}.$$

Let us define the function $h(z)$ by

$$h(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha g(t) dt, \qquad f \in \mathcal{A},\ g \in \mathcal{P}.$$

The function f is indeed regular in $\mathbb{U}$ and satisfies the following normalization condition:

$$f(0)=f'(0)-1=0.$$

Now, calculating the derivatives of $h(z)$ of the first and second orders, we readily obtain

$$h'(z)=\left(\frac{f(z)}{z}\right)^{\alpha}g(z) \tag{2.3}$$

and

$$h''(z)=\alpha\left(\frac{f(z)}{z}\right)^{\alpha-1}\left(\frac{zf'(z)-f(z)}{z^2}\right)g(z)+\left(\frac{f(z)}{z}\right)^{\alpha}g'(z). \tag{2.4}$$

We easily find from (2.3) and (2.4) that

$$\frac{zh''(z)}{h'(z)}=\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+\frac{zg'(z)}{g(z)}, \tag{2.5}$$

which readily shows that

$$\begin{aligned}&\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|\\&\quad=\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{1}{\beta}\left(\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+\frac{zg'(z)}{g(z)}\right)\right|\\&\quad\leqq|c|+\frac{1}{|\beta|}\left(|\alpha|\left(\left|\frac{z^2f'(z)}{[f(z)]^2}\right|\cdot\left|\frac{f(z)}{z}\right|+1\right)+\left|\frac{zg'(z)}{g(z)}\right|\right).\end{aligned} \tag{2.6}$$

Furthermore, from the hypothesis (2.1) of Theorem 2.1, we have

$$|f(z)|<M,\quad z\in\mathbb{U}\qquad\text{and}\qquad\left|\frac{zg'(z)}{g(z)}\right|\leqq N,\quad z\in\mathbb{U}.$$

By applying the General Schwarz Lemma, we thus obtain

$$|f(z)|\leqq M\,|z|,\qquad z\in\mathbb{U}.$$

Next, by making use of (2.6), we have

$$\begin{aligned}
&\left| c\,|z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)} \right| \\
&\quad \leqq |c| + \frac{1}{|\beta|}\left(|\alpha|\left(\left|\frac{z^2 f'(z)}{[f(z)]^2}\right| M + 1\right) + N\right) \\
&\quad \leqq |c| + \frac{1}{|\beta|}\Bigg(|\alpha|\Bigg(\left|\left(\frac{z^2 f'(z)}{[f(z)]^2} - 1\right) - \frac{m-1}{2}\,|z|^{m+1}\right| M \\
&\qquad\qquad + \left(1 + \frac{m-1}{2}\,|z|^{m+1}\right) M + 1\Bigg) + N\Bigg) \\
&\quad \leqq |c| + \frac{1}{|\beta|}\left(|\alpha|\left(\frac{m+1}{2}\,|z|^{m+1} M + \left(1+\frac{m-1}{2}\,|z|^{m+1}\right) M + 1\right) + N\right) \\
&\quad \leqq |c| + \frac{1}{|\beta|}\left[|\alpha|\left[(m+1)\,M+1\right] + N\right] \\
&\quad \leqq |c| + \frac{1}{\Re(\beta)}\left[|\alpha|\left[(m+1)\,M+1\right] + N\right] \\
&\quad \leqq 1, \;\; z \in \mathbb{U},
\end{aligned}$$

where we have also used the hypothesis (2.2) of Theorem 2.1.
Finally, by applying Theorem 1.3, we conclude that the function $F_{\alpha,\beta}(z)$ defined by (1.2) is analytic and univalent in $\mathbb{U}$. This evidently completes the proof of Theorem 2.1. □

3. Applications of Theorem 2.1

First of all, upon setting $m = 1$ in Theorem 2.1, we immediately arrive at the following application of Theorem 2.1.

Corollary 3.1. *Let the function $f \in \mathcal{A}$ satisfy the condition* (1.3) *and suppose that M, N are real positive numbers, $m > 0$ and $g \in \mathcal{P}$. Also let*

(3.1) $$\Re(\beta) \geqq [|\alpha|\,(2M+1) + N], \qquad \alpha, \beta \in \mathbb{C}.$$

If

(3.2) $$|f(z)| < M, \quad z \in \mathbb{U}, \qquad \left|\frac{zg'(z)}{g(z)}\right| \leqq N, \quad z \in \mathbb{U}$$

and

(3.3) $$|c| \leqq 1 - \frac{1}{\Re(\beta)}\,|\alpha|\,(2M+1) - \frac{1}{\Re(\beta)} N, \qquad c \in \mathbb{C},$$

then the function $F_{\alpha,\beta}(z)$ defined by (1.2) *is analytic and univalent in* $\mathbb{U}$.

We next set

$$g(z) = 1, \qquad z \in \mathbb{U}$$

in Theorem 2.1, and thus obtain the following interesting consequence of Theorem 2.1.

Corollary 3.2. *Let the function* $f \in \mathcal{A}$ *satisfy the condition* (1.3) *and suppose that* M *is a real positive number. Also let*

$$\Re(\beta) \geqq |\alpha| \, [(m+1)\, M + 1], \qquad \alpha, \beta \in \mathbb{C}. \tag{3.4}$$

If

$$|f(z)| < M, \qquad z \in \mathbb{U} \tag{3.5}$$

and

$$|c| \leqq 1 - \frac{1}{\Re(\beta)} \, |\alpha| \, [(m+1)\, M + 1], \qquad c \in \mathbb{C}, \tag{3.6}$$

then the function

$$F_{\alpha,\beta}(z) = \left(\beta \int_0^z t^{\beta-\alpha-1} [f(t)]^{\alpha} dt \right)^{\frac{1}{\beta}}$$

is analytic and univalent in $\mathbb{U}$.

Finally, upon setting

$$m = 1 \qquad \text{and} \qquad g(z) = 1, \quad z \in \mathbb{U}$$

in Theorem 2.1, we obtain the following consequence of Theorem 2.1.

Corollary 3.3. *Let the function* $f \in \mathcal{A}$ *satisfy the condition* (1.3) *and suppose that* M *is a real positive number. Also let*

$$\Re(\beta) \geqq |\alpha| \, (2M + 1)\,, \qquad \alpha, \beta \in \mathbb{C}. \tag{3.7}$$

If

$$|f(z)| < M, \qquad z \in \mathbb{U} \tag{3.8}$$

and

$$|c| \leqq 1 - \frac{1}{\Re(\beta)} \, |\alpha| \, (2M + 1)\,, \qquad c \in \mathbb{C}, \tag{3.9}$$

then the function

$$F_{\alpha,\beta}(z) = \left(\beta \int_0^z t^{\beta-\alpha-1} [f(t)]^{\alpha} dt \right)^{\frac{1}{\beta}}$$

is analytic and univalent in $\mathbb{U}$.

Acknowledgements

This work was partially supported by the Strategic Project POSDRU 107/1.5/S/77265 under POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People.

References

[1] Nehari, Z., Conformal Mapping. New York, Toronto, London: McGraw-Hill Book Company, 1952.

[2] Ozaki, S., Nunokawa, M., The Schwarzian derivate and univalent functions. Proc. Amer. Math. Soc. 33 (1972), 392–394.

[3] Pescar, V., A new generalization of Ahlfors's and Becker's criterion of univalence. Bull. Malaysian Math. Soc. (Ser. 2) 19 (1996), 53–54.

[4] Pescar, V., On the univalence of an integral operator. Novi Sad J. Math. 40 (2010), 29–36.

[5] Răducanu, D., Radomir, I., Gageonea, M. E., Pascu, N. R., A generalization of Ozaki-Nunokawa's univalence criterion. J. Inequal. Pure Appl. Math. 5 (4) (2004), Article 95, 1–4 (electronic).

[6] Srivastava, H. M., Deniz, E., Orhan, H., Some general univalence criteria for a family of integral operators. Appl. Math. Comput. 215 (2010), 3696–3701.

Received by the editors October 14, 2011