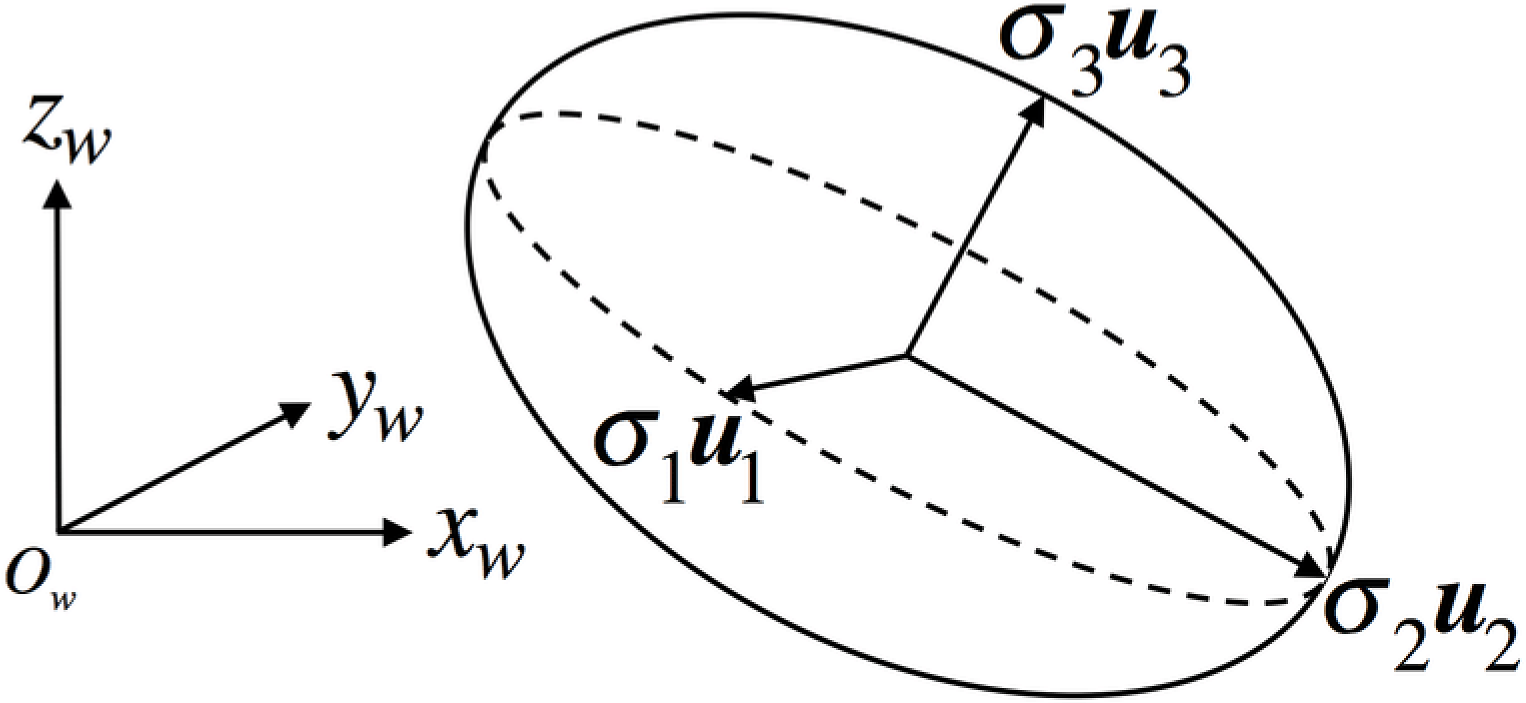
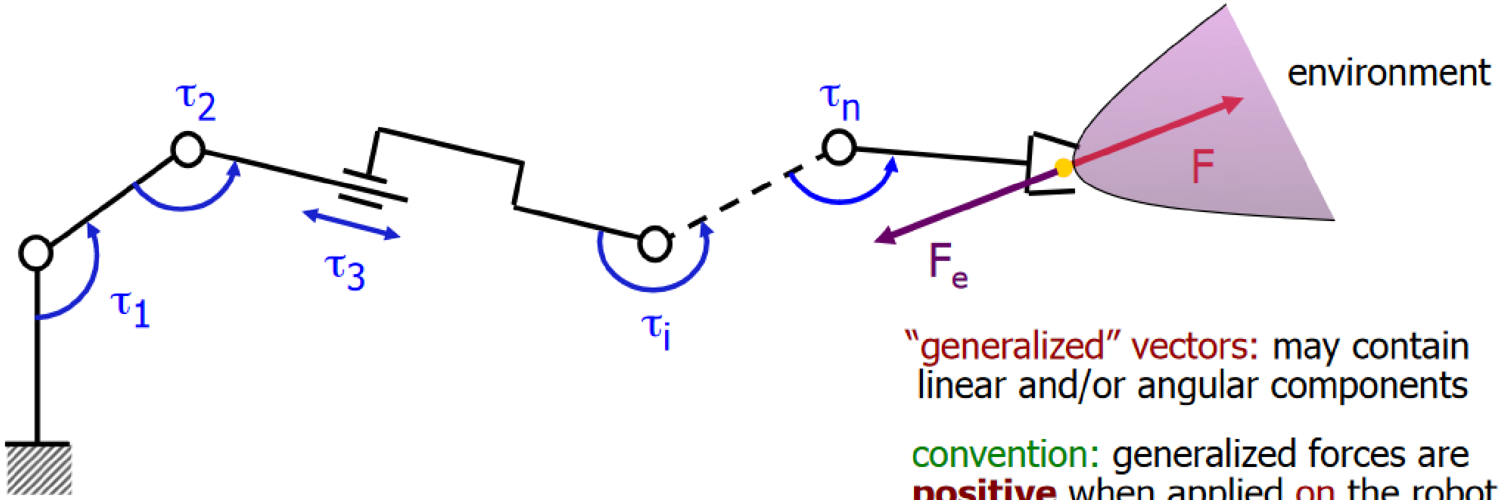
Manipulability Ellipsoids



https://robotacademy.net.au/

Gravagne, Ian A. y Walker, Ian D., Manipulability, Force and Compliance Analysis for Planar Continuum Manipulators, IEEE Transactions on Robotics and Automation. No. 3, v. 18 (2002), p 263-273.

Generalized forces and torques



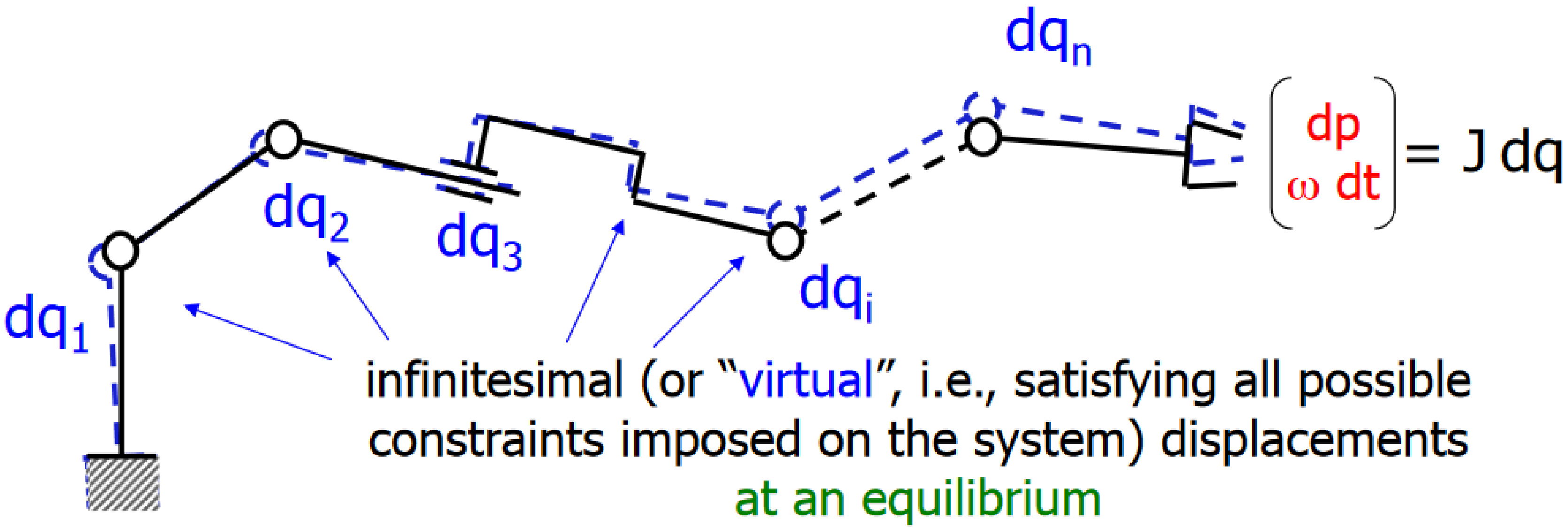
"generalized" vectors: may contain linear and/or angular components

convention: generalized forces are **positive** when applied on the robot

- τ = forces/torques exerted by the motors at the robot joints
- F = equivalent forces/torques exerted at the robot end-effector
- F_e = forces/torques exerted by the environment at the end-effector
- principle of action and reaction: $F_e = -F$

reaction from environment is equal and opposite to the robot action on it

Virtual displacements and works

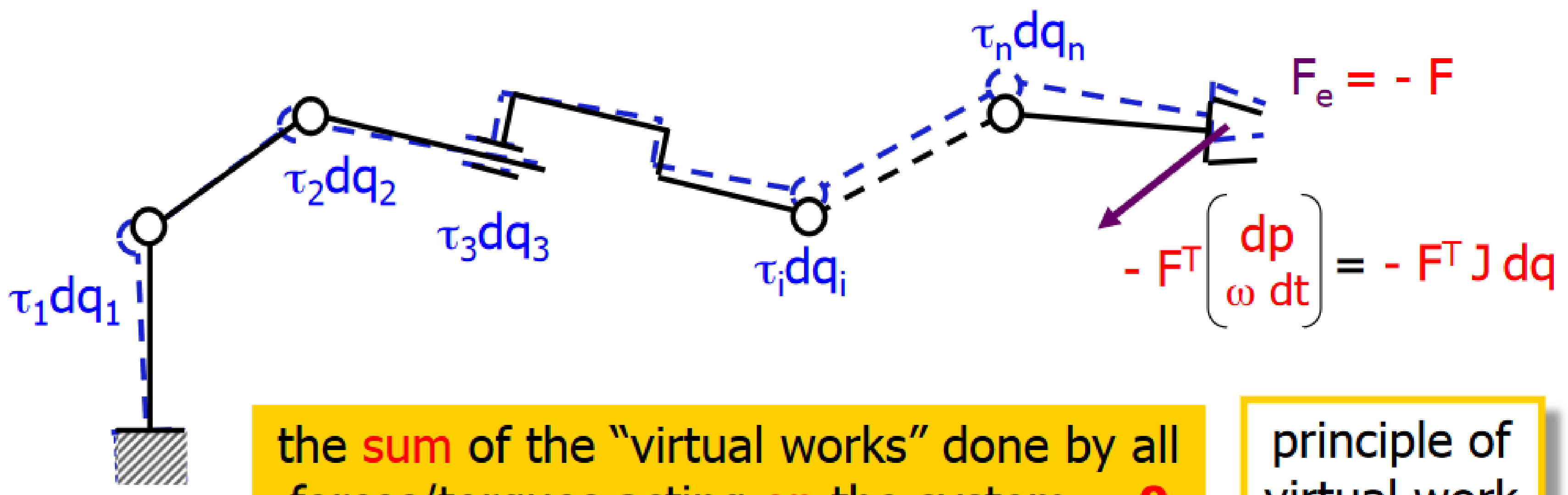


infinitesimal (or "virtual", i.e., satisfying all possible constraints imposed on the system) displacements at an equilibrium

No motion at the EE ➡

- without kinetic energy variation (zero acceleration)
- without dissipative effects (zero velocity)

the "virtual work" is the work done by all forces/torques acting on the system for a given virtual displacement



the sum of the "virtual works" done by all forces/torques acting on the system = 0

principle of virtual work

$$\tau^T dq - F^T \begin{bmatrix} dp \\ \omega dt \end{bmatrix} = \tau^T dq - F^T J dq = 0 \quad \forall dq$$

$$\Rightarrow \quad \tau = J^T(q) F$$

Duality between velocity and force

J(q)

velocity $\dot{q}$
(or displacement dq)
in the joint space

generalized velocity v
(or e-e displacement $\begin{bmatrix} dp \\ \omega dt \end{bmatrix}$)
in the Cartesian space

forces/torques τ
at the joints

generalized forces F
at the Cartesian e-e

$J^T(q)$

the singular configurations
for the velocity map are the same
as those for the force map

$\rho(J) = \rho(J^T)$

Dual subspaces of velocity and force

summary of definitions

$$\mathcal{R}(J) = \{v \in I\!R^m : \exists \dot{q} \in I\!R^n, J\dot{q} = v\}$$

$$\mathcal{N}(J^T) = \{F \in I\!R^m : J^T F = 0\}$$

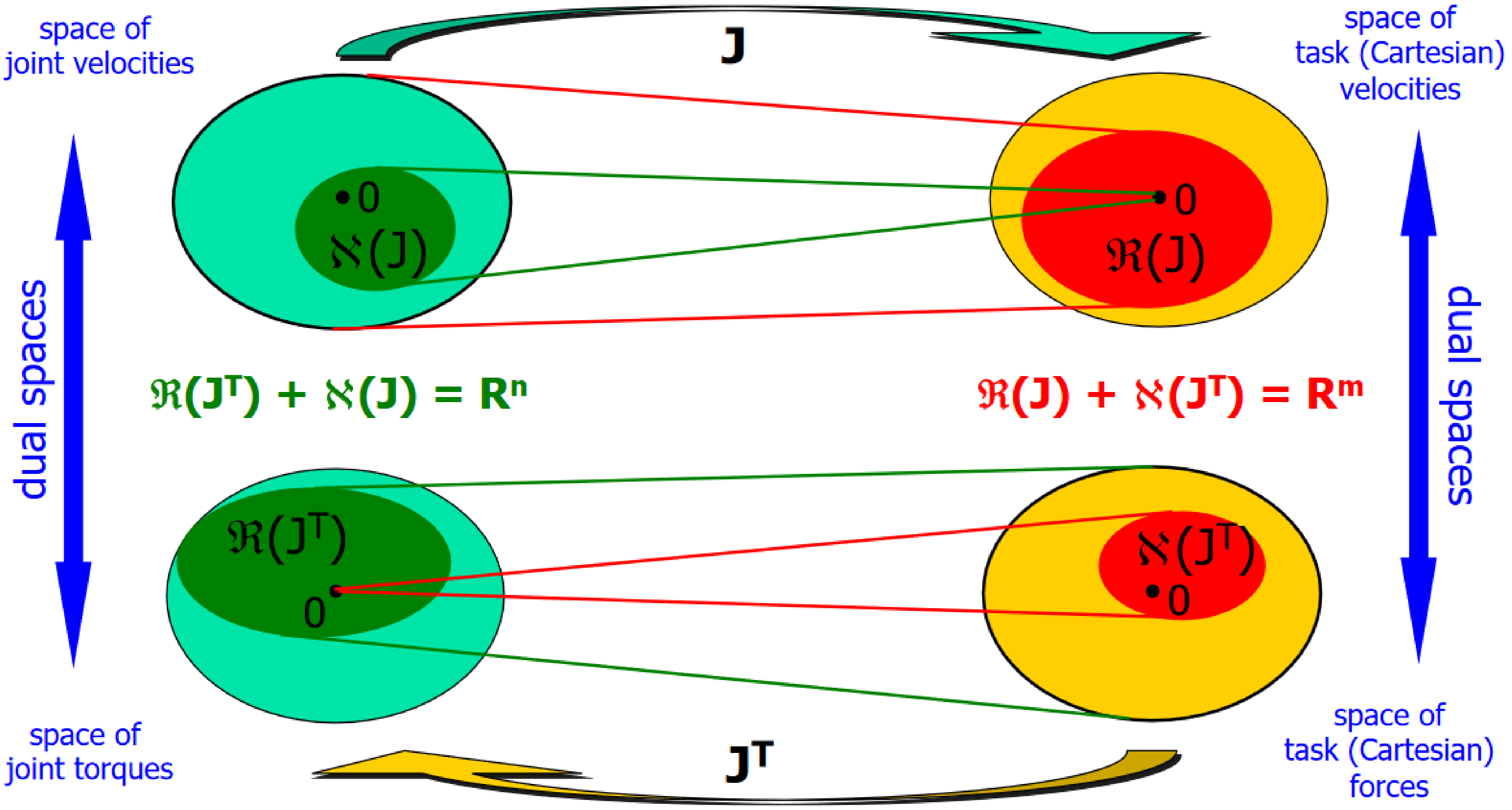
$$\mathcal{R}(J) + \mathcal{N}(J^T) = I\!R^m$$

$$\mathcal{R}(J^T) = \{\tau \in I\!R^n : \exists F \in I\!R^m, J^T F = \tau\}$$

$$\mathcal{N}(J) = \{\dot{q} \in I\!R^n : J\dot{q} = 0\}$$

$$\mathcal{R}(J^T) + \mathcal{N}(J) = I\!R^n$$

Kinetostatic Duality



(in a given configuration q)

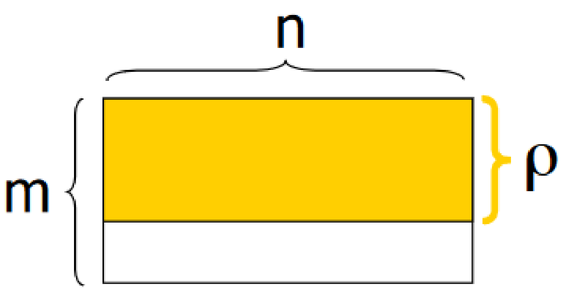
Velocity and force singularities

list of possible cases

$$\rho = \text{rank}(J) = \text{rank}(J^T) \leq \min(m,n)$$

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{\gamma}_e$$

$$\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$



1. $\rho = m$ REDUNDANCY

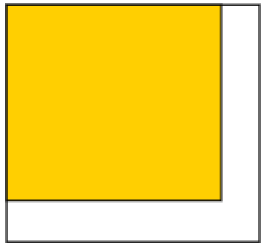
$$\exists \dot{q} \neq 0 : \; J\dot{q} = 0$$

$$\mathcal{N}(J^T) = \{0\}$$

2. $\rho < m$ SINGULARITY

$$\exists \dot{q} \neq 0 : \; J\dot{q} = 0$$

$$\exists F \neq 0 : \; J^T F = 0$$



1. $\det J \neq 0$

$$\mathcal{N}(J) = \{0\}$$

$$\mathcal{N}(J^T) = \{0\}$$

2. $\det J = 0$ SINGULARITY

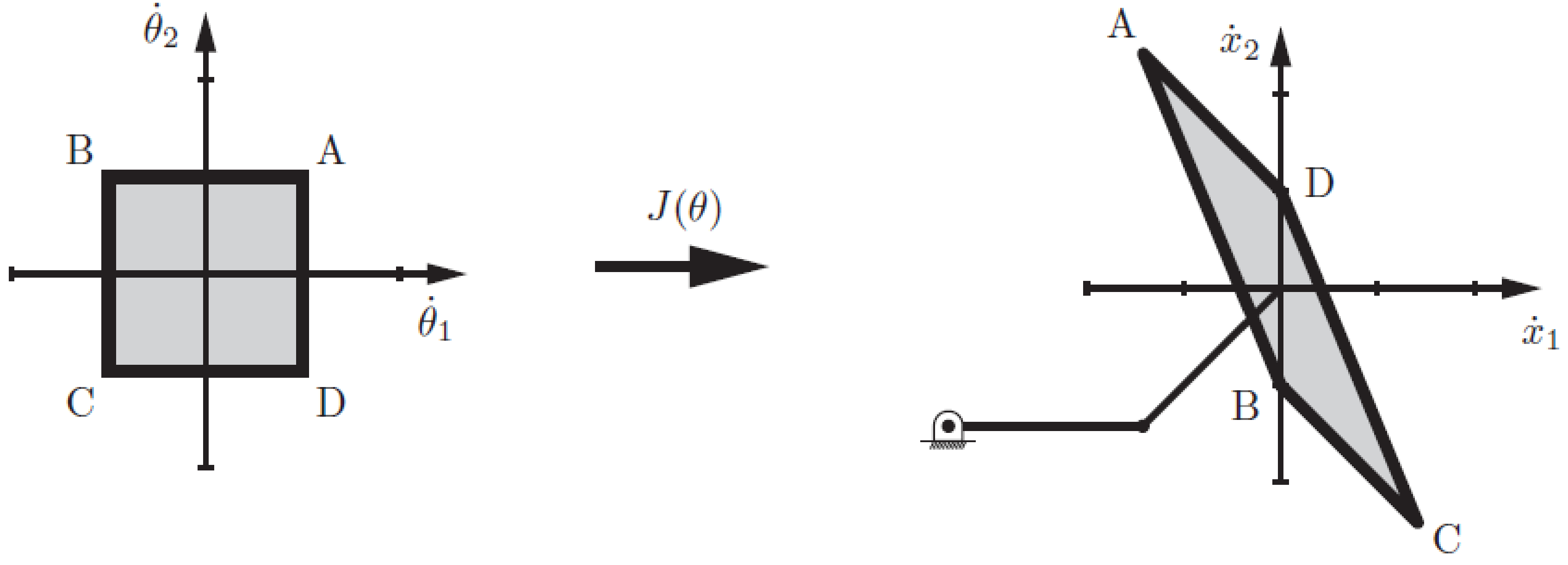
$$\exists \dot{q} \neq 0 : \; J\dot{q} = 0$$

$$\exists F \neq 0 : \; J^T F = 0$$

Different null space configurations

https://www.youtube.com/watch?v=vYho21M44Lw

Transformation joint to EE velocity



Mapping the set of possible joint velocities, represented as a square in the $\dot{\theta}_1$–$\dot{\theta}_2$ space, through the Jacobian to find the parallelogram of possible end-effector velocities. The extreme points A, B, C, and D in the joint velocity space map to the extreme points A, B, C, and D in the end-effector velocity space.

Velocity manipulability

- in a given configuration, we wish to evaluate how "effective" is the mechanical transformation between joint velocities and end-effector velocities
  - "how easily" can the end-effector be moved in the various directions of the task space
  - equivalently, "how far" is the robot from a singular condition

- we consider all end-effector velocities that can be obtained by choosing joint velocity vectors of unit norm

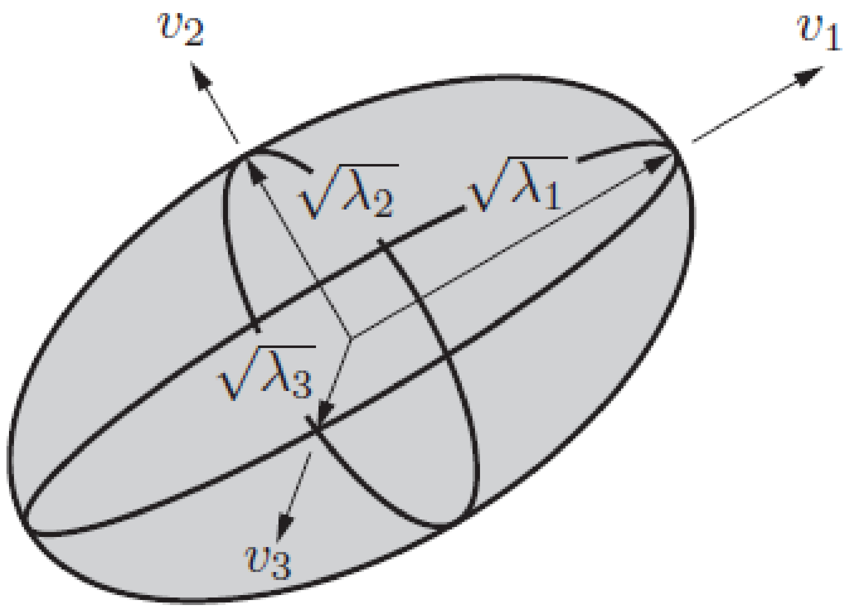
$$\dot{q}^T \dot{q} = 1 \quad \Rightarrow \quad v^T J^{\#T} J^{\#} v = 1$$

task velocity manipulability ellipsoid

$(JJ^T)^{-1}$ if $\rho = m$

note: the "core" matrix of the ellipsoid equation $v^T A^{-1} v=1$ is the matrix A!

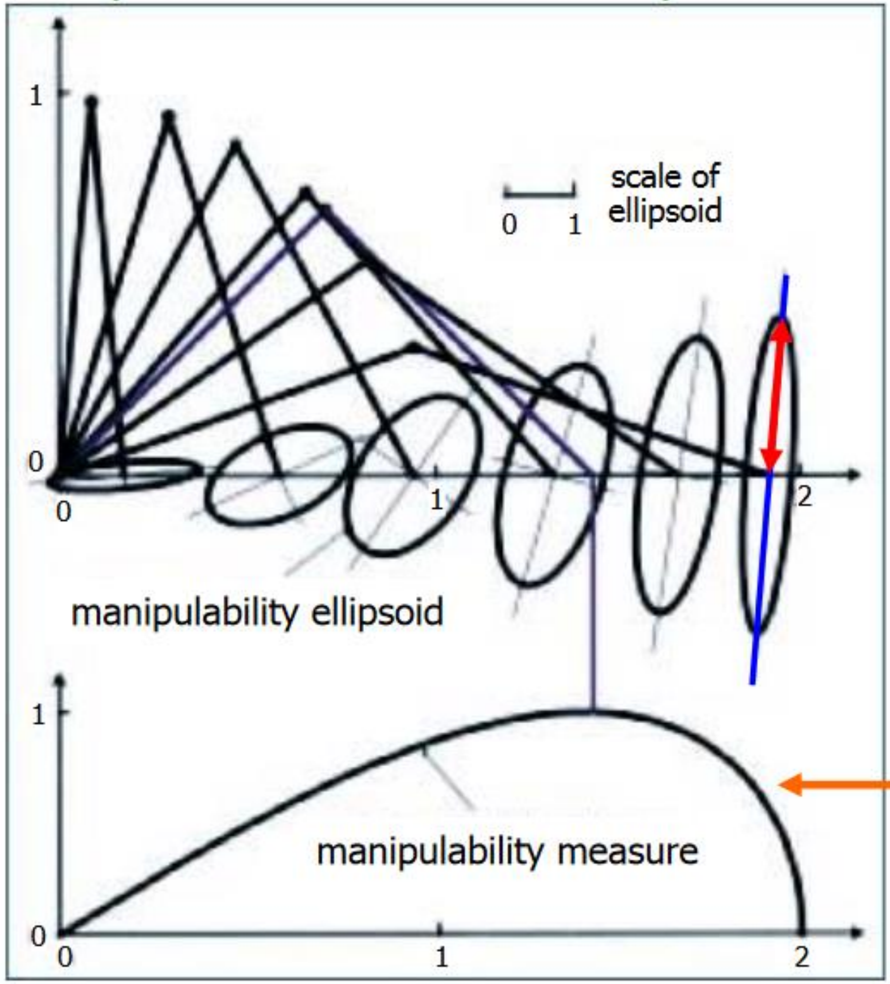
What is an ellipsoid



An ellipsoid visualization of $\dot{q}^{\mathrm{T}} A^{-1} \dot{q} = 1$ in the $\dot{q}$ space $\mathbb{R}^3$, where the principal semi-axis lengths are the square roots of the eigenvalues λ_i of A and the directions of the principal semi-axes are the eigenvectors v_i.

Manipulability ellipsoid
in velocity

planar 2R arm with unitary links



length of principal (semi-)axes:
singular values of J (in its SVD)

$$\sigma_i\{J\} = \sqrt{\lambda_i\{JJ^T\}} \geq 0$$

in a singularity, the ellipsoid loses a dimension
(for m=2, it becomes a segment)

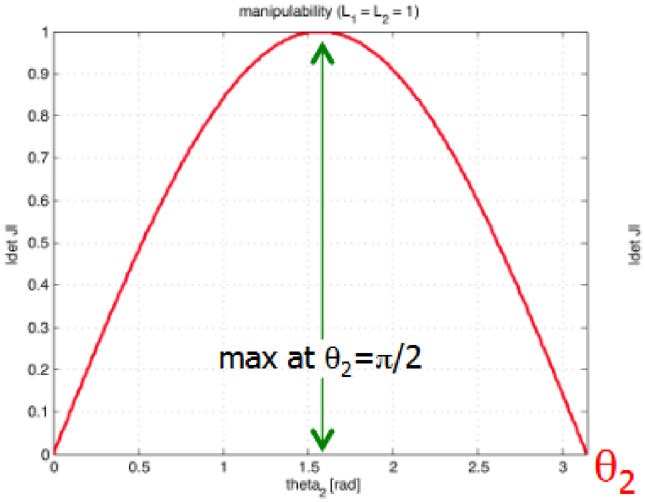
direction of principal axes:
(orthogonal) eigenvectors associated to λ_i

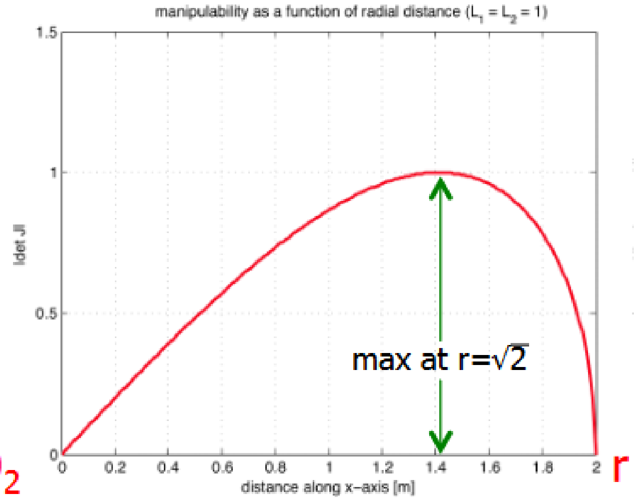
$$w = \sqrt{\det JJ^T} = \prod_{i=1}^{m} \sigma_i \geq 0$$

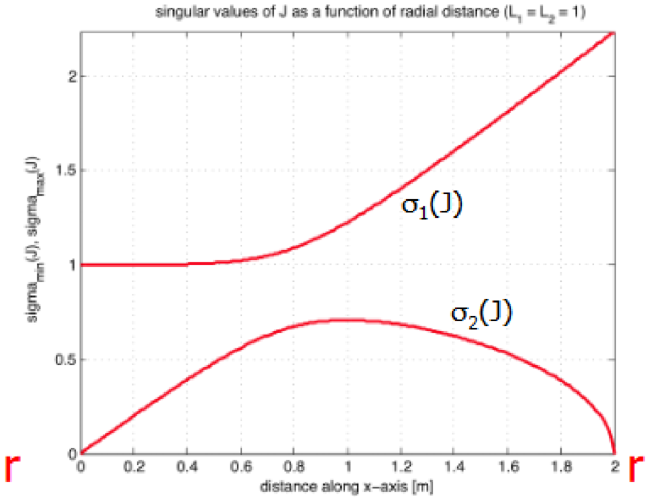
proportional to the volume of the ellipsoid (for m=2, to its area)

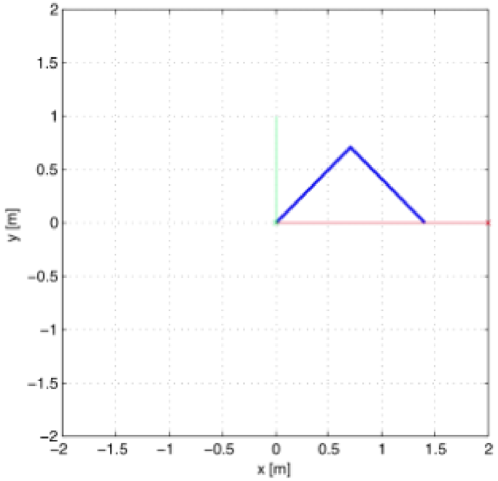
Manipulability measure

planar 2R arm with unitary links: Jacobian J is square ➡ $\sqrt{\det\left(JJ^T\right)} = \sqrt{\det J \cdot \det J^T} = \left|\det J\right| = \prod_{i=1}^{2} \sigma_i$







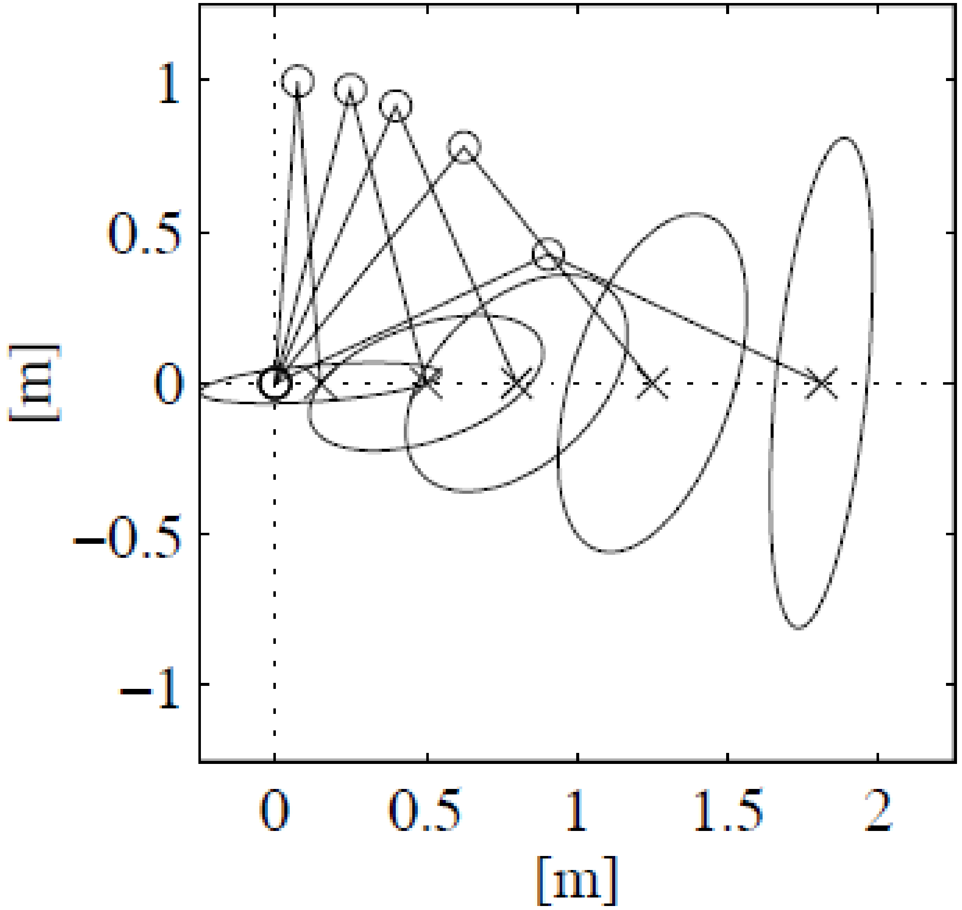


best posture for manipulation
(similar to a human arm!)

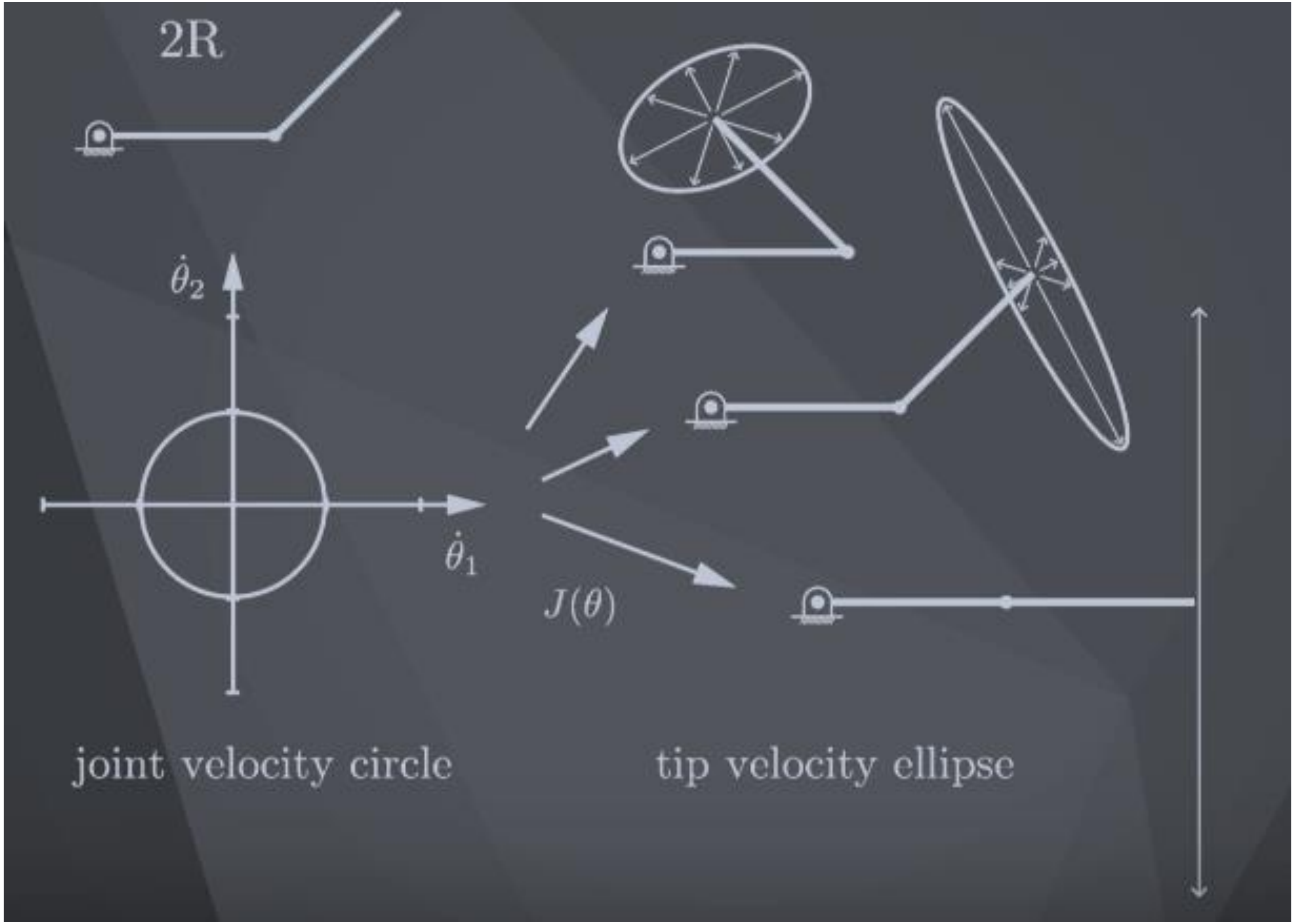
full isotropy is never obtained
in this case, since it always $\sigma_1 \neq \sigma_2$

Velocity manipulability ellipsoid

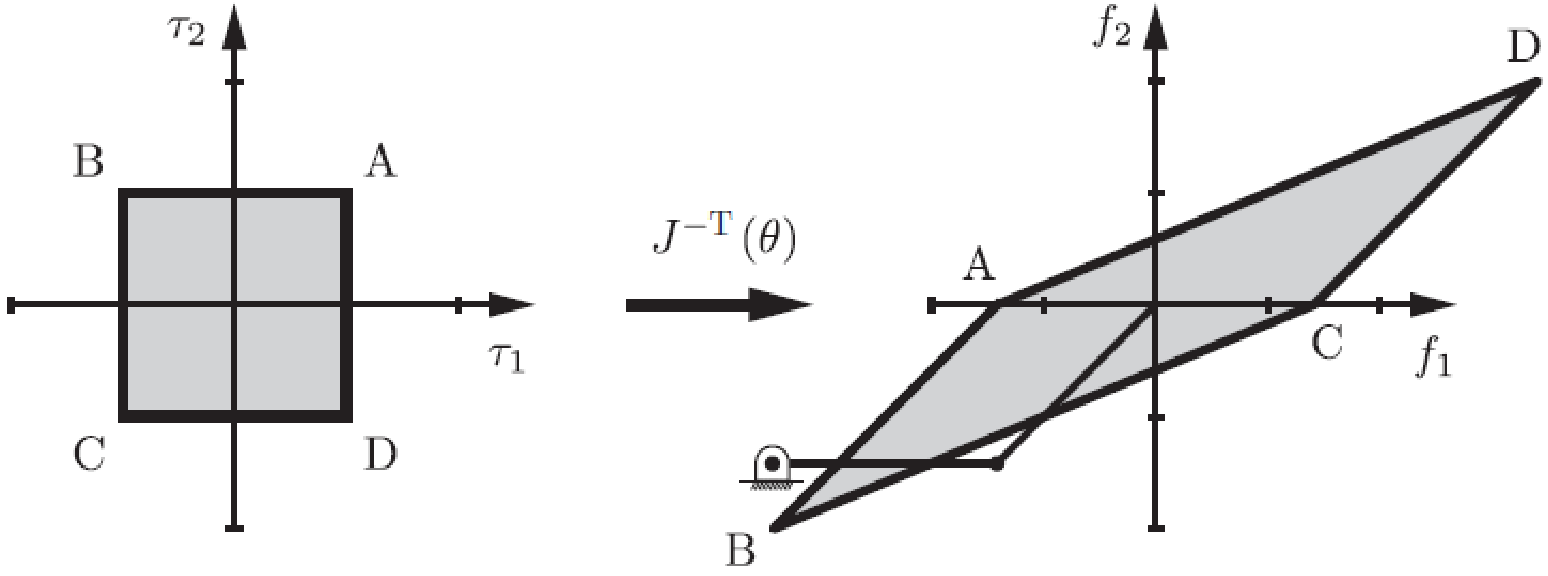
Example two link planar arm



Velocity manipulability ellipsoid



Transformation joint torque to EE force



Mapping joint torque bounds to tip force bounds.

Force manipulability

- in a given configuration, evaluate how "effective" is the transformation between joint torques and end-effector forces
  - "how easily" can the end-effector apply generalized forces (or balance applied ones) in the various directions of the task space
  - in singular configurations, there are directions in the task space where external forces/torques are balanced by the robot without the need of any joint torque

- we consider all end-effector forces that can be applied (or balanced) by choosing joint torque vectors of unit norm

$$\tau^T \tau = 1 \quad \Rightarrow \quad F^T J J^T F = 1$$

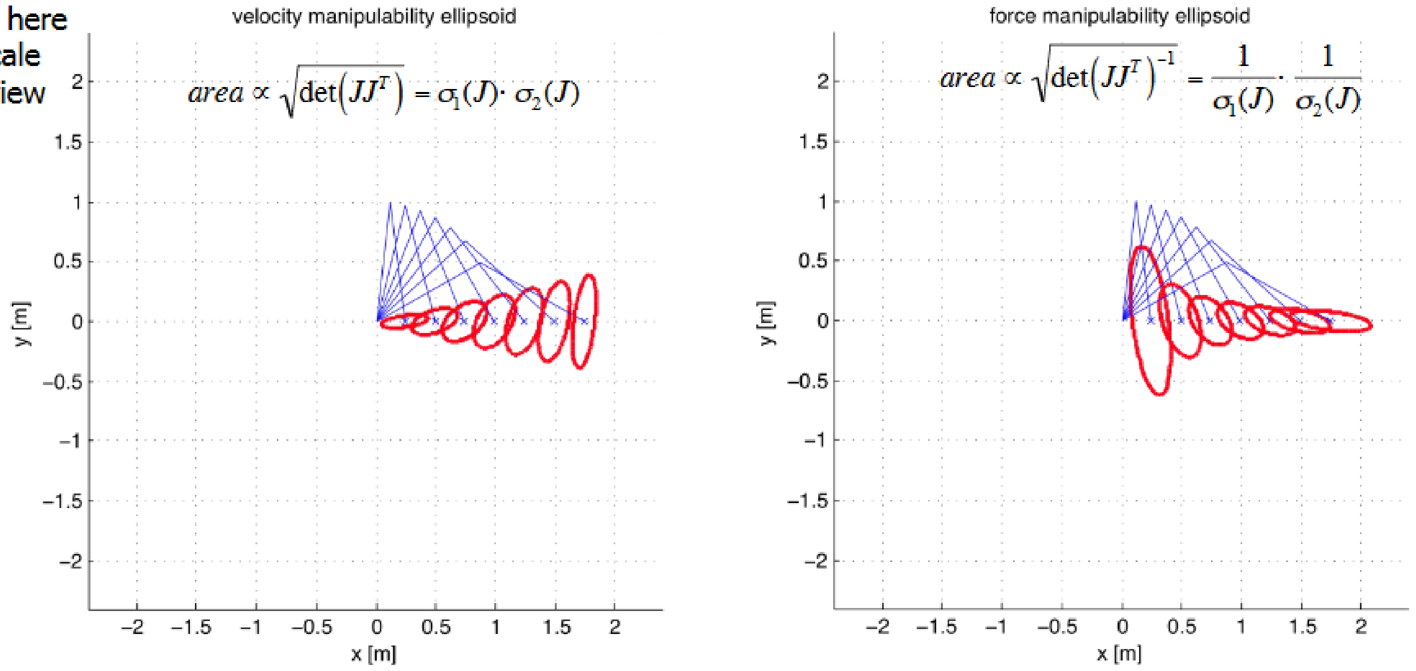
same directions of the principal axes of the velocity ellipsoid, but with semi-axes of inverse lengths → task force manipulability ellipsoid

Velocity and force manipulability

dual comparison of actuation vs. control

note:
velocity and force ellipsoids have here a different scale for a better view

planar 2R arm with unitary links

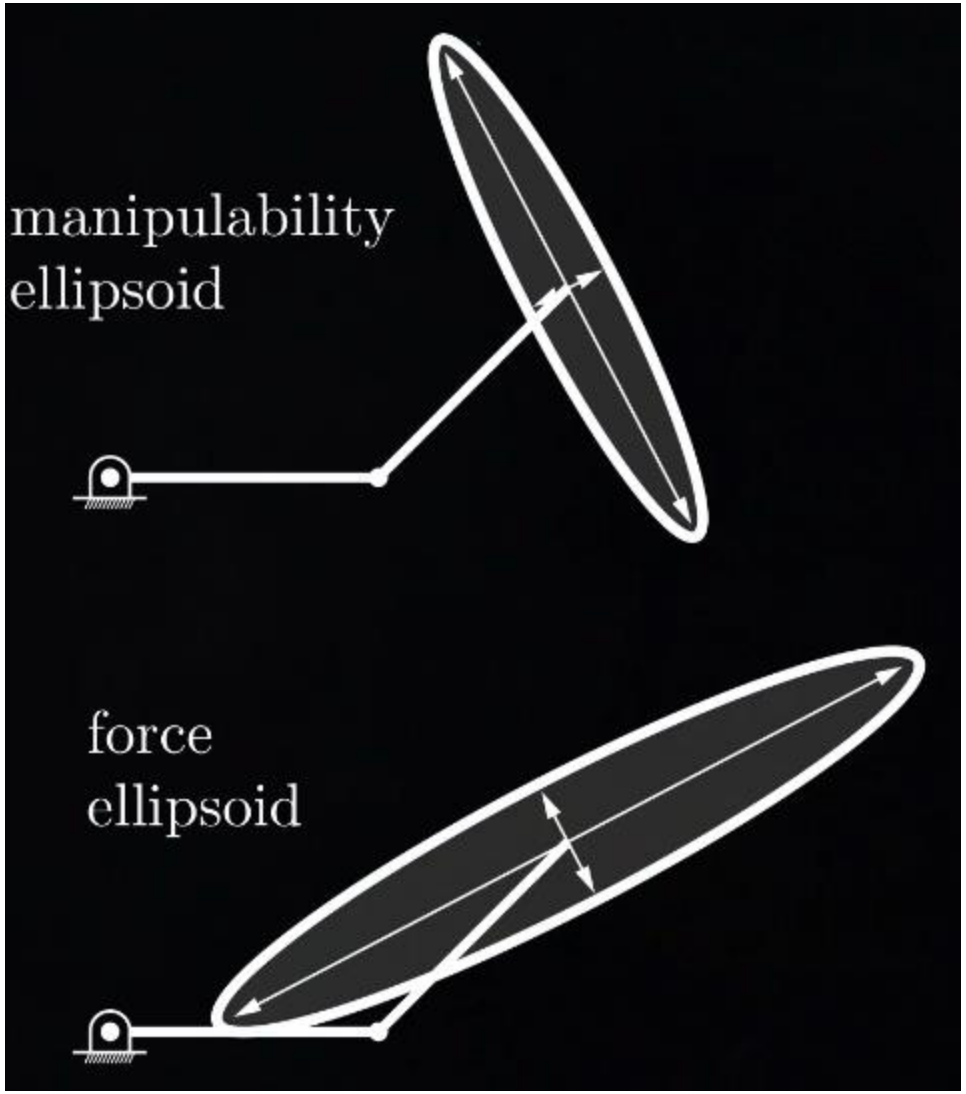


Cartesian **actuation** task (high joint-to-task transformation ratio):
preferred velocity (or force) directions are those where the ellipsoid *stretches*

⬍

Cartesian **control** task (low transformation ratio = high resolution):
preferred velocity (or force) directions are those where the ellipsoid *shrinks*

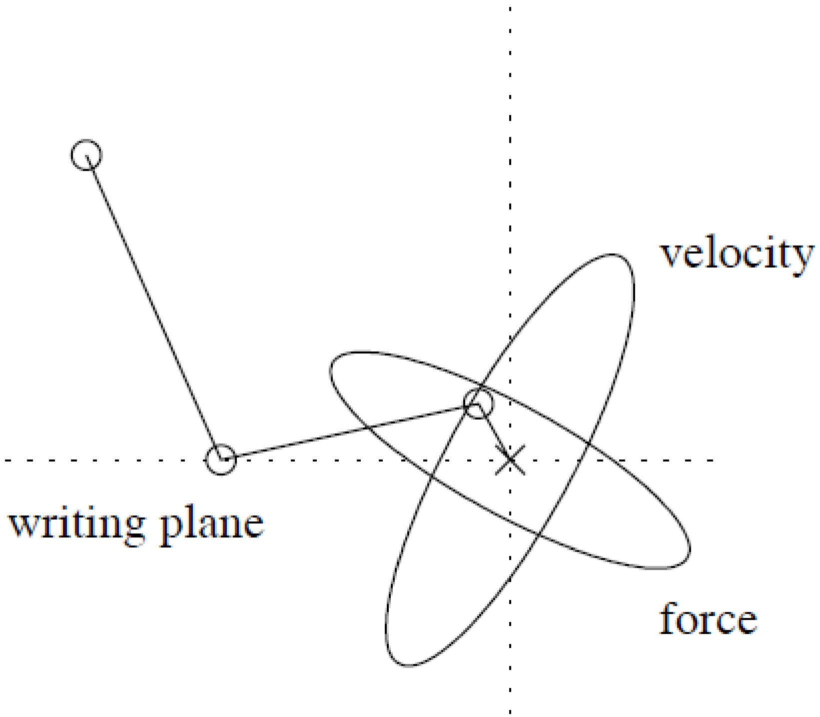
Velocity Vs Force manipulability ellipsoids

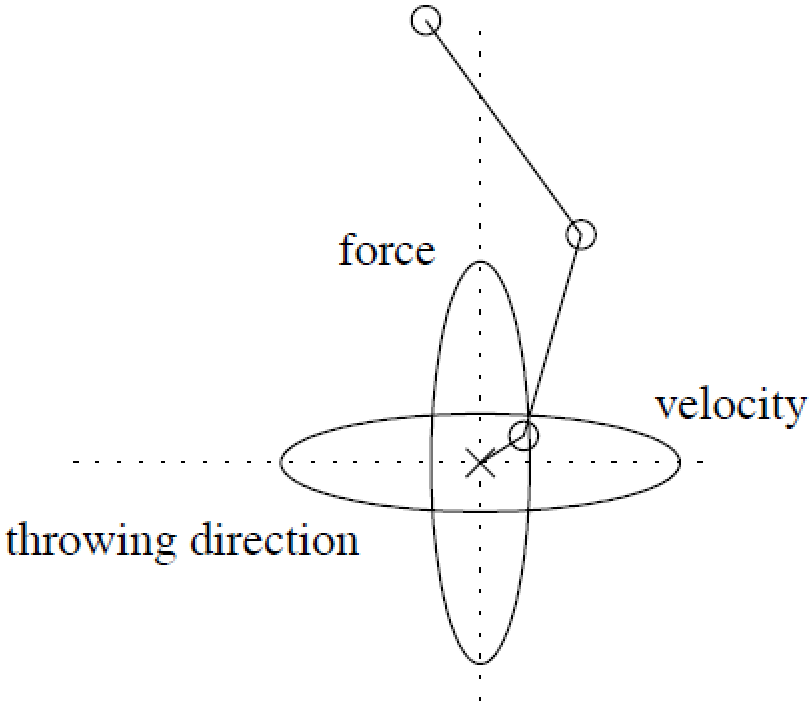


Only small force can be applied in the directions where high velocities can be obtained and viceversa.

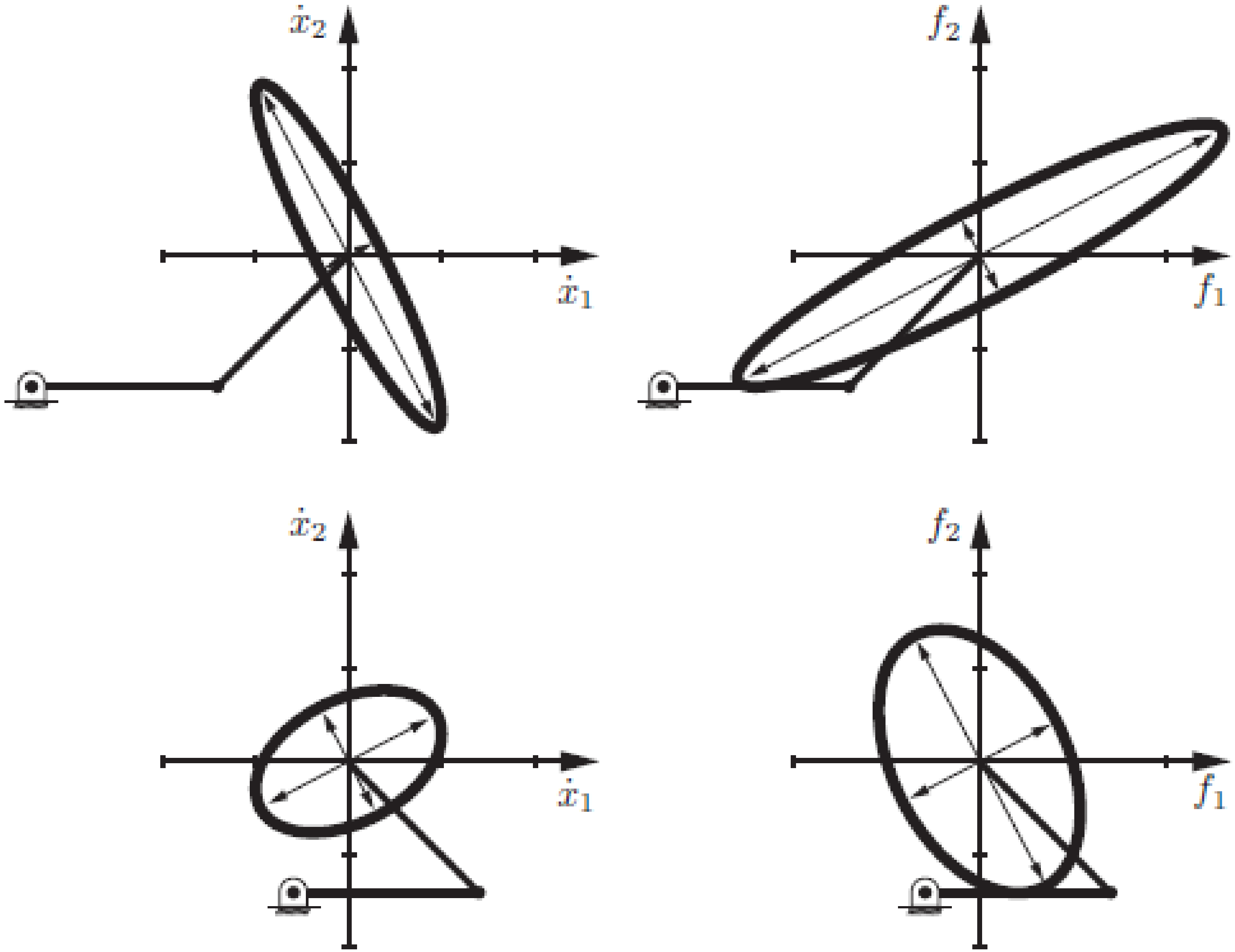
Kineto-static dualism in manipulability

Therefore, according to the concept of force/velocity duality, a direction along which good velocity manipulability is obtained is a direction along which poor force manipulability is obtained, and vice versa.





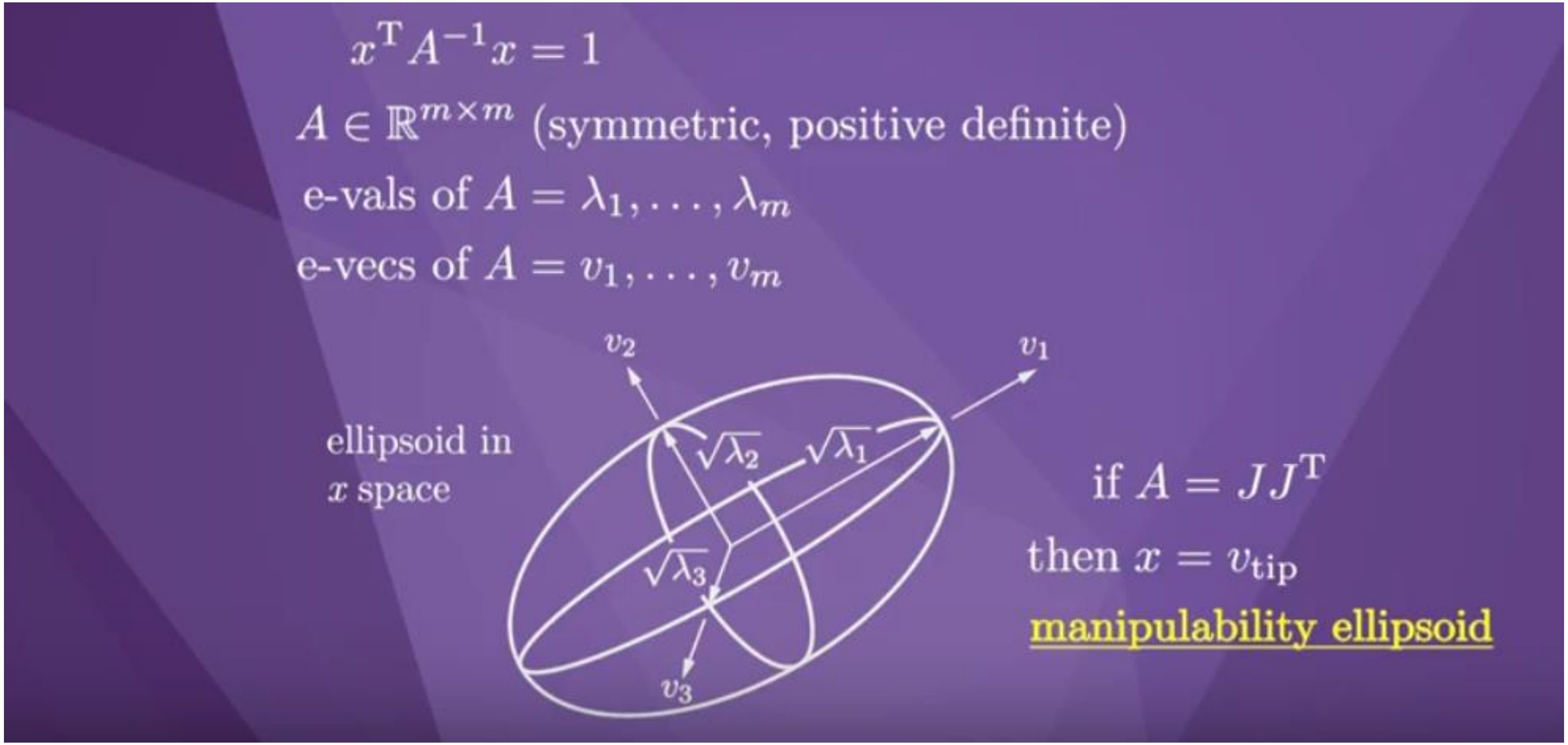
Kineto-static dualism in manipulability



Velocity manipulability ellipsoid

the shape and orientation of the velocity ellipsoid are determined by the core of its quadratic form and then by the matrix ***A=J(q)J(q)^T*** which is in general a function of the manipulator configuration.

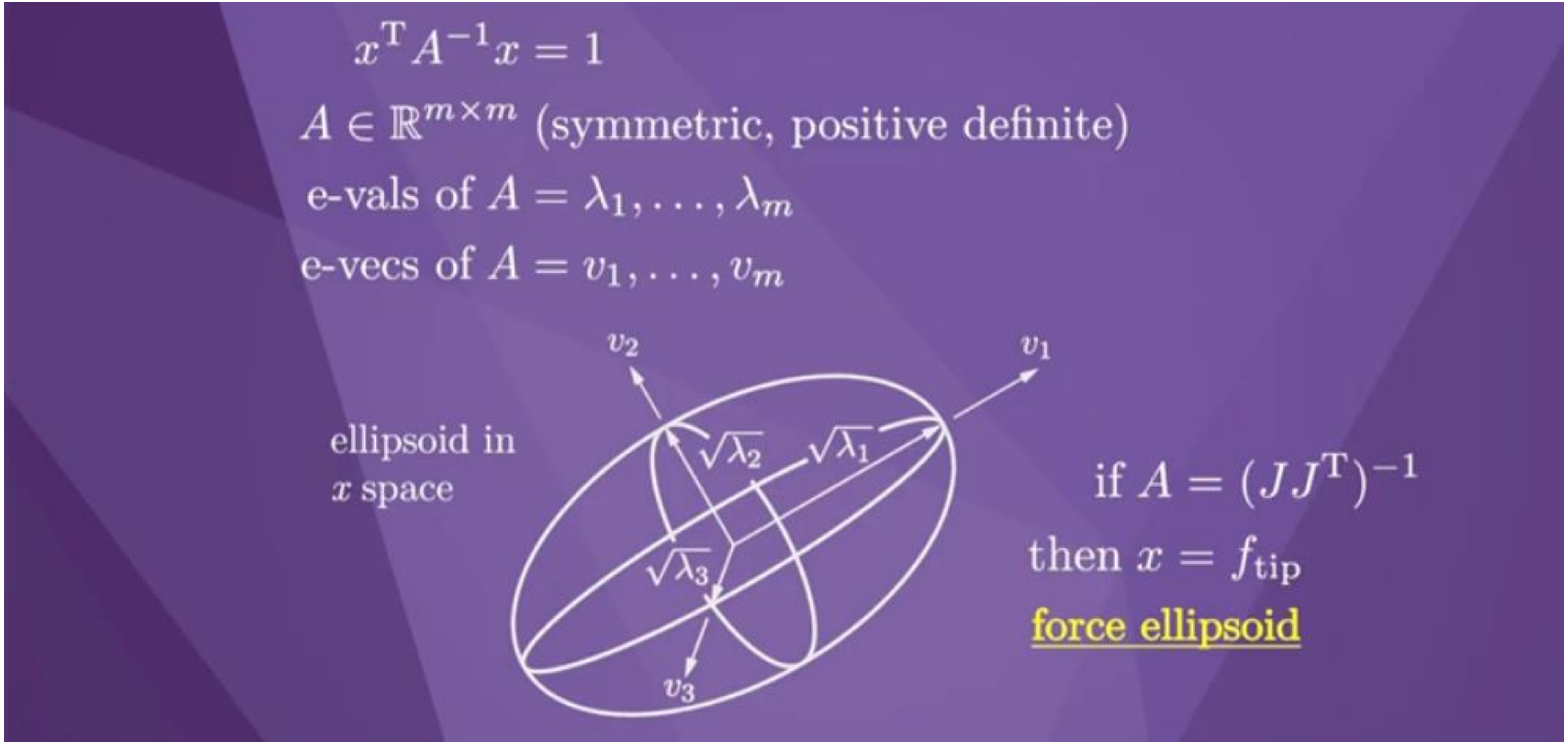
$$v_e^T \left(J(q)J^T(q)\right)^{-1} v_e = 1,$$



Force manipulability ellipsoid

the shape and orientation of the force ellipsoid are determined by the core of its quadratic form and then by the matrix ***$A=(J(q)J(q)^T)^{-1}$*** which is in general a function of the manipulator configuration.

$$\gamma_e^T \left(J(q)J^T(q)\right)\gamma_e = 1$$



The end!