HAUSDORFF'S PARADOX — OR TWO FOR ONE*

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1. Introduction

We shall use the symbol $\dot\cup$ to denote disjoint unions. Thus we write $Z = X \dot\cup Y$ to denote that the set Z is the union of X and Y and further that X and Y are disjoint.

If A, B are subsets of Euclidean 3-space $\mathbb{R}^3$, we write $A \cong B$ if the sets are congruent, i.e. if the one can be obtained from the other by translation, rotation and possibly reflection. Another way of expressing this is to say that there is an *isometry,* a map f from A onto B which preserves distances i.e. $d(f(a), f(a')) = d(a,a')$ for all $a, a' \in A$. In this lecture we shall be interested in the following generalization of this notion. We shall write $A \stackrel{n}{\equiv} B$ if there are disjoint decompositions of A and B into n sets

$$A = A_1 \dot\cup A_2 \dot\cup \ldots \dot\cup A_n, \quad B = B_1 \dot\cup B_2 \dot\cup \ldots \dot\cup B_n$$

such that $A_i \cong B_i$ $(1 \leqslant i \leqslant n)$. We say that A and B are *equivalent* by *finite decomposition* if $A \stackrel{n}{\equiv} B$ for some positive integer n.

The idea of geometrical equivalence by decomposition (viz. paper-cutting) is a familar one from elementary geometry, but here we have a much more precise notion. For example, if the unit square S is cut along the diagonal AC, the two pieces can be re-assembled to make a right-angled isosceles triangle T having unit area (Fig. 1).

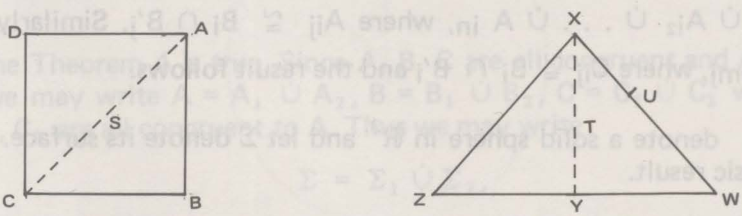

Fig. 1

However, this is too crude to show that $S \stackrel{2}{\equiv} T$. Indeed, it is not immediately obvious that $S \stackrel{n}{\equiv} T$ for any integer n. Let us examine the above cutting more carefully. When cut along the diagonal AC we must specify to which of the two triangles the common side AC is to belong! As a trial we see that $S = S_1 \dot\cup S_2 \dot\cup S_3$, where S_1 is the triangle ABC including interior and edges, S_2 is the interior of triangle ADC together with the open ended segment CD, and S_3 is the segment DA which includes D but not A. These three disjoint pieces are respectively congruent to T_1, the closed triangle XYZ; T_2, the interior of triangle XYW together with the open segment YW;

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and T_3, the segement WU which includes W but not U. Thus $S \overset{3}{\cong} T \setminus P$, when P is the segment UX of length $\sqrt{2} - 1$ including U but excluding X. In fact, $S \overset{6}{\cong} T$ and to show this it is enough to show that $S \overset{2}{\cong} S \setminus Q$, where Q is a segment of length $\sqrt{2} - 1$ open at one end and closed at the other. The idea is simple and is contained in some of the other results we shall prove and so we leave this as an exercise for the reader.

There is a very delightful little book [1] by Sierpinski which is in the library of the N.U.S. and which gives a fairly complete discussion of this notion of equivalence by finite decomposition and states various unsolved problems. I shall consider only one special result of this kind, the so-called paradoxes associated with the names of Hausdorff, Banach & Tarski. A fuller discussion of these results can be found in [1].

2. Preliminaries

One immediate consequence of the definition is the following.

THEOREM 1. *If* $A \overset{m}{\cong} B$ *and* $B \overset{n}{\cong} C$, *then* $A \overset{m \cdot n}{\cong} C$.

PROOF. By hypothesis there are decompositions

$$A = A_1 \dot{\cup} A_2 \dot{\cup} \ldots \dot{\cup} A_m, \quad B = B_1 \dot{\cup} B_2 \dot{\cup} \ldots \dot{\cup} B_m$$

$$B = B'_1 \dot{\cup} B'_2 \dot{\cup} \ldots \dot{\cup} B'_n, \quad C = C_1 \dot{\cup} C_2 \dot{\cup} \ldots \dot{\cup} C_n$$

such that $A_i \cong B_i$ $(1 \leqslant i \leqslant m)$, $B'_j \cong C_j$ $(1 \leqslant j \leqslant n)$.

Since $A_i \cong B_i \cong (B_i \cap B'_1) \dot{\cup} (B_i \cap B'_2) \dot{\cup} \ldots \dot{\cup} (B_i \cap B'_n)$, we may write $A_i \cong A_{i1} \dot{\cup} A_{i2} \dot{\cup} \ldots \dot{\cup} A_{in}$, where $A_{ij} \cong B_i \cap B'_j$. Similarly, $C_j = C_{1j} \dot{\cup} C_{2j} \dot{\cup} \ldots \dot{\cup} C_{mj}$, where $C_{ij} \cong B_i \cap B'_j$ and the result follows.

Let Ω denote a solid sphere in $\mathcal{R}^3$ and let Σ denote its surface. We need the following basic result.

THEOREM 2. *If* D *is a countable set of points of* Σ, then $\Sigma \overset{2}{\cong} \Sigma \setminus D$.

PROOF. Since D is countable we can choose a diameter ZZ′ of Σ so that Z, Z′ $\notin$ D. For any point $p \in \Sigma$, let $\theta(p)$ denote the angle the plane ZPZ′ makes with some reference plane ZXZ′ (Fig. 2). Since D is countable we can choose a real number α so that

$$\alpha \neq (2\pi k + \theta(p_1) - \theta(p_2))/\ell$$

for any points $p_1, p_2 \in \Sigma$ and any integers ℓ $(\neq 0)$ and k. Let $D(\beta)$ denote the subset of Σ obtained by rotating the set D through an angle β about the line ZZ′. Then, by our choice of α, the sets D, $D(\alpha)$, $D(2\alpha)$, $D(3\alpha)$, . . . are pairwise disjoint. Now put $A = D \dot{\cup} D(\alpha) \dot{\cup} D(2\alpha) \dot{\cup} \ldots$, $B = \Sigma \setminus A$. Then $\Sigma = A \dot{\cup} B$. Since $A \cong A(\alpha) = A \setminus D$, it follows that $\Sigma \overset{2}{\cong} \Sigma \setminus D$.

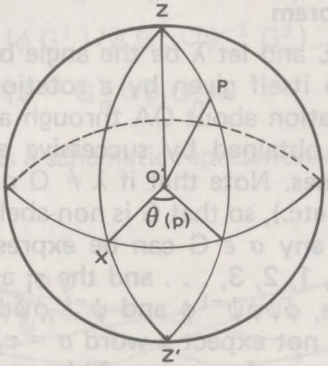


Fig. 2

The same proof also gives

THEOREM 3. *If* $p \in \Omega$, *then* $\Omega \overset{2}{\cong} \Omega \setminus \{p\}$.

PROOF. Choose a sphere $\Omega' \subseteq \Omega$ whose surface Σ' contains p. By the argument used in Theorem 2, there is a countable set $A \subseteq \Sigma'$ so that $p \in A$ and by a rotation about some axis $A \cong A \setminus \{p\}$. Put $B = \Omega \setminus A$. Then $\Omega = A \,\dot\cup\, B$, $\Omega \setminus \{p\} = (A \setminus \{p\}) \,\dot\cup\, B$.

In this lecture we will give an outline proof of the following famous result of Hausdorff (1914).

THEOREM 4. (Hausdorff's Paradox) *There is a partition of* Σ *into four sets;* $\Sigma = A \,\dot\cup\, B \,\dot\cup\, C \,\dot\cup\, D$ *such that* D *is a denumerable set and* $A \cong B$, $A \cong C$ *and* $A \cong B \,\dot\cup\, C$.

Assume Theorem 4 is true. Since A, B, C are all congruent and A is congruent to $B \dot\cup C$, we may write $A = A_1 \,\dot\cup\, A_2$, $B = B_1 \,\dot\cup\, B_2$, $C = C_1 \,\dot\cup\, C_2$ where $A_1, A_2, B_1, B_2, C_1, C_2$ are all congruent to A. Thus we may write

$$\Sigma = \Sigma_1 \,\dot\cup\, \Sigma_2,$$

where $\Sigma_1 = A_1 \,\dot\cup\, B_1 \,\dot\cup\, C_1 \,\dot\cup\, D$ and $\Sigma_2 = A_2 \,\dot\cup\, B_2 \,\dot\cup\, C_2$. Clearly $\Sigma_1 \overset{4}{\cong} \Sigma$ and $\Sigma_2 \overset{3}{\cong} \Sigma \setminus D$. By Theorem 2, $\Sigma \setminus D \overset{2}{\cong} \Sigma$ and therefore, by Theorem 1, $\Sigma_2 \overset{6}{\cong} \Sigma$. In other words, the surface of a sphere may be decomposed into 10 parts, 4 of these can be reassembled to give the surface of an equal sphere, and the remaining 6 can also be reassembled to give a similar spherical surface.

If instead of points on the surface of a sphere we consider the corresponding radii of a solid sphere with its centre deleted, we see that such a body can be partitioned into 10 parts which can be reassembled to form two similar bodies. Moreover, since by Theorem 3 a solid sphere $\Omega \overset{2}{\cong} \Omega \setminus \{p\}$ for any point $p \in \Omega$, we easily deduce that $\Omega_O \overset{20}{\cong} \Omega_1 \,\dot\cup\, \Omega_2$ where $\Omega_O, \Omega_1, \Omega_2$ are disjoint solid spheres having the same radius. This is the paradox of Banach & Tarski. In fact they proved the more general result: *If* A, B *are two bounded subsets of* $\mathbb{R}^3$ *and if each of them contains a solid sphere, then* $A \overset{n}{\cong} B$ *for some integer* n.

3. Proof of Hausdorff's Theorem

Let OZ, OA be radii of Σ and let λ be the angle between these. Let ϕ denote the transformation of Σ onto itself given by a rotation through π about the axis OZ and let ψ denote the rotation about OA through an angle $2\pi/3$. Let G be the group of all transformations obtained by successive applications of ϕ and ψ an arbitrary finite number of times. Note that if $\lambda \neq 0$ or π then $\phi\psi \neq \phi\psi$ (for any point $p \in \Sigma$, $\phi\psi(p) = \phi(\psi(p))$ etc.), so that G is non-abelian. Also, since $\phi^2 = \psi^3 = 1$ (the identity transformation) any $\sigma \in G$ can be expressed as a word of the form $\sigma = \tau_1\tau_2 \ldots \tau_n$ where n = 0, 1, 2, 3, . . . and the τ_i are alternatively ϕ and either ψ or $\psi^2 = \psi^{-1}$. For example, $\phi\psi\phi\psi^{-1}\phi$ and $\psi^{-1}\phi\psi\phi\psi\phi$ are typical members of G. Now in general we would not expect a word $\sigma = \tau_1\tau_2 \ldots \tau_n$ of length $n \geqslant 1$ to coincide with the identity transformation. This could happen (for example, if $\lambda = \frac{1}{2}\pi$ then $\phi\psi\phi\psi = 1$), but in order that such a word represent the identity transformation the angle λ has to chosen in a very special way. In fact, it can be shown that there are at most a finite number of different values of λ which would entail that a particular word, of the form $\sigma = \tau_1\tau_2 \ldots \tau_n$ with $n \geqslant 1$, represents the identity transformation. (For an algebraic justification of this remark see [1]). Since G has only denumerably many different members it follows that we can choose the angle λ suitably so that every word of the above form with $n \geqslant 1$ is different from the identity. It follows, of course, that two different words represent different members of G, so that each member of G has a unique representation as a word of the above form.

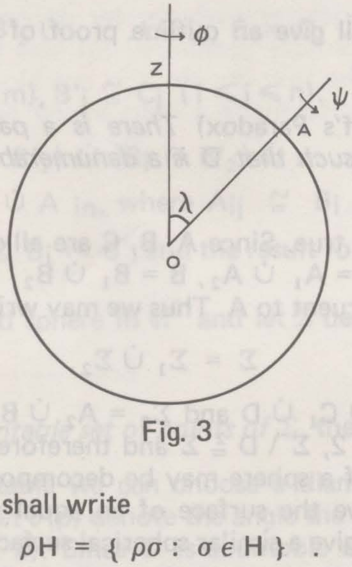


Fig. 3

If $H \subseteq G$ and $\rho \in G$, we shall write

$$\rho H = \{ \rho\sigma : \sigma \in H \} .$$

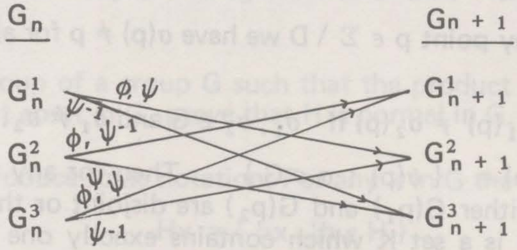
Let G_n denote the set of all members of G which have length n. Thus $G_0 = \{ 1 \}$, $G_1 = \{ \phi, \psi, \psi^{-1} \}$, $G_2 = \{ \phi\psi, \phi\psi^{-1}, \psi\phi, \psi^{-1}\phi \}$, $G_3 = \{ \phi\psi\phi, \phi\psi^{-1}\phi, \psi\phi\psi, \psi\phi\psi^{-1}, \psi^{-1}\phi\psi, \psi^{-1}\phi\psi^{-1} \}$ etc. A member of G_{n+1} either has the form (i) $\phi\tau$ where $\tau \in G_n$ does not begin with ϕ or (ii) $\psi^{\epsilon}\tau$ where $\epsilon = \pm 1$, $\tau \in G_n$ and τ does not begin with ψ or ψ^{-1}. We shall define, by induction on n, a partition of G_n into three disjoint sets G_n^1, G_n^2, G_n^3. Put $G_0^1 = \{ 1 \}$ $G_0^2 = G_0^3 = \emptyset$. Now suppose that G_n^i (i = 1,2,3) has already been defined, then we define

$$G^1_{n+1} = G_{n+1} \cap (\phi\,(G^2_n \cup G^3_n) \cup \psi^{-1}\, G^2_n \cup \psi\, G^3_n),$$

$$G^2_{n+1} = G_{n+1} \cap (\phi\, G^1_n \cup \psi\, G^1_n \cup \psi^{-1} G^3_n),$$

$$G^3_{n+1} = G_{n+1} \cap (\psi^{-1}\, G^1_n \cup \psi\, G^2_n).$$

The following diagram gives a schematic representation of this:



Thus we have defined a partition of $G = G^1 \dot\cup G^2 \dot\cup G^3$, where $G^i = \dot\bigcup_n G^i_n$ $(i = 1,2,3)$.

We claim that:

(1) $\phi G^1 = G^2 \dot\cup G^3$,

(2) $\psi G^1 = G^2$,

(3) $\psi^{-1} G^1 = G^3$.

From the above definitions we immediately see that if $\sigma \in G^1_n$ then $\phi\,\sigma \in G^2_{n+1}$ if σ does not begin with ϕ; also if $\sigma = \phi\tau$ where $\tau \in G_{n-1}$ then $\phi\sigma = \tau \in G^2_{n-1} \dot\cup G^3_{n-1}$ This shows that

$$\phi G^1_n \subseteq G^2_{n+1} \dot\cup G^2_{n-1} \dot\cup G^3_{n-1}$$

and hence

(4) $\phi G^1 \subseteq G^2 \dot\cup G^3$.

Similarly, $\phi G^2_n \subseteq G^1_{n+1} \dot\cup G^1_{n-1}$, $\phi G^3_n \subseteq G^1_{n-1}$, and so

(5) $\phi(G^2 \dot\cup G^3) \subseteq G^1$.

(1) follows from (4) and (5). Similar arguments give

(6) $\psi G^1_n \subseteq G^2_{n-1} \dot\cup G^2_n \dot\cup G^2_{n+1}$,

(7) $\psi^{-1} G^2_n \subseteq G^1_{n-1} \dot\cup G^1_n \dot\cup G^1_{n+1}$,

(8) $\psi^{-1} G^1_n \subseteq G^3_{n-1} \dot\cup G^3_n \dot\cup G^3_{n+1}$,

(9) $\psi G^3_n \subseteq G^1_{n-1} \dot\cup G^1_n \dot\cup G^1_{n+1}$,

and (2) follows from (6) and (7) and (3) follows from (8) and (9).

It is well-known (a theorem of Euler) that if a solid body is moved from one position to another so that one point O remains fixed, then the transformation is the result of a rotation about some axis through O. Thus each $\sigma \in G \setminus \{1\}$ is a rotation about some diameter of Σ and there are just two diametrically opposite points of Σ which remain fixed under σ. Thus the set

$$D = \{ p \in \Sigma : \sigma(p) = p \text{ for some } \sigma \in G \setminus \{1\} \}$$

is denumerable. For any point $p \in \Sigma \setminus D$ we have $\sigma(p) \neq p$ for all $\sigma \in G \setminus \{1\}$ and hence

$$\sigma_1(p) \neq \sigma_2(p) \text{ if } \sigma_1, \sigma_2 \in G \text{ and } \sigma_1 \neq \sigma_2 .$$

For $p \in \Sigma \setminus D$, let $G(p) = \{ \sigma(p) : \sigma \in G \}$. Then for any two points $p_1, p_2 \in \Sigma \setminus D$ we have that either $G(p_1)$ and $G(p_2)$ are disjoint or they coincide. By the axiom of choice there is a set K which contains exactly one point from each of the different sets $G(p)$ $(p \in \Sigma \setminus D)$. Thus

$$\Sigma \setminus D = \dot{\bigcup_{p \in K}} G(p)$$

and $G(p_1) \cap G(p_2) = \emptyset$ for $p_1, p_2 \in K$ with $p_1 \neq p_2$. Put $\sigma(K) = \{ \sigma(p) : p \in K \}$ for $\sigma \in G$. Then $\sigma_1(K) \cap \sigma_2(K) = \emptyset$ if σ_1, σ_2 are different members of G. For, if $p_1, p_2 \in K$ and $\sigma_1(p_1) = \sigma_2(p_2)$, then $p_2 = \sigma_2^{-1} \sigma_1(p_1)$ and so $p_1 = p_2$ since K contains a single point from the set $G(p_1)$. But this implies that p_1 is a fixed point of $\sigma_2^{-1}\sigma_1$ and hence $\sigma_2 = \sigma_1$ since $p_1 \in K \subseteq \Sigma \setminus D$. Thus the set $\sigma(K)$ are pairwise disjoint for $\sigma \in G$ and

$$\Sigma \setminus D = \dot{\bigcup_{\sigma \in G}} \sigma(K).$$

Now put $A = \dot{\bigcup}_{\sigma \in G^1} \sigma(K)$, $B = \dot{\bigcup}_{\sigma \in G^2} \sigma(K)$, $C = \dot{\bigcup}_{\sigma \in G^3} \sigma(K)$. Then $\Sigma = A \,\dot{\cup}\, B \,\dot{\cup}\, C \,\dot{\cup}\, D$ and in view of the equations (1), (2), (3) we have $\phi(A) = B \,\dot{\cup}\, C$, $\psi(A) = B$ and $\psi^{-1}(A) = C$. This completes the proof.

4. Concluding Remark

Note that, by a rather more complicated argument, R. M. Robinson [2] proved that a solid sphere Ω can be decomposed into 5 pieces (one piece consisting of a single point) and these can be reassembled to form two disjoint equal spheres, and the number 5 is best possible.

References

1. W. Sierpinski, *On the congruence of sets and their equivalence by finite decomposition,* Lucknow University Studies, No. X X (1954).

2. R. M. Robinson, *On the decomposition of Spheres,* Fund. Math. 34 (1947), p 246 – 260.