A seminormal form for partition algebras

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NY Workshop on the Symmetric Group & Related Topics

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Outline

1. Partition algebras and Schur–Weyl duality.

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2. The partition monoid and the diagram presentation for partition algebras.

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Partition algebras

Partition algebras are a family of diagram algebras

$$0 \subseteq \mathcal{A}_{\frac{1}{2}}(n) \subseteq \mathcal{A}_1(n) \subseteq \mathcal{A}_{1+\frac{1}{2}}(n) \subseteq \cdots \qquad (\text{for } n \in \mathbb{Z}_{\geq 0})$$

which arose in the work of P. Martin [Mar] and V.F.R. Jones [Jo] in connection with the Potts model and statistical mechanics.

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Results of Jones [Jo] show that there exist homomorphisms

$$\Phi_k : \mathcal{A}_k(n) \twoheadrightarrow \mathrm{End}_{\mathbb{Q}\mathfrak{S}_n}(V^{\otimes k}) \quad \text{and} \quad \Psi_k : \mathbb{Q}\mathfrak{S}_k \twoheadrightarrow \mathrm{End}_{\mathcal{A}_k(n)}(V^{\otimes k}).$$

As an $(\mathbb{Q}\mathfrak{S}_n, \mathcal{A}_k(n))$–bimodule,

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \hat{A}_k(n)} S^\lambda \otimes A_k(n)^\lambda$$

where

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Martin established an analogous relationship between $\mathcal{A}_{k+\frac{1}{2}}(n)$ and the subalgebra $\mathbb{Q}\mathfrak{S}_{n-1} \subseteq \mathbb{Q}\mathfrak{S}_n$.

The partition monoid

For $k \in \mathbb{Z}_{\geq 0}$, let

$$A_k = \{\text{set partitions of } \{1, 2, \ldots, k, 1', 2', \ldots, k'\}\}, \qquad \text{and,}$$
$$A_{k-\frac{1}{2}} = \{d \in A_k \mid k \text{ and } k' \text{ are in the same block of } d\}.$$

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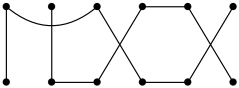
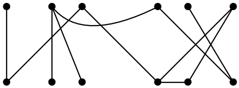
Elements of A_k may be represented diagrammatically. For example,

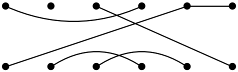
$$\rho = \{\{1, 1', 3, 4', 5', 6\}, \{2, 2', 3', 4, 5, 6'\}\}$$

can be represented as:

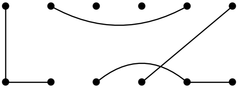
$\rho =$  or $\rho =$ 

The partition monoid

Example. The concatenation product $\rho_1 \circ \rho_2$ of $\rho_1, \rho_2 \in A_6$. Let

$\rho_1 =$ 

and

$\rho_2 =$  .

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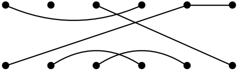
$\rho_1 =$ 

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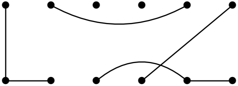
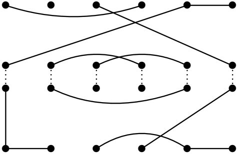
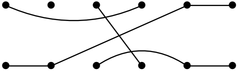
$\rho_2 =$  .

Then

$\rho_1 \circ \rho_2 =$  $=$ 

Partition algebras as diagram algebras

Let z be an indeterminant and $\kappa = \mathbb{Q}(z)$. The partition algebra $\mathcal{A}_k(z)$ is the κ vector space freely generated by A_k, with the product

$$\rho_1\rho_2 = z^{\ell}\rho_1 \circ \rho_2, \qquad \text{for } \rho_1, \rho_2 \in A_k,$$

where ℓ is the number of blocks removed from the middle row in constructing the composition $\rho_1 \circ \rho_2$.

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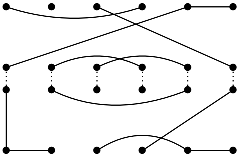
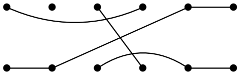


$= z$



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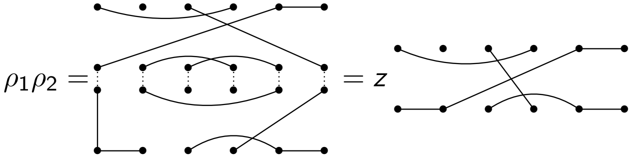
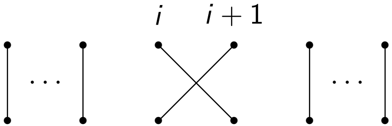
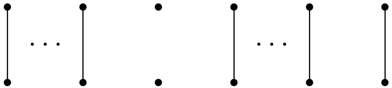


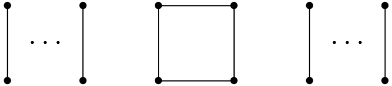
Let $\mathcal{A}_{k-\frac{1}{2}}(z)$ denote the subalgebra of $\mathcal{A}_k(z)$ spanned by $A_{k-\frac{1}{2}}$.

A presentation for $\mathcal{A}_k(z)$ has been given (Halverson and Ram [HR] and East [Ea]) in terms of generators:

$s_i =$ 

$(i = 1, \ldots, k-1)$

$p_i =$ 
$(i = 1, \ldots, k)$,

$p_{i+\frac{1}{2}} =$ 
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Using Schur–Weyl duality, Halverson and Ram [HR] gave a combinatorial definition in terms of the diagram basis of commuting elements $L_0, L_{\frac{1}{2}}, L_1, L_{1+\frac{1}{2}}, \ldots$ that play a role analogous to the Jucys–Murphy elements in the representation theory of the symmetric group:

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- $L_i \in \mathcal{A}_i(z)$ and $L_{i+\frac{1}{2}} \in \mathcal{A}_{i+\frac{1}{2}}(z)$ for $i = 0, 1, \ldots$.

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The definition of Jucys–Murphy elements given by [HR] is not recursive.

Jucys–Murphy elements

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Define $(\sigma_{i+1} : i = 1, 2, \ldots)$ and $(L_i : i = 0, 1, \ldots)$ by

$$L_{i+1} = -s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i s_i + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i s_i + \sigma_{i+1},$$

where, for $i = 2, 3, \ldots,$

$$\sigma_{i+1} = s_{i-1} s_i \sigma_i s_i s_{i-1} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} - s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i.$$

Similarly, define $(\sigma_{i+\frac{1}{2}} : i = 1, 2, \dots)$ and $(L_{i+\frac{1}{2}} : i = 0, 1, \dots)$ by

$$L_{\frac{1}{2}} = 0, \qquad \sigma_{\frac{1}{2}} = 1, \qquad \text{and,} \qquad \sigma_{1+\frac{1}{2}} = 1,$$

and, for $i = 1, 2, \dots,$

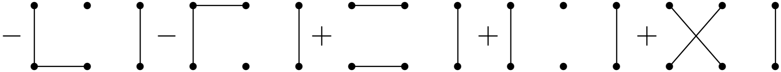
$$L_{i+\frac{1}{2}} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} + s_i L_{i-\frac{1}{2}} s_i + \sigma_{i+\frac{1}{2}},$$

where, for $i = 2, 3, \dots,$

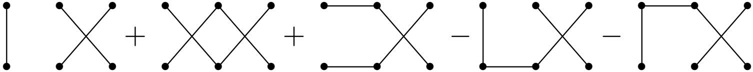
$$\begin{aligned}\sigma_{i+\frac{1}{2}} = s_{i-1} s_i \sigma_{i-\frac{1}{2}} s_i s_{i-1} + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\ - p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} - s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i.\end{aligned}$$

Jucys–Murphy elements

For example,

$L_2 =$ 

and

$\sigma_3 =$ 

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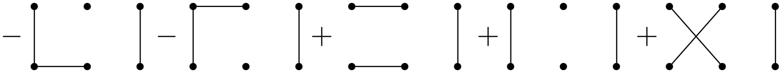
$L_2 =$ 

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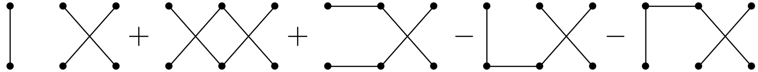
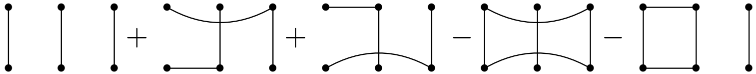
$\sigma_3 =$ 

and

$\sigma_{2+\frac{1}{2}} =$ 

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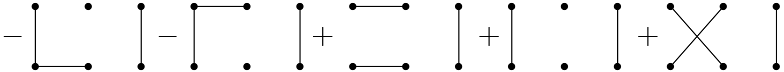
$L_2 =$ 

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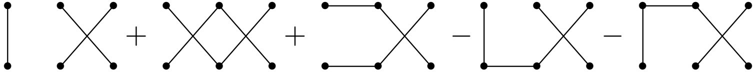
$\sigma_3 =$ 

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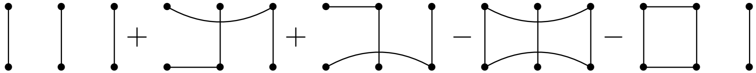
$\sigma_{2+\frac{1}{2}} =$ 

Lemma

For $i = 1, 2, \dots,$

$$\sigma_{i+\frac{1}{2}} s_i = s_i \sigma_{i+\frac{1}{2}} = \sigma_{i+1} \qquad \textit{and} \qquad \sigma_{i+\frac{1}{2}}^2 = \sigma_{i+1}^2 = 1.$$

We therefore obtain a new presentation for the partition algebras:

Theorem

The partition algebra $\mathcal{A}_k(z)$ is presented by the generators

$$p_1, p_{1+\frac{1}{2}}, p_2, \ldots, p_{k-\frac{1}{2}}, p_k, \sigma_2, \sigma_{2+\frac{1}{2}}, \sigma_3, \ldots, \sigma_{k-\frac{1}{2}}, \sigma_k,$$

and the following relations:

Theorem

1. *(Involutions)*
   - a $\sigma_{i+\frac{1}{2}}^2 = 1$, *for* $i = 2, \ldots, k-1$.
   - b $\sigma_{i+1}^2 = 1$, *for* $i = 1, \ldots, k-1$.
2. *(Braid–like relations)*
   - a $\sigma_{i+1}\sigma_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}}\sigma_{i+1}$, *if* $j \neq i+1$.
   - b $\sigma_i\sigma_j = \sigma_j\sigma_i$, *if* $j \neq i+1$.
   - c $\sigma_{i+\frac{1}{2}}\sigma_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}}\sigma_{i+\frac{1}{2}}$, *if* $j \neq i+1$.
   - d $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, *for* $i = 1, \ldots, k-2$, *where*

   $$s_\ell = \begin{cases} \sigma_{\ell+1}, & \text{if } \ell = 1, \\ \sigma_{\ell+\frac{1}{2}}\sigma_{\ell+1}, & \text{if } \ell = 2, \ldots, k-1, \end{cases}$$

   are the Coxeter generators for the symmetric group $\mathfrak{S}_k$.
3. *(Idempotent relations)*
   - a $p_i^2 = z p_i$, *for* $i = 1, \ldots, k$.
   - b $p_{i+\frac{1}{2}}^2 = p_{i+\frac{1}{2}}^2$, *for* $i = 1, \ldots, k-1$.
   - c $\sigma_{i+1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+1} = p_{i+\frac{1}{2}}$, *for* $i = 1, \ldots, k-1$.
   - d $\sigma_{i+\frac{1}{2}} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$, *for* $i = 1, \ldots, k-1$, *. . . , etc.*

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The irreducible representations of $\mathcal{A}_k(z)$ and $\mathcal{A}_{k+\frac{1}{2}}(z)$ are indexed by the same set:

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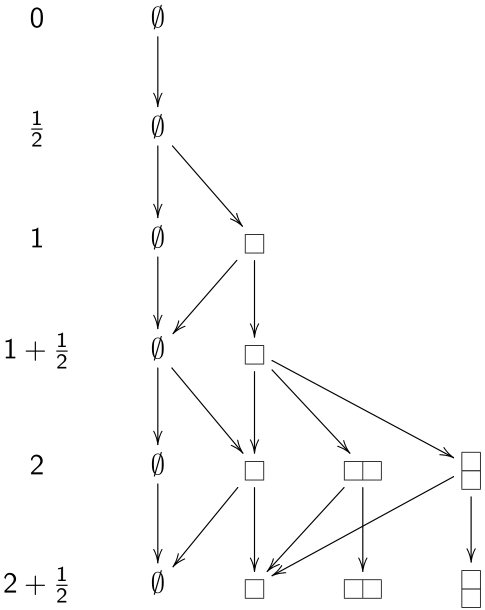
0
$\frac{1}{2}$
1
$1 + \frac{1}{2}$
2
$2 + \frac{1}{2}$
∅
∅
∅
∅
∅
∅

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The elements of $\hat{A}_k^{(\lambda,\ell)}$ are the partition algebra analogues to the standard tableaux in the representation theory of the symmetric group.

Theorem

There exist

$$\left\{ p_{\mathfrak{t}} = p_{\mathfrak{t}}^{(k)} \in \mathcal{A}_k(z) \mid \mathfrak{t} \in \hat{A}_k^{(\lambda,\ell)}, (\lambda, \ell) \in \hat{A}_k \right\},$$

and $f_{(\lambda,\ell)}^{(k)} \in \mathcal{A}_k(z)$, *for* $(\lambda, \ell) \in \hat{A}_k$, *such that*

$$\mathscr{A}_k = \left\{ p_{\mathfrak{t}}^* f_{(\lambda,\ell)}^{(k)} p_{\mathfrak{t}} \,\middle|\, \mathfrak{s}, \mathfrak{t} \in \hat{A}_k^{(\lambda,\ell)},\ (\lambda, \ell) \in \hat{A}_k,\ \textit{and } \ell = 0, 1, \ldots k \right\}$$

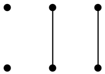
is a Murphy–type cellular basis for $\mathcal{A}_k(z)$.

For $k = 3$, $\lambda = (1,1)$ and $\ell = 1$. The association

$$\mathfrak{t} \mapsto f^{(k)}_{(\lambda,\ell)} p_{\mathfrak{t}}, \qquad\qquad \text{for } \mathfrak{t} \in \hat{A}^{(\lambda,\ell)}_k$$

is as follows:

$$(\emptyset, \emptyset, \emptyset, \emptyset, \Box, \Box, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) \mapsto$$



$= f^{(k)}_{(\lambda,\ell)}$

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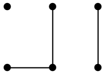

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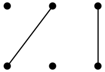

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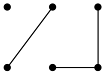

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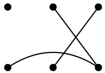

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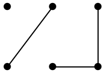

Cell modules

Order the elements of $\hat{A}_k$ by writing $(\lambda, \ell) \rhd (\mu, m)$ if either

1. $\ell > m$, or
2. $\ell = m$ and $\lambda \rhd \mu$ as partitions of $k - \ell$.

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If $(\lambda, \ell) \in \hat{A}_k$, let

$$\mathcal{A}_k^{\rhd(\lambda,\ell)} = \mathrm{Span}\left\{ p_{\mathfrak{s}}^* f_{(\mu,m)}^{(k)} p_{\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \hat{A}_k^{(\mu,m)}, (\mu, m) \rhd (\lambda, \ell) \right\}.$$

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Lemma

If $(\lambda, \ell) \in \hat{A}_k$ and $\mathfrak{t} \in \hat{A}_k^{(\lambda,\ell)}$, then there exist $r_{\mathfrak{u}} \in \kappa$, for $\mathfrak{u} \in \hat{A}_k^{(\lambda,\ell)}$, such that

$$p_{\mathfrak{s}}^* f_{(\lambda,\ell)}^{(k)} p_{\mathfrak{t}} p = \sum_{\mathfrak{u} \in \hat{A}_k^{\rhd(\lambda,\ell)}} r_{\mathfrak{u}} p_{\mathfrak{s}}^* f_{(\lambda,\ell)} p_{\mathfrak{u}} \mod \mathcal{A}_k^{(\lambda,\ell)} \quad \textit{for all } \mathfrak{s} \in \hat{A}_k^{(\lambda,\ell)}.$$

Cell modules

Let $(\lambda, \ell) \in \hat{A}_k$. The cell module $A_k^{(\lambda,\ell)}$ is the right $\mathcal{A}_k(z)$–submodule of $\mathcal{A}_k(z)/A_k^{\rhd(\lambda,\ell)}$ with κ–basis

$$f_{(\lambda,\ell)}^{(k)} p_{\mathfrak{t}} + A_k^{\rhd(\lambda,\ell)}$$

and right $\mathcal{A}_k(z)$ action determined by

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Consequently, we can show that the Jucys–Murphy elements act as triangular operators on $\mathcal{A}_k(z)$.

For $(\lambda,\ell)\in\hat{A}_k$, let $\succ$ denote the *reverse lexicographic order* on $\hat{A}_k^{(\lambda,\ell)}$ induced by the order $\rhd$ on $\hat{A}_k$.

Lemma

If $(\lambda,\ell)\in\hat{A}_k$ and $\mathfrak{t}\in\hat{A}_k^{(\lambda,\ell)}$, then there exist $r_{\mathfrak{u}}\in\kappa$, for $\mathfrak{u}\in\hat{A}_k^{(\lambda,\ell)}$, such that

$$f_{(\lambda,\ell)}p_{\mathfrak{t}}L_i\equiv c_{\mathfrak{t}}(i)f_{(\lambda,\ell)}p_{\mathfrak{t}}+\sum_{\mathfrak{u}\succ\mathfrak{t}}r_{\mathfrak{u}}f_{(\lambda,\ell)}p_{\mathfrak{u}}\mod \mathcal{A}_k^{(\lambda,\ell)}.$$

Similarly there exist $r'_{\mathfrak{u}}\in\kappa$, for $\mathfrak{u}\in\hat{A}_k^{(\lambda,\ell)}$, such that

$$f_{(\lambda,\ell)}p_{\mathfrak{t}}L_{i+\frac{1}{2}}\equiv c_{\mathfrak{t}}(i+\tfrac{1}{2})f_{(\lambda,\ell)}p_{\mathfrak{t}}+\sum_{\mathfrak{u}\succ\mathfrak{t}}r'_{\mathfrak{u}}f_{(\lambda,\ell)}p_{\mathfrak{u}}\mod \mathcal{A}_k^{(\lambda,\ell)}.$$

If $a = (i, j)$ is a node, let $c(a) = j - i$ be the content of a. Let $(\mu, m) \in \hat{A}_k$. For $\mathfrak{t} = (\mu^{(0)}, \mu^{(\frac{1}{2})}, \ldots, \mu^{(k)}) \in \hat{A}_k^{(\mu,m)}$, and $i = 1, \ldots, k$, define

$$c_{\mathfrak{t}}(i - \tfrac{1}{2}) = \begin{cases} |\mu^{(i-1)}|, & \text{if } \mu^{(i-1)} = \mu^{(i-\frac{1}{2})}; \\ z - c(a), & \text{if } \mu^{(i-\frac{1}{2})} = \mu^{(i-1)} \setminus \{a\}, \text{ and} \end{cases}$$

$$c_{\mathfrak{t}}(i) = \begin{cases} z - |\mu^{(i)}|, & \text{if } \mu^{(i)} = \mu^{(i-\frac{1}{2})}, \\ c(a), & \text{if } \mu^{(i)} = \mu^{(i-\frac{1}{2})} \cup \{a\}. \end{cases}$$

A seminormal basis

The triangular action of the Jucys–Murphy elements on the Murphy basis for $\mathcal{A}_k(z)$ allows us to diagonalize the Jucys–Murphy elements and obtain a seminormal basis for $\mathcal{A}_k(z)$ and each cell module for $\mathcal{A}_k(z)$.

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Let

$$\left\{ f_{\mathfrak{t}}^{(k)} \mid \mathfrak{t} \in \hat{A}_k^{(\lambda,\ell)} \right\}$$

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Let $\sigma_{i+\frac{1}{2}}(\mathfrak{s},\mathfrak{t})$ denote the $(\mathfrak{s},\mathfrak{t})$–entry of the matrix of representing $\sigma_{i+\frac{1}{2}}$ in the seminormal representation.

Let $(\lambda, \ell) \in \hat{A}_k$ and

$$\mathfrak{t} = (\mathfrak{t}^{(0)}, \mathfrak{t}^{(\frac{1}{2})}, \ldots, \mathfrak{t}^{(k-\frac{1}{2})}, \mathfrak{t}^{(k)}) \in \hat{A}_k^{(\lambda,\ell)}.$$

Theorem

If $\mathfrak{t}^{(i-1)} \neq \mathfrak{t}^{(i)}$ and $\mathfrak{t}^{(i-\frac{1}{2})} \neq \mathfrak{t}^{(i+\frac{1}{2})}$, and $\mathfrak{t}\sigma_{i+\frac{1}{2}}$ exists, then

$$\sigma_{i+\frac{1}{2}}(\mathfrak{s}, \mathfrak{t}) = \begin{cases} \dfrac{1}{c_\mathfrak{t}(i+\frac{1}{2}) - c_\mathfrak{t}(i-\frac{1}{2})}, & \text{if } \mathfrak{s} = \mathfrak{t}, \\ 1 - \dfrac{1}{(c_\mathfrak{t}(i+\frac{1}{2}) - c_\mathfrak{t}(i-\frac{1}{2}))^2}, & \text{if } \mathfrak{s} = \mathfrak{t}\sigma_{i+\frac{1}{2}} \text{ and } \mathfrak{s} \succ \mathfrak{t}, \\ 1, & \text{if } \mathfrak{s} = \mathfrak{t}\sigma_{i+\frac{1}{2}} \text{ and } \mathfrak{t} \succ \mathfrak{s}, \\ 0, & \text{otherwise.} \end{cases}$$

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If $\mathfrak{t}^{(i-\frac{1}{2})} \neq \mathfrak{t}^{(i+\frac{1}{2})}$ *and* $\mathfrak{t}^{(i)} \neq \mathfrak{t}^{(i+1)}$, *and* $\mathfrak{t}\sigma_{i+1}$ *exists, then*

$$\sigma_{i+1}(\mathfrak{s},\mathfrak{t}) = \begin{cases} \dfrac{1}{c_{\mathfrak{t}}(i+1) - c_{\mathfrak{t}}(i)}, & \text{if } \mathfrak{s} = \mathfrak{t}, \\ 1 - \dfrac{1}{(c_{\mathfrak{t}}(i+1) - c_{\mathfrak{t}}(i))^2}, & \text{if } \mathfrak{s} = \mathfrak{t}\sigma_{i+1} \text{ and } \mathfrak{s} \succ \mathfrak{t}, \\ 1, & \text{if } \mathfrak{s} = \mathfrak{t}\sigma_{i+1} \text{ and } \mathfrak{t} \succ \mathfrak{s}, \\ 0, & \text{otherwise.} \end{cases}$$

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- We have explicit combinatorial expressions for all the terms $\sigma_{i+\frac{1}{2}}(\mathfrak{s},\mathfrak{t})$, $\sigma_{i+1}(\mathfrak{s},\mathfrak{t})$, $p_{i+\frac{1}{2}}(\mathfrak{s},\mathfrak{t})$ and $p_{i+1}(\mathfrak{s},\mathfrak{t})$.

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- Complete statements of these results can be found in the preprints arXiv:1009.1939 and arXiv:1102.2047.

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