# Logicism: Frege

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Logicism says that mathematics is just logic. It purports to explain how we know mathematics.

## 1. Frege's logicism

### 1.1 The need for definitions of general arithmetic concepts

Leibniz and Mill's approaches only applied to sums of particular numbers. To account for general claims about all numbers, logical definitions of mathematical concepts are needed.

Mathematicians tell us that:

> '$m$ is prime' means that $m > 1$, and the only natural numbers that evenly divide $m$ are 1 and $m$ itself
>
> '$m > n$' means that for some natural number, $p \neq 0$, $m = n + p$
>
> '$m$ is divisible by $n$' means that $n \neq 0$, and there exists some natural number $p$ such that $m = n \times p$

but not what 'natural number', '0', '×, and '+' mean.

### 1.2 Number-of, and Hume's Principle

Frege's crucial phrase:

> *number of*

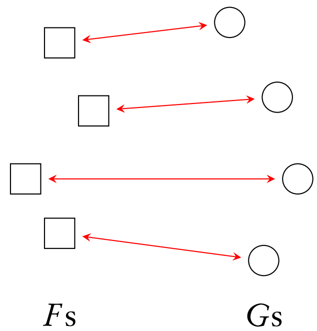
as in "the number of fingers on my right hand = 5". Frege's crucial principle:

**Hume's Principle** For any concepts $F$ and $G$, the number of $F$s = the number of $G$s if and only if $F$ is equinumerous to $G$.

'Equinumerous' does not mean having the same number; it is defined logically:

> If a waiter wishes to be certain of laying exactly as many knives on a table as plates, he has no need to count either of them; all he has to do is to lay immediately to the right of every plate a knife, taking care that every knife on the table lies immediately to the right of a plate. Plates and knives are thus correlated one to one…(Frege, 1884, pp. 81–2).

The instances of equinumerous concepts can be “correlated one-to-one”:




**Official definition of equinumerosity** $F$ is equinumerous to $G$ if and only if for some $R$, each $F$ bears $R$ to exactly one $G$, and for each $G$ there is exactly one $F$ that bears $R$ to it. In symbols:

$$\exists R\Big(\forall x\Big(Fx \rightarrow \exists y\Big(Gy \,\&\, Rxy \,\&\, \forall z\big((Gy \,\&\, Rxz) \rightarrow y = z\big)\Big)\Big) \,\&$$

$$\forall x\Big(Gx \rightarrow \exists y\Big(Fy \,\&\, Ryx \,\&\, \forall z\big((Gy \,\&\, Rzx) \rightarrow y = z\big)\Big)\Big)\Big)$$

## 1.3 Definitions of arithmetic concepts

### 1.3.1 Numerals

‘0’ means: ‘the number of things that are not self-identical’

‘1’ means ‘the number of things that are identical to 0’

‘2’ means ‘the number of things that are either identical to 0 or identical to 1’

### 1.3.2 Successor

‘$n$ is a successor of $m$’ means that for some $F$ and some $x$:

i) $Fx$

ii) $n =$ the number of $F$s, and

iii) $m =$ the number of $F$s-that-are-not-identical-to-$x$.

Proof that 1 is a successor of 0:

1. 0 = the number of things that are not self-identical (def of '0')
2. 1 = the number of things that are identical to 0 (def of '1')
3. 0 = 0 (logic)
4. The concept thing-that-is-not-self-identical is equinumerous with the concept thing-that-is-identical-to-0-and-not-identical-to-0 (logic)
5. The number of things that are not self-identical = the number of things that are identical to 0 and not identical to 0 (iv, Hume's Principle)
6. 0 = the number of things that are identical to 0 and not identical to 0 (i, v, logic)
7. 0 = 0, and 1 = the number of things that are identical to 0, and 0 = the number of things that are identical to 0 and not identical to 0 (iii, ii, vi, logic)
8. For some $F$ and for some $x$: $Fx$, and 1 = the number of $F$s, and 0 = the number of things that are $F$ and not identical to $x$ (vii, logic)
9. 1 is a successor of 0 (viii, def of 'successor')

### 1.3.3 Natural number

'$n$ is a natural number' means that for any $F$, if 0 has $F$, and if whenever some $m$ has $F$ so does every successor of $m$, then $n$ is $F$.

### 1.3.4 Definitions of other concepts

Frege gives "recursive" definitions of other arithmetic concepts. Here are two equations governing addition:

$$m + 0 = m$$
$$m + \text{succ}(n) = \text{succ}(m + n)$$

The first specifies the result of adding 0; the second specifies the result of adding the successor of $n$, in terms of the result of adding $n$. Frege shows how to use second-order logic to transform these equations into a definition of '+'.

### 1.4 Proving the axioms of arithmetic

Frege proved, using his logic, that all of the axioms of arithmetic (the second order Peano axioms) follow from his definitions plus Hume's Principle.

### 1.5 Defining number-of

But what about Hume's Principle?

Could it be a definition of 'number of'?

No, Frege says. It only defines sentences of the form "the number of $F$s = the number of $G$s", and tells us nothing about, e.g., "the number of fish in the ocean = Julius Caesar".

Instead, Frege defined 'number of' as follows:

> 'The number of $F$s' means 'the extension of the concept: *being a concept, G, such that G is equinumerous to F*'

The *extension* of a concept is basically the set of things falling under the concept. Frege regarded extensions as logical objects, and included the following principle governing them in his logic:

**Frege's Basic Law V** The extension of $F$ = the extension of $G$ if and only if: for any object $x$, $x$ has $F$ if and only if $x$ has $G$

Proof of Hume's Principle:

> The number of $F$s = the number of $G$s …
>
> …if and only if the extension of *being equinumerous to F* = the extension of *being equinumerous to G* (definition of 'number of')
>
> …if and only if for any concept, $H$, $F$ is equinumerous to $H$ if and only if $G$ is equinumerous to $H$ (Basic Law V)
>
> …if and only if $F$ and $G$ are equinumerous (equinumerosity is reflexive and symmetric)

### 1.6 Russell's objection

Consider this concept $R$:

$R =$ the concept of *being the extension of some concept that is not possessed by that extension*

Thus:

(*) $x$ has $R$ if and only if for some concept $F$, $x =$ the extension of $F$ and $x$ does not have $F$

Now let $r =$ the extension of $R$. Does $r$ have $R$?

1. Suppose that $r$ *does* have $R$.
   (a) Then for some $F$, $r =$ the extension of $F$ and $r$ does not have $F$ (2, *)
   (b) For any $y$, $y$ has $R$ if and only if $y$ has $F$ (1, 2a, Basic Law V)
   (c) $y$ has $F$. Contradiction. (2, 2b)
2. Suppose instead that $r$ does not have $R$.
   (a) $r =$ the extension of $R$ and $r$ does not have $R$ (1, 3)
   (b) For some concept $F$ (namely: $R$), $r =$ the extension of $F$ and $r$ does not have $F$ (3a)
   (c) $r$ has $R$. Contradiction again! (3b, *)

So the basis for Frege's entire system is contradictory! Frege's reply to Russell:

> Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic... [The matter is] all the more serious since, with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish... In any case your discovery is very remarkable and will perhaps result in a great advance in logic, unwelcome as it may seem at first glance. (van Heijenoort, 1967, pp. 127–8)

## References

Frege, Gottlob (1884). *The Foundations of Arithmetic*. 2nd edition. Oxford: Blackwell, 1953. Translated by J. L. Austin.

van Heijenoort, J. (1967). *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*. Harvard University Press.